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ON THE RANGE OF A REGENERATIVE SEQUENCE

Peter W. GLYNN

Department of Industrial Engineering, University of Wisconsin-Madison, Madison, Wisconsin 53706, USA

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For a given countable partition of the range of a regenerative sequence $\{X_n: n \geq 0\}$, let R_n be the number of distinct sets in the partition visited by X up to time n. We study convergence issues associated with the range sequence $\{R_n: n \geq 0\}$. As an application, we generalize a theorem of Chosid and Isaac to Harris recurrent Markov chains.

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1. Introduction

Let $\{X_n: n \geq 0\}$ be a stochastic sequence taking values in a measurable space $(E, \cdot)$. For a given family $\{A_n: n \geq 0\}$ of $\mathscr{E}$-measurable sets partitioning E, set $Y_n = k$ if $X_n \in A_k$. Let $\phi(i, i) = 1$ and set

$$\phi(i,j) = I(Y_i \neq Y_j, Y_{i+1} \neq Y_j, \ldots, Y_{j-1} \neq Y_j), \qquad R(i,j) = \sum_{k=i}^{j} \phi(i,k), \quad (1.1)$$

for $i < j$, where $I(A)$ is 1 or 0 depending on whether or not $\omega \in A$. Note that $R(i,j)$ is the number of distinct sites visited by $\{Y_n\}$ during the interval $[i,j]$. The process $\{R(0, n): n \geq 0\}$, which is called the *range sequence* associated with $\{X_n: n \geq 0\}$, counts the number of distinct sets A_k visited by X up to time n. The range process $\{R(0, n): n \geq 0\}$ has been extensively studied, in the case that $\{X_n: n \geq 0\}$ is a random walk with stationary increments; see, for example, Dvoretzky and Erdos [6], p. 35-40 of Spitzer [12], and Jain and Pruitt [8]. Chosid and Isaac [1, 2] have considered the problem when $\{X_n: n \geq 0\}$ is a recurrent countable state Markov chain.

In this paper, we wish to study the range process in the case that $\{X_n: n \geq 0\}$ is a delayed regenerative sequence, with associated regeneration times $T_1, T_2, \ldots$ (see p. 298-302 of Cinlar [4] for the definition). Our starting point is a simple decomposition formula for the range. In Section 2, we use this formula to improve the main result of Chosid and Isaac. Section 3 discusses the application of the results to

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Harris recurrent Markov chains, and considers a generalization to the case where $\{A_n: n \geq 0\}$ is not a partition of E.

2. Convergence results for the range

Let $R_{ik} = R(T_i, T_k - 1)$ for $i < k$, $W_i = R_{1,i+1} - R_{1i}$, and set $S(n) = \inf\{j \geq T_1: Y_j = n\}$. Then,

$$W_i = \sum_{k=T_i}^{T_{i+1}-1} \phi(T_1, k) = \sum_{k=T_i}^{T_{i+1}-1} \phi(T_1, k) \cdot \sum_{n=0}^{\infty} I(Y_k = n)$$

$$= \sum_{n=0}^{\infty} \sum_{k=T_i}^{T_{i+1}-1} I(Y_{T_1} \neq n, \ldots, Y_{k-1} \neq n, Y_k = n)$$

$$= \sum_{n=0}^{\infty} I(T_i \leq S(n) < T_{i+1}). \tag{2.1}$$

The decomposition formula (2.1) plays an important role in our development.

(2.2) Lemma. (i) *Set $\tau_i = T_{i+1} - T_i$. Then $E(W_i/\tau_i) \to 0$ as $i \to \infty$.*
(ii) *If $EW_1^m < \infty$ and $m \geq 1$, then $EW_i^m \to 0$ as $i \to \infty$.*

