RECURSIVE MOMENT FORMULAS FOR REGENERATIVE SIMULATION

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1. Introduction

Let f be a real-valued function defined on the state space of a regenerative process $\underset{\sim}{X} = \{X(t) : t \geqslant 0\}$ with regeneration times $0 = T_0 < T_1 < \ldots$, and suppose that

$$r_t = \frac{1}{t} \int_0^t f(X(s))ds \to r \qquad \text{a.s.} \tag{1.1}$$

as $t \to \infty$. The problem of estimating r via a simulation of $\underset{\sim}{X}$ is called the steady state simulation problem.

Relation (1.1) implies that r_t is a strongly consistent point estimator for r. To obtain confidence intervals for r, set (for $k \geqslant 1$)

$$Y_k(f) = \int_{T_{k-1}}^{T_k} f(X(s))ds$$

$$\tau_k = T_k - T_{k-1}$$

$$Z_k = Y_k(f) - r\tau_k .$$

The regenerative structure of $\underset{\sim}{X}$ guarantees that $r = E\{Y_1(f)\}/E\{\tau_1\}$ and that $\{(Y_k(f), \tau_k) : k \geq 1\}$ is a sequence of i.i.d. random vectors. Standard arguments (see Crane and Iglehart, 1975) show that if $E\{Y_1^2(f) + \tau_1^2\} < \infty$, then

$$\sqrt{t}(r_t - r) \Rightarrow \sigma N(0,1) \tag{1.2}$$

as $t \to \infty$, where $\sigma^2 = \sigma^2\{Z_1\}/E\{\tau_1\}$. To use (1.2) for confidence intervals, the regenerative cycle structure of $\underset{\sim}{X}$ is exploited to obtain a strongly consistent estimator v_t for σ^2.

These confidence intervals, while asymptotically correct, often have poor small-sample behavior. For example, such confidence intervals often tend to significantly undercover the parameter r. Several recent studies have examined this problem. Glynn (1982), in considering regenerative confidence intervals on the time scale of regenerative cycles, obtained asymptotic expansions for the coverage error which indicated that skewness/kurtosis effects play a significant role in determining quality of the confidence interval. To be more precise, the error, to a first approximation, is determined by the magnitude of quantities of the form $E\{Z_1^m \tau_1^n\}$ for $m + n \leq 4$. Glynn and Iglehart (1984) obtain expressions for the asymptotic covariance between r_t and v_t, and the variance of v_t; these expressions also involve mixed moments of the form $E\{Z_1^m \tau_1^n\}$. Consequently, in studying small-sample behavior of regenerative confidence intervals, it is of some interest to be able to calculate the exact values of the mixed moments for some "test case" stochastic models. Hordijk, Iglehart and Schassberger (1976) showed how to do this for $m + n \leq 2$, when X is a discrete or continuous time Markov chain with countably many states. In this note, we show how to calculate such quantities when X is a semi-Markov process with countably many states; discrete and continuous time Markov chain results follow as special cases.

2. Statement of the Recursive Moment Formulas

Let $\underset{\sim}{X} = \{X(t) : t \geq 0\}$ be an irreducible non-explosive regenerative semi-Markov process on countable state space E. Thus, X(t) may be represented as

$$X(t) = \sum_{k=0}^{\infty} R_k \, I(S_k \leq t < S_{k+1}) ,$$

where :

(i) $\underset{\sim}{R} = \{R_n : n \geq 0\}$ is a discrete-time Markov chain on E with transition matrix $\underset{\sim}{P} = (p_{xy} : x, y \in E)$

(ii) $\underset{\sim}{S} = \{S_n : n \geq 0\}$ is an increasing sequence of jump times with $S_0 = 0$ and differences $\alpha_n = S_{n+1} - S_n$ which are conditionally independent r.v.'s given $\underset{\sim}{R}$.

