An Efficiency Theorem for Incompletely Known Preferences

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Abstract

There are n agents who have von Neumann-Morgenstern utility functions on a finite set of alternatives A. Each agent i's utility function is known to lie in the nonempty, convex, relatively open set U_i. Suppose L is a lottery on A that is undominated, meaning that there is no other lottery that is guaranteed to Pareto dominate L no matter what the true utility functions are. Then, there exist utility functions $u_i \in U_i$ for which L is Pareto efficient. This result includes the ordinal efficiency welfare theorem as a special case.

Keywords: Dominance; ordinal efficiency welfare theorem; Pareto efficiency; unknown preferences

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1 Introduction

The purpose of this note is to prove an theorem on efficiency when agents' preferences are not perfectly known. Loosely put, the result says that when agents have expected-utility preferences over lotteries, any lottery that is not guaranteed to be Pareto dominated by some specific other lottery may in fact be Pareto efficient. To help provide context, we begin by discussing a special case, the ordinal efficiency welfare theorem.

Let A be a finite set of possible policies, and suppose each of n agents, $i = 1, \ldots, n$, has a known preference ordering $\succsim_i$ over these policies, where indifferences are allowed. Given two lotteries L, L' over A, we say that L' *ordinally dominates* L if L' first-order stochastically dominates L with respect to every agent's preference ordering, with strict domination for at least one agent. That is, writing $L(a)$ for the probability of a under lottery L, we say that L' ordinally dominates L if, for every agent i and policy $a \in A$,

$$\sum_{b \succsim_i a} L'(b) \geq \sum_{b \succsim_i a} L(b)$$

and the inequality is strict for at least one choice of i, a. Equivalently, for every choice of von Neumann-Morgenstern utilities for the agents consistent with the preference orderings $\succsim_i$, L' Pareto dominates L (in ex ante expected utility). We say that L is *ordinally efficient* if it is not ordinally dominated by any other lottery. There has been much interest in ordinal efficiency in the particular context of the random assignment problem, in which there are n indivisible objects to be given out to the n agents, and the set A consists of all $n!$ possible assignments of objects to agents, e.g. [5, 2, 9]. However, the definition applies much more generally.

The *ordinal efficiency welfare theorem* says that if L is ordinally efficient, then there exists some choice of von Neumann-Morgenstern utility functions, consistent with the preference orderings, for which L is Pareto efficient. This result was first proved by McLennan [10] for the random assignment problem (though his proofs easily extend to the general setting). Manea [8] gave another proof specific to the random assignment problem.

The ordinal efficiency welfare theorem can be interpreted as follows. Suppose a social planner knows that the agents evaluate lotteries according to expected utilities, but she does not know the agents' utility functions; she only knows each agent's (ordinal) ranking of the alternatives in A. A lottery L is ordinally dominated by L' if L' is guaranteed to Pareto dominate L no matter what the true utility functions are. The theorem says that, if L is ordinally efficient (not dominated by any L'), then it may in fact be Pareto efficient.

In this interpretation, each agent i has a set U_i of "plausible" utility functions, namely those functions that are consistent with the ordering $\succsim_i$. Our result generalizes this theorem by allowing freer specifications for the planner's knowledge about agents' preferences: We allow each U_i to be any nonempty, convex, relatively open set. The result will again be that, if there is no L' that is guaranteed to Pareto dominate L for every choice of utility functions $u_i \in U_i$, then there exist utility functions $u_i \in U_i$ such that L is Pareto efficient. This provides a justification for using undominatedness as a natural efficiency criterion when preferences are incompletely known.

There are various situations where one might reasonably suppose that the planner's incomplete knowledge about an agent's utility function consists of something other than the agent's ordinal ranking of alternatives. It could happen that even the ordinal ranking is not fully known; for example, the planner might only know the agent's top k choices (as in the school choice mechanisms used by some school districts [1, 7, 12]). On the other hand, some cardinal information might be available. For example, the agents may have expressed preferences by making some pairwise choices between lotteries. Or there might be a priori reasons to impose other restrictions on the plausible utility functions: for instance, when allocating multiple objects to agents, where some of the objects are identical, one might assume that agents have decreasing marginal utility for identical goods.

