Bayesian persuasion with multiple senders and rich signal spaces

Matthew Gentzkow and Emir Kamenica*
Stanford University and University of Chicago

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Abstract

A number of senders with no *ex ante* private information publicly choose signals whose realizations they observe privately. Senders then convey verifiable messages about their signal realizations to a receiver who takes a non-contractible action that affects the welfare of all players. The space of available signals includes all conditional distributions of signal realizations and allows any sender to choose a signal that is arbitrarily correlated with signals of others. We characterize the information revealed in pure strategy equilibria, and show that greater competition tends to increase the amount of information revealed.

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1 Introduction

In Kamenica and Gentzkow (2011), we study the problem of "Bayesian persuasion" where a single sender chooses what information to gather and communicate to a receiver who then takes a non-contractible action that affects the welfare of both parties. In this paper, we extend the analysis to multiple senders. We introduce a new way to formalize the notion of a *signal,* which provides an easy way to capture any correlation in signal realizations across multiple senders.[1] The new formalization endows the set of signals with an algebraic structure that allows us to add various senders' signals together and thus easily examine their joint informational content. Combining this modeling approach with the insights from Kamenica and Gentzkow (2011) yields a tractable framework for studying multi-sender communication.

In our model, several senders, who have no *ex ante* private information, simultaneously conduct costless experiments about an unknown state of the world. The set of possible experiments is rich; it includes all conditional distributions of signal realizations given the state, and it allows arbitrary correlation among the senders' signals. Each sender privately observes the outcome of his experiment and then sends a verifiable message about the outcome to a third party. Receiver observes what experiments were conducted and the messages about their outcomes sent by the senders. She then takes a non-contractible action that affects the welfare of all players. The state space is finite. Receiver and each of the senders have arbitrary, state-dependent, utility functions over the Receiver's action and the state of the world. Throughout the paper we focus on pure-strategy equilibria of the game.[2]

The information revealed in an equilibrium of this game can be succinctly summarized by the distribution of Receiver's posterior beliefs (Blackwell 1953). We refer to such a distribution as an *outcome* of the game and order outcomes by informativeness according to the usual Blackwell criterion.

We begin our analysis by establishing the following lemma: if the senders other than i together induce some outcome τ', sender i can unilaterally deviate to induce some other outcome τ if and only if τ is more informative than τ'. The "only if" part of this lemma is trivial and captures a basic

[1] Specifically, we define a signal as a partition of the state space crossed with the unit interval. The unit interval is the support of the random variable that drives the stochastic realization of the signal conditional on the state.

[2] In Section 5, we briefly discuss the complications that arise with mixed strategies.

property of information: an individual sender can unilaterally increase the amount of information revealed, but can never decrease it below the informational content of the other senders' signals. The "if" part of the lemma is more substantive, and draws on the assumption that senders have access to a rich set of possible signals. (In the language of Gentzkow and Kamenica (2016a), this result means that the rich signal space we consider in this paper is "Blackwell-connected." We discuss the relationship of our results to Gentzkow and Kamenica (2016a) in greater detail below.) Our lemma implies that no outcome can be a pure-strategy equilibrium if there exists a more informative outcome preferred by any sender. This property is the fundamental reason why competition tends to increase information revelation in our model.

Our main characterization result provides an algorithm for finding the full set of pure-strategy equilibrium outcomes. We consider each sender i's value function over Receiver's beliefs $\hat{v}_i$ and its concavification V_i (the smallest concave function everywhere above $\hat{v}_i$). Kamenica and Gentzkow (2011) show that when there is a single sender $i = 1$, any belief μ that Receiver holds in equilibrium must satisfy $\hat{v}_1(\mu) = V_1(\mu)$. We extend this result and establish that, when there are two or more senders, a distribution of posteriors is an equilibrium outcome *if and only if* every belief μ in its support satisfies $\hat{v}_i(\mu) = V_i(\mu)$ for all i. Identifying the set of these "coincident" beliefs for a given sender is typically straightforward. Any given outcome is an equilibrium if and only if its support lies in the the intersection of these sets.

We then turn to the impact of competition on information revelation. We consider three ways of increasing competition among senders: (i) moving from collusive to non-cooperative play, (ii) introducing additional senders, and (iii) decreasing the alignment of senders' preferences. Since there are typically many equilibrium outcomes, we state these results in terms of set comparisons that modify the strong and the weak set orders introduced by Topkis (1978). We show that, for all three notions of increasing competition, more competition never makes the set of outcomes less informative (under either order). We also show that if the game is zero-sum for any subset of senders, full revelation is the unique equilibrium outcome whenever the value functions are sufficiently nonlinear.

The next section discusses the relationship between our paper and the existing literature. Section 3 develops mathematical preliminaries. Section 4 presents the model. Section 5 develops our main characterization result. Section 6 presents comparative statics. Section 7 presents applications to persuasion in courtrooms and product markets. Section 8 concludes.

2 Relationship to existing literature

As mentioned earlier, we extend the analysis in Kamenica and Gentzkow (2011) to the case of multiple senders. The earlier paper establishes conditions under which (single sender) persuasion is possible and characterizes optimal signals. This paper's primary focus is strategic interaction of the multiple senders.

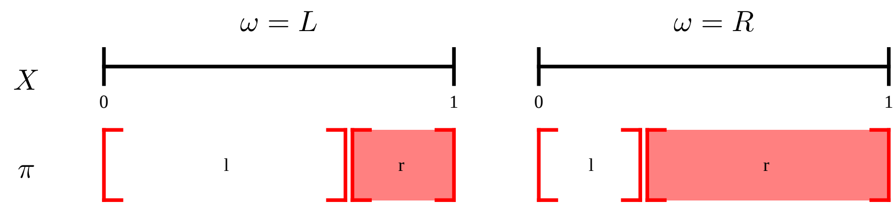
In Gentzkow and Kamenica (2016a), we analyze the same comparative statics as in this paper, but with more general signal spaces and more general preferences.[3] In this paper, we study the special case of rich signal spaces in more detail. This focus allows us to draw on the analytical methods from Aumann and Maschler (1995) and Kamenica and Gentzkow (2011) to geometrically characterize equilibria in terms of specific beliefs that are induced. Moreover, the focus allows us to extend the results to settings where senders observe the outcomes of their experiments privately and then convey verifiable messages about those outcomes (cf: Gentzkow and Kamenica 2016b).

There are also two other papers that examine symmetric information games with multiple senders. Brocas *et al.* (2012) and Gul and Pesendorfer (2012) study models where two senders with opposed interests generate costly signals about a binary state of the world. Unlike those papers, we assume signals are costless, but consider a more general setting with an arbitrary state space, arbitrary preferences, and arbitrary signals. Neither Brocas *et al.* (2012) nor Gul and Pesendorfer (2012) examine the impact of competition on information revelation.

Finally, our paper also broadly relates to work on multi-sender communication (e.g., Milgrom and Roberts 1986; Krishna and Morgan 2001; Battaglini 2002) and advocacy (e.g., Shin 1998; Dewatripont and Tirole 1999). In contrast to the former, we make experts' information endogenous while in contrast to the latter, we make signals costless which shuts down the issue of moral hazard central to that literature.

[3] Here we allow for arbitrary preferences over Receiver's action and the state. In Gentzkow and Kamenica (2016a), we allow for arbitrary preferences over the resultant distribution of posteriors. The former implies that expected utility is a linear function of the distribution of posteriors so the latter is more general.

Figure 1: A signal



3 Mathematical preliminaries

3.1 State space and signals

Let Ω be a finite state space. A state of the world is denoted by $\omega \in \Omega$. A belief is denoted by μ. The prior distribution on Ω is denoted by μ_0. Let X be a random variable that is independent of ω and uniformly distributed on $[0,1]$ with typical realization x. We model signals as deterministic functions of ω and x. Formally, a *signal* π is a finite partition of $\Omega \times [0,1]$ s.t. $\pi \subset S$, where S is the set of non-empty Lebesgue measurable subsets of $\Omega \times [0,1]$. We refer to any element $s \in S$ as a *signal realization.*

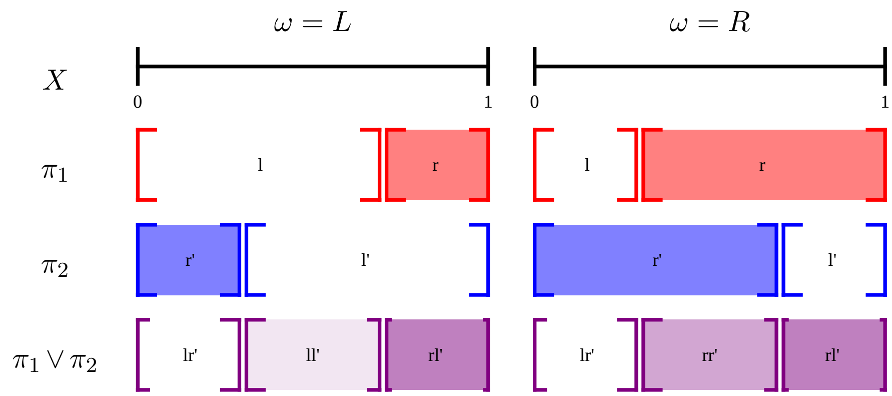
With each signal π we associate an S-valued random variable that takes value $s \in \pi$ when $(\omega, x) \in s$. Let $p(s|\omega) = \lambda(\{x | (\omega, x) \in s\})$ and $p(s) = \sum_{\omega \in \Omega} p(s|\omega) \mu_0(\omega)$ where $\lambda(\cdot)$ denotes the Lebesgue measure. For any $s \in \pi$, $p(s|\omega)$ is the conditional probability of s given ω and $p(s)$ is the unconditional probability of s.

We let Π denote the set of all signals and we refer to it as the *rich signal space.*

Our definition of a signal is somewhat non-standard because we model the source of noise in signal realizations (the random variable X) explicitly. This is valuable for studying multiple senders because for any two signals π_1 and π_2, our definition pins down not only their marginal distributions on S but also their joint distribution on $S \times S$. The joint distribution is important as it captures the extent to which observing both π_1 and π_2 reveals more information than observing only π_1 or π_2 alone. A definition of a signal, which takes the marginal distribution on S conditional on ω as the primitive, leaves the joint informational content of two or more signals unspecified.

Our definition of a signal is illustrated in Figure 1. In this example, $\Omega = \{L, R\}$ and $\pi = \{l, r\}$ where $l = (L, [0, 0.7]) \cup (R, [0, 0.3])$ and $r = (L, [0.7, 1]) \cup (R, [0.3, 1])$. The signal is a partition of $\Omega \times [0,1]$ with marginal distribution $\Pr(l|L) = \Pr(r|R) = 0.7$.

Figure 2: The join of two signals



3.2 Lattice structure

The formulation of a signal as a partition has the additional benefit of inducing an algebraic structure on the set of signals. This structure allows us to "add" signals together and thus easily examine their joint information content. Let $\trianglerighteq$ denote the refinement order on Π, that is, $\pi_1 \trianglerighteq \pi_2$ if every element of π_1 is a subset of an element of π_2. The pair $(\Pi, \trianglerighteq)$ is a lattice and we let $\vee$ and $\wedge$ denote join and meet, respectively. For any finite set (or vector)[4] P we denote the join of all its elements by $\vee P$. We write $\pi \vee P$ for $\pi \vee (\vee P)$.

Note that $\pi_1 \vee \pi_2$ is a signal that consists of signal realizations s such that $s = s_1 \cap s_2$ for some $s_1 \in \pi_1$ and $s_2 \in \pi_2$. Hence, $\pi_1 \vee \pi_2$ is the signal that yields the same information as observing both signal π_1 and signal π_2. In this sense, the binary operation $\vee$ "adds" signals together. The join of two signals is illustrated in Figure 2.

