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Source: *Mathematics of Operations Research*, Vol. 15, No. 1 (Feb., 1990), pp. 1-16

Published by: INFORMS

Stable URL: <http://www.jstor.org/stable/3689929>

Accessed: 21/07/2010 03:40

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## SIMULATION OUTPUT ANALYSIS USING STANDARDIZED TIME SERIES\*†

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The method of standardized time series (STS) was proposed by Schruben as an approach for constructing asymptotic confidence intervals for the steady-state mean from a single simulation run. The STS method “cancels out” the variance constant while other methods attempt to consistently estimate the variance constant. Our goal in this paper is to generalize the STS method and to study some of its basic properties. Starting from a functional central limit theorem (FCLT) for the sample mean of the simulated process, a class of mappings of  $C[0, 1]$  to  $\mathbb{R}$  is identified, each of which leads to a STS confidence interval. One of these mappings leads to the batch means method. A lower bound is obtained for the expected length of the asymptotic (as the run size becomes large) STS confidence intervals. This lower bound is not attained, but can be approached arbitrarily closely, by STS confidence intervals. Methods that consistently estimate the variance constant do realize this lower bound. The variance of the length of a STS confidence interval is of larger order (in the run length) than is that for the regenerative method.

**1. Introduction.** A principal problem in the simulation literature is to construct asymptotic (as the run length becomes large) confidence intervals for steady-state parameters of the simulation output process from a single simulation run. There are two basic approaches to this problem. The first is to consistently estimate the variance constant in the relevant central limit theorem. This is the approach used in the regenerative, spectral, and autoregressive methods. The second approach, proposed by Schruben (1983), is based on standardized time series (STS), and essentially “cancels out” the variance constant in a manner reminiscent of the  $t$ -statistic. For other work on STS see Chen and Sargent (1985), Goldsman and Schruben (1984), Glynn and Iglehart (1985), and Nozari (1986).

Our goal in this paper is to generalize the method of STS and to study some of its basic properties. The starting point for the STS method is the existence of a functional central limit theorem (FCLT) for the sample mean of the simulated process. These FCLT's exist for stationary (and some nonstationary)  $\phi$ -mixing processes, strictly stationary strongly mixing processes, associated strictly stationary processes, and regenerative processes. We identify a class of mappings from  $C[0, 1]$  to  $\mathbb{R}$ , each of which leads to a STS confidence interval. One of these mappings yields the batch means method. We study the asymptotic length of these STS confidence intervals and develop a lower bound for the expected length; see (4.16). This lower bound is attained by the methods which consistently estimate the variance constant. While this lower

\*Received June 26, 1985; revised August 22, 1988.

AMS 1980 subject classification. primary: 65C05. Secondary: 62M10.

IAOR 1973 subject classification. Main: Simulation. Cross references: Statistics: inference.

OR/MS Index 1978 subject classification. Primary: 767 Simulation/Statistical analysis. Secondary: 805 Statistics/time series.

Key words. Batch means, confidence intervals, functional central limit theorem, weak convergence of probability measures, simulation output analysis, standardized time series, steady-state simulation.

†This work was supported by Army Research Office Contract DAAG29-84-K-0030 and National Science Foundation Grants MCS-82-3483, DCR-85-09668, and ECS-84-04809.

bound is not attained by any of the STS confidence intervals, it can be approached arbitrarily closely using the batch means mapping with a sufficiently large number of batches. We also study the standard deviation of the length of a STS confidence interval, and show that it is of order  $n^{-1/2}$  (where  $n$  is the run length), whereas for the regenerative method the order is  $n^{-1}$ ; see (4.33) and (4.38). While these results show that STS confidence intervals are asymptotically inferior to those constructed by consistent estimation methods, this does not preclude the possibility that STS confidence intervals may be superior in certain small sample contexts. This question should be explored in a future numerical study.

As indicated above, the main result of this paper is that STS confidence intervals are asymptotically less desirable than those obtained via procedures which consistently estimate the variance parameter. However, it should be emphasized that obtaining consistent estimators for the variance constant is a nontrivial task. In particular, consistent estimators are available only for processes enjoying certain special stochastic properties. For example, it is known (see Anderson 1971, pp. 522–534) that under certain moment and correlation assumptions, spectral density estimators are consistent for stationary processes. Due to the strength of the hypotheses imposed, it is possible that there exist processes for which spectral density estimators are inconsistent, and yet a FCLT is in force. This would suggest that STS methods are, in some sense, more robust than consistent estimation procedures.