The conditional distribution of α_n is given by $F(R_n, R_{n+1}, dt) = P\{\alpha_n \in dt | \underset{\sim}{R}\}$, where $F(x,y,0) = 0$ for all x, $y \in E$. Note that $S_n \to \infty$ a.s., since X is non-explosive by assumption. Fix $z \in E$ as the regenerative state; let $T(z) = \inf\{t > 0 : X(t-) \neq z, X(t) = z\}$ and set

$$Y(u) = \int_0^{T(z)} u(X(t))dt$$

where $u : E \to \underset{\sim}{R}$ is an arbitrary function. We wish to study mixed moments of the form

$$a_{ij}(x) = E_x\{Y(g)^i \, Y(h)^j\}$$

for $x \in E$, $0 \leq i \leq m$, and $0 \leq j \leq n$, when g and h are fixed functions, and m and n are non-negative integers. Throughout the paper we shall use $P_x\{\cdot\}$ and $E_x\{\cdot\}$ to denote conditional probabilities and expectations, given $X(0) = R_0 = x$. Note that by choosing $g(\cdot) = f(\cdot) - r$

and $h(\cdot) = 1$, $a_{ij}(z)$ yields $E_z\{Z_1^i \tau_1^j\}$.

To state our result, let $a \vee b$ denote $\max(a,b)$ and set $b_{mn}(x) = E_x\{Y(|g|) \vee 1)^m (Y(|h|) \vee 1)^n\}$. Let G_n and β_n be the matrix and function, respectively, defined by

$$G_n(x,y) = \begin{cases} p_{xy}\,\mu(x,y) ; & y \neq z \\ 0 ; & y = z \end{cases}$$

$$\beta_n(x) = \sum_{y \in E} \mu_n(x,y)\, p_{xy}$$

where $\mu_n(x,y) = \int_0^\infty t^n F(x,y,dt)$. Also, we shall identify real-valued functions $u(\cdot)$ on E with column vectors $\underset{\sim}{u}$, and shall use the notation $\underset{\sim}{u} \circ \underset{\sim}{v}$ to denote the vector with x^{th} component $(\underset{\sim}{u} \circ \underset{\sim}{v})(x) = u(x)v(x)$. Set $\underset{\sim}{u}^0(\cdot) = 1$ and $\underset{\sim}{u}^{n+1} = (\underset{\sim}{u} \circ \underset{\sim}{u}^n)$ for $n \geq 0$.

Let C denote the class of all $m \times n$ matrix-valued functions, $\underset{\sim}{C}$, on E. Then set

$$C_{mn} = \{\underset{\sim}{C} \in C : \sum_{k=0}^{i} \sum_{\ell=0}^{j} |g(x)|^{i-k} \cdot |h(x)|^{j-\ell}$$

$$\cdot \sum_{y \in E} G_{i+j-k-\ell}(x,y)\, |c_{k\ell}(y)| < \infty$$

$$\text{and } (G_0^k\, c_{ij})(x) \to 0 \text{ as } k \to \infty ,$$

$$\text{for all } x \in E,\ 0 \leq i \leq m, \text{ and } 0 \leq j \leq n\}.$$

2.1. Theorem

If $b_{mn}(z) < \infty$, then $\underset{\sim}{A} = \{\underset{\sim}{a}_{ij} : 0 \leq i \leq m,\ 0 \leq j \leq n\}$ is the unique solution in C_{mn} to the system :

$$\underset{\sim}{c}_{ij} = \underset{\sim}{g}^{j} \circ \underset{\sim}{h}^{j} \circ \underset{\sim}{\beta}_{i+j} + \sum_{(k,\ell) \in B_{ij}} \binom{i}{k}\binom{j}{\ell}(\underset{\sim}{g}^{i-k} \circ \underset{\sim}{h}^{j-\ell} \circ \underset{\sim}{G}^{i+j-k-\ell} \underset{\sim}{c}_{k\ell}) \tag{2.2}$$

where $0 \leqslant i \leqslant m$, $0 \leqslant j \leqslant n$, and $B_{ij} = \{(k,\ell) : 0 \leqslant k \leqslant i,\ 0 \leqslant \ell \leqslant j,\ k+\ell > 0\}$.