The proof of our theorem uses the same separating-hyperplane machinery as in McLennan.

2 The result

Now let's formally state our result. Let A be a finite set, whose elements are called *policies*. A utility function is a function $u : A \to \mathbb{R}$. Evidently the space of possible utility functions can be identified with $\mathbb{R}^{|A|}$, so it makes sense, for example, to talk about convex sets of utility functions. Let $\mathcal{L}(A)$ be the set of all lotteries (probability distributions) on A. If u is a utility function and L a lottery on A, we can naturally define $u(L) = \sum_{a \in A} L(a)u(a)$, where $L(a)$ is the probability assigned to a by L.

For each of n agents, $i = 1, \ldots, n$, let U_i be a set of utility functions. We require that U_i be nonempty, convex, and relatively open. (A subset of Euclidean space is *relatively open* if it is open in the relative topology of an affine subspace containing it.) We say that a lottery L' *dominates* lottery L (with respect to the U_i) if, for every choice of a utility function $u_i \in U_i$ for each i, we have $u_i(L') \geq u_i(L)$ for all i, with strict inequality for some i. [1]

Theorem 1 *Let A be a finite set, and $U_1, \ldots, U_n$ nonempty, convex, relatively open sets of utility functions on A. Suppose L is a lottery on A that is not dominated by any lottery L'. Then, there exist functions $u_i \in U_i$ and positive weights λ_i such that*

$$L \in \operatorname{argmax}_{\tilde{L} \in \mathcal{L}(A)} \sum_{i=1}^{n} \lambda_i u_i(\tilde{L}),$$

that is, L is Pareto efficient in $\mathcal{L}(A)$.[2]

There is also a converse, which is trivial: if there exist functions $u_i \in U_i$ and weights λ_i for which L maximizes the sum $\sum_i \lambda_i u_i$, then L is not dominated by any other lottery L'.

Once again, we point out that the ordinal efficiency welfare theorem, as stated in the introduction, is a special case of Theorem 1. Suppose that for each agent i, we are given a preference relation $\succsim_i$, a weak ordering on A. Let U_i be the set of utility functions consistent with $\succsim_i$: $U_i = \{u_i : A \to \mathbb{R} \mid u_i(a) \geq u_i(b) \Leftrightarrow a \succsim_i b\}$. One easily checks that U_i is nonempty and convex. U_i is also relatively open, since it is an open subset of the affine (in fact, linear) space $\{u_i : A \to \mathbb{R} \mid u_i(a) = u_i(b) \text{ whenever } a \sim_i b\}$. It is straightforward to check that $u_i(L') \geq u_i(L)$ for all $u_i \in U_i$ if and only if L' weakly stochastically dominates L with respect to the ordering $\succsim_i$, and $u_i(L') > u_i(L)$ for all u_i if and only if L' strictly stochastically dominates L with respect to $\succsim_i$. Therefore, L' dominates L (with respect to all the sets U_i) if and only if it ordinally dominates L with respect to the preference orders $\succsim_i$. Consequently, we can apply Theorem 1 in this context, and we recover the ordinal efficiency welfare theorem.

Threre are some interesting connections between Theorem 1 and the existing literature on multi-utility representations of incomplete preferences over lotteries. We discuss these connections in the concluding section.

[1] Or, equivalently: for all i and all $u_i \in U_i$, we have $u_i(L') \geq u_i(L)$; and there is some i such that for all $u_i \in U_i$, $u_i(L') > u_i(L)$.

[2] The referee points out that this form of optimality is properly credited to Negishi [11]. In our context, Negishi optimality for some positive weights λ_i is equivalent to Pareto efficiency if the true utility functions are indeed the u_i.

3 The proof

We need a little bit of geometric machinery, for which we largely follow the exposition in McLennan [10]. For sets $S, T \subseteq \mathbb{R}^d$, let $S + T = \{s + t \mid s \in S, t \in T\}$.