This remark requires some additional qualifications, however. As shown in Glynn and Iglehart (1988), the regenerative method applies to processes which do not necessarily obey a FCLT. Therefore, no universal statement can be made as to whether STS procedures apply to a larger class of processes than do consistent estimation algorithms (and are consequently more robust). It seems reasonable to argue that the class of processes for which both classes of methods apply is extremely large, and covers most cases of practical interest; the exceptional class of processes for which only one class applies is of less practical interest, but deserves further attention. The reader is referred to Glynn and Iglehart (1985) for a more complete discussion of the domain of applicability of the FCLT.

In §2 the method of STS is introduced and its basic properties are investigated. §3 gives examples of STS, while §4 discusses the asymptotic behavior of the method.

**2. Standardized time series.** Let  $\mathbf{Y} = \{Y(t): t \geq 0\}$  be a real-valued (measurable) stochastic process representing the output of a simulation. Discrete-time processes  $\{Y_n: n \geq 0\}$  can be handled in the usual way by setting  $Y(t) = Y_{[t]}$ . To apply the method of STS to the output process  $\mathbf{Y}$ , we make the following assumption:

(2.1) There exist finite constants  $\mu$  and  $\sigma$  ( $\sigma$  positive) such that  $X_n \Rightarrow \sigma B$  as  $n \rightarrow \infty$ , where  $B$  is a standard Brownian motion, and

$$X_n(t) = n^{1/2}(\bar{Y}_n(t) - \mu t) \quad \text{with}$$

$$\bar{Y}_n(t) = \int_0^{nt} Y(s) ds/n, \quad \text{for } 0 \leq t \leq 1.$$

Note that  $X_n$  and  $B$  are both processes whose sample paths lie in  $C[0, 1]$ , even though  $\mathbf{Y}$  may not be, so that the weak convergence required by (2.1) is assumed to take place in the function space  $C[0, 1]$ . A variety of different output processes satisfy (2.1). We shall simply list these processes with appropriate references.

(i) Stationary (measurable)  $\phi$ -mixing processes; see pp. 178–179 of Billingsley (1968) for a proof of (2.1) plus pp. 179–182 for extensions to nonstationary processes.

(ii) Strictly stationary, strongly mixing sequences; see Hall and Heyde (1980, p. 132), for a proof of (2.1).

(iii) Associated sequences of strictly stationary random variables; see Newman and Wright (1981) for a proof of (2.1).

(iv) Delayed (and nondelayed) regenerative processes; see Freedman (1967) for a proof of (2.1) for the Markov chain case which can easily be extended to the general regenerative setting.

The continuous mapping theorem (see Theorem 5.1 of Billingsley 1968) applied to (2.1) yields

$$(2.2) \quad \begin{aligned} X_n(1) &\Rightarrow \sigma B(1), \quad \text{or} \\ n^{1/2}(\bar{Y}_n(1) - \mu) &\Rightarrow \sigma B(1) \end{aligned}$$

as  $n \rightarrow \infty$ . Application of a standard converging-together argument (see p. 93 of Chung 1974) yields  $\bar{Y}_n(1) \Rightarrow \mu$ .

Thus, (2.1) suffices to guarantee that the steady-state estimation problem for  $Y$  makes sense;  $\mu$  is the steady-state parameter which the simulator wishes to estimate.

Note that the CLT (2.2) could be used to obtain confidence intervals for  $\mu$ , provided that  $\sigma$  were known. As Schruben (1983) points out, the principle underlying standardized time series is to "cancel out" the  $\sigma$ . To carry out the cancellation procedure in the next theorem involves choosing a function  $g$  from the class  $\mathcal{M}$ ;  $\mathcal{M}$  is the class of (measurable) functions  $g: C[0, 1] \rightarrow \mathbb{R}$  such that:

$$(2.3) \quad \begin{aligned} \text{(i)} \quad &g(\alpha x) = \alpha g(x) \text{ for } \alpha > 0, x \in C[0, 1] \\ \text{(ii)} \quad &g(x - \beta k) = g(x) \text{ for } \beta \in \mathbb{R} \text{ and } x \in C[0, 1], \text{ where } k(t) = t, \\ \text{(iii)} \quad &P\{g(b) > 0\} = 1, \\ \text{(iv)} \quad &P\{B \in D(g)\} = 0. \end{aligned}$$