Proof. For (i), observe that

$$E(W_i/\tau_i) = \sum_{n=0}^{\infty} E\{1/\tau_i;\, T_i \leq S(n) < T_{i+1}\}$$

$$= \sum_{n=0}^{\infty} P\{S(n) \geq T_2\}^{i-1} E\{1/\tau_1;\, S(n) < T_2\} \tag{2.3}$$

where the second equality follows from the regenerative property of X. Since

$$\sum_{n=0}^{\infty} E\{1/\tau_1;\, S(n) < T_2\} = E(W_1/\tau_1) \leq 1,$$

one may apply bounded convergence to (2.3) to obtain (i). For (ii), note that

$$EW_i^m = \sum_{n_1,\ldots n_m} P\{T_i \leq S(n_1), \ldots, S(n_m) < T_{i+1}\}$$

$$= \sum_{n_1,\ldots,n_m} P\{S(n_1) \geq T_2, \ldots, S(n_m) \geq T_2\}^{i-1}$$

$$\cdot P\{S(n_1) < T_2, \ldots, S(n_m) < T_2\}. \tag{2.4}$$

The bounded convergence theorem then applies to (2.4) provided that

$$\sum_{n_1,\ldots,n_m} P\{S(n_1) < T_2, \ldots, S(n_m) < T_2\} = EW_1^m < \infty. \qquad \square$$

When $m \geq 1$, Minkowski's inequality yields

$$(E(R_{1n}/n)^m)^{1/m} \leq \sum_{i=1}^{n-1} (E(W_i/n)^m)^{1/m} = \frac{1}{n} \sum_{i=1}^{n-1} (EW_i^m)^{1/m}$$

so the following corollary is immediate.

(2.5) Corollary. *If $EW_1^m < \infty$ and $m \geq 1$, then $E(R_{1n}/n)^m \to 0$ as $n \to \infty$.*

The following lemma shows that the process $\{R_{ik}: 0 \leq i < k\}$ $(T_0 = 0)$ obeys a subadditive inequality.

(2.6) Lemma. *If $0 \leq i < j < k$, then $R_{ik} \leq R_{ij} + R_{jk}$.*

Proof. If $0 \leq i < j < k$, then (1.1) implies that

$$\begin{aligned} R_{ij} &= R_{ij} + \sum_{l=T_j}^{T_k - 1} \phi(T_i, l) = R_{ij} + \sum_{l=T_j}^{T_k - 1} I(Y_{T_i} \neq Y_l, \ldots, Y_{l-1} \neq Y_l) \\ &\leq R_{ik} + \sum_{l=T_j}^{T_k - 1} I(Y_{T_j} \neq Y_l, \ldots, Y_{l-1} \neq Y_l) = R_{ij} + R_{jk}. \qquad \square \end{aligned}$$

The following theorem now follows easily.

(2.7) Theorem. *If $EW_1 < \infty$, then $R_{0n}/n \to 0$ a.s.*

Proof. The regenerative structure of X implies that the distribution of $\{R_{k+1,j+1}: 1 \leq k < j\}$ is identical to that of $\{R_{kj}: 1 \leq k < j\}$. By virtue of Corollary 2.5 ($m = 1$), the nonnegativity of R_{kj}, and Lemma 2.6, this implies that $\{R_{kj}: 1 \leq k < j\}$ satisfies the postulates of the subadditive ergodic theorem (see Kingman (1973)). Hence, $R_{1n}/n \to 0$ as $n \to \infty$ a.s. But by Lemma 2.6,

$$0 \leq R_{0n}/n \leq R_{01}/n + R_{1n}/n \to 0 \quad \text{a.s.}$$

as $n \to \infty$. $\square$

We wish to point out that the subadditive ergodic theorem has been previously used to analyze the mean range in a different setting; see Derriennic [5].

Corollary 2.5, Lemma 2.6, and the observation that $R(0, n) \leq R_{0,n+1}$ yield the next result.

