Online Appendix for
"Policy Decay and Political Competition"

November 22, 2025

Contents

A1 Definitions

In each period $t = 1, 2, 3, ...$, two policymakers, L (him) and R (her), bargain over policy. The policy space is two-dimensional. The first dimension is an ideological continuum represented by the real line, $\mathbb{R}$. The second dimension captures the quality of policy. Each policy has a maximum quality that we set to be zero, and quality is unbounded below. Thus, the policy space is $\mathbb{R} \times \mathbb{R}^-$.

Policymakers have a common preference over quality but differ in their ideological preferences. In the ideological space, L's ideal point is 0 and R's is π, such that their ideal policy positions in the two-dimensional space are $(0, 0)$ and $(\pi, 0)$, respectively. We assume that per-period utility is separable across dimensions, linear in quality, and quasiconcave in ideology. A common functional form that satisfies these requirements is quadratic-loss utility over policy.

We assume that decay λ_t arrives each period iid from an exponential distribution with rate parameter r. We denote the CDF of λ by $F(\lambda)$ and the corresponding density function by $f(\lambda)$. Only the proof of Lemma 4 relies on specific properties of the exponential distribution;

for the other lemmas it is sufficient that F has full support on the positive reals and finite expectation:

$$E[\lambda] = \bar{\lambda} < \infty$$

.

Section A4 shows that the numerical solution is very similar when we substitute alternative distributional assumptions in place of the exponential, suggesting that the specific shape of this distribution is not crucial.

Finally, we assume that proposals must be on the efficient frontier; i.e., all proposed policies take the form $(x, 0)$. This assumption is needed to guarantee existence of an optimal proposal and rule out cases where Proposer, recognizing that she cannot retain power, offers a policy infinitesimally below the frontier to avoid realizing decay that period.

Our equilibrium concept is Markov Perfect Equilibrium (MPE). We will look for equilibrium strategies that condition only on the current-period values of policy (x), quality (q) and the identity of the proposer (either L or R). We further select equilibria with continuous proposal strategies: equilibrium proposals $x^*(x, q, P)$ are continuous in (x, q) for all $x \in [0, 1], P \in \{L, R\}$.[1] We define the value functions $v_L(x, q, P), v_R(x, q, P)$ for $P \in \{L, R\}$, where P denotes the identity of the Proposer. The value functions give the expected discounted future utility for each player along the equilibrium path of play beginning from the point (x, q, P). We define the total utility of any point as $U_i(x, q, P)$, where $U_i(x, q, P) = u_i(x, q) + \delta v_i(x, q, P)$ and $\delta < 1$ is the common discounting parameter.

A2 Proofs

Proposition 1 in the main text is Proposition 4 in Callander and Martin (2017). The proof is given in the appendix to that paper.

To prove Proposition 2, we first show that the policy space can be restricted without loss to a bounded subset of $\mathbb{R} \times \mathbb{R}^-$.

Lemma 1. *Let $S^\star$ be the set of all points (x, q) visited in equilibrium with positive probability. $S^\star \subseteq [0, \pi] \times [0, -B]$ where $0 < B < \infty$.*

Proof. Suppose that R is Proposer and L in the Opposition role. An identical argument will apply in the opposite case. Note that the worst possible path for L following R's ideal point $(\pi, 0)$ is decay forever from that point, which yields total utility for L in expectation of

[1] The continuity property is needed for Lemma 2.

$$\sum_{t=0}^{\infty} \delta^t u_L(\pi, 0) - \sum_{t=1}^{\infty} t\bar{\lambda}\delta^t$$
$$= \frac{1}{1-\delta} u_L(\pi, 0) - \bar{\lambda} \frac{\delta}{(1-\delta)^2} > -\infty$$

Since the total utility for L here is finite, at some sufficiently negative decay shock he will accept a proposal from R of $(\pi, 0)$ over allowing the utility from the decayed status quo to be realized, regardless of what he expects to follow from that point or from $(\pi, 0)$. R will prefer proposing this point to allowing decay to take hold for the same reason. Hence there must be a finite lower bound below which no status quo is realized in equilibrium. □

Lemma 1 implies that one-period utilities are bounded. This property is sufficient to show that the value functions exist, are continuous, and are unique given the strategy of the other player; uniqueness of the value functions implies that there is always a unique best proposal and hence that a MPE in continuous strategies exists.