(2.4) THEOREM. *Suppose that  $g \in \mathcal{M}$ . Under Assumption (2.1),*

$$(2.5) \quad \frac{\bar{Y}_n(1) - \mu}{g(\bar{Y}_n)} \Rightarrow \frac{B(1)}{g(B)} \quad \text{as } n \rightarrow \infty.$$

PROOF. Let  $h: C[0, 1] \rightarrow \mathbb{R}$  be the mapping defined by  $h(x) = x(1)/g(x)$  for  $g(x) \neq 0$  (and zero elsewhere). Assumptions (2.3iii) and (2.3iv) allow one to verify that  $P\{\sigma B \in D(h)\} = 0$ . Thus, Assumption (2.1) plus the continuous mapping theorem guarantees that  $h(X_n) \Rightarrow h(\sigma B)$  as  $n \rightarrow \infty$ . By (2.3i),  $h(\sigma B) = B(1)/g(B)$  (recall that  $\sigma^2 > 0$ ). Furthermore,

$$\begin{aligned} h(X_n) &= \frac{n^{1/2}(\bar{Y}_n(1) - \mu)}{g(n^{1/2}(\bar{Y}_n - k\mu))} \\ &= \frac{\bar{Y}_n(1) - \mu}{g(\bar{Y}_n - k\mu)} \\ &= \frac{\bar{Y}_n(1) - \mu}{g(\bar{Y}_n)}, \end{aligned}$$

where the last equality is due to (2.3ii). These observations immediately yield the theorem.

The proof indicates that Assumption (2.3i) is used to cancel out  $\sigma$ , Assumption (2.3ii) guarantees that  $g(X_n)$  does not depend on the unknown parameter  $\mu$ , and Assumptions (2.3iii) and (2.3iv) are technical assumptions required to invoke the continuous mapping theorem.

To construct confidence intervals based on (2.5), we need to learn more about the limit RV  $B(1)/g(B)$ . We start by obtaining an alternative description of  $\mathcal{M}$ . Let  $\Gamma: C[0, 1] \rightarrow C[0, 1]$  be the map defined by

$$(\Gamma x)(t) = x(t) - tx(1).$$

Let  $\mathcal{N}$  be the class of functions  $b: C[0, 1] \rightarrow \mathbb{R}$  which satisfy:

- (2.6) (i)  $b(\alpha x) = \alpha b(x)$  for  $\alpha > 0$ ,  $x \in C[0, 1]$ ,  
(ii)  $P\{(b \circ \Gamma)(B) > 0\} = 1$ ,  
(iii)  $P\{B \in D(b \circ \Gamma)\} = 0$ .

Set  $\mathcal{M}^* = \{g: g = b \circ \Gamma, b \in \mathcal{N}\}$ .

(2.7) PROPOSITION.  $\mathcal{M}^* = \mathcal{M}$ .

PROOF. We first show that  $\mathcal{M}^* \subseteq \mathcal{M}$ . Suppose that  $g = b \circ \Gamma$ , where  $b \in \mathcal{N}$ . Clearly,  $g$  satisfies (2.3i), (2.3iii), and (2.3iv). For (2.3ii), observe that

$$\begin{aligned} (\Gamma(x - \beta k))(t) &= x(t) - \beta k(t) - t(x(1) - \beta k(1)) \\ &= x(t) - tx(1) = (\Gamma x)(t) \end{aligned}$$

so  $\Gamma(x - \beta k) = \Gamma x$ ; hence,  $g(x - \beta k) = g(x)$ .

To prove that  $\mathcal{M} \subseteq \mathcal{M}^*$ , consider  $g \in \mathcal{M}$ . We claim that  $g$  can be represented in the form  $g = b \circ \Gamma$ , by setting  $b = g$ . Recall that  $g(x) = g(x - \beta k)$  for all  $\beta \in \mathbb{R}$ . In particular, setting  $\beta = x(1)$ , we see that  $g(x) = g(\Gamma x)$ , proving our assertion.

We can now obtain the following result.

(2.8) PROPOSITION. *If  $g \in \mathcal{M}$ , then  $B(1)$  is independent of  $g(B)$ .*

PROOF. It is well known that the process  $B(t) - tB(1)$  ( $0 \leq t \leq 1$ ) is independent of  $B(1)$  (see p. 84 of Billingsley 1968, for example). In other words,  $\Gamma B$  is independent of  $B(1)$ , which, of course, implies that  $g(B) = (b \circ \Gamma)(B)$  is independent of  $B(1)$ . (Also see p. 1096 of Schruben 1983.)