3.3 Distributions of posteriors

A *distribution of posteriors*, denoted by τ, is an element of $\Delta(\Delta(\Omega))$ that has finite support.[5] A distribution of posteriors τ is *Bayes-plausible* if $\mathrm{E}_\tau[\mu] = \mu_0$.

[4] Given a vector of signals $\boldsymbol{\pi} = (\pi_1, ..., \pi_n)$, we write $\vee\boldsymbol{\pi}$ for $\vee\{\pi_i\}_{i=1}^n$.

[5] The fact that distributions of posteriors have finite support follows from the assumption that each signal has finitely many realizations. The focus on such signals is without loss of generality under the maintained assumption that Ω is finite.

Observing a signal realization s s.t. $p(s) > 0$ generates a unique posterior belief[6]

$$\mu_s(\omega) \quad = \quad \frac{p(s|\omega)\,\mu_0(\omega)}{p(s)} \text{ for all } \omega.$$

Note that the expression above does not depend on the signal; observing s from any signal π leads to the same posterior μ_s.

Each signal π induces a Bayes-plausible distribution of posteriors. We write $\langle\pi\rangle$ for the distribution of posteriors induced by π. It is easy to see that $\tau = \langle\pi\rangle$ assigns probability $\tau(\mu) = \sum_{\{s\in\pi:\mu_s=\mu\}} p(s)$ to each μ. Kamenica and Gentzkow (2011) establish that the image of the mapping $\langle\cdot\rangle$ is the set of all Bayes-plausible τ's:

Lemma 1. *(Kamenica and Gentzkow 2011) For any Bayes-plausible distribution of posteriors τ, there exists a $\pi \in \Pi$ such that $\langle\pi\rangle = \tau$.*

We define a *conditional distribution of posteriors* $\langle\pi|s\rangle$ to be the distribution of posteriors induced by observing signal π after having previously observed some signal realization s with $p(s) > 0$. This distribution assigns probability $\sum_{\{s'\in\pi:\mu_{s\cap s'}=\mu\}} \frac{p(s\cap s')}{p(s)}$ to each belief μ. For any π and s with $p(s) > 0$, we have $\mathrm{E}_{\langle\pi|s\rangle}[\mu] = \mu_s$. Lemma 1 can easily be extended to conditional distributions of posteriors:

Lemma 2. *For any s s.t. $p(s) > 0$ and any distribution of posteriors τ s.t. $\mathrm{E}_\tau[\mu] = \mu_s$, there exists a $\pi \in \Pi$ such that $\tau = \langle\pi|s\rangle$.*

Proof. Given any s s.t. $p(s) > 0$ and any distribution of posteriors τ s.t. $\mathrm{E}_\tau[\mu] = \mu_s$, let S' be a partition of s constructed as follows. For each ω, let $s_\omega = \{x|\,(\omega, x) \in s\}$. Now, partition each s_ω into $\{s^\mu_\omega\}_{\mu\in\mathrm{Supp}(\tau_s)}$ with $\lambda(s^\mu_\omega) = \frac{\mu(\omega)\tau(\mu)}{\mu_s(\omega)}\lambda(s_\omega)$. This is possible because $\mathrm{E}_\tau[\mu] = \mu_s$ implies $\sum_{\mu\in\mathrm{Supp}(\tau)} \mu(\omega)\,\tau(\mu) = \mu_s(\omega)$; hence, $\sum_{\mu\in\mathrm{Supp}(\tau)} \lambda(s^\mu_\omega) = \lambda(s_\omega)$. For each $\mu \in \mathrm{Supp}(\tau)$, let $s^\mu = \cup_\omega s^\mu_\omega$. Note that $S' = \{s^\mu|\mu \in \mathrm{Supp}(\tau)\}$ is a partition of s. Let $\pi = S' \cup \{\{\Omega \times [0,1] \setminus \{s\}\}\}$. It is easy to check that $\tau = \langle\pi|s\rangle$. $\square$

Note that Lemma 1 is a corollary of Lemma 2 as we can set s in the statement of Lemma 2 to $\Omega \times [0,1]$ so that $\mu_s = \mu_0$.

[6]For those s with $p(s) = 0$, set μ_s to be an arbitrary belief.

3.4 Informativeness

We order distributions of posteriors by informativeness in the sense of Blackwell (1953). We say that τ is *more informative* than τ', denoted $\tau \succsim \tau'$, if for some π and π' s.t. $\tau = \langle\pi\rangle$ and $\tau' = \langle\pi'\rangle$, there exists a *garbling* $g : S \times S \to [0,1]$ such that $\sum_{s' \in \pi'} g(s', s) = 1$ for all $s \in \pi$, and $p(s'|\omega) = \sum_{s \in \pi} g(s', s) p(s|\omega)$ for all ω and all $s' \in \pi'$. The relation $\succsim$ is a partial order. We refer to the minimum element as *no revelation*, denoted $\underline{\tau}$. Distribution $\underline{\tau}$ places probability one on the prior. We refer to the maximum element as *full revelation*, denoted $\overline{\tau}$. Distribution $\overline{\tau}$ has only degenerate beliefs in its support.[7]

The refinement order on the space of signals implies the informativeness order on the space of distributions of posteriors:

Lemma 3. $\pi \trianglerighteq \pi' \Rightarrow \langle\pi\rangle \succsim \langle\pi'\rangle$.

Proof. Define $g(s', s)$ equal to 1 if $s \subset s'$, and equal to 0 otherwise. Given any π and π' s.t. $\pi \trianglerighteq \pi'$, we know that for each $s \in \pi$, there is exactly one $s' \in \pi'$ s.t. $s \subset s'$. Hence, for all s, $\sum_{s' \in \pi'} g(s', s) = 1$. Moreover, $\pi \trianglerighteq \pi'$ implies that $\cup \{s \in \pi : s \subset s'\} = s'$. Hence, for any ω and any $s' \in \pi'$, $\{x | (\omega, x) \in \cup \{s \in \pi : s \subset s'\}\} = \{x | (\omega, x) \in s'\}$. This in turn implies $p(s'|\omega) = \sum_{s \in \pi} g(s', s) p(s|\omega)$. □

Note that it is not true that $\langle\pi\rangle \succsim \langle\pi'\rangle \Rightarrow \pi \trianglerighteq \pi'$.[8] Note also that Lemma 3 implies $\langle\pi_1 \vee \pi_2\rangle \succsim \langle\pi_1\rangle, \langle\pi_2\rangle$.

We establish one more relationship between $\trianglerighteq$ and $\succsim$.

Lemma 4. *For any τ, τ', and π s.t. $\tau' \succsim \tau$ and $\langle\pi\rangle = \tau$, $\exists \pi'$ s.t. $\pi' \trianglerighteq \pi$ and $\langle\pi'\rangle = \tau'$.*

Proof. Consider any τ, τ', and π s.t. $\tau' \succsim \tau$ and $\langle\pi\rangle = \tau$. By Lemma 1, there is a $\hat{\pi}$ such that $\langle\hat{\pi}\rangle = \tau'$. Hence, by definition of $\succsim$, there is a garbling g such that $p(s|\omega) = \sum_{\hat{s} \in \hat{\pi}} g(s, \hat{s}) p(\hat{s}|\omega)$ for all $s \in \pi$ and ω. Define a new signal $\pi' \trianglerighteq \pi$ as follows. For each $s \in \pi$, for each $\omega \in \Omega$, let $s_\omega = \{x | (\omega, x) \in s\}$. Now, define a partition of each s_ω such that each element of the partition, say $s'(s, \hat{s}, \omega)$, is associated with a distinct $\hat{s} \in \hat{\pi}$ and has Lebesgue measure $g(s, \hat{s}) p(\hat{s}|\omega)$. This

[7] A belief is degenerate if it places positive probability only on a single state.

[8] For example, suppose that there are two states L and R. Suppose π is a perfectly informative signal with two realizations, and π' is an uninformative signal with ten realizations, each of which is equally likely in state L and state R. Then $\langle\pi\rangle \succsim \langle\pi'\rangle$, but π cannot be finer than π' because π' has more elements.

is possible since the sum of these measures is $p(s|\omega) = \lambda(s_\omega)$. Let $s'(s,\hat{s}) = \cup_\omega s'(s,\hat{s},\omega)$. Let $\pi' = \{s'(s,\hat{s}) | \hat{s} \in \hat{\pi}, s \in \pi\}$. For any $s, \hat{s}, \omega_1, \omega_2$, we have

$$\frac{p(s'(s,\hat{s})|\omega_1)}{p(s'(s,\hat{s})|\omega_2)} = \frac{g(s,\hat{s})\,p(\hat{s}|\omega_1)}{g(s,\hat{s})\,p(\hat{s}|\omega_2)} = \frac{p(\hat{s}|\omega_1)}{p(\hat{s}|\omega_2)},$$

which implies $\langle\pi'\rangle = \langle\hat{\pi}\rangle = \tau'$. $\square$

Note that it is not true that for any $\tau' \succsim \tau$ and $\langle\pi'\rangle = \tau'$, $\exists\pi$ s.t. $\pi' \trianglerighteq \pi$ and $\langle\pi\rangle = \tau$.

3.5 Orders on sets

We will frequently need to compare the informativeness of sets of outcomes. Topkis (1978; 1998) defines two orders on subsets of a lattice. Given two subsets Y and Y' of a lattice $(\mathcal{Y}, \geq)$, consider two properties of a pair $y, y' \in \mathcal{Y}$:

$$\begin{aligned} &(S) \quad y \vee y' \in Y \text{ and } y \wedge y' \in Y' \\ &(W) \quad \exists \hat{y} \in Y : \hat{y} \geq y' \text{ and } \exists \hat{y}' \in Y' : y \geq \hat{y}' \end{aligned}$$

Topkis defines Y to be strongly above Y' (denoted $Y \geq_s Y'$) if property S holds for any $y \in Y$ and $y' \in Y'$, and to be weakly above Y' (denoted $Y \geq_w Y'$) if property W holds for any $y \in Y$ and $y' \in Y'$.

The set of outcomes is not generally a lattice under the Blackwell order (Müller and Scarsini 2006), so the strong set order is not well defined on this set, but this will not be an issue for the modification of the strong set order that we introduce below. The weak set order is of course well defined on any poset, whether or not it is a lattice. Given two sets of outcomes T and T', we say T is *weakly more informative* than T' if $T \succsim_w T'$.

Some of our results will establish that a particular set cannot be strictly less informative than another set. To simplify the statement of those propositions, we say that T is *no less informative* than T' if T is not strictly less informative than T' in the weak order.

Both the strong and the weak order are partial. Broadly speaking, there are two ways that sets Y and Y' can fail to be ordered. The first arises when one set has elements that are ordered both above and below the elements of the other set. For example, suppose that $\max(Y) > \max(Y')$ but

$\min(Y) < \min(Y')$. Then, sets Y and Y' are not comparable in either the strong or the weak order, as seems intuitive. The second way that two sets can fail to be comparable arises when individual elements of the two sets are themselves not comparable. For example, suppose that $Y \geq_s Y'$ and $\tilde{y} \in \mathcal{Y}$ is not comparable to any element of $Y \cup Y'$. Then $Y \cup \tilde{y}$ is not comparable to Y' in either the strong or the weak order. The intuitive basis for calling $Y \cup \tilde{y}$ and Y' unordered may seem weaker than in the first case, and in some contexts we might be willing to say that $Y \cup \tilde{y}$ is above Y'.