A *polyhedron* in $\mathbb{R}^d$ is a subset of $\mathbb{R}^d$ that can be represented as the intersection of finitely many closed half-spaces. A *polytope* is the convex hull of a finite set of points in $\mathbb{R}^d$. A *polyhedral cone* is a set of the form $\{a_1x_1 + \cdots + a_rx_r \mid a_i \geq 0\}$, where $x_1, \ldots, x_r$ are some fixed elements of $\mathbb{R}^d$. (Taking $r = 0$, the set $\{0\}$ is a polyhedral cone.) The "main theorem" on polyhedra (credited to Motzkin in [13]) states:

Theorem 2 *[13, p. 30] A nonempty set $P \subseteq \mathbb{R}^d$ is a polyhedron if and only if it is of the form $Q + C$, where Q is a polytope and C is a polyhedral cone.*

If $S \subseteq \mathbb{R}^d$, the *affine hull* of S is the set of all affine combinations of elements of S:

$$\mathrm{aff}(S) = \{a_1x_1 + \cdots + a_rx_r \mid x_i \in S;\ a_1 + \cdots + a_r = 1\}.$$

Equivalently, this is the smallest affine space containing S. For any convex $S \subseteq \mathbb{R}^d$, the *relative interior* of S is the interior of S in the relative topology of $\mathrm{aff}(S)$. S is relatively open if and only if it equals its own relative interior.

We will need a strengthened version of the separating hyperplane theorem for polyhedra:

Lemma 3 *Let P be a polyhedron and $x \in P$. Suppose x is not in the relative interior of P. Then there is a hyperplane H containing x, such that P is contained in one of the closed half-spaces bounded by H, and moreover x lies in the relative interior of $P \cap H$.*

This follows immediately from Lemma 2 and Theorem 2 of McLennan [10]. (It can also be proven using classical facts about the face lattice of a polytope, see [13, pp. 51-61].)

We will use this result in a way similar to the second proof of the ordinal efficiency welfare theorem in [10] (which is credited to Zhou). We will also need a simple lemma on relatively open sets:

Lemma 4 *(a) If $U \subseteq \mathbb{R}^d$ is relatively open, then the set $U^+ = \{\lambda u \mid \lambda > 0, u \in U\}$ is also relatively open. If U is convex, then so is U^+.*

(b) If $U_1, \ldots, U_n \subseteq \mathbb{R}^d$ are relatively open, then $U_1 + \cdots + U_n$ is also relatively open. If $U_1, \ldots, U_n$ are convex, then so is $U_1 + \cdots + U_n$.

The proof of this lemma is straightforward and is omitted.

The essence of the proof of Theorem 1 lies in the following geometric result. Here the symbol $\cdot$ denotes the usual Euclidean inner product.

Lemma 5 *Let $U, V \subseteq \mathbb{R}^d$ be nonempty convex sets such that U is relatively open and V is a polyhedron. Let $v_0 \in V$. Suppose that for every $v \in V$, there exists $u \in U$ such that $u \cdot (v - v_0) \leq 0$. Then, there exists a single $u \in U$ such that $u \cdot (v - v_0) \leq 0$ for all $v \in V$.*

Before proving this result, we will first show how it is used to prove the main theorem. Notice that we can think of lotteries on A and utility functions on A both as elements of $\mathbb{R}^{|A|}$, in which case the expected utility $u(L)$ equals the inner product $u \cdot L$.

Proof of Theorem 1: Let $V = \mathcal{L}(A)$, the set of lotteries over A. V can be naturally represented as a polytope in $\mathbb{R}^{|A|}$, so is a polyhedron by Theorem 2. Let $U \subseteq \mathbb{R}^{|A|}$ be the set of all Pareto-weighted sums of plausible utility functions:

$$U = \{\lambda_1 u_1 + \cdots + \lambda_n u_n \mid \lambda_i > 0;\ u_i \in U_i\}.$$

U is nonempty because each U_i is. And each U_i is convex and relatively open, so by Lemma 4, so is U.

Take $v_0 \in V$ to be the lottery L that is assumed undominated. For every $v \in V$, the fact that v does not dominate v_0 means that there are utility functions $u_i \in U_i$ such that either $u_i \cdot v_0 > u_i \cdot v$ for some i, or $u_i \cdot v_0 = u_i \cdot v$ for all i. In the former case, choose arbitrary $u_j \in U_j$ for $j \neq i$ and let $u = \sum_j \lambda_j u_j$ where λ_i is sufficiently large relative to all the other λ_j; then $u \cdot v_0 > u \cdot v$. In the latter case, let $u = \sum_j u_j$ and we have $u \cdot v_0 = u \cdot v$. Either way, we have $u \in U$ such that $u \cdot (v - v_0) \leq 0$.