Lemma 2. *The value functions v_L and v_R are continuous in x and q and are unique. There exists a MPE in continuous proposal strategies.*

Proof. We apply theorem 9.6 of Stokey and Lucas (1989), which requires four assumptions to hold.

The first (Assumption 9.4), which requires that the policy space is a bounded subset of $\mathbb{R}^l$, is satisfied by Lemma 1.

The second (Assumption 9.5) imposes a technical condition on how the probability measure over the exogenous state variables — in our case, the current-period decay realization λ — can evolve over time. Our assumption that the distribution F is constant over time satisfies this.

The third (Assumption 9.6) requires that the correspondence that maps realizations of decay λ into a feasible set for the endogenous variables (x, q, and P) be nonempty, compact-valued, and continuous in λ. The feasible set in our case is the union of two sets: the intersection of Opposition's and Proposer's acceptance sets with no change in proposal power, and the point $(x, q - \lambda)$ with the current period Opposition in the Proposer role. Nonemptiness is satisfied because the decayed status quo $(x, q - \lambda)$ is always available. Continuity of the acceptance sets at any point where Proposer's offer is accepted is ensured by the continuity of the proposal strategies; an acceptance set that changed discontinuously in λ would imply that some future Proposer's proposal strategies are discontinuous in either x or q. Moreover, decay can occur only at points where the intersection of the acceptance sets is empty (otherwise, the Proposer would propose a point inside, and it would be accepted). Hence the feasible set everywhere changes continuously. Compactness follows because both acceptance sets are closed and bounded (above by the efficient frontier, and below by the inequality constraint defining the acceptance region).

The fourth (Assumption 9.7) imposes that the one-period utility function u is bounded and continuous on the feasible set. u is continuous by assumption. The upper bound on one-period utility is zero, and the lower bound is finite per Lemma 1. □

We now show several useful properties of the value functions $v_L(x, q, P), v_R(x, q, P)$. We first show that the value functions are monotone for both players along the ideological dimension. We can write the value functions as

$$v_i(x, q, P) = \int_0^\infty U_i(g^*(x, q - \lambda, P)) f(\lambda) d\lambda \tag{1}$$

where $g^*(x, q - \lambda, P)$ is the equilibrium outcome resulting from status quo (x, q, P) when the realization of decay is λ. Defining the Opposition's acceptance set $A = \{(x', q', P) : U_{-P}(x', q', P) \geq U_{-P}(x, q - \lambda, -P)\}$, we have:

$$g^*(x, q - \lambda, P) = \begin{cases} \arg\max_A U_P(x', q', P) & \max_A U_P(x', q', P) \geq U_P(x, q - \lambda, -P) \\ (x, q - \lambda, -P) & \text{o.w.} \end{cases} \tag{2}$$

With that notation in place, we can state:

Lemma 3. *The value functions are monotone in x for $x \in [0, \pi]$. $v_L(x, \cdot)$ is decreasing in x, and $v_R(x, \cdot)$ is increasing.*

Proof. We apply Theorem 9.7 of Stokey and Lucas (1989). u_R and u_L are monotone in the directions proposed. The remaining condition to verify is that the acceptance sets A are monotone in the sense defined in Stokey and Lucas' Assumption 9.9. We consider the case with R proposing, L receiving, with current status quo of (x, q) or (x', q), with $x' > x$, and current-period decay (not yet realized) of λ. An identical argument will apply for the case of L proposing, with the directions of inequalities and set containment operations reversed. The needed monotonicity condition is $A(x') \supset A(x)$. The acceptance sets are defined by the utility the Opposition would get after allowing decay to manifest and taking power:

$$A(x') = \{(\tilde{x}, \tilde{q}) \ : \ u_L(\tilde{x}, \tilde{q}) + \delta v_L(\tilde{x}, \tilde{q}, R) \geq u_L(x', q - \lambda) + \delta v_L(x', q - \lambda, L)\}$$
$$A(x) = \{(\tilde{x}, \tilde{q}) \ : \ u_L(\tilde{x}, \tilde{q}) + \delta v_L(\tilde{x}, \tilde{q}, R) \geq u_L(x, q - \lambda) + \delta v_L(x, q - \lambda, L)\}$$