In the analysis below, we will frequently encounter sets that fail to be ordered only in the latter sense. It will therefore be useful to distinguish these cases from those where sets are unordered even when we restrict attention to their comparable elements. A *chain* is a set in which any two elements are comparable, and a chain is *maximal* if it is not a strict subset of any other chain. We say that Y *is strongly (weakly) above* Y' *along chains* if for any maximal chain $C \subset \mathcal{Y}$ that intersects both Y and Y', $Y \cap C \geq_s Y' \cap C$ ($Y \cap C \geq_w Y' \cap C$).[9] In other words, orders along chains only require that conditions S and W hold for comparable pairs y and y'.

Given two sets of outcomes T and T', we say T is *strongly (weakly) more informative than* T' *along chains* if T is strongly (weakly) above T' along chains under the Blackwell order. Note that the strong (as well as the weak) order along chains is well defined on any poset. This is why it will not matter that the set of outcomes is not generally a lattice.

To gain more intuition about orders along chains, consider again properties S and W. When Y is strongly (weakly) above Y', property S (W) holds for any $y \in Y$ and $y' \in Y'$. When Y is strongly (weakly) above Y' along chains, property S (W) holds for any *comparable* y and y'.

Orders along chains also arise naturally in decision theory. The standard result on monotone comparative statics (Milgrom and Shannon 1994) states that, given a lattice $\mathcal{Y}$ and a poset Z, $\arg\max_{y \in Y} f(y, z)$ is monotone nondecreasing in z in the strong set order if and only if $f(\cdot, \cdot)$ satisfies the single-crossing property[10] and $f(\cdot, z)$ is quasisupermodular[11] for any z. It turns out that if we drop the requirement of quasisupermodularity, we obtain monotone comparative statics

[9] Given any two sets Y and Y', the following three statements are equivalent: (i) for any maximal chain C, $Y \cap C \geq_s Y' \cap C$, (ii) for any chain C, $Y \cap C \geq_s Y' \cap C$, and (iii) for any chain C s.t. $|C| = 2$, $Y \cap C \geq_s Y' \cap C$.

[10] Function $f : Y \times Z \to \mathbb{R}$ satisfies the single-crossing property if $y > y'$ and $z > z'$ implies that $f(y, z') \geq f(y', z') \Rightarrow f(y, z) \geq f(y', z)$ and $f(y, z') > f(y', z') \Rightarrow f(y, z) > f(y', z)$.

[11] Function $f : Y \to \mathbb{R}$ is quasisupermodular if $f(y) \geq f(y \wedge y') \Rightarrow f(y \vee y') \geq f(y')$ and $f(y) > f(y \wedge y') \Rightarrow f(y \vee y') > f(y')$.

in the strong order along chains:[12]

Remark 1. Given a poset $\mathcal{Y}$ and a poset Z, $\arg\max_{y \in Y} f(y,z)$ is monotone nondecreasing in z in the strong set order along chains if and only if $f(\cdot,\cdot)$ satisfies the single-crossing property.

4 The model

4.1 The baseline model

Receiver has a continuous utility function $u(a,\omega)$ that depends on her action $a \in A$ and the state of the world $\omega \in \Omega$. For each belief μ, we assume that there is a unique action that maximizes Receiver's expected utility.

There are $n \geq 1$ senders indexed by i. Each sender i has a continuous utility function $v_i(a,\omega)$ that depends on Receiver's action and the state of the world. All senders and Receiver share the prior μ_0. The action space A is compact.

The timing is as follows:

1. Each sender simultaneously chooses a signal π_i from Π.
2. Each sender privately observes the realization s_i of his own signal.
3. Each sender simultaneously sends a message $m_i \in M(s_i)$ to Receiver.
4. Receiver observes the signals chosen by the experts and the messages they sent.
5. Receiver chooses an action.

Function $M(\cdot)$ specifies the set of messages that are feasible upon observing a given signal realization. Let $\mathscr{M} = \cup_{s \in S} M(s)$ denote the set of all possible messages. For each $m \in \mathscr{M}$, let $T(m) = \{s \in S | m \in M(s)\}$ denote the set of signal realizations under which m could have been sent. We say that a message m *verifies* s if $T(m) = \{s\}$. For each $s \in S$, we assume there exists a message in $M(s)$ that verifies it.

[12] We thank John Quah for this observation.

4.2 Simplifying the model

Studying the equilibria of the baseline model requires keeping track of what signals senders choose, what messages they send for every signal realizations, and how Receiver forms beliefs, not only to the equilibrium messages, but also to off-equilibrium behavior by senders. Luckily, however, the analysis can be greatly simplified by noting that, precisely because senders have access to rich signal spaces, the equilibrium outcomes would be the same if Receiver were to directly observe the signal realizations, rather than rely on senders' messages about them.

Recall that we define the distribution of Receiver's beliefs as the *outcome* of the game. (It is easy to see that Receiver's distribution of posteriors determines the distribution of Receiver's actions and the payoffs of all the players.) We then have the following result:

Proposition 1. *(Gentzkow and Kamenica 2016b) The set of pure strategy equilibrium outcomes is unchanged if Receiver directly observes the realizations of the signals that are chosen.*

Given this result and given that our focus is on equilibrium outcomes rather than particular strategies employed by senders, from here on we consider a simpler game where each sender simply chooses a signal and Receiver directly observes the signal realizations. Throughout the paper, we focus on pure strategy equilibria.

Receiver forms her posterior using Bayes' rule; hence her belief after observing the signal realizations is μ_s where $s = \cap_{i=1}^{n} s_i$. She chooses the unique action that maximizes $\mathrm{E}_{\mu_s} u(a, \omega)$ which we denote by $a^*(\mu)$.

We denote sender i's expected utility when Receiver's belief is μ by $\hat{v}_i(\mu)$:

$$\hat{v}_i(\mu) \equiv \mathrm{E}_\mu v_i(a^*(\mu), \omega).$$

We denote senders' strategy profile by $\boldsymbol{\pi} = (\pi_1, ..., \pi_n)$ and let $\boldsymbol{\pi}_{-i} = (\pi_1, ...\pi_{i-1}, \pi_{i+1}, ..., \pi_n)$. A profile $\boldsymbol{\pi}$ is an *equilibrium* if

$$\mathrm{E}_{\langle \vee \boldsymbol{\pi} \rangle} \hat{v}_i(\mu) \geq \mathrm{E}_{\langle \pi_i' \vee \boldsymbol{\pi}_{-i} \rangle} \hat{v}_i(\mu) \;\; \forall \pi_i' \in \Pi \; \forall i.$$

We say a belief μ is *induced* in an equilibrium if it is in the support of the equilibrium outcome.

4.3 Discussion of the model

Our model makes several strong assumptions.

First, we assume that signals are costless and that each sender can choose any signal whatsoever, i.e., that each sender has access to the rich signal space. This assumption would be violated if different senders' information were conditionally independent, if some sender had comparative advantage in accessing certain kinds of information, if there were some information that senders could not avoid learning, or if the experimental technology were coarse.

Second, our model implicitly allows each sender to choose a signal whose realizations are arbitrarily correlated, conditional on ω, with the signal realizations of the other senders. This would not be possible if signal realizations were affected by some idiosyncratic noise. One way to motivate our assumption is to consider a setting in which there is an exogenous set of experiments about ω and each sender's strategy is simply a deterministic mapping from the outcomes of those experiments to a message space. In that case, each sender can make his messages correlated with those of other senders.

Third, senders do not have any private information that could be inferred from their choice of the signal.

Fourth, our Receiver is a classical Bayesian who can costlessly process all information she receives. The main import of this assumption is that no sender can drown out the information provided by others, say by sending many useless messages. From Receiver's point of view, the worst thing that any sender can do is to provide no information. Hence, unlike in a setting with costly information processing, our model induces an asymmetry whereby each sender can add to but not detract from the information provided by others.

The four assumptions above not only make the model more tractable, but are required for our main results to hold. In contrast, we also make several assumptions that are not necessary for the results, but greatly simplify the exposition.

First, we present the model as if there were a single Receiver, but an alternative way to interpret our setting is to suppose there are several receivers $j = 1, .., m$, each with a utility function $u_j(a_j, \omega)$, with receiver j taking action $a_j \in A_j$, and all receivers observing the realizations of all senders' signals. Even if each sender's utility $v_i(a, \omega)$ depends in an arbitrary way on the full vector of receivers' actions $a = (a_1, ..., a_m)$, our analysis still applies directly since, from senders' perspective, the situation is exactly the same as if there were a single Receiver maximizing $u(a, \omega) = \sum_{j=1}^{m} u_j(a_j, \omega)$.

Second, it is easy to extend our results to situations where Receiver has private information. Suppose that, at the outset of the game, Receiver privately observes a realization r from some signal $\xi(\cdot|\omega)$. In that case, Receiver's action, $a^*(s, r)$, depends on the realization of her private signal and is thus stochastic from senders' perspective. However, given a signal realization s, each sender simply assigns the probability $\xi(r|\omega)\mu_s(\omega)$ to the event that Receiver's signal is r and the state is ω. Hence, sender i's expected payoff given s is $\hat{v}_i(\mu_s) = \sum_\omega \sum_r v(a^*(s, r), \omega)\xi(r|\omega)\mu_s(\omega)$. All the results then apply directly with respect to the re-formulated $\hat{v}_i$'s.

5 Characterizing equilibrium outcomes

In this section, we characterize the set of equilibrium outcomes. As a first step, consider the set of distributions of posteriors that a given sender can induce given the strategies of the other senders. It is immediate that he can only induce a distribution of posteriors that is more informative than the one induced by his opponents' signals alone. The following lemma establishes that he can induce *any* such distribution.[13]

Lemma 5. *Given a strategy profile $\boldsymbol{\pi}$ and a distribution of posteriors τ, for any sender i there exists a $\pi'_i \in \Pi$ such that $\langle \pi'_i \vee \boldsymbol{\pi}_{-i} \rangle = \tau$ if and only if $\tau \succsim \langle \vee \boldsymbol{\pi}_{-i} \rangle$.*

Proof. Suppose $\tau \succsim \langle \vee \boldsymbol{\pi}_{-i} \rangle$. By Lemma 4, there exists a $\pi'_i \trianglerighteq \vee \boldsymbol{\pi}_{-i}$ s.t. $\langle \pi'_i \rangle = \tau$. Since $\pi'_i = \pi'_i \vee \boldsymbol{\pi}_{-i}$, we know $\langle \pi'_i \vee \boldsymbol{\pi}_{-i} \rangle = \langle \pi'_i \rangle = \tau$. The converse follows from Lemma 3. $\square$

Lemma 5 draws on our assumption that each sender can choose a signal whose realizations are arbitrarily correlated, conditional on ω, with the signal realizations of the other senders. As a result, if senders were to play mixed strategies, the analogue of this lemma does not hold – it is easy to construct an example where senders other than i are playing mixed strategies $\tilde{\boldsymbol{\pi}}_{-i}$, there is a distribution of posteriors $\tau \succsim \langle \vee \tilde{\boldsymbol{\pi}}_{-i} \rangle$, and there is no π'_i such that $\langle \pi'_i \vee \tilde{\boldsymbol{\pi}}_{-i} \rangle = \tau$.[14] Consequently,

[13] In terminology of Gentzkow and Kamenica (2016a), this means that a situation where each sender has access to the rich signal space is "Blackwell-connected."

[14] Here, we extend the notation $\langle \cdot \rangle$ to denote the distribution of posteriors induced by a mixed strategy profile.

the approach we develop below cannot be used to characterize mixed strategy equilibria. Li and Norman (2015) provide a more extensive discussion of mixed strategy equilibria in our setting.