Set $\tau = \inf\{n \geqslant 1 : R_n = z\}$ and observe that

$$G_0^k(x,y) = P_x\{R_k = y,\ \tau > k\}.$$

Hence, $\underset{\sim}{G}_0^k \to 0$ as $k \to \infty$. It follows that if E has a finite number of elements, $C_{mn} \equiv C$, so that the $\underset{\sim}{a}_{ij}$'s are unique in the class of all possible solutions to (2.2). Also, in the presence of a finite state space, it is well known that $(\underset{\sim}{I} - \underset{\sim}{G}_0)^{-1}$ exists, so that (2.2) may be re-written as

$$\underset{\sim}{c}_{ij} = (\underset{\sim}{I} - \underset{\sim}{G}_0)^{-1}\{\underset{\sim}{g}^{i} \circ \underset{\sim}{h}^{i} \circ \underset{\sim}{\beta}_{i+j}$$

$$+ \sum_{(k,\ell) \in A_{ij}} \binom{i}{k}\binom{j}{\ell}(\underset{\sim}{g}^{i-k} \circ \underset{\sim}{h}^{j-\ell} \circ (\underset{\sim}{G}_{i+j-k-\ell}\underset{\sim}{c}_{k\ell}))\}, \tag{2.3}$$

where $0 \leqslant i \leqslant m$, $0 \leqslant j \leqslant n$, $i + j > 0$, and $A_{ij} = \{(k,\ell) : 0 \leqslant k \leqslant i,\ 0 \leqslant \ell \leqslant j,\ 0 < k+\ell < i+j\}$. Also observe that $\underset{\sim}{c}_{00} \equiv 1$. Note that the system of equations (2.3) is recursive in i+j, in the sense that the $\underset{\sim}{c}_{ij}$'s may be solved in terms of the $\underset{\sim}{c}_{k\ell}$'s, where $k+\ell < i+j$. By successively solving for the $\underset{\sim}{c}_{k\ell}$'s with fixed $k+\ell$ on each iteration, one eventually obtains $\underset{\sim}{c}_{mn}$.

Formula (2.3) can be further simplified when X has special structure. Note that if $\underset{\sim}{X}$ is a continuous time Markov chain, then

$$F(x,y,dt) = \lambda(x)\ \exp(-\lambda(x)t)dt$$

for $t > 0$, so that

$$\mu_n(x,y) = n!/\lambda(x)^n \equiv \eta_n(x) \ .$$

We find that (2.3) can be re-written as

$$\mathbf{c}_{ij} = (\mathbf{I} - \mathbf{G}_0)^{-1}\{\mathbf{g}^i \circ \mathbf{h}^j \circ \boldsymbol{\eta}_{i+j}$$

$$+ \sum_{(k,\ell) \in A_{ij}} \binom{i}{k}\binom{j}{\ell}(\mathbf{g}^{i-k} \circ \mathbf{h}^{j-\ell} \circ \boldsymbol{\eta}_{i+j-k-\ell} \circ (\mathbf{G}_0\ \mathbf{c}_{k\ell}))\} \ ,$$

where $0 \leqslant i \leqslant m$, $0 \leqslant j \leqslant n$, and $i+j > 0$.

For discrete time Markov chains, $\boldsymbol{\beta}_n \equiv 1$ and $\mathbf{G}_n = \mathbf{G}_0$, so (2.3) takes the form

$$\mathbf{c}_{ij} = (\mathbf{I} - \mathbf{G}_0)^{-1}\{\mathbf{g}^i \circ \mathbf{h}^j$$

$$+ \sum_{(k,\ell) \in A_{ij}} \binom{i}{k}\binom{j}{\ell}(\mathbf{g}^{i-k} \circ \mathbf{h}^{k-\ell} \circ (\mathbf{G}_0\ \mathbf{c}_{k\ell}))\} \ ,$$

where $0 \leqslant i \leqslant m$, $0 \leqslant j \leqslant m$, and $i+j > 0$.