Let  $\Phi(x) = P\{B(1) \leq x\}$ ,  $G(x) = P\{g(B) \leq x\}$ , and  $H(x) = P\{B(1)/g(B) \leq x\}$ . Then

$$(2.9) \quad H(x) = \int_0^\infty \Phi(xy)G(dy)$$

by Proposition (2.8). The continuity of  $\Phi(\cdot)$  and the bounded convergence theorem imply that the right-hand side of (2.9) is continuous everywhere in  $x$ . Thus, by (2.5), it follows that under the conditions of (2.1),

$$P\{(\bar{Y}_n(1) - \mu)/g(\bar{Y}_n) \leq x\} \rightarrow H(x)$$

as  $n \rightarrow \infty$  for all  $x \in \mathbb{R}$ . Hence, to obtain a  $100(1 - \delta)\%$  confidence interval, one selects  $\alpha$  and  $\beta$  such that  $H(\beta) - H(\alpha) = 1 - \delta$  (such  $\alpha, \beta$  exist since  $H(\cdot)$  is

continuous; also, (2.9) implies that  $H(\cdot)$  is strictly increasing). Then the interval

$$(2.10) \quad [\bar{Y}_n(1) - g(\bar{Y}_n)\beta, \bar{Y}_n(1) - g(\bar{Y}_n)\alpha]$$

is an asymptotic  $100(1 - \delta)\%$  confidence interval for  $\mu$ .

The process  $(\bar{Y}_n - \mu k)/g(\bar{Y}_n)$  is called a STS. Theorem 2.4 and Proposition 2.7 show that every  $b \in \mathcal{N}$  gives rise to a particular STS procedure; (2.10) is then the corresponding confidence interval for  $\mu$ .

**3. Examples of standardized time series.** Our first example of a STS captures a methodology which has been extensively studied in the simulation literature, namely the method of batch means.

(3.1) EXAMPLE. Let  $b_m: C[0, 1] \rightarrow \mathbb{R}$  be defined by

$$b_m(x) = \left[ \left( \frac{m}{m-1} \right) \sum_{i=1}^m (x(i/m) - x((i-1)/m))^2 \right]^{1/2},$$

for  $m \geq 2$ . It is easily verified that  $b_m \in \mathcal{N}$ , so that  $g_m = b_m \circ \Gamma \in \mathcal{M}$  (see Proposition 2.7). But

$$g_m(x) = \left[ \left( \frac{m}{m-1} \right) \sum_{i=1}^m (\Delta_m x(i/m) - x(1)/m)^2 \right]^{1/2},$$

where  $\Delta_h x(t) = x(t) - x(t - 1/h)$ . Note that  $\Delta_m B(i/m)$  ( $i = 1, \dots, m$ ) are increments of standard Brownian motion, and are therefore independent and identically distributed normal RV's with mean zero and variance  $1/m$ . Also,  $B(1)/m$  is the sample mean of these increments. Hence  $B(1)/g_m(B)$  has a Student's  $t$  distribution with  $m - 1$  degrees of freedom.

On the other hand,

$$g_m(\bar{Y}_n) = m^{-1/2} \left[ \frac{1}{m-1} \sum_{i=1}^m \left( Z_i(n) - \frac{1}{m} \sum_{j=1}^m Z_j(n) \right)^2 \right]^{1/2}, \quad \text{where}$$

$$Z_i(n) = \int_{(i-1)n/m}^{in/m} Y(s) ds / (n/m)$$

is the  $i$ th batch mean of the process  $\{Y(t): 0 \leq t \leq n\}$ . Specializing Theorem 2.4 to our example therefore allows us to conclude that

$$\sqrt{m} \left( \frac{1}{m} \sum_{i=1}^m Z_i(n) - \mu \right) \left/ \left[ \frac{1}{m-1} \sum_{i=1}^m \left( Z_i(n) - \frac{1}{m} \sum_{j=1}^m Z_j(n) \right)^2 \right]^{1/2} \right. \Rightarrow t_{m-1}$$

as  $n \rightarrow \infty$ , where  $t_{m-1}$  is a Student's  $t$  RV with  $m - 1$  degrees of freedom. To summarize, we have just shown that the method of batch means, with the number of batches fixed at  $m \geq 2$ , is asymptotically valid under condition (2.1). This result complements a similar theorem due to Brillinger [4].