We next turn to the question of when a given sender would wish to deviate to some more informative τ. For each i, let V_i be the *concavification* of $\hat{v}_i$:

$$V_i(\mu) \equiv \sup\{z | (\mu, z) \in \text{co}(\hat{v}_i)\},$$

where $\text{co}(\hat{v}_i)$ denotes the convex hull of the graph of $\hat{v}_i$. Note that each V_i is concave by construction. In fact, it is the smallest concave function that is everywhere weakly greater than $\hat{v}_i$. Kamenica and Gentzkow (2011) establish that when there is only a single sender i, $V_i(\mu_0)$ is the greatest payoff that the sender can achieve:

Lemma 6. *(Kamenica and Gentzkow 2011) For any belief μ, $\hat{v}_i(\mu) = V_i(\mu)$ if and only if $\mathrm{E}_\tau[\hat{v}_i(\mu')] \leq \hat{v}_i(\mu)$ for all τ such that $\mathrm{E}_\tau[\mu'] = \mu$.*

We refer to beliefs μ such that $\hat{v}_i(\mu) = V_i(\mu)$ as *coincident.* We let M_i denote the set of coincident beliefs for sender i.

The lemma above establishes that, if there is a single sender, any belief induced in equilibrium has to be coincident for that sender. Our main characterization result shows that when $n \geq 2$, any belief induced in equilibrium has to be coincident for all senders. Moreover, unlike in the single sender case, this condition is not only necessary but sufficient: for any Bayes-plausible τ whose support lies in the intersection $M = \bigcap_{i=1}^{n} M_i$, there exists an equilibrium that induces τ.

Proposition 2. *Suppose $n \geq 2$. A Bayes-plausible distribution of posteriors τ is an equilibrium outcome if and only if each belief μ in its support is coincident for each sender, i.e., $\hat{v}_i(\mu) = V_i(\mu)$ for all i.*

Gentzkow and Kamenica (2016a) establish a related result. We say there that an outcome τ is *unimprovable* for sender i if sender i's payoff is lower at every outcome that is more informative than τ. Under that definition, A distribution of posteriors is an equilibrium outcome if and only if it is unimprovable for each sender. In context of that result, Proposition 2 implies that an outcome is unimprovable if and only if every belief in its support is coincident. In practice, this additional result can dramatically simplify the process of computing the equilibrium set.

We provide a sketch of the proof here; a more detailed argument is in the Appendix. Suppose that τ is an equilibrium outcome. If there were some $\mu \in \text{Supp}(\tau)$ such that $\hat{v}_i(\mu) \neq V_i(\mu)$ for some sender i, Lemmas 5 and 6 imply that sender i could profitably deviate by providing additional information when the realization of τ is μ. Conversely, suppose that τ is a Bayes-plausible distribution of beliefs such that for each $\mu \in \text{Supp}(\tau)$, $\hat{v}_i(\mu) = V_i(\mu)$ for all i. Consider the strategy profile where all senders send the same signal π with $\langle \pi \rangle = \tau$. No sender can then deviate to induce any $\tau' \prec \tau$. Moreover, the fact that all beliefs in the support of τ are coincident means that no sender would want to deviate to any $\tau' \succ \tau$ (cf: Lemma 10 in the Appendix). Thus, this strategy profile is an equilibrium.

An important feature of Proposition 2 is that it provides a way to solve for the informational content of equilibria simply by inspecting each sender's preferences in turn, without worrying about fixed points or strategic considerations. This is particularly useful because identifying the set of coincident beliefs for each sender is typically straightforward. In Section 7, we will use this characterization to develop some applications. For now, Figure 3 illustrates how Proposition 2 can be applied in a simple example with hypothetical value functions. In this example, there are two senders, A and B. Panel (a) displays $\hat{v}_A$ and V_A, while Panel (b) displays $\hat{v}_B$ and V_B. Panel (c) shows the sets of coincident beliefs M_A and M_B, as well as their intersection M. Any distribution of beliefs with support in M is an equilibrium outcome. A belief such as μ_1 cannot be induced in equilibrium because sender A would have a profitable deviation. A belief such as μ_2 cannot be induced in equilibrium because sender B would have a profitable deviation.

Observe that full revelation is an equilibrium in the example of Figure 3 (both $\mu = 0$ and $\mu = 1$ are in M). This is true whenever there are multiple senders, because degenerate beliefs are always coincident. This also implies that an equilibrium always exists.[15]

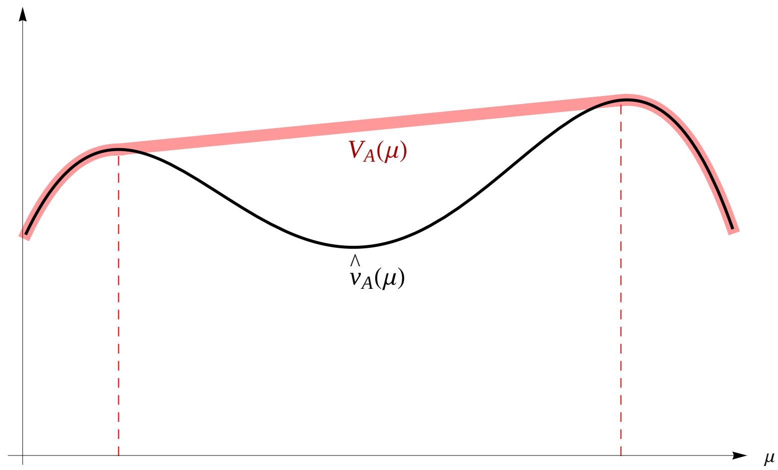
Corollary 1. *If $n \geq 2$, full revelation is an equilibrium outcome.*

As Sobel (2013) discusses, the existence of fully revealing equilibria under weak conditions is a common feature of multi-sender strategic communication models. In many of these models, as in ours, full revelation can be an equilibrium outcome even if all senders have identical preferences and

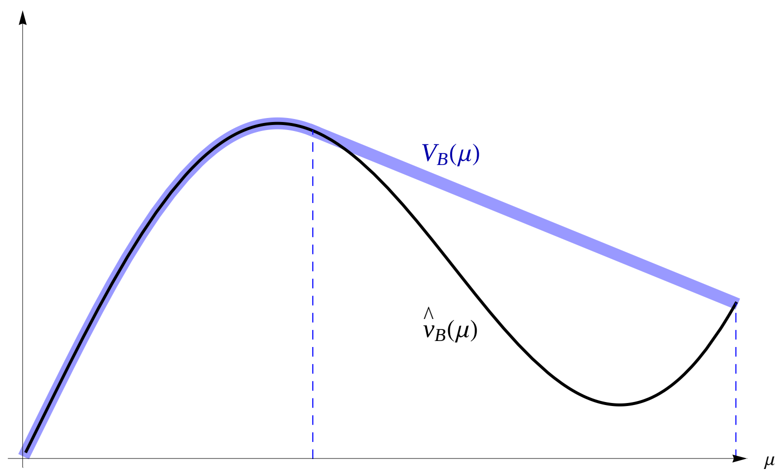
[15]Kamenica and Gentzkow (2011) establish existence for the case $n = 1$. Consider an $a^*(\cdot)$ where Receiver takes a sender-preferred optimal action at each belief. Such an $a^*(\cdot)$ guarantees that $\hat{v}_i$ is upper semicontinuous and thus that an equilibrium exists.

Figure 3: Characterizing equilibrium outcomes

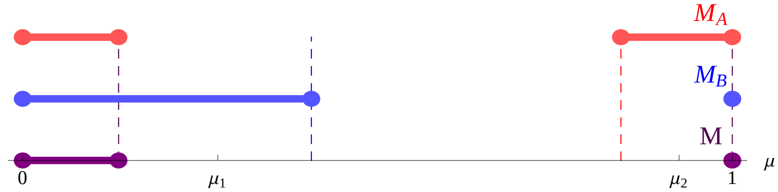
(a) $\hat{v}$ and V functions for sender A



(b) $\hat{v}$ and V functions for sender B



(c) Sets of coincident beliefs $\{\mu : \hat{v}(\mu) = V(\mu)\}$



strictly prefer no information disclosure to all other outcomes – a seemingly unappealing prediction.

One response would be to introduce a selection criterion that eliminates such equilibria. Given any two comparable equilibrium outcomes, every sender weakly prefers the less informative one. Hence, while the appropriate selection criterion might depend on the setting, selection criteria that always pick out a minimally informative equilibrium are appealing. We discuss the implications of such a selection criterion in Section 6.4 below. The approach we take in our formal results, however, is to focus on set comparisons of the full range of equilibrium outcomes.

6 Competition and information revelation

In this section we consider the impact of competition on the amount of information revealed. Our analysis here closely parallels that in Gentzkow and Kamenica (2016a). We consider the same three notions of "more competition." Moreover, while the results here are not direct corollaries of the propositions in the other paper,[16] the basic economic intuition is similar. Nonetheless, the results below offer further insights into rich signal spaces and illustrate the applicability of our notion of orders along chains.

6.1 Comparing competitive and collusive outcomes

One way to vary the extent of competition is to compare the set of non-cooperative equilibria to what senders would choose if they could get together and collude. This might be the relevant counterfactual for analyzing media ownership regulation or the effect of mergers on disclosure.

We say an outcome τ is *collusive* if it is Pareto optimal from the senders' perspective. This definition implies that we can remain agnostic about whether senders reach the collusive outcome by maximizing a (potentially weighted) sum of their payoffs, or though Nash bargaining, or through other means.

Proposition 3. *Let T^* be the set of equilibrium outcomes and T^c the set of collusive outcomes. T^* is no less informative than T^c. Moreover, T^* is strongly more informative than T^c along chains.*

[16] Gentzkow and Kamenica (2016a) consider comparisons of sets only in settings where any two feasible outcomes are comparable.

If there is a single sender, the proposition holds trivially as $T^* = T^c$, so suppose throughout this subsection that $n \geq 2$. We begin the proof with the following Lemma.

Lemma 7. *If $\tau^* \in T^*$, $\tau^c \in T^c$, and $\tau^c \succsim \tau^*$, then $\tau^c \in T^*$ and $\tau^* \in T^c$.*

Proof. Suppose $\tau^* \in T^*$, $\tau^c \in T^c$, and $\tau^c \succsim \tau^*$. By Lemma 5, we know $E_{\tau^c}[\hat{v}_i(\mu)] \leq E_{\tau^*}[\hat{v}_i(\mu)]$ for all i; otherwise, the sender i for whom $E_{\tau^c}[\hat{v}_i(\mu)] > E_{\tau^*}[\hat{v}_i(\mu)]$ could profitably deviate to τ^c. Since $\tau^c \in T^c$, we know $E_{\tau^c}[\sum \hat{v}_i(\mu)] \geq E_{\tau^*}[\sum \hat{v}_i(\mu)]$ for all i. Therefore, $E_{\tau^c}[\hat{v}_i(\mu)] = E_{\tau^*}[\hat{v}_i(\mu)]$ for all i which implies $\tau^* \in T^c$.