(2.8) Corollary. (i) *If $EW_1 < \infty$, then $R(0, n)/n \to 0$ a.s.*

(ii) *If $E(R_{01} + W_1)^m < \infty$, and $m \geq 1$, then $E(R_{0n}/n)^m \to 0$ as $n \to \infty$.*

Recall that any irreducible recurrent Markov chain $\{X_n: n \geq 0\}$ on $\{0, 1, \ldots\}$ can be regarded as a regenerative sequence. In particular, Corollary 2.8 shows that if

$$\alpha(i) \triangleq E\left\{ \sum_{j=0}^{T_1(i)-1} \phi(0,j) \,\middle|\, X_0 = i \right\} < \infty$$

$(T_n(i) = \inf\{m > T_{n-1}(i): X_m = i\}$, $T_0(i) = 0)$ for some i, then $R(0, n)/n \to 0$ a.s. This is Theorem 1 of [1]. Note that if $\{X_n: n \geq 0\}$ is positive recurrent, then $\alpha(i) \leq E\{T_1(i) \mid X(0) = i\} < \infty$, so that the mean range then automatically converges to zero a.s. Our results are somewhat stronger than those of [1] in the null recurrent case. Theorem 2.7 proves that $R_{0n}/n \to 0$ a.s., whereas [1] only proves that R_{0n}/n is a.s. bounded under the hypothesis $EW_1 < \infty$. In the same spirit, Corollary 2.8(ii) and Lemma 2.2(i) are more complete than those of [1].

A natural question, in the context of the Markov chains discussed above, is whether finiteness of $\alpha(i)$ is a solidarity property; in other words, if $\alpha(i)$ is finite for one i, need $\alpha(i)$ be finite for all i. Of course, if $R(0, n)/n \to 0$ a.s. were to imply that $\alpha(i) < \infty$, we would be done. However, as pointed out in [2], a symmetric nearest neighbor random walk on the integers obeys $R(0, n)/n \to 0$ a.s., and yet $\alpha(i) = \infty$ for all i.

(2.9) Proposition. *Let X be an irreducible recurrent Markov chain. Then, if $\alpha(i) < \infty$ for one i, $\alpha(i)$ is finite for all i.*

The proof of this result is similar to an argument appearing in Chung (1967, p. 84), in connection with a solidarity problem concerning moments of certain functionals.

As indicated in [1], the mean range $R(0, n)/n$ can display a wide range of different limit behavior, in the absence of the moment condition $EW_1 < \infty$.

(2.10) Example. Let $\tau_1, \tau_2, \ldots$ be a sequence of positive independent identically-distributed integer-valued random variables. Suppose their common distribution F is in the domain of attraction of a stable law with parameter $0 < \alpha < 1$. Let $S_n = \tau_1 + \cdots + \tau_n$ and set $l(n) = \max\{k: S_k \leq n\}$. The process $X_n = S_{l(n)+1} - n$ is an irreducible recurrent Markov chain on $\{0, 1, 2, \ldots\}$. Letting $A_n = \{n\}$, we consider the mean range on the subsequence $\{S_n: n \geq 1\}$. It is easily checked that

$$R(0, S_n - 1)/S_n = \max_{1 \leq k \leq n} \tau_k / S_n$$

so that

$$R(0, S_n - 1)/S_n \Rightarrow Z(\alpha)$$

as $n \to \infty$, where $Z(\alpha)$ is a nondegenerate r.v. and $\Rightarrow$ denotes weak convergence (see Feller (1971, p. 465)). We conclude that $R(0, n)/n$ does not converge to zero a.s. along the subsequence S_n.

So far, our study of convergence has centered on normalizing $R(0, n)$ by n. In view of the geometric factors in (2.3) and (2.4), it is natural to investigate whether normalizing by $n^{-\alpha}$ $(0<\alpha<1)$ is adequate for $ER(0, n)$.

(2.11) Example. Let $\{X_n: n \geq 0\}$ be a sequence of nonnegative independent identically distributed integer-valued random variables. Then, X is regenerative with $T_n = n$, and $EW_1 \leq 1$. Suppose that $n^{-\alpha}ER(0, n) \to 0$ as $n \to \infty$ where $\alpha < 1$. Then,

$$\sum_{k=1}^{n} E\phi(0, k)/k = \sum_{k=1}^{n} (ER(0, k) - ER(0, k-1))/k$$

$$= \sum_{k=1}^{n} ER(0, k)/k(k+1) + R(0, n)/(n+1) - R(0, 0); \tag{2.12}$$

letting $n \to \infty$, we see that the second term vanishes and the first is summable since we are assuming that $ER(0, n) = 0(n^{\alpha})$ for $\alpha < 1$. We will now show that (2.12) diverges in general, proving that $ER(0, n) = O(n^{\alpha})$ is not valid without further assumptions. Using (2.1), we see that