Relation (2.4) expresses $\mathbf{c}_{ij}$ in terms of $\mathbf{G}_0\ \mathbf{c}_{k\ell}$, where $(k,\ell) \in A_{ij}$. Equation (2.4) can be re-written, when $\mathbf{g} = \mathbf{h}$, so that the $\mathbf{c}_{ij}$'s are written directly in terms of the $\mathbf{c}_{k\ell}$'s. If $\mathbf{g} = \mathbf{h}$, we write $\mathbf{c}_{ij}$ as $\mathbf{c}_{i+j}$, and observe that (2.4) takes the form

$$\mathbf{c}_i = (\mathbf{I} - \mathbf{G}_0)^{-1}\{\mathbf{g}^i + \sum_{k=1}^{i-1} \binom{i}{k}(\mathbf{g}^{i-k} \circ \mathbf{G}_0\ \mathbf{c}_k)\}, \quad 1 \leqslant i \leqslant n. \quad (2.5)$$

Recall also that from (2.2), $\mathbf{c}_0 \equiv 1$. We claim that the system (2.5) can be re-written as

$$\underset{\sim}{c}_i = (\underset{\sim}{I} - \underset{\sim}{G}_0)^{-1}\{\sum_{k=1}^{i} (-1)^{k+1}\binom{i}{k} \underset{\sim}{g}^k \circ \underset{\sim}{c}_{i-k}\},\ 1 \leq i \leq n. \quad (2.6)$$

The proof is by induction. For n = 1, the result is obvious, so suppose (2.5) and (2.6) are equivalent systems for n = m. To check the (m+1)'st equation in the (m+1)'st system, observe that a solution of (2.5) satisfies

$$\underset{\sim}{c}_{m+1} = (\underset{\sim}{I} - \underset{\sim}{G}_0)^{-1}\{\underset{\sim}{g}^{m+1} + \sum_{i=1}^{m} \binom{m+1}{i} (\underset{\sim}{g}^{m+1-i} \circ \underset{\sim}{G}_0\ \underset{\sim}{c}_i)\}. \quad (2.7)$$

By the inductive hypothesis, (2.6) shows that

$$\underset{\sim}{G}_0\ \underset{\sim}{c}_i = -\sum_{k=0}^{i} (-1)^{k+1}\binom{i}{k}\underset{\sim}{g}^k \circ \underset{\sim}{c}_{i-k} \quad (2.8)$$

for $i \leq m$. Substituting (2.8) into (2.7), we get that $(\underset{\sim}{I} - \underset{\sim}{G}_0)\underset{\sim}{c}_{m+1}$ equals

$$\underset{\sim}{g}^{m+1} + \sum_{i=1}^{m}\sum_{k=0}^{i} \binom{m+1}{i}\binom{i}{k}(-1)^{i-k}\ \underset{\sim}{g}^{m+1-k} \circ \underset{\sim}{c}_k$$

$$= \sum_{k=0}^{m} \binom{m+1}{k}(\underset{\sim}{g}^{m+1-k} \circ \underset{\sim}{c}_k) \sum_{j=1}^{m+1-k} \binom{m+1-k}{j}(-1)^{m+1-k-j}$$

$$= \sum_{k=0}^{m} \binom{m+1}{k}(\underset{\sim}{g}^{m+1-k} \circ \underset{\sim}{c}_k)(-1)^{m+2-k}$$

$$= \sum_{\ell=1}^{m+1} \binom{m+1}{\ell}(\underset{\sim}{g}^{\ell} \circ \underset{\sim}{c}_{m+1-\ell})(-1)^{\ell+1},$$

which is equivalent to the (m+1)'st relation of (2.6) (the binomial identity was used for the third equality). The steps being reversible, this proves the claimed result. We remark that (2.6) yields the equations of A. Hordijk, D.L. Iglehart and R. Schassberger, Discrete-Time Methods of Stimulating Continuous-Time Markov Chains, for i = 1,2.

3. Proof of the Theorem

We proceed via a series of lemmas.