As we have already seen, the fundamental assumption of the method of standardized time series is that the output process may be approximated by a Brownian motion. Intuitively, then, it should follow that the increments of the output process can be

approximated by the increments of Brownian motion. This suggests that one might try to extend the power of the method of standardized time series by applying the procedure separately to each increment of the output process, and then “patching” the increments together. In some sense, this phenomenon occurs in the method of batch means, and is related to the somewhat arbitrary nature of the parameter  $m$ . Before presenting our next examples we need to characterize two other classes of functions that belong to  $\mathcal{M}$ ; these examples are related to several presented in Schruben [18].

Let  $\Lambda_i: C[0, 1] \rightarrow C[0, 1]$  be the map defined by

$$(\Lambda_i x)(t) = x((i+t)/m) - x(i/m), \quad 0 \leq t \leq 1$$

for  $0 \leq i < m$  ( $m \geq 1$ ). The next result characterizes the first class of functions in  $\mathcal{M}$ .

(3.2) PROPOSITION. *If  $b \in \mathcal{N}$ , then  $g_m^* \in \mathcal{M}$  ( $m \geq 1$ ), where*

$$g_m^* = \sum_{i=0}^{m-1} b \circ \Gamma \circ \Lambda_i.$$

PROOF. We shall show  $g_m^*$  can be represented as  $g_m^* = b_m \circ \Gamma$ , where  $b_m \in \mathcal{N}$ , thereby proving that  $g_m^* \in \mathcal{M}^*$ . Let  $\Psi_i: C[0, 1] \rightarrow C[0, 1]$  be given by

$$(\Psi_i x)(t) = x((i+t)/m) - x(i/m)(1-t) - tx((i+1)/m)$$

for  $0 \leq t \leq 1$  ( $0 \leq i < m$ ). The following relations are easily verified:

$$(3.3) \quad \Psi_i = \Gamma \circ \Lambda_i, \quad \Psi_i = \Psi_i \circ \Gamma.$$

It is evident from (3.3) that

$$g_m^* = \sum_{i=1}^m b \circ \Psi_i = \sum_{i=1}^m b \circ \Psi_i \circ \Gamma,$$

so that if

$$b_m = \sum_{i=1}^m b \circ \Psi_i,$$

we have a representation of  $g_m^*$  of the form  $g_m^* = b_m \circ \Gamma$ .

Clearly,  $b_m$  satisfies (2.6i). For (2.6ii), observe that  $\Lambda_i B$  is a Brownian motion so (2.6ii) implies that  $(b \circ \Gamma \circ \Lambda_i)(B) > 0$  a.s. for  $0 \leq i < m$ , thus yielding (2.6ii) for  $b_m$ .

For (2.6iii), note that the continuity of  $\Lambda_i$  implies that

$$D(g_m^*) \subset \bigcup_{i=0}^{m-1} \{x: \Lambda_i x \in D(b \circ \Gamma)\}$$

so that

$$P\{B \in D(g_m^*)\} \leq \sum_{i=0}^{m-1} P\{\Lambda_i B \in D(b \circ \Gamma)\}.$$

But  $\Lambda_i B$  is a Brownian motion so that (2.6iii) for  $b$  shows that  $P\{\Lambda_i B \in D(b \circ \Gamma)\} = 0$  for  $0 \leq i < m$ , yielding (2.6iii) for  $b_m$ . Hence  $b_m \in \mathcal{N}$  and we are through.

It is of some interest to consider the behavior of  $g_m^*(B)$  for large  $m$ .

(3.4) PROPOSITION.  $g_m^*(B)/m^{1/2} \Rightarrow E(b \circ \Gamma \circ B)$  as  $m \rightarrow \infty$ .

PROOF. Note that

$$g_m^*(B)/m^{1/2} = \frac{1}{m} \sum_{i=0}^{m-1} b \circ \Gamma \circ (m^{1/2}\Lambda_i B).$$

But  $\{m^{1/2}\Lambda_i B: 0 \leq i < m\}$  has the same distribution as a collection of  $m$  independent standard Brownian motions  $\{B_i: 0 \leq i < m\}$ , so

$$g_m^*(B)/m^{1/2} \stackrel{\mathcal{D}}{=} \frac{1}{m} \sum_{i=0}^{m-1} b \circ \Gamma \circ B_i$$

( $\stackrel{\mathcal{D}}{=}$  denotes equality in distribution). Of course, the strong law of large numbers guarantees that

$$\frac{1}{m} \sum_{i=0}^{m-1} b \circ \Gamma \circ B_i \rightarrow E(b \circ \Gamma \circ B) \quad \text{a.s.}$$

as  $m \rightarrow \infty$ , proving the result.