So we have verified all the hypotheses of Lemma 5. Therefore, there exists some $u \in U$ such that $u \cdot v \leq u \cdot v_0$ for all $v \in V$. Expressing things back in terms of utility functions, with $u = \sum_i \lambda_i u_i$, this says exactly that $\sum_i \lambda_i u_i(L') \leq \sum_i \lambda_i u_i(L)$ for all lotteries L' — the desired result. □

It remains to prove Lemma 5.

The idea behind the proof is as follows. We can think of U as a set of linear utility functions on $\mathbb{R}^d$. Let W be the set of utility functions for which v_0 is optimal in V. We wish to show that U and W intersect. If they fail to intersect, we can use the separating hyperplane theorem to find a direction x that provides decreasing utility under every function in U but increasing utility under every function in W. The former property implies that starting at v_0 and walking along the ray with direction $-x$ takes us outside of V. But then we can use another hyperplane separating V from this ray to produce a utility function $y \in W$ for which x is a direction of decreasing utility — contradicting the construction of x.

Proof of Lemma 5: By translating V, we may assume that $v_0 = 0$, so we will do so henceforth. We first prove the lemma under the extra assumption that U is not contained in any proper linear subspace of $\mathbb{R}^d$.

Let $W = \{u \mid u \cdot v \leq 0 \text{ for all } v \in V\}$. Note that W is nonempty, because it contains 0, and it is convex. Our goal is to show that $U \cap W$ is nonempty.

Suppose $U \cap W$ is empty. By the usual separating hyperplane theorem, there is a hyperplane separating U from W — that is, there exists some nonzero vector $x \in \mathbb{R}^d$ and some constant c such that $u \cdot x \leq c$ for all $u \in U$, and $u \cdot x \geq c$ for all $u \in W$. From the facts that W is invariant under positive scalar multiplication and $0 \in W$, it follows that we may take $c = 0$.

Now we consider two cases, depending on whether or not V contains any negative multiple of x.

- Suppose $ax \notin V$ for all real $a < 0$. Let $V' = V + \{ax \mid a \geq 0\}$. Then V' is a polyhedron. (Proof: By Theorem 2, $V = Q + C$ where Q is a polytope and C is a polyhedral cone. Then $C' = C + \{ax \mid a \geq 0\}$ is again a polyhedral cone, and applying Theorem 2 again gives that $V' = Q + C'$ is a polyhedron.) And $0 \in V \subseteq V'$. Moreover, the affine hull of V' contains the line $L = \{ax \mid a \in \mathbb{R}\}$, and V' does not contain any neighborhood of 0 in L (because, by assumption, V contains no negative multiple of x). So, 0 is not in the relative interior of V'.

  Therefore, by Lemma 3, there is some hyperplane H passing through 0 such that V' lies on one side of H, and 0 lies in the relative interior of $V' \cap H$. This implies $x \notin H$: otherwise 0 could not be in the relative interior of $V' \cap H$, by the same reasoning as in the preceding paragraph. Now let y be a vector perpendicular to H, signed so that $v \cdot y \leq 0$ for $v \in V'$. In particular, $x \in V'$ and $x \notin H$ means that in fact $x \cdot y < 0$. On the other hand, we know that for all $v \in V \subseteq V'$, $v \cdot y \leq 0$, so $y \in W$. This implies $x \cdot y \geq 0$, by the construction of x. So we have $x \cdot y < 0$ and $x \cdot y \geq 0$ — a contradiction.

- Suppose there does exist $a < 0$ with $ax \in V$. Now, the hypotheses of the lemma imply that there exists $u \in U$ with $u \cdot ax \leq 0$, therefore $u \cdot x \geq 0$. But we also know $u \cdot x \leq 0$ since $u \in U$; therefore $u \cdot x = 0$.