3.1. Lemma

If $b_{mn}(z) < \infty$, then the r.v.'s $Y(g)^i\, Y(h)^j$ are integrable under the probability distribution P_x for $0 \leq i \leq m$, $0 \leq j \leq n$, $x \in E$.

Proof.

Note that

$$(Y(|g|) \vee 1)^i \; (Y(|h|) \vee 1)^j \leq (Y(|g|) \vee 1)^m \; (Y(|h|) \vee 1)^n$$

so that $b_{ij}(z) < \infty$ for $0 \leq i \leq m$, $0 \leq j \leq n$. Since $\underset{\sim}{X}$ is irreductible, it follows that $P_z\{T(y) < T(z)\} > 0$ for all $y \in E$, for otherwise the regenerative property guarantees that $P_z\{T(y) = \infty\} = 1$, which violates our irreducibility assumption.

Then, by the strong Markov property applied at time $T(y)$,

$$E_z\{(\textstyle\int_{T(y)}^{T(z)} |g(X(s))|ds \vee 1)^i \; (\int_{T(y)}^{T(z)} |h(X(s))|ds \vee 1)^j;\; T(y) < T(z)\}$$

$$= b_{ij}(y)\; P\{T(y) < T(z) | X(0) = z\} \leq b_{ij}(z)$$

so that $b_{ij}(y) < \infty$ for all $y \in E$, which proves the result. □

3.2 Proposition

If $b_{mn}(z) < \infty$, then $a_{ij}(x)$ exists and is finite for $0 \leq i \leq m$, $0 \leq j \leq n$, $x \in E$. Furthermore, $\underset{\sim}{A}$ solves (2.2).

Proof

The first part follows immediately from lemma 3.1. For the second part, the integrability of $Y(g)^i\, Y(h)^j$ ensures that the following manipulations of conditional expectations are valid :

$$a_{ij}(x) = E_x\{Y(g)^i\, Y(h)^j\}$$

$$= E_x\{Y(g)^i\, Y(h)^j;\ \tau = 1\} + E_x\,\{Y(g)^i\, Y(h)^j;\ \tau > 1\}$$

$$= g^i(x)\, h^j(x)\, \mu_{i+j}(x,z)\, p_{xz}$$

$$+ E_x\,\{(g(R_0)\,\alpha_0 + \int_{S_1}^{T(z)} g(X(t)dt)^i\,(h(R_0)\,\alpha_0$$

$$+ \int_{S_1}^{T(z)} h(X(t))dt)^j;\ \tau > 1\}$$

$$= \sum_{k=0}^{i}\sum_{\ell=0}^{j} \binom{i}{k}\binom{j}{\ell} E_x\{g(R_0)^{i-k}\, h(R_0)^{j-\ell}(\int_{S_1}^{T(z)} g(X(t))dt)^k$$

$$\cdot\,(\int_{S_1}^{T} h(X(t))dt)^{\ell};\ \tau > 1\}$$

$$= g^i(x)\, h^j(x)\, \mu_{i+j}(x,z)\, p_{xz}$$

$$+ \sum_{k=0}^{i}\sum_{\ell=0}^{j} \binom{i}{k}\binom{j}{\ell} E_x\{g(R_0)^{i-k}\, h(R_0)^{j-\ell}\alpha_0^{i+j-k-\ell}$$

$$(\int_{S_1}^{T(z)} g(X(t))dt)^k \cdot (\int_{S_1}^{T} h(X(t))dt)^{\ell};\ \tau > 1\}$$

$$= g^i(x)\, h^j(x)\, \mu_{i+j}(x,z)\, p_{xz}$$

$$+ \sum_{k=0}^{i}\sum_{\ell=0}^{j} \binom{i}{k}\binom{j}{\ell} g^{i-k}(x) h(x)^{j-\ell} \sum_{y\neq z} p_{xy}\, \mu_{i+j-k-\ell}(x,y)\, a_{k\ell}(y)$$

$$= (\underset{\sim}{g}^i \circ \underset{\sim}{h}^i \circ \underset{\sim}{\beta}_{i+j})(x) + \sum_{(k,\ell)\in B_{ij}} \binom{i}{k}\binom{j}{\ell} (\underset{\sim}{g}^{i-k} \circ \underset{\sim}{h}^{j-\ell}$$

$$\circ\ (\underset{\sim}{G}_{i+j-k-\ell}\, \underset{\sim}{a}_{k\ell})(x))\ ;$$

the strong Markov property at time S_1 was used to obtain the second last equality. □

3.3. Lemma

If $b_{mn}(z) < \infty$, then $\underset{\sim}{A} \in C_{mn}$.