Thus, if  $E(b \circ \Gamma \circ B) < \infty$ , we observe that

$$\frac{m^{1/2}(\bar{Y}_n(1) - \mu)}{g_m^*(\bar{Y}_n)/E(b \circ \Gamma \circ B)} \Rightarrow \frac{m^{1/2}B(1)}{g_m^*(B)/E(b \circ \Gamma \circ B)}$$

as  $n \rightarrow \infty$ , were the limit RV, for large  $m$ , is the normally distributed quantity  $B(1)$ . Hence, as the method of standardized time series is extended to more and more increments, the corresponding confidence intervals converge to those associated with a normal approximation. This phenomenon is consistent with that observed in the method of batch means, where it is known that as  $m \rightarrow \infty$ , the Student's  $t$  distribution approaches a normal.

A second class of functions in  $\mathcal{M}$  involves defining the class

$$\mathcal{N}_2 = \{b \in \mathcal{N}: P\{B \in D(b^2 \circ \Gamma)\} = 0\},$$

where  $b^2(x) = b(x) \cdot b(x)$ . The following proposition has a proof similar to that of Proposition 3.2 and will be omitted.

(3.5) PROPOSITION. If  $b \in \mathcal{N}_2$ , then  $\tilde{g}_m \in \mathcal{M}$  ( $m \geq 1$ ), where

$$\tilde{g}_m = \left( \sum_{i=0}^{m-1} b^2 \circ \Gamma \circ \Lambda_i \right)^{1/2}.$$

The analogue to Proposition 3.4 is then given by

(3.6) PROPOSITION.  $\tilde{g}_m(B) \Rightarrow (E(b^2 \circ \Gamma \circ B))^{1/2}$  as  $m \rightarrow \infty$ .

Thus, confidence intervals based on  $\tilde{g}_m(\cdot)$  will, for large  $m$ , correspond to that associated with a normal approximation. We now turn to some specific examples of  $g_m^*$ 's and  $\tilde{g}_m$ 's.

(3.7) EXAMPLE. Let  $b: C[0, 1] \rightarrow \mathbb{R}$  be defined by  $b(x) = \left| \int_0^1 x(t) dt \right|$ . To calculate the distribution of  $g_m^*(B)$  and  $\tilde{g}_m(B)$ , it is convenient to first find the distribution of



$(b \circ \Gamma)(B)$ . Note that the continuity of  $B$  implies that

$$(3.8) \quad \frac{1}{m} \sum_{i=1}^m (\Gamma B)(i/m) \rightarrow \int_0^1 (\Gamma B)(t) dt$$

as  $m \rightarrow \infty$ , a.s. The left-hand side of (3.8) is normally distributed with mean zero and variance

$$\begin{aligned} & \frac{1}{m^2} \sum_{k=1}^m \sum_{l=1}^m \text{cov}[(\Gamma B)(k/m), (\Gamma B)(l/m)] \\ &= \frac{1}{m^2} \sum_{k=1}^m \sum_{l=1}^m \left[ \min\left(\frac{k}{m}, \frac{l}{m}\right) - \frac{kl}{m^2} \right] \triangleq v(m). \end{aligned}$$

Note that  $v(m)$  is a Riemann approximation to the integral

$$\int_0^1 \int_0^1 [\min(s, t) - st] ds dt$$

which has value  $1/12$ . Thus,  $v(m) \rightarrow 1/12$ ; hence, taking characteristic functions of both sides of (3.8) shows that the right-hand side of (3.8) is normally distributed with mean-zero and variance  $1/12$ . Since  $\Lambda_0 B, \dots, \Lambda_{m-1} B$  are independent Brownian motions, it follows that

$$\sqrt{12m} g_m^*(B) \stackrel{\mathcal{D}}{=} \sum_{i=0}^{m-1} |B_i(1)|$$

where  $B_0, \dots, B_{m-1}$  are independent standard Brownian motions; on the other hand,