By Proposition 2, we know $\tau^c \in T^*$ unless there is a sender i and a distribution of posteriors $\tau' \succsim \tau^c$ s.t. $E_{\tau'}[\hat{v}_i(\mu)] > E_{\tau^c}[\hat{v}_i(\mu)]$. But since $E_{\tau^c}[\hat{v}_i(\mu)] = E_{\tau^*}[\hat{v}_i(\mu)]$ and $\tau' \succsim \tau^c \succsim \tau^*$, sender i could then profitably deviate from τ^* to τ^c which would contradict $\tau^* \in T^*$. □

Lemma 7 establishes one sense in which competition increases the amount of information revealed: no non-collusive equilibrium outcome is less informative than a collusive outcome, and no equilibrium outcome is less informative than a non-equilibrium collusive outcome. The lemma also plays a central role in the proof of Proposition 3:

Proof. Suppose $T^c \succsim_w T^*$. To establish that T^* is no less informative than T^c, we need to show this implies $T^* \succsim_w T^c$. For any $\tau^c \in T^c$, we know by Corollary 1 there exists $\tau^* \in T^*$ such that $\tau^* \succsim \tau^c$. For any $\tau^* \in T^*$, $T^c \succsim_w T^*$ implies there is a $\tau' \in T^c$ s.t. $\tau' \succsim \tau^*$. By Lemma 7, we must then have $\tau^* \in T^c$. Thus, there is a $\tau^c \in T^c$, namely τ^*, s.t. $\tau^c \precsim \tau^*$. Now, consider any chain C that intersects T and T'. Consider any $\tau^* \in T^* \cap C$ and any $\tau^c \in T^c \cap C$. By Lemma 7, $\tau^* \vee \tau^c \in T^* \cap C$ and $\tau^* \wedge \tau^c \in T^c \cap C$. Therefore, T^* is strongly more informative than T^c along chains. □

Note that the proposition allows for T^* to be non-comparable to T^c. The two sets can indeed be non-comparable in both the strong and the weak order. We will discuss the importance of these caveats below when we analyze whether competition necessarily makes Receiver better off.

6.2 Varying the number of senders

A second way to vary the extent of competition is to compare the set of equilibria with many senders to the set of equilibria with fewer senders. This might be the relevant counterfactual for assessing the impact of lowering barriers to entry on equilibrium advertising in an industry.

Proposition 4. *Let T and T' be the sets of equilibrium outcomes when the sets of senders are J and $J' \subset J$, respectively. T is no less informative than T'. Moreover, T is weakly more informative than T' if $|J'| > 1$, and weakly more informative than T' along chains if $|J'| = 1$.*

As suggested by the statement of the proposition, the basic intuition behind this result is somewhat different when we consider a change from many senders to more senders (i.e., when $|J'| > 1$), and when we consider a change from a single sender to many senders (i.e., when $|J'| = 1$).

In the former case, Proposition 2 implies that $T \subset T'$. In other words, adding senders causes the set of equilibrium outcomes to shrink. But, Corollary 1 implies that, even as the set of equilibrium outcomes shrinks, full revelation must remain in the set. Hence, loosely speaking, adding senders causes the set of equilibrium outcomes to shrink "toward" full revelation. We formalize this intuition in the following lemma, which will also be useful in proving Proposition 5 below.

Lemma 8. *Suppose T and T' are sets of outcomes with $T \subset T'$ and $\overline{\tau} \in T$. Then T is weakly more informative than T'.*

Proof. Suppose T and T' are sets of outcomes s.t. $T \subset T'$ and $\overline{\tau} \in T$. For any $\tau' \in T'$ there exists a $\tau \in T$, namely $\overline{\tau}$, s.t. $\tau \succsim \tau'$. For any $\tau \in T$ there exists a $\tau' \in T'$, namely τ, s.t. $\tau \succsim \tau'$. $\square$

In the latter case ($|J'| = 1$), the key observation is that no $\tau \in T \setminus T'$ can be less informative than a $\tau' \in T'$. Otherwise, the single sender in J' would prefer to deviate from τ to τ'. We now turn to the formal proof of Proposition 4.

Proof. If J is a singleton, the proposition holds trivially, so suppose that $|J| \geq 2$. First, consider there case where $|J'| > 1$. By Proposition 2, $T \subset T'$, and by Corollary 1, $\overline{\tau} \in T$. Hence, the proposition follows from Lemma 8. Second, consider the case where $|J'| = 1$. Let i denote the sender in J'. To establish that T is no less informative than T', we need to show that $T' \succsim_w T$ implies $T \succsim_w T'$. Suppose $T' \succsim_w T$. By Corollary 1, for any $\tau' \in T'$, we know there exists $\tau \in T$, namely $\overline{\tau}$, such that $\tau \succsim \tau'$. Given any $\tau \in T$, $T' \succsim_w T$ implies there is a $\tau' \in T'$ s.t. $\tau' \succsim \tau$. But, then it must be the case that τ is also individually optimal for sender i, i.e., $\tau \in T'$; otherwise, by Lemma 5, sender i could profitably deviate to τ' and hence τ would not be an equilibrium. Now, consider any maximal chain C that intersects T'. Since C is maximal, it must include $\overline{\tau}$. Moreover, $\overline{\tau} \in T$. Hence, for any $\tau' \in T' \cap C$ there is a $\tau \in T \cap C$, namely $\overline{\tau}$, s.t. $\tau \succsim \tau'$. It remains to show that for any $\tau \in T \cap C$ there is a $\tau' \in T' \cap C$ s.t. $\tau \succsim \tau'$. Given any $\tau \in T \cap C$, since C is a chain, every element of $T' \cap C$ is comparable to τ. Consider any $\tau' \in T' \cap C$. Since T' intersects C, there must be some such τ'. If $\tau' \precsim \tau$, we are done. Suppose $\tau' \succsim \tau$. Then, it must be the case that τ is also individually optimal for sender i, i.e., $\tau \in T'$; otherwise, by Lemma 5, sender i could profitably deviate to τ' and hence τ would not be an equilibrium. □

6.3 Varying the alignment of senders' preferences

A third way to vary the extent of the competition is to make senders' preferences more or less aligned. This counterfactual sheds lights on the efficacy of adversarial judicial systems and advocacy more broadly (Shin 1998; Dewatripont and Tirole 1999).

Given that senders can have any arbitrary state-dependent utility functions, the extent of preference alignment among senders is not easy to parameterize in general. Hence, we consider a specific form of preference alignment: given any two functions $f, g : A \times \Omega \to \mathbb{R}$ we let $\left\{\boldsymbol{v}^b\right\}_{b \in \mathbb{R}_+}$ denote a collection of preferences where some two senders, say j and k, have preferences of the form

$$\begin{aligned} v_j(a,\omega) &= f(a,\omega) + bg(a,\omega) \\ v_k(a,\omega) &= f(a,\omega) - bg(a,\omega) \end{aligned}$$

while preferences of Receiver and of other senders are independent of b. The parameter b thus captures the extent of preference misalignment between two of the senders.

Proposition 5. *Let T and T' be the sets of equilibrium outcomes when preferences are $\boldsymbol{v}^b$ and $\boldsymbol{v}^{b'}$, respectively, where $b > b'$. T is weakly more informative than T'.*

Proof. For each i, let M_i and M_i' denote the sets of coincident beliefs for sender i when preferences are $\boldsymbol{v}^b$ and $\boldsymbol{v}^{b'}$, respectively. Let $M = \cap_i M_i$ and $M' = \cap_i M_i'$. Let $\tilde{M} = M_j \cap M_k$ and $\tilde{M}' = M_j' \cap M_k'$. Let $\hat{f}(\mu) = \mathrm{E}_\mu[f(a^*(\mu),\omega)]$ and $\hat{g}(\mu) = \mathrm{E}_\mu[g(a^*(\mu),\omega)]$. Consider any $\mu \in \tilde{M}$. For any τ s.t. $\mathrm{E}_\tau[\mu'] = \mu$, we know that $\mu \in \tilde{M}_j$ implies $\mathrm{E}_\tau\left[\hat{f}(\mu') + b\hat{g}(\mu')\right] \le \hat{f}(\mu) + b\hat{g}(\mu)$ and $\mu \in \tilde{M}_k$ implies $\mathrm{E}_\tau\left[\hat{f}(\mu') - b\hat{g}(\mu')\right] \le \hat{f}(\mu) - b\hat{g}(\mu)$. Combining these two inequalities, we get $\hat{f}(\mu) - \mathrm{E}_\tau\left[\hat{f}(\mu')\right] \ge$

$b\left|\hat{g}\left(\mu\right)-\mathrm{E}_{\tau}\left[\hat{g}\left(\mu'\right)\right]\right|$, which means $\hat{f}\left(\mu\right)-\mathrm{E}_{\tau}\left[\hat{f}\left(\mu'\right)\right]\geq b'\left|\hat{g}\left(\mu\right)-\mathrm{E}_{\tau}\left[\hat{g}\left(\mu'\right)\right]\right|$. This last inequality implies $\mathrm{E}_{\tau}\left[\hat{f}\left(\mu'\right)+b'\hat{g}\left(\mu'\right)\right]\leq\hat{f}\left(\mu\right)+b'\hat{g}\left(\mu\right)$ and $\mathrm{E}_{\tau}\left[\hat{f}\left(\mu'\right)-b\hat{g}\left(\mu'\right)\right]\leq\hat{f}\left(\mu\right)-b\hat{g}\left(\mu\right)$. Since these two inequalities hold for any τ s.t. $\mathrm{E}_{\tau}\left[\mu'\right]=\mu$, we know $\mu\in\tilde{M}'$. Hence, $\tilde{M}\subset\tilde{M}'$. Therefore, since $M_i=M_i'$ for all $i\notin\{j,k\}$, we know $M\subset M'$. This in turn implies $T\subset T'$. By Corollary 1, we know $\overline{\tau}\in T$. Hence, the proposition follows directly from Lemma 8. □

Note that proofs of both Proposition 4 and Proposition 5 rely on the fact that, as competition increases (whether through adding senders or increasing misalignment of their preferences), the set of equilibrium outcomes shrinks. This is worth noting since it suggests another sense, not fully captured by the propositions, in which competition increases information revelation. Specifically, $T\subset T'$ implies that the set of coincident beliefs is smaller when there is more competition; hence, with more competition there are fewer prior beliefs such that no revelation is an equilibrium outcome.

Proposition 5 establishes that as preference misalignment b grows, the set of equilibrium outcomes shrinks and the extent of information revealed in equilibrium increases. A natural conjecture, therefore, may be that in the limit where two senders have fully opposed preferences, full revelation becomes the only equilibrium.

Specifically, suppose there are two senders j and k s.t. $v_j=-v_k$. Does the presence of two such senders guarantee full revelation? It turns out the answer is no. For example, if $\hat{v}_j$ is linear, and j and k are the only 2 senders, then $M_j=M_k=\Delta\left(\Omega\right)$ and any outcome is an equilibrium. Moreover, it will not be enough to simply assume that $\hat{v}_j$ is non-linear; as long as it is linear along some dimension of $\Delta\left(\Omega\right)$, it is possible to construct an equilibrium that is not fully revealing along that dimension. Accordingly, we say that $\hat{v}_j$ is *fully non-linear* if it is non-linear along every edge of $\Delta\left(\Omega\right)$, i.e., if for any two degenerate beliefs μ_ω and $\mu_{\omega'}$, there exist two beliefs μ_l and μ_h on the segment $\left[\mu_\omega,\mu_{\omega'}\right]$ such that for some $\gamma\in[0,1]$, $\hat{v}_j\left(\gamma\mu_l+(1-\gamma)\mu_h\right)\neq\gamma\hat{v}_j\left(\mu_l\right)+(1-\gamma)\hat{v}_j\left(\mu_h\right)$. If $v_j=-v_k$ and $\hat{v}_j$ is fully non-linear, then full revelation is indeed the unique equilibrium outcome. Proposition 6 establishes the analogous result for the more general case where there is some subset of senders for whom the game is zero-sum.