Given this definition, $A(x') \supset A(x)$ iff:

$$u_L(x', q-\lambda) + \delta v_L(x', q-\lambda, L) \leq u_L(x, q-\lambda) + \delta v_L(x, q-\lambda, L) \;\; \forall \;\; \lambda \tag{3}$$
$$\Rightarrow \delta(v_L(x', q-\lambda, L) - v_L(x, q-\lambda, L)) \leq u_L(x, 0) - u_L(x', 0) \tag{4}$$

Note that the right-hand side of (4) is strictly positive because u_L is strictly decreasing in x. Suppose this condition is not satisfied, and instead:

$$v_L(x', q-\lambda, L) - v_L(x, q-\lambda, L) > \frac{1}{\delta}(u_L(x,0) - u_L(x',0)) > 0$$

for some value of $\lambda > 0$. Since $v_L(x', q-\lambda, L) = \max U_L(y)$ s.t. $U_R(y) \geq E_{\lambda'}[U_R(x', q-\lambda-\lambda', R)]$, this implies that there is some value of $\lambda' > 0$ such that

$$U_R(x, q-\lambda-\lambda', R) > U_R(x', q-\lambda-\lambda', R)$$

which implies that

$$v_R(x, q-\lambda-\lambda', R) - v_R(x', q-\lambda-\lambda', R) > \frac{1}{\delta}(u_R(x', 0) - u_R(x, 0)) > 0$$

Repeated application of this logic eventually yields a contradiction because we know from Lemma 1 that for some sufficiently negative value on the vertical dimension, $\underline{q}$, the Opposition accepts Proposer's ideal point from any point on the x dimension and hence $v_P(x, \underline{q}, P) - v_P(x', \underline{q}, P) = 0$.

This shows $v_R(\cdot, R)$ is monotone in x. To show $v_R(\cdot, L)$ is also monotone, note that because L will optimally exactly satisfy R's participation constraint, we have that

$$\begin{aligned} v_R(x, q, L) & \\ &= \int_0^\infty U_R(g^*(x, q-\lambda, L)) f(\lambda) d\lambda \\ &= \int_0^\infty U_R(x, q-\lambda, R) f(\lambda) d\lambda \\ &= \int_0^\infty \left(u_R(x, q-\lambda) + \delta v_R(x, q-\lambda, R)\right) f(\lambda) d\lambda \end{aligned}$$

And because u_R and $v_R(\cdot, R)$ are both monotone in x, $v_R(\cdot, L)$ must be as well. □

We next show that under the exponential distribution, proposal power is always valuable in equilibrium.

Lemma 4. $v_i(x, q, i) > v_i(x, q, -i) \;\; \forall \; (x, q) \in S^\star$.

Proof. Recall that

$$v_i(x, q, -i) = \int_0^\infty U_i(g^*(x, q - \lambda, -i))f(\lambda)d\lambda$$

Maximization by the Proposer implies that $g^*(x, q-\lambda, -i)$ sets $U_i(g^*(x, q-\lambda, -i)) = U_i(x, q-\lambda, i)$. Hence

$$\begin{aligned} v_i(x, q, -i) &= \int_0^\infty U_i(x, q - \lambda, i)f(\lambda)d\lambda \\ &= \int_0^\infty (u_i(x, q - \lambda) + \delta v_i(x, q - \lambda, i))\, f(\lambda)d\lambda \\ &= u_i(x, 0) + q - \bar{\lambda} + \delta E_\lambda[v_i(x, q - \lambda, i)] \end{aligned}$$

which implies

$$E_\lambda[v_i(x, q - \lambda, i)] = \frac{v_i(x, q, -i) - u_i(x, 0) - q + \bar{\lambda}}{\delta} > v_i(x, q, -i) \tag{5}$$

Where the inequality holds because u_i and q are weakly negative, $\bar{\lambda} > 0$, and $\delta < 1$. In words, the expected value of being in power at all points on the vertical line below (x, q) — taking expectations over the distribution of decay — must be strictly greater than the value of being out of power at (x, q).