$$\sqrt{12m} \tilde{g}_m(B) = \left( \sum_{i=0}^{m-1} B_i^2(1) \right)^{1/2} = (\chi_m^2)^{1/2},$$

where  $\chi_m^2$  denotes a chi-square RV with  $m$  degrees of freedom. The chi-square property of  $\tilde{g}_m(B)$  makes standardized time series based on  $\tilde{g}_m$  particularly attractive, since in that case

$$\frac{B(1)}{(12m)^{1/2} \tilde{g}_m(B)} \stackrel{\mathcal{D}}{=} t_m$$

where  $t_m$  is the Student's  $t$  distribution with  $m$  degrees of freedom; the limit theorem (2.5) can then be used to construct confidence intervals for  $\mu$ . These confidence intervals, which were suggested by Schruben [18], are based on the so-called *standardized sum process*  $(\bar{Y}_n(1) - \mu/(12m)^{1/2} \tilde{g}_m(\bar{Y}_n(\cdot)))$ .

(3.9) EXAMPLE. The map  $b: C[0, 1] \rightarrow \mathbb{R}$  defined by

$$b(x) = \int_0^1 |x(t)| dt$$

also lies in the class  $\mathcal{N}$ . Furthermore, the distribution of  $(b \circ \Phi)(B)$  is known; see Johnson and Killeen [14]. However, the distributions of both  $g_m^*(B)$  and  $\tilde{g}_m(B)$  are quite complicated, and this would appear to limit the applicability of this method.

(3.10) EXAMPLE. Let  $b: C[0, 1] \rightarrow \mathbb{R}$  be defined by

$$b(x) = x(t^*) / (t^*(1 - t^*))^{1/2}$$

where  $t^* = \inf\{t \geq 0: x(t^*) = M^*\}$ ,  $M^* = \max\{x(t): 0 \leq t \leq 1\}$ . Schruben [17] showed that  $(b^2 \circ \Gamma)(B)$  has a chi-square distribution with 3 degrees of freedom. Consequently,

$$\sqrt{m} \tilde{g}_m(B) \stackrel{\mathcal{D}}{=} (\chi_{3m}^2)^{1/2}$$

so that

$$\frac{B(1)}{\sqrt{m} \tilde{g}_m(B)} \stackrel{\mathcal{D}}{=} t_{3m},$$

where  $t_{3m}$  is a Student's  $t$  RV with  $3m$  degrees of freedom. Confidence intervals based on  $\tilde{g}_m(B)$  as defined above are related to the *standardized maximum intervals* of [18].

**4. Asymptotics for standardized confidence intervals.** In this section, we study certain asymptotic properties of standardized confidence intervals. In particular, we consider the asymptotics of the expected length of such confidence intervals, as well as the end-point variability of these intervals.

Now, from (2.10), it is clear that the width of the interval (2.10) is given by  $L_n = g(\bar{Y}_n) \cdot (\beta - \alpha)$ .

(4.1) PROPOSITION. Assume  $g \in \mathcal{M}$ , and that (2.1) holds.

(a) If  $g$  is nonnegative, then

$$\liminf_{n \rightarrow \infty} n^{1/2} EL_n \geq \sigma E g(B) \cdot (\beta - \alpha).$$

(b) If  $\{g(X_n): n \geq 1\}$  is uniformly integrable, then

$$\lim_{n \rightarrow \infty} n^{1/2} EL_n = \sigma E g(B) \cdot (\beta - \alpha).$$

PROOF. Using the properties of  $g$ , it is easy to show that  $n^{1/2} L_n = g(X_n) \cdot (\beta - \alpha)$ . Assumption (2.1) and the continuous mapping theorem guarantees that if  $g \in \mathcal{M}$ , then

$$(4.2) \quad g(X_n) \Rightarrow \sigma g(B),$$

as  $n \rightarrow \infty$ . If  $g$  is nonnegative, then Fatou's lemma (see Billingsley [2, p. 32]) can be applied to (4.2) to conclude that

$$E g(B) \leq \liminf_{n \rightarrow \infty} E g(X_n),$$

proving (a). On the other hand, it is well known (see Chung [6, p. 96]) that uniform integrability implies that

$$E g(B) = \lim_{n \rightarrow \infty} E g(X_n),$$

proving (b).

Clearly, it is desirable to obtain confidence intervals with as small an expected length as possible. From Proposition 4.1, it seems reasonable to therefore choose  $