$$\sum_{k=1}^{\infty} E\phi(0, k)/k = \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} P\{X_i \neq n\}^{k-1} P\{X_i = n\}/k$$

$$= -\sum_{n=0}^{\infty} \log(P\{X_i = n\}) P\{X_i = n\} \tag{2.13}$$

where we interpret $\log(0) \cdot 0 = 0$. Choosing the mass function $P\{X_i = n\}$ to be $C/n(\log n)^2$ (some constant C) for $n \geq 2$ causes (2.13) to diverge.

We therefore have the following conclusion. If $\{X_n: n \geq 0\}$ is a regenerative sequence for which $E\tau_1 < \infty$, then $ER(0, n)/n$ converges to zero regardless of the partition used. On the other hand, one needs conditions on the partition to ensure convergence to $ER(0, n)/n^{\alpha}$ to zero, for $\alpha < 1$.

(2.14) Proposition. *Suppose that* $\sum_{n=0}^{\infty} n^{\gamma} P\{S(n) < T_2\} < \infty$ *for some* $\gamma > 1$. *Then,* $n^{-\alpha} R_{0n} \to 0$ *a.s. and* $n^{-\alpha} ER_{1n} \to 0$, *as* $n \to \infty$, *where* $\alpha = 1/\gamma$.

Proof. Let $Z_n = \max\{k: S(k) < T_{n+1}\}$ and observe that $R_{1n} \leq Z_n$. But $Z_n = \max\{V_1, \ldots, V_n\}$, where $\{V_i: i \geq 1\}$ is the independent and identically distributed sequence defined by

$$V_i = \max\{k: T_i \leq S(k) < T_{i+1}\}. \tag{2.15}$$

Noting that

$$EV_1^{\gamma} = \sum_{n=0}^{\infty} n^{\gamma} P\{S(n) < T_2;\ S(m) \geq T_2 \text{ for } m > n\}$$

$$\leq \sum_{n=0}^{\infty} n^{\gamma} P\{S(n) < T_2\}, \tag{2.16}$$

it follows that $EV_i^\gamma < \infty$ under our hypotheses. Hence, $\sum_{k=1}^{n} V_k^\gamma / n$ converges to a finite quantity, which implies that $Z_n^\gamma / n \to 0$ a.s. Thus, $n^{-\alpha} R_{0n} \to 0$ a.s.

For the convergence of $n^{-\alpha} ER_{1n}$, we use the finiteness of EV_k^γ to conclude that $EZ_n^\gamma / n \to 0$ as $n \to \infty$ (see p. 90 of Chung [3]). But

$$0 \leq (ER_{1n}/n^\alpha)^\gamma \leq (EZ_n/n^\alpha)^\gamma \leq EZ_n^\gamma / n, \tag{2.17}$$

where the last inequality is a statement of the fact that $(E|X|^r)^{1/r}$ is a nondecreasing function of r for any random variable X (Feller [7, p. 155]). Relation (2.17) implies our result. □

Recalling that $R(0, n) \leq R_{01} + R_{1n}$, we obtain the following corollary.

(2.18) Corollary. *Assume that the conditions of Proposition* 2.14 *are in force. If* $ER_{01} < \infty$, *then* $n^{-\alpha} ER(0, n) \to 0$ *as* $n \to \infty$, *for* $\alpha \geq 1/\gamma$.

To conclude this section, recall that if $\{X_n : n \geq 0\}$ is a delayed regenerative sequence for which $E\tau_1 < \infty$, then

$$\frac{1}{n} \sum_{k=1}^{n} P\{X_k \in B\} \to \pi(B) \triangleq \frac{1}{E\tau_1} P\{X_{T_1+k} \in B;\, \tau_1 > k\}$$

as $n \to \infty$ (see [4, p. 299]); the probability π is called the ergodic measure for the sequence. Note that

$$P\{S(n) < T_2\} \leq \sum_{k=0}^{\infty} P\{X_{T_1+k} \in A_n;\, \tau_1 > k\} = \pi(A_n) E\tau_1.$$

Hence, if $E\tau_1 < \infty$, a sufficient condition for $n^{-\alpha} ER_{1n} \to 0$ is to require that the partition satisfy $\sum_{n=0}^{\infty} n^\gamma \pi(A_n) < \infty$ for some $\gamma \geq 1/\alpha$, where π is the ergodic measure of X.