By assumption, U is not contained in the hyperplane through 0 perpendicular to x, so $\operatorname{aff}(U)$ contains some vector u' with $u' \cdot x \neq 0$. Since U is relatively open, $u + \epsilon(u' - u) \in U$ as long as $|\epsilon|$ is sufficiently small. But by choosing ϵ small and of the appropriate sign so that $\epsilon u' \cdot x > 0$, we get $(u + \epsilon(u' - u)) \cdot x > 0$. This contradicts the fact that every element of U has nonpositive inner product with x.

So in both cases, we get a contradiction. This completes the proof of Lemma 5 under the assumption that U is not contained in any proper linear subspace of $\mathbb{R}^d$.

For the general case, let S be the linear span of U, let $\pi : \mathbb{R}^d \to S$ be the orthogonal projection, and let $\overline{V} = \pi(V)$. By Theorem 2, write $V = Q + C$ where Q is a polytope and C a polyhedral cone; one easily sees that $\pi(Q)$ is a polytope and $\pi(C)$ is a polyhedral cone, so $\overline{V} = \pi(Q) + \pi(C)$ is again a polyhedron. For any $\overline{v} \in \overline{V}$, choose v with $\pi(v) = \overline{v}$. There exists $u \in U$ with $u \cdot v \leq 0$. But the definition of π gives $u \cdot (v - \pi(v)) = 0$, so $u \cdot \overline{v} = u \cdot v \leq 0$. It follows that the hypotheses of the lemma hold with $U, V \subseteq \mathbb{R}^d$ replaced by $U, \overline{V} \subseteq S$. And now U is not contained in any proper linear subspace of S, so the case we have already proven applies: there exists $u \in U$ such that $u \cdot \overline{v} \leq 0$ for all $\overline{v} \in \overline{V}$. Therefore, for any $v \in V$, we have $u \cdot v = u \cdot \pi(v) \leq 0$, as desired.

□

4 Comments

None of the hypotheses of Theorem 1 is dispensable. To see that the convexity assumption is required, let $A = \{a, b\}$, and consider just one agent, with $U_1 = \{u_1 \mid u_1(a) \neq u_1(b)\}$. Then the lottery $L = (1/2, 1/2)$ placing probability 1/2 on each of a and b is undominated — for any $L' \neq L$, L is preferred by a utility function that favors the policy that is less likely under L'. But there is no one utility function in U_1 for which L is optimal.

To see that relative openness is required, let $A = \{a, b, c\}$, and consider one agent with

$$U_1 = \{u_1 \mid u_1(a) < u_1(b) \qquad \text{or} \qquad u_1(a) = u_1(b) < u_1(c)\}.$$

This set is convex, but not relatively open. To see that the conclusion of Theorem 1 fails in this case, check that the lottery $L = (1, 0, 0)$ is undominated. Indeed, consider any $L' \neq L$. If L' places positive probability on c, then by choosing u_1 with $u_1(a) < u_1(b)$ and $u_1(c)$ sufficiently low, we get $u_1(L') < u_1(L)$. And if L' places zero probability on c, then we can choose u_1 with $u_1(a) = u_1(b)$, which ensures indifference between L and L'. Thus, in neither case can L' dominate L. However, for any choice of $u_1 \in U_1$ there is some lottery L' which is preferred to L: if $u_1(a) < u_1(b)$ then take $L' = (0, 1, 0)$, and if $u_1(a) = u_1(b) < u_1(c)$ then take $L' = (0, 0, 1)$.

It is possible to drop the relative openness hypothesis of Theorem 1 by strengthening the non-dominance hypothesis. Namely, suppose each set U_i is merely nonempty and convex, and for each lottery $L' \neq L$ there exist some i and $u_i \in U_i$ with $u_i(L') < u_i(L)$. Then the conclusion of Theorem 1 holds. The proof is almost the same as that we have given: in the proof of (the analogue of) Lemma 5, in the case where V contains ax for some $a < 0$, the hypothesis implies that U contains some u with $u \cdot ax < 0$, and the contradiction with $u \cdot x \leq 0$ is reached immediately.