Proof

For the absolute summability observe that $d_{ij}(x) = E_x\{Y(|g|)^i\ Y(|h|)^j\}$ satisfies

$$\underset{\sim}{d}_{ij} = |g|^i \circ |\underset{\sim}{h}|^j \circ \underset{\sim}{\beta}_{i+j}$$

$$+ \sum_{(k,\ell) \in B_{ij}} \binom{i}{k}\binom{j}{\ell}(|\underset{\sim}{g}|^{i-k} \circ |\underset{\sim}{h}|^{j-\ell} \circ (\underset{\sim}{G}_{i+j-k-\ell}\ \underset{\sim}{d}_{k\ell})).$$

But $d_{ij}(x) \leq b_{ij}(x) < \infty$, so $|\underset{\sim}{g}|^{i-k} \circ |\underset{\sim}{h}|^{j-\ell} \circ (\underset{\sim}{G}_{i+j-k-\ell}\underset{\sim}{d}_{k\ell})$ is finite, proving the first part, since $|\underset{\sim}{a}_{ij}| \leq \underset{\sim}{d}_{ij}$. For the second,

$$(\underset{\sim}{G}_0^k\ \underset{\sim}{a}_{ij})(x) = E_x\{(\int_{S_k}^{T(z)} Y(g))^i\ (\int_{S_k}^{T(z)} Y(h))^j;\ \tau > k\},$$

which tends to zero by integrability of $Y(|g|)^i\ Y(|h|)^j$. □

3.4. Lemma

If $b_{mn}(z) < \infty$, then $\underset{\sim}{A}$ is the unique solution to (2.2) in C_{mn}.

Proof

Suppose $\{\underset{\sim}{a}_{rs} : 0 \leq r \leq k,\ 0 \leq s \leq \ell\}$ is unique in $C_{k\ell}$ for $k+\ell < i+j$, where $0 \leq i \leq m$, $0 \leq j \leq n$. We shall prove uniqueness in C_{ij}; "bootstrapping" this result yields the lemma. Since the $\underset{\sim}{a}_{rs}$'s are unique in $C_{k\ell}$, any solution to the (i,j) equation must satisfy

$$\underset{\sim}{c}_{ij} = \underset{\approx}{g}^{i} \circ \underset{\sim}{h}^{j} \circ \underset{\sim}{\beta}_{i+j} + \sum_{(k,\ell) \in A_{ij}} \binom{i}{k}\binom{j}{\ell} (\underset{\approx}{g}^{i-k} \circ \underset{\sim}{h}^{j-\ell} \circ \underset{\sim}{G}_{i+j-k-\ell}\, \underset{\sim}{a}_{k\ell}) + \underset{\sim}{G}_0\, \underset{\sim}{c}_{ij}. \tag{3.5}$$

By the absolute summability, any two solutions $\underset{\sim}{c}_{ij}$, $\underset{\sim}{c}'_{ij}$ of (3.5) must satisfy $(\underset{\sim}{c}_{ij} - \underset{\sim}{c}'_{ij}) = \underset{\sim}{G}_0(\underset{\sim}{c}_{ij} - \underset{\sim}{c}'_{ij})$. Since $\underset{\sim}{G}_0^k(\underset{\sim}{c}_{ij} - \underset{\sim}{c}'_{ij}) \to 0$ as $k \to \infty$, it follows that $\underset{\sim}{c}_{ij} = \underset{\sim}{c}'_{ij}$, proving uniqueness in C_{ij}. □

These above results prove all the assertions of the theorem.

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