Using the memoryless property of the exponential, we can write the expectation on the left hand side as:

$$\begin{aligned} E_\lambda[v_i(x, q - \lambda, i)] &= \int_0^\infty v_i(x, q - \lambda, i)f(\lambda)d\lambda \\ &= \int_0^\epsilon v_i(x, q - \lambda, i)f(\lambda)d\lambda + \int_\epsilon^\infty v_i(x, q - \lambda, i)f(\lambda)d\lambda \\ &= \int_0^\epsilon v_i(x, q - \lambda, i)f(\lambda)d\lambda + (1 - F(\epsilon)) \int_0^\infty v_i(x, q - \lambda, i)f(\lambda)d\lambda \end{aligned}$$

Differentiating with respect to ϵ and taking the limit as $\epsilon \to 0$ yields the relation:

$$v_i(x, q, i) = E_\lambda[v_i(x, q - \lambda, i)]$$

and thus $v_i(x, q, i) > v_i(x, q, -i)$. □

We can now state the main characterization result (Proposition 2 in the main text).

Proposition 2. *A Markov Perfect Equilibrium in continuous proposal strategies exists. In any such equilibrium, there exist functions* $\lambda_1, \lambda_2, \lambda_3$*, with* $\lambda_j : [0, \pi] \to [0, \infty)$*,* $j \in \{1, 2, 3\}$*, and a policy threshold* $x' \in (0, \pi)$ *such that when* δ *is sufficiently close to 1, the following properties hold for any status quo* $x \in [0, \pi]$*:*

1. *For* $\lambda_t < \lambda_1(x)$*, with* $\lambda_1(x) > 0$ *if and only if* $x \leq x'$*, L rejects R's proposal such that delay is experienced and power transitions.*

2. *For* $\lambda_t \leq \lambda_2(x)$ *with* $\lambda_2(x) \geq \lambda_1(x)$*, R concedes on policy offering* $x_P < x$ *or L rejects the proposal.*

3. *For* $\lambda_t \geq \lambda_3(x)$ *with* $\lambda_3(x) \geq \lambda_2(x)$*, R proposes* $x_P > x$ *and this proposal is accepted.*

Behavior is symmetric when L is the Proposer.

Proof. Existence is Lemma 2.

To show part 1), note that decay must occur in equilibrium as a consequence of Lemma 4. Since Opposition (strictly) prefers to take power, when the realization of the decay shock is small and the status quo x is close to Opposition's ideal, the proposer does not have enough policy concessions to give to convince Opposition to remove decay and allow Proposer to retain power. The maximum concession that the Proposer can give is to offer Opposition's ideal point, which (in the R-receiving case) has utility to R of $\delta v_R(\pi, 0, L)$. The Opposition will accept this offer if:

$$\delta v_R(\pi, 0, L) \geq u_R(x, -\lambda) + \delta v_R(x, 0, R)$$

By continuity and monotonicity of the value functions and the full-support assumption on $F(\lambda)$, there exist x sufficiently close to π and λ sufficiently close to 0 that this constraint cannot be satisfied.

To show part 2), suppose the path is currently at $(x, 0)$ and the current realization of decay is λ. The Opposition will accept Proposer's offer of $(x', 0)$ if $U_i(x', 0, -i) \geq U_i(x, -\lambda, i)$, i.e. that:

$$u_i(x', 0) + \delta v_i(x', 0, -i) \geq u_i(x, 0) - \lambda + \delta v_i(x, -\lambda, i)$$

Rearranging:

$$u_i(x', 0) - u_i(x, 0) + \lambda \geq \delta(v_i(x, -\lambda, i) - v_i(x', 0, -i))$$

Plugging in $x' = x$, we get:

$$\lambda \geq \delta(v_i(x, -\lambda, i) - v_i(x, 0, -i))$$

Suppose a return to the frontier is acceptable to Opposition for all realizations of λ from $(x, 0)$. Then the inequality above also holds in expectation:

$$\bar{\lambda} \geq \delta(E_\lambda[v_i(x, -\lambda, i)] - v_i(x, 0, -i))$$

Plugging in from equation (5) we have:

$$(1 - \delta)v_i(x, 0, -i) \leq u_i(x, 0)$$

where $u_i(x, 0) \leq 0$, with the inequality strict at all points other than Opposition's ideal, so there is some δ close enough to 1 that this is a contradiction for all such x. Hence, either decay occurs or Proposer concedes.