The hypothesis that A is finite is also necessary, since the application of Lemma 5 requires the hypothesis that V is a polyhedron. For a counterexample with infinitely many policies, let A be the interval $[0, 1] \subseteq \mathbb{R}$, and consider one agent with

$$U_1 = \{u^{a,b} \mid a, b > 0\} \qquad \text{where} \qquad u^{a,b}(x) = ax - bx^2.$$

We can take $\mathcal{L}(A)$ to be (say) the space of all Borel probability measures on A. The lottery L putting probability 1 on the policy 0 is undominated: For any other lottery L', we have $E_{L'}[x], E_{L'}[x^2] > 0$, where $E_{L'}$ denotes expectation with respect to L'. So there exist $a, b > 0$ with $E_{L'}[ax - bx^2] < 0$, showing that L' does not dominate L. However, for any choice of $u_1 = u^{a,b}$, we can consider the lottery L' which puts probability 1 on some x with $0 < x < a/b$, and then u_1 prefers L' over L.

As mentioned above, Theorem 1 is related to the literature on representing incomplete preferences over lotteries, pioneered by Aumann [3, 4] and recently placed on firm foundations by Dubra, Maccheroni, and Ok [6]. We have here taken the sets of utility functions as primitive; by contrast, that literature takes incomplete preference relations over lotteries as primitive and derives utility functions from them. In particular, [6] shows that a reflexive, transitive (but not necessarily complete) preference relation $\succsim$ on $\mathcal{L}(A)$, satisfying independence and continuity axioms, can be represented by a *set* of utility functions on A, such that $L' \succsim L$ if and only if $u(L') \geq u(L)$ for every u in the set; and this representation is unique up to appropriately defined equivalence. The connections with our work are twofold.

First, as suggested by the referee, an alternative proof of Theorem 1 can be obtained via these results. Define a preference relation $\succsim$ on $\mathcal{L}(A)$ by $L' \succsim L$ if either L' dominates L, or $u_i(L') = u_i(L)$ for all i and $u_i \in U_i$. Theorem B of [4] shows that if L is undominated, then there is a linear utility function u on $\mathcal{L}$ such that L maximizes u. By applying the uniqueness theorem of [6], one can then show that u is representable in the form $\sum_i \lambda_i u_i$, although a little extra work is needed to show that the λ_i are strictly positive. This proof is ultimately not so different from ours, since the results of [4] and [6] are proven using similar separation arguments. We have chosen not to present this alternative proof here in favor of a more self-contained approach, but the details are available from the author.

Second, our Theorem 1 can be easily translated into a context where preference orders are taken as the primitives. Suppose each agent i has an (incomplete) preference relation $\succsim_i$ on $\mathcal{L}(A)$ satisfying the independence and continuity axioms, and suppose L is a lottery that is not dominated with respect to these preference relations. Applying the results of [6], we can find a maximal set $\overline{U}_i$ of utility functions that represents $\succsim_i$, and each $\overline{U}_i$ is convex. Let U_i be the relative interior of $\overline{U}_i$ and apply Theorem 1, then translate the utility functions u_i back into preference orders $\succsim'_i$. We thus obtain *completions* $\succsim'_i$ of the relations $\succsim_i$, again satisfying independence and continuity, such that L is Pareto efficient with respect to the $\succsim'_i$. Moreover, relative interiority ensures that these completions preserve strict preference: if $L' \succ_i L$ then $L' \succ'_i L$.

Finally, it is worth mentioning that the problem we have considered in this note can be viewed as a special case of a more general problem: Given a set C of choices and sets $U_1, \ldots, U_n$ of plausible utility functions over C for agents $1, \ldots, n$ respectively, suppose some choice $c \in C$ has the property that no other choice c' is guaranteed to Pareto dominate c. What conditions on C and the U_i suffice to ensure that there must exist utility functions $u_i \in U_i$ for which c is Pareto efficient? (Even more generally, one can again ask this question when the U_i are sets of incomplete preference relations rather than utility functions.)

We have addressed this problem in the context of von Neumann-Morgenstern utility functions over lotteries, but it would be interesting to find non-expected utility models, or more general social choice models, in which analogous results hold. Since the approach used in this paper depends crucially on linearity, it seems that attacking more general models would require other techniques.

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