Proposition 6. *Suppose there is a subset of senders $J\subset\{1,...,n\}$ s.t. (i) for any a and ω,*

$\sum_{i \in J} v_i (a, \omega) = 0$*, and (ii) there exists* $i \in J$ *s.t.* $\hat{v}_i$ *is fully non-linear. Then, full revelation is the unique equilibrium outcome.*

Detailed proof is in the Appendix.

6.4 Does competition make Receiver better off?

Propositions 3, 4, and 5 establish a sense in which moving from collusion to non-cooperative play, adding senders, and making senders' preferences less aligned all tend to increase information revelation. Since more information must weakly increase Receiver's utility, increasing competition thus tends to make Receiver better off.

To make this observation more precise, we translate our set comparisons of the informativeness of outcomes into set comparisons of Receiver's utilities. Given two ordered sets $(\mathcal{Y}, \succeq)$ and $(\mathcal{Z}, \geq)$, a function $f : \mathcal{Y} \to \mathcal{Z}$ is said to be *increasing* if $y \succeq y'$ implies $f(y) \geq f(y')$. Moreover, if the domain of f is a chain, then an increasing f preserves the set order. We state this formally in the following Lemma:

Lemma 9. *If* $f : (\Delta(\Delta(\Omega)), \succsim) \to (\mathbb{R}, \geq)$ *is increasing, then for any chain* $C \subset \Delta(\Delta(\Omega))$, $\forall T, T' \subset C$, $T \succsim_{s(w)} T' \Rightarrow f(T) \geq_{s(w)} f(T')$.

Proof. First consider the strong order. Consider any $y \in f(T)$ and $y' \in f(T')$. If $y \geq y'$, then $y \vee y' \in f(T)$. Suppose $y' > y$. Let τ and τ' be any elements of $f^{-1}(y) \subset T$ and $f^{-1}(y') \subset T'$, respectively. Since f is increasing and $y > y'$, we know $\tau' > \tau$. Hence, since $T \succsim T'$, it must be the case that $\tau' \in T$. Hence, $y \wedge y' = y' = f(\tau') \in f(T)$. Now consider the weak order. Given $y \in f(T)$, consider any $\tau \in f^{-1}(y)$. Since $T \succsim_w T'$ there is a $\tau' \in T'$ s.t. $\tau \succsim \tau'$. Let $y' = f(\tau')$. Since f is increasing, $y \geq y'$. Given $y' \in f(T')$, consider any $\tau' \in f^{-1}(y')$. Since $T \succsim_w T'$ there is a $\tau \in T$ s.t. $\tau \succsim \tau'$. Let $y = f(\tau)$. Since f is increasing, $y \geq y'$. □

By Blackwell's Theorem (1953), the function $f_u : (\Delta(\Delta(\Omega)), \succsim) \to (\mathbb{R}, \geq)$, which maps distributions of posteriors into the expected utility of a decision-maker with a utility function u, is increasing for any u. Hence, Lemma 9 allows us to translate the results of the previous three subsections into results about Receiver's payoff.

Corollary 2. *Let T^* be the set of equilibrium outcomes and T^c be the set of collusive outcomes. Let T and T' be the sets of equilibrium outcomes when the sets of senders are J and $J' \subset J$, respectively. Let T^b and $T^{b'}$ be the sets of equilibrium outcomes when preferences are $\boldsymbol{v}^b$ and $\boldsymbol{v}^{b'}$, respectively, where $b > b'$. For any maximal chain C that intersects T':*

1. *Receiver's payoffs under $T^* \cap C$ are strongly greater than under $T^c \cap C$*
2. *Receiver's payoffs under $T \cap C$ are weakly greater than under $T' \cap C$*
3. *Receiver's payoffs under $T^b \cap C$ are weakly greater than under $T^{b'} \cap C$*

By the definition of Blackwell informativeness, Corollary 2 applies not only to Receiver, whom senders are trying to influence, but also to any third-party who observes the signal realizations and whose optimal behavior depends on ω.[17]

An alternative to comparing sets of Receiver's payoffs is to consider a selection criterion that picks out a particular outcome from the overall set. As mentioned in Section 5, selection criteria that always pick out a minimally informative equilibrium may be appealing. Under any such criterion, there is a strong sense in which competition makes Receiver better off. Proposition 3 implies that any minimally informative equilibrium gives Receiver a weakly higher payoff than any comparable collusive outcome. Propositions 4 and 5 imply that any minimally informative equilibrium with more senders or less aligned preferences gives Receiver a weakly higher payoff than any comparable minimally informative equilibrium with fewer senders or more aligned sender preferences.

Whether we consider the entire equilibrium set or a particular selection rule, however, our results apply only to mutually comparable outcomes. This is a substantive caveat. If the outcomes under more and less competition are non-comparable, it is possible that the outcome with more competition makes Receiver worse off.

For example, suppose there are two dimensions of the state space, horizontal and vertical. Senders benefit by providing information only about the vertical dimension but strongly dislike providing information about both dimensions. In this case, competition could lead to a coordination failure; there can exist an equilibrium in which senders provide only horizontal information, even though all senders and Receiver would be strictly better off if only vertical information were

[17] In the statement of Corollary 2, we do not need to assume that C intersects T^* or T^c because an empty set is strongly above and below any set and we do not need to assume that C intersects T, T^b, or $T^{b'}$ because all these sets contain $\bar{\tau}$ so any maximal chain must intersect them.

provided:

Example 1. The state space is $\Omega = \{l, r\} \times \{u, d\}$. The action space is $A = \{l, m, r\} \times \{u, d\}$. Denote states, beliefs, and actions by ordered pairs (ω_x, ω_y), (μ_x, μ_y), and (a_x, a_y), where the first element refers to the *l*-*r* dimension and the second element refers to the *u*-*d* dimension. The prior is $\mu_0 = \left(\frac{1}{2}, \frac{1}{2}\right)$. Receiver's preferences are $u(a, \omega) = \frac{1}{100} u_x(a_x, \omega_x) + u_y(a_y, \omega_y)$, where $u_x(a_x, \omega_x) = \frac{2}{3} I_{\{a_x = m\}} + I_{\{a_x = \omega_x\}}$ and $u_y = I_{\{a_y = \omega_y\}}$. There are two senders with identical preferences: $v_1(a, \omega) = v_2(a, \omega) = I_{\{a_x = m\}} I_{\{a_y = \omega_y\}}$. A distribution of posteriors τ^* with support on beliefs $\left(0, \frac{1}{2}\right)$ and $\left(1, \frac{1}{2}\right)$ is an equilibrium outcome. The set of collusive outcomes, T^c, is the same as the set of equilibrium outcomes with a single sender, T'. Each of these sets consists of distributions of posteriors with support on $\left(\left[\frac{1}{3}, \frac{2}{3}\right] \times \{0\}\right) \cup \left(\left[\frac{1}{3}, \frac{2}{3}\right] \times \{1\}\right)$. It is easy to see that Receiver is strictly better off under any outcome in $T^c \cup T'$ than she is under τ^*.

7 Applications

7.1 A criminal trial

In Kamenica and Gentzkow (2011), we introduce the example of a prosecutor trying to persuade a judge that a defendant is guilty. Here, we extend that example to include two senders, a prosecutor (p) and a defense attorney (d).

There are two states, innocent ($\omega = 0$) and guilty ($\omega = 1$). The prior is $\Pr(\omega = 1) = \mu_0 = 0.3$. Receiver (the judge) can choose to either acquit ($a = 0$) or convict ($a = 1$). Receiver's utility is $u(a, \omega) = I_{\{a = \omega\}}$. The prosecutor's utility is $v_p(a, \omega) = a$. The defense attorney's utility is $v_d(a, \omega) = -a$.

If the prosecutor were playing this game by himself, his optimal strategy would be to choose a signal that induces a distribution of posteriors with support $\left\{0, \frac{1}{2}\right\}$ that leads 60% of defendants to be convicted. If the defense attorney were playing this game alone, his optimal strategy would be to gather no information, which would lead the judge to acquit everyone. Because $v_p + v_d = 0$, all outcomes in this game are collusive outcomes.

When the attorneys compete, the unique equilibrium outcome is full revelation. This follows directly from Proposition 6, since $v_p = -v_d$ and the $\hat{v}_i$'s are fully non-linear. Thus, the equilibrium outcome is more informative than every collusive outcome and more informative than the two outcomes each sender would implement on his own, consistent with Propositions 3 and 4. In this example, competition clearly makes Receiver better off.

To make the analysis more interesting, we can relax the assumption that the two senders' preferences are diametrically opposed. In particular, suppose that the defendant on trial is a confessed terrorist. Suppose that the only uncertainty in the trial is how the CIA extracted the defendant's confession: legally ($\omega = 1$) or through torture ($\omega = 0$).

Any information about the CIA's methods released during the trial will be valuable to terrorist organizations; the more certain they are about whether the CIA uses torture or not, the better they will be able to optimize their training methods. Aside from the attorneys' respective incentives to convict or acquit, both prefer to minimize the utility of the terrorists.

Specifically, we assume there is a second receiver, a terrorist organization.[18] The organization must choose a fraction $a_T \in [0,1]$ of its training to devote to resisting torture. The organization's utility is $u_T(a_T,\omega) = -(1-a_T-\omega)^2$. The attorneys' utilities are $v_p(a,\omega) = a - cu_T$ and $v_d(a,\omega) = -a - cu_T$. The parameter $c \in [4,25]$ captures the social cost of terrorism internalized by the attorneys.[19]

If the prosecutor were playing this game alone, his optimal strategy would be to choose a signal that induces a distribution of posteriors $\left\{\frac{1}{2} - \frac{1}{\sqrt{c}}, \frac{1}{2}\right\}$. If the defense attorney were playing this game alone, his optimal strategy would still be to gather no information. The unique collusive outcome is no revelation. To identify the set of equilibrium outcomes, we apply Proposition 2. Panel (a) of Figure 4 plots $\hat{v}_p$ and V_p. We can see that $M_p = \{\mu | \hat{v}_p(\mu) = V_p(\mu)\} = \left[0, \frac{1}{2} - \frac{1}{\sqrt{c}}\right] \cup \left[\frac{1}{2}, 1\right]$. Panel (b) plots $\hat{v}_d$ and V_d. We can see that $M_d = \{\mu | \hat{v}_d(\mu) = V_d(\mu)\} = [0, \frac{1}{2}) \cup \left[\frac{1}{2} + \frac{1}{\sqrt{c}}, 1\right]$. Hence, as panel (c) shows, $M = M_p \cap M_d = \left[0, \frac{1}{2} - \frac{1}{\sqrt{c}}\right] \cup \left[\frac{1}{2} + \frac{1}{\sqrt{c}}, 1\right]$. The set of equilibrium outcomes is the set of τ's whose support lies in this M.

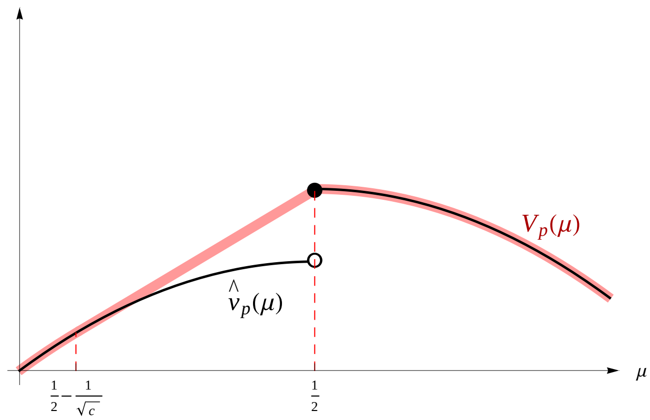
Competition between the attorneys increases information revelation. Every equilibrium outcome is more informative than the collusive outcome (cf: Proposition 3) and more informative than what either sender would reveal on his own (cf: Proposition 4). Moreover, when the extent of shared

[18] As discussed in Section 4.3, our model is easily reinterpreted to allow multiple receivers.

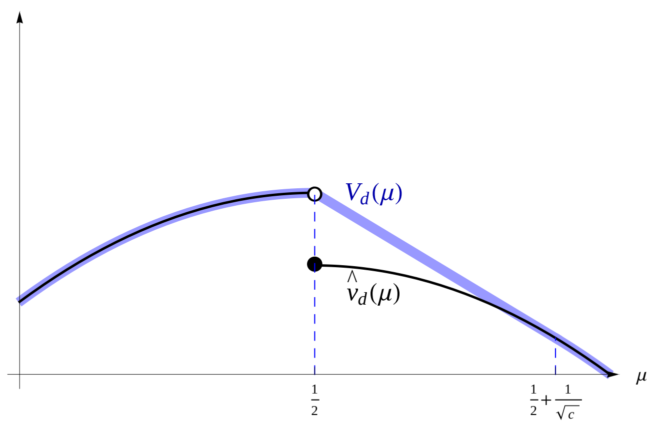
[19] If $c < 4$, the outcome is the same as when $c = 0$; the preferences of the two senders are sufficiently opposed that full revelation is the unique equilibrium outcome. If $c > 25$, both senders are so concerned about giving information to the terrorists that neither wishes to reveal anything.

Figure 4: Characterizing equilibrium outcomes for the criminal trial example

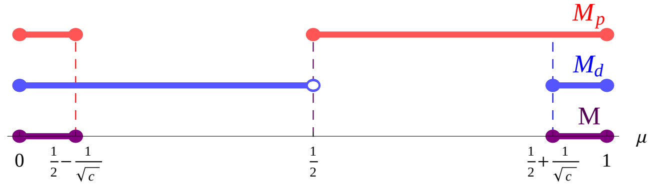
(a) $\hat{v}$ and V functions for sender p



(b) $\hat{v}$ and V functions for sender d



(c) Sets of coincident beliefs $\{\mu : \hat{v}(\mu) = V(\mu)\}$



interest by the two attorneys is greater, i.e., when c is greater, the set of equilibrium outcomes becomes weakly less informative (cf: Proposition 5).

7.2 Advertising of quality by differentiated firms

There are two firms $i \in \{1, 2\}$ which sell differentiated products. The prices of these products are fixed exogenously and normalized to one, and marginal costs are zero. The uncertain state ω is a two-dimensional vector whose elements are the qualities of firm 1's product and firm 2's product. Receiver is a consumer whose possible actions are to buy neither product ($a = 0$), buy firm 1's product ($a = 1$), or buy firm 2's product ($a = 2$) . We interpret the senders' choice of signals as a choice of verifiable advertisements about quality.[20]

There are three possible states: (i) both products are low quality ($\omega = (-5, -5)$), (ii) firm $1's$ product is low quality and firm 2's product is high quality ($\omega = (-5, 5)$), or (iii) both products are high quality ($\omega = (5, 5)$). Let $\mu_1 = \Pr(\omega = (-5, 5))$ and $\mu_2 = \Pr(\omega = (5, 5))$.

The firms' profits are $v_1 = I_{\{a=1\}}$ and $v_2 = I_{\{a=2\}}$. Receiver is a consumer whose utility depends on a, $\omega = (\omega_1, \omega_2)$ and privately observed shocks $\epsilon = (\epsilon_0, \epsilon_1, \epsilon_2)$:[21]

$$
\begin{aligned}
u(a = 0, \omega, \epsilon) &= \epsilon_0 \\
u(a = 1, \omega, \epsilon) &= \omega_1 + \epsilon_1 \\
u(a = 2, \omega, \epsilon) &= \omega_2 + \epsilon_2
\end{aligned}
$$

We assume that the elements of ϵ are distributed i.i.d. type-I extreme value. Senders' expected payoffs at belief μ are thus

$$
\begin{aligned}
\hat{v}_1(\mu) &= \frac{\exp[\mathrm{E}_\mu(\omega_1)]}{1 + \exp[\mathrm{E}_\mu(\omega_1)] + \exp[\mathrm{E}_\mu(\omega_2)]} \\
\hat{v}_2(\mu) &= \frac{\exp[\mathrm{E}_\mu(\omega_2)]}{1 + \exp[\mathrm{E}_\mu(\omega_1)] + \exp[\mathrm{E}_\mu(\omega_2)]}.
\end{aligned}
$$

Figure 5 applies Proposition 2 to solve for the set of equilibrium outcomes. Panel (a) shows $\hat{v}_1$

[20] Note that in this setting, our model allows for firms' advertisements to provide information about the competitor's product as well as their own. This is a reasonable assumption in certain industries. For example, pharmaceutical companies occasionally advertise clinical trials showing unpleasant side-effects or delayed efficacy of a rival product.

[21] As discussed in Section 4.3, our model is easily reinterpreted to allow Receiver to have private information.

and $\hat{v}_2$. Panel (b) shows V_1 and V_2. Panel (c) shows the sets of coincident beliefs M_1 and M_2 and their intersection M. The set of equilibrium outcomes is the set of τ's with supports in M.

Competition between the firms increases information revelation. The set of equilibrium outcomes is weakly more informative than what either firm would reveal on its own (cf: Proposition 4). Although not immediately apparent from Figure 5, the set of equilibrium outcomes is also weakly more informative than the set of collusive outcomes, and is strongly more informative along chains (cf: Proposition 3). The functional form of senders' utilities does not allow us to apply Proposition 5.

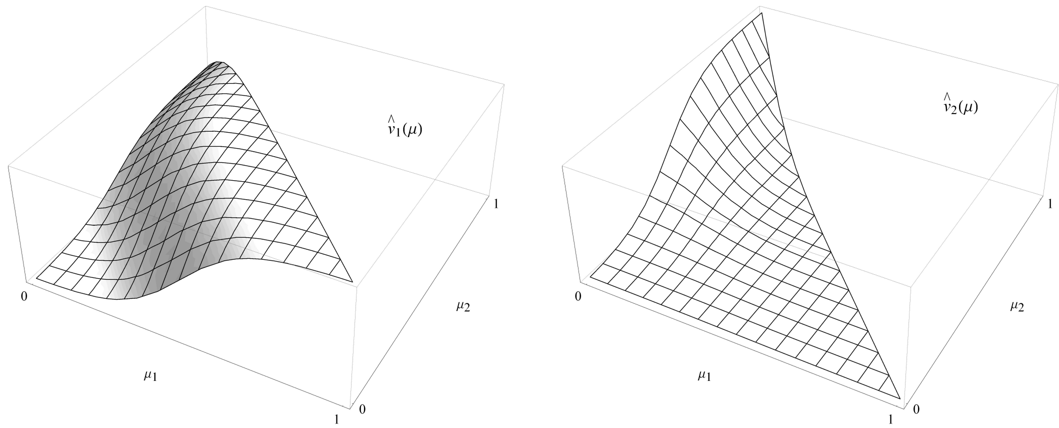
To understand the set of equilibria in this example, it is useful to consider the following two simpler settings. First, suppose $\mu_1 = 0$, so the only possible states are $\omega = (-5, -5)$ and $\omega = (5, 5)$. In this case, the two firms' preferences are aligned: they both want to convince the consumer that $\omega = (5, 5)$. The equilibrium outcomes, which one can easily identify by looking at the μ_2-edges in panel (c), involve partial information revelation. Next, suppose $\mu_2 = 0$, so the only possible states are $\omega = (-5, -5)$ or $\omega = (-5, 5)$. Here, senders' preferences are opposed: sender 2 would like to convince Receiver that $\omega = (-5, 5)$, while sender 1 would like to convince the consumer that $\omega = (-5, -5)$. The unique equilibrium outcome, which one can easily identify by looking at the μ_1-edges in panel (c), is full revelation. This is the case even though each firm on its own would prefer a partially revealing signal.[22] Finally, suppose that $\mu_1 + \mu_2 = 1$, so the only possible states are $\omega = (-5, 5)$ or $\omega = (5, 5)$. The firms' preferences are again opposed, and the unique equilibrium outcome, which one can read off the hypotenuses in panel (c), is again full revelation. This is the case despite the fact that firm 1 would strictly prefer no revelation.

In the full three-state example, the equilibrium involves full revelation along the dimensions where senders' preferences are opposed and partial revelation along the dimension where they are aligned. Consequently, the consumer learns for certain whether or not the state is $\omega = (-5, 5)$, but may be left uncertain whether the state is $\omega = (-5, -5)$ or $\omega = (5, 5)$.

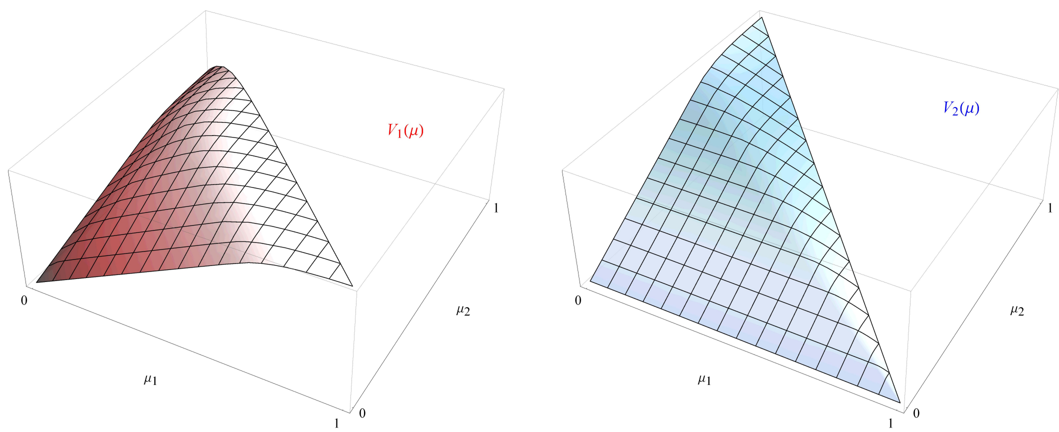
[22] The gain to firm 2 from increasing μ_1 is much larger than the corresponding loss to firm 1; for this reason, at the scale of Figure 5, $\hat{v}_1$ appears flat with respect to μ_1 despite the fact that it is actually decreasing.

Figure 5: Characterizing equilibrium outcomes for the advertising example

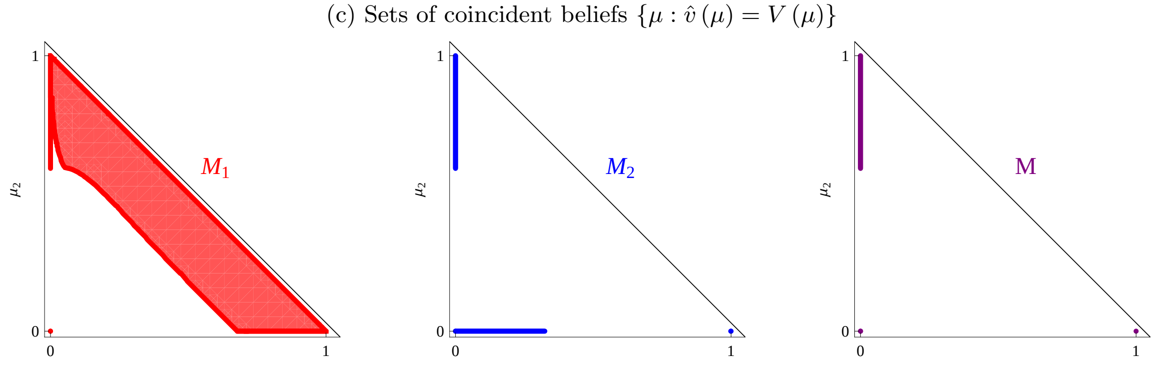
(a) $\hat{v}$ functions for senders 1 and 2



(b) V functions for senders 1 and 2



(c) Sets of coincident beliefs $\{\mu : \hat{v}(\mu) = V(\mu)\}$



8 Conclusion

We study Bayesian persuasion by multiple senders when the set of possible signals is rich. The rich signal space has a lattice structure that makes it possible to combine and compare sets of signals in an intuitive way. In this rich signal space setting, we show that Kamenica and Gentzkow (2016a)'s analysis of competition in persuasion can be extended in two important ways. First, we obtain a more intuitive characterization of equilibrium by drawing on the analytical methods from Aumann and Maschler (1995) and Kamenica and Gentzkow (2011). Second, we relax the assumption that the Receiver observes signal realizations directly, allowing senders to observe realizations privately and potentially conceal information *ex post*. Taken together, the results suggest a strong sense in which competition tends to increase information revelation when signals are rich.

9 Appendix

9.1 Proof of Proposition 2

Lemma 10. *For any sender i and any distribution of posteriors τ:*

$$\hat{v}_i(\mu) = V_i(\mu)\,\forall\mu \in \text{Supp}(\tau) \Leftrightarrow \text{E}_{\tau'}[\hat{v}_i(\mu)] \leq \text{E}_\tau[\hat{v}_i(\mu)]\,\forall\tau' \succsim \tau.$$

Proof. Consider any i and any τ s.t. $\hat{v}_i(\mu) = V_i(\mu)\,\forall\mu \in \text{Supp}(\tau)$. Consider any $\tau' \succsim \tau$ and π' such that $\langle\pi'\rangle = \tau'$. For any s s.t. $\mu_s \in \text{Supp}(\tau)$, consider the conditional distribution of posteriors $\langle\pi'|s\rangle$. We know $\text{E}_{\langle\pi'|s\rangle}[\mu] = \mu_s$. Hence, by Lemma 6, $\text{E}_{\langle\pi'|s\rangle}[\hat{v}_i(\mu)] \leq \hat{v}_i(\mu_s)$. Therefore, $\text{E}_{\tau'}[\hat{v}_i(\mu)] = \sum_{s\in\{s|\mu_s\in\text{Supp}(\tau)\}} p(s)\,\text{E}_{\langle\pi'|s\rangle}[\hat{v}_i(\mu)] \leq \sum_{s\in\{s|\mu_s\in\text{Supp}(\tau)\}} p(s)\,\hat{v}_i(\mu_s) = \text{E}_\tau[\hat{v}_i(\mu)]$.

Conversely, suppose $\exists\mu_s \in \text{Supp}(\tau)$ such that $\hat{v}_i(\mu_s) \neq V(\mu_s)$. By Lemma 6, we know there exists a distribution of posteriors τ'_s with $\text{E}_{\tau'_s}[\mu] = \mu_s$ and $\text{E}_{\tau'_s}[\hat{v}_i(\mu)] > \hat{v}_i(\mu_s)$. By Lemma 2, there exists a π' s.t. $\tau'_s = \langle\pi'|s\rangle$. Let π be any signal s.t. $\langle\pi\rangle = \tau$. Let π'' be the union of $\pi \setminus \{s\}$ and $\{s \cap s' : s' \in \pi'\}$. Then $\langle\pi''\rangle \succsim \langle\pi\rangle = \tau$ and $\text{E}_{\langle\pi''\rangle}[\hat{v}_i(\mu)] = p(s)\,\text{E}_{\tau'_s}[\hat{v}_i(\mu)] + \sum_{\tilde{s}\in\pi\setminus\{s\}} p(\tilde{s})\,\hat{v}_i(\mu_{\tilde{s}}) > p(s)\,\hat{v}_i(\mu_s) + \sum_{\tilde{s}\in\pi\setminus\{s\}} p(\tilde{s})\,\hat{v}_i(\mu_{\tilde{s}}) = \text{E}_\tau[\hat{v}_i(\mu)]$ □

With Lemma 10, it is straightforward to establish Proposition 2.

Proof. Suppose $n \geq 2$. Suppose $\hat{v}_i(\mu) = V_i(\mu)\ \forall i\ \forall \mu \in \text{Supp}(\tau)$. By Lemma 1, there is a π such that $\langle \pi \rangle = \tau$. Consider the strategy profile $\boldsymbol{\pi}$ where $\pi_i = \pi\ \forall i$. Since $n \geq 2$, we know that $\vee \boldsymbol{\pi}_{-i} = \vee \boldsymbol{\pi}$. Hence, for any $\pi_i' \in \Pi$ we have $\pi_i' \vee \boldsymbol{\pi}_{-i} = \pi_i' \vee \boldsymbol{\pi} \trianglerighteq \vee \boldsymbol{\pi}$. Hence, by Lemma 3, $\langle \pi_i' \vee \boldsymbol{\pi}_{-i} \rangle \succsim \langle \vee \boldsymbol{\pi} \rangle$. Lemma 10 thus implies $\text{E}_{\langle \vee \boldsymbol{\pi} \rangle} \hat{v}_i(\mu) \geq \text{E}_{\langle \pi_i' \vee \boldsymbol{\pi}_{-i} \rangle} \hat{v}_i(\mu)$. Hence, $\boldsymbol{\pi}$ is an equilibrium.

Conversely, consider any equilibrium $\boldsymbol{\pi}$. Consider any $\tau' \succsim \langle \vee \boldsymbol{\pi} \rangle$. By Lemma 5, for any sender i there exists $\pi_i' \in \Pi$ such that $\langle \pi_i' \vee \boldsymbol{\pi}_{-i} \rangle = \tau'$. Since $\boldsymbol{\pi}$ is an equilibrium, this means $\text{E}_{\langle \vee \boldsymbol{\pi} \rangle}[\hat{v}_i(\mu)] \geq [\text{E}_{\tau'} \hat{v}_i(\mu)]$ for all i. Lemma 10 then implies that $\hat{v}_i(\mu) = V_i(\mu)\ \forall i\ \forall \mu \in \text{Supp}(\langle \vee \boldsymbol{\pi} \rangle)$. $\square$

9.2 Proof of Proposition 6

We build the proof through the following three lemmas.

Lemma 11. *If there is a subset of senders $J \subset \{1, ..., n\}$ s.t. for any a and ω, $\sum_{i \in J} v_i(a, \omega) = 0$, then for any belief μ^* induced in an equilibrium, for any τ s.t. $\text{E}_\tau[\mu] = \mu^*$, we have $\text{E}_\tau[\hat{v}_i(\mu)] = \hat{v}_i(\mu^*)$ for all $j \in J$.*

Proof. Consider J s.t. $\sum_{i \in J} v(a, \omega) = 0\ \forall a, \omega$ and any μ^* induced in an equilibrium. We must have $\hat{v}_i(\mu^*) = V_i(\mu^*)\ \forall i$, and thus, by Lemma 6, $\text{E}_\tau[\hat{v}_i(\mu)] - \hat{v}_i(\mu^*) \leq 0\ \forall i$. We also have $\sum_{i \in J} \hat{v}(\mu) = 0\ \forall \mu$, which implies $\sum_{i \in J} \text{E}_\tau[\hat{v}_i(\mu)] = 0$. Hence, $\sum_{i \in J} [\text{E}_\tau[\hat{v}_i(\mu)] - \hat{v}_i(\mu^*)] = 0\ \forall i \in J$. Combining this with the earlier inequality, we obtain that $\text{E}_\tau[\hat{v}_i(\mu)] - \hat{v}_i(\mu^*) = 0\ \forall i \in J$. $\square$

Lemma 12. *If $\hat{v}_j$ is non-linear, for any $\mu^* \in \text{int}(\Delta(\Omega))$ there exists a τ s.t. $\text{E}_\tau[\mu] = \mu^*$ and $\text{E}_\tau[\hat{v}_j(\mu)] \neq \hat{v}_j(\mu^*)$.*

Proof. If $\hat{v}_j$ is non-linear, there exist $\{\mu_t\}_{t=1}^T$ and weights β_t s.t. $\sum \beta_t \hat{v}_j(\mu_t) \neq \hat{v}_j(\sum_t \beta_t \mu_t)$. Consider any $\mu^* \in \text{int}(\Delta(\Omega))$. There exists some μ_l and $\gamma \in [0, 1)$ s.t. $\mu^* = \gamma \mu_l + (1 - \gamma) \sum \beta_t \mu_t$. If $\hat{v}_j(\mu^*) \neq \gamma \hat{v}_i(\mu_l) + (1 - \gamma) \sum \beta_t \hat{v}_j(\mu_t)$, we are done. So, suppose that $\hat{v}_j(\mu^*) = \gamma \hat{v}_j(\mu_l) + (1 - \gamma) \sum \beta_t \hat{v}_i(\mu_t)$. Now, consider the distribution of posteriors τ equal to μ_l with probability γ and equal to belief $\sum \beta_t \mu_t$ with probability $1 - \gamma$. We have that $\text{E}_\tau[\mu] = \mu^*$ and $\hat{v}_j(\mu^*) = \gamma \hat{v}_j(\mu_l) + (1 - \gamma) \sum \beta_t \hat{v}_j(\mu_t) \neq \gamma \hat{v}_j(\mu_l) + (1 - \gamma) \hat{v}_j(\sum \beta_t \mu_t) = \text{E}_\tau[\hat{v}_j(\mu)]$. $\square$

Lemma 13. *If $\hat{v}_j$ is fully non-linear, then the restriction of $\hat{v}_j$ to any n-dimensional face of $\Delta(\Omega)$ is non-linear if $n \geq 1$.*

Proof. The definition of fully non-linear states that the restriction of $\hat{v}_j$ to any one-dimensional face of $\Delta(\Omega)$ is non-linear. For any $n \geq 1$, every n-dimensional face of $\Delta(\Omega)$ includes some $(n-1)$-dimensional face of $\Delta(\Omega)$ as a subset. Hence, if the restriction of $\hat{v}_j$ to every $(n-1)$-dimensional face is non-linear, so is the restriction of $\hat{v}_j$ to every n-dimensional face. Hence, by induction on n, the restriction of $\hat{v}_j$ to any n-dimensional face of $\Delta(\Omega)$ is non-linear if $n \geq 1$. □

With these lemmas, the proof of Proposition 6 follows easily.

Proof. Suppose there is a subset of senders $J \subset \{1, ..., n\}$ s.t. the conditions of the proposition hold. Let j be the sender in J for whom $\hat{v}_j$ is fully non-linear. Let μ^* be a belief induced in an equilibrium. Lemmas 11 and 12 jointly imply that μ^* must be at the boundary of $\Delta(\Omega)$. Hence, μ^* is on some n-dimensional face of $\Delta(\Omega)$. But, by Lemma 13, if $n > 0$, the restriction of $\hat{v}_j$ to this n-dimensional face is non-linear. Hence, Lemmas 11 and 12 imply that μ^* must be on the boundary of this n-dimensional face, i.e., it must be on some $(n-1)$-dimensional face. Since this holds for all $n > 0$, we know that μ^* must be on a zero-dimensional face, i.e., it must be an extreme point of $\Delta(\Omega)$. Hence, any belief induced in an equilibrium is degenerate. □

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