Part 3) follows directly from the logic of Lemma 1. No matter the status quo on the policy dimension, because the expected future utility is bounded, there exists some current-period decay large enough to induce the Opposition to accept an offer that moves the status quo towards Proposer, regardless of what path of play she anticipates will follow from then on.

□

A3 Probabilistic transition rules

All properties of the equilibrium given in Proposition 2 extend to the case where the transition rule is probabilistic rather than deterministic as in our baseline case, under two assumptions:

1. The probability of transition is a function only of the quality of implemented policy and not of the policy location: $\mathbb{P}(P_t \neq P_{t-1}) \equiv \rho(q_{t-1})$.

2. $\lim_{q \to 0} \rho(q) \neq 0$

Lemma 1 is unchanged by this modification.

In Lemma 2, only the demonstration of Stokey and Lucas' Assumption 9.6 requires slight modification, that the feasible set now contains an additional point, $(x, q - \lambda, P)$, in addition to $(x, q - \lambda, -P)$. This addition preserves the required properties of nonemptiness, compactness, and continuity.

To show Lemma 3, modify Equation (4) to the case with probabilistic transition, yielding:

$$\begin{aligned}
&\delta(\rho(q-\lambda)v_L(x',q-\lambda,L)+(1-\rho(q-\lambda))v_L(x',q-\lambda,R)\\
&-\rho(q-\lambda)v_L(x,q-\lambda,L)-(1-\rho(q-\lambda))v_L(x,q-\lambda,R))\\
&\qquad\qquad\qquad\qquad\leq u_L(x,0)-u_L(x',0)
\end{aligned}$$

Rearranging,

$$\begin{aligned}
\rho(q-\lambda)\left(v_L(x',q-\lambda,L)-v_L(x,q-\lambda,L)\right)+\\
(1-\rho(q-\lambda))\left(v_L(x',q-\lambda,R)-v_L(x,q-\lambda,R)\right)\\
\leq \frac{u_L(x,0)-u_L(x',0)}{\delta}
\end{aligned}$$

Analogously to the proof with deterministic transition, if this property does not hold (a necessary condition for $v_R(\cdot,R)$ to be non-monotone in the direction proposed) then either $v_L(x',q-\lambda,L)-v_L(x,q-\lambda,L)>0$ or $v_L(x',q-\lambda,R)-v_L(x,q-\lambda,R)>0$ for some $\lambda>0$.

As in the proof with deterministic transition, the former condition implies $v_R(x,q-\lambda-\lambda',R)-v_R(x',q-\lambda-\lambda',R)$ for some $\lambda'>0$, and so on until we reach the same contradiction that for some sufficiently large q the value functions are constant in x.

The latter condition leads to exactly the same contradiction, with the only difference being that the logical chain starts one additional step below the x-axis: $v_L(x',q-\lambda,R)-v_L(x,q-\lambda,R)>0$ implies $U_L(x',q-\lambda-\lambda',R)>U_L(x,q-\lambda-\lambda',R)$ for some $\lambda'>0$, which implies $v_L(x',q-\lambda-\lambda',L)-v_L(x,q-\lambda-\lambda',L)>0$.

To show Lemma 4, modify equation 5 to the case with probabilistic transition as follows:

$$E_\lambda[\rho(\lambda)v_i(x,q-\lambda,i)+(1-\rho(\lambda))v_i(x,q-\lambda,-i)]=\frac{v_i(x,q,-i)-u_i(x,0)-q+\bar{\lambda}}{\delta}>v_i(x,q,-i) \tag{6}$$