3. Some extensions

The analysis of Section 2 can be easily extended to cover the case in which $\{X_n : n \geq 0\}$ is a Harris recurrent Markov chain (see Revuz [11, p. 75] for the definition). For any initial distribution for X_0, a probability space can be constructed which supports both the Markov chain $\{X_n : n \geq 0\}$ and a sequence $\{T_n : n \geq 1\}$ of random times, and which satisfy:

(i) $\{X_{T_n+k} : k \geq 0\}$ has an identical distribution for each $n \geq 1$,

(ii) $\mathcal{B}(X_j : j < T_n)$ is independent of $\mathcal{B}(X_j : j \geq T_{n+1})$ for $n \geq 1$;

see Niemi and Nummelin (1982) for details of the construction. The Markov chain X can therefore be analyzed as a stochastic process analog of a 1-dependent sequence of identically distributed random variables. To extend the results of Section 2 to our current setting, it is necessary to obtain a moment bound similar to (2.4).

Retaining the notation of Section 2, we note that the 1-dependence yields the bound

$$\begin{aligned} EW_{2m+1} &= \sum_{n=0}^{\infty} P\{T_{2m+1} \leq S(n) < T_{2m+2}\} \\ &\leq \sum_{n=0}^{\infty} \prod_{j=0}^{m-1} P\{Y_{T_{2j}} \neq n, \ldots, Y_{T_{2j+1}} \neq n\} \cdot P\{S(n) < T_2\} \\ &= \sum_{n=0}^{\infty} P\{S(n) \geq T_2\}^m P\{S(n) < T_2\}. \end{aligned} \tag{3.1}$$

Since Lemma 2.6 clearly continues to hold, systematic application of (3.1) proves that Theorem 2.7 and its Corollary 2.8 remain valid in the Harris chain setting. Further verification, based also on (3.1), proves that Proposition 2.18 also works for Harris chains.

Our second extension concerns the case where the collection $\Gamma = \{A_n : n \geq 0\}$ is not a partition of E. For a given family Γ of sets satisfying $\bigcup_{A \in \Gamma} A = E$, let

$$\phi(\Gamma, \Lambda) = \text{card}\{\Lambda \cap A : A \in \Gamma\} \tag{3.2}$$

for $\Lambda \subseteq E$. If Γ is a countable partition of E and $\Lambda_n = \{X_0, \ldots, X_n\}$, then $\phi(\Gamma, \Lambda_n) = R(0, n) + 1$, where $R(0, n)$ is the range process associated with the partition Γ. Thus, (3.2) legitimately generalizes the range sequence studied in Section 2. In the case that Γ is not a partition, $\phi(\Gamma, \Lambda_n)$ can potentially be of magnitude 2^n, and hence it is not reasonable to expect that $\phi(\Gamma, \Lambda_n)/n$ will always converge. Instead, it is natural to study $\log \phi(\Gamma, \Lambda_n)/n$.

Set $\phi_{ik} = \phi(\Gamma, \Lambda_{ik})$ for $0 \leq i < k$ where $\Lambda_{ik} = \{X_{T_i}, \ldots, X_{T_k - 1}\}$.

(3.3) Lemma. *For* $0 \leq i < j < k$, $\log \phi_{ik} \leq \log \phi_{ij} + \log \phi_{jk}$.

Proof. The argument follows that used by Steele (1978). Note that $A \cap \Lambda_{ik} = (A \cap \Lambda_{ij}) \cup (A \cap \Lambda_{jk})$, so that there are fewer sets of the form of the form $A \cap \Lambda_{ik}$ than pairs of sets $A \cap \Lambda_{ij}$, $A \cap \Lambda_{jk}$. It follows that $\phi_{ik} \leq \phi_{ij}\phi_{jk}$. □

The following theorem is an easy consequence.

(3.4) Theorem. *Let X be a delayed regenerative sequence satisfying* $E \log \phi_{12} < \infty$. *Then,* $(\log \phi_{1n})/n \to V(\Gamma)$ *a.s. as* $n \to \infty$, *where* $V(\Gamma)$ *is a finite constant.*

Proof. Since $\{\log \phi_{ik} : 1 \leq i < k\}$ has the same distribution as $\{\log \phi_{i+1,k+1} : 1 \leq i < k\}$, Lemma 3.3 and our moment hypothesis allow one to apply the subadditive ergodic theorem. As a consequence, $(\log \phi_{in})/n \to V(\Gamma)$ a.s. as $n \to \infty$, for each $i \geq 1$. Since the limit holds for each $i \geq 1$, it follows that $V(\Gamma)$ is a tail random variable, in the sense that $V(\Gamma)$ is independent of $\beta(X_j : j < T_i)$ for each $i \geq 1$. Hence, a zero-one law applies and $V(\Gamma)$ is constant a.s. □

(3.5) Corollary. *Let X be a delayed regenerative sequence satisfying* $E\tau_1 < \infty$. *Then* $(\log \phi(\Gamma, \Lambda_n))/n \to V(\Gamma)/E\tau_1$ *a.s. as* $n \to \infty$.

Proof. Since $\log \phi_{12} \leq \tau_1 \log 2$, the moment hypothesis of Theorem 3.4 is satisfied, so

$$(\log \phi_{1n})/n \to V(\Gamma) \quad \text{a.s.} \tag{3.6}$$

as $n \to \infty$.

We now show that if $W_1 \subseteq W_2$, then $\phi(\Gamma, W_1) \leq \phi(\Gamma, W_2)$. Observe that two sets $A_i \cap W$ are distinct if and only if $(A_1 \Delta A_2) \cap W \neq \emptyset$, where Δ denotes symmetric difference. Thus, if $W_1 \subseteq W_2$, there are more sets of the form $A \cap W_2$ than $A \cap W_1$, proving that $\phi(\Gamma, W_1) \leq \phi(\Gamma, W_2)$. Since $\Lambda_{0n} \supseteq \Lambda_{1n}$, it follows that $\phi_{0n} \geq \phi_{1n}$. Thus, Lemma 3.3 proves that $\log \phi_{0n}$ can be squeezed by $\log \phi_{1n}$:

$$\log \phi_{1n} \leq \log \phi_{0n} \leq \log \phi_{01} + \log \phi_{1n}. \tag{3.7}$$

Let $N(m) = \max\{k\colon T_k \leq m\}$ and observe that

$$\Lambda_{T_{N(m)-1}} \subseteq \Lambda_m \subseteq \Lambda_{T_{N(m)+1}-1}. \tag{3.8}$$

Relations (3.7) and (3.8) allows us to 'squeeze' $\log \phi(\Gamma, \Lambda_m)$ via

$$\log \phi_{1,N(m)} \leq \log \phi(\Gamma, \Lambda_m) \leq \log \phi_{01} + \log \phi_{1,N(m)+1}. \tag{3.9}$$

Dividing through in (3.9) by $N(m)$, using (3.6), and exploiting the fact that $N(m)/m \to 1/E\tau_1$ a.s. yields the desired conclusion. □

To conclude this section, we wish to point out that the constant $V(\Gamma)/E\tau_1$ has been extensively studied in a rather different context. Vapnik and Chervonenkis (1971) showed that if $\{X_n\colon n \geq 0\}$ is a sequence of independent and identically distributed variates, then the empirical discrepancy function associated with Γ, namely

$$\frac{1}{n} \sup_{A \in \Gamma} \left| \sum_{i=1}^{n} I(X_i \in A) - nP\{X_1 \in A\} \right|$$

converges to zero a.s. if and only if $V(\Gamma)/E\tau_1$ vanishes (see also [13]); the constant $V(\Gamma)/E\tau_1$ is called the Vapnik–Chervonenkis entropy associated with Γ.

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