The expectation $E_\lambda[\rho(\lambda)v_i(x,q-\lambda,i)+(1-\rho(\lambda))v_i(x,q-\lambda,-i)]$ on the left hand side in (6) is:

$$
\begin{aligned}
&= \int_0^\infty \left(\rho(\lambda)v_i(x,q-\lambda,i) + (1-\rho(\lambda))v_i(x,q-\lambda,-i)\right) f(\lambda)d\lambda \\
&= \int_0^\epsilon \left(\rho(\lambda)v_i(x,q-\lambda,i) + (1-\rho(\lambda))v_i(x,q-\lambda,-i)\right) f(\lambda)d\lambda \\
&\quad + \int_\epsilon^\infty \left(\rho(\lambda)v_i(x,q-\lambda,i) + (1-\rho(\lambda))v_i(x,q-\lambda,-i)\right) f(\lambda)d\lambda \\
&= \int_0^\epsilon \left(\rho(\lambda)v_i(x,q-\lambda,i) + (1-\rho(\lambda))v_i(x,q-\lambda,-i)\right) f(\lambda)d\lambda \\
&\quad + (1-F(\epsilon)) \int_0^\infty \left(\rho(\lambda)v_i(x,q-\lambda,i) + (1-\rho(\lambda))v_i(x,q-\lambda,-i)\right) f(\lambda)d\lambda
\end{aligned}
$$

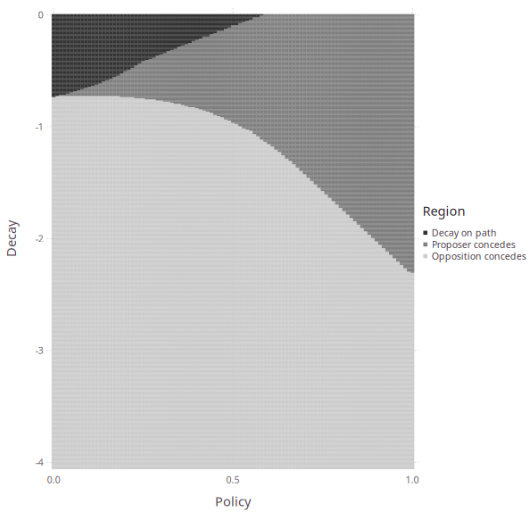
Again differentiating both sides with respect to ϵ and taking limits as $\epsilon \to \infty$, we have the similar relation to that derived in Lemma 4:

$$
\lim_{q\to 0} \rho(q)v_i(x,q,i) + (1 - \lim_{q\to 0} \rho(q))v_i(x,q,-i) = E_\lambda[\rho(\lambda)v_i(x,q-\lambda,i) + (1-\rho(\lambda))v_i(x,q-\lambda,-i)]
$$

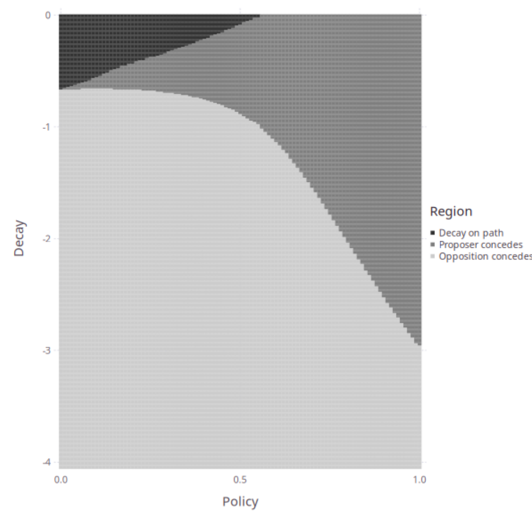
and hence $\lim_{q\to 0} \rho(q)v_i(x,q,i) + (1 - \lim_{q\to 0} \rho(q))v_i(x,q,-i) > v_i(x,q,-i)$, which simplifies to $v_i(x,q,i) > v_i(x,q,-i)$ as desired for any $\lim_{q\to 0} \rho(0) \neq 0$.

A4 Equilibrium under alternative distributional assumptions

Although the proof of Proposition 2 uses a property of the exponential distribution, it does not appear that this property is critical to the result. We solve numerically for the value functions with two alternative distributions of decay, uniform and lognormal, holding the mean of the distribution at 1 as in the baseline case. The results are qualitatively very similar to the baseline case.



(a) Uniform on $[0, 2]$.



(b) Lognormal ($\mu = -0.5, \sigma = 1.0$)

Figure A1: Equilibrium regions under alternative assumptions about the distribution of decay. $E[\lambda] = 1$ in both, as in the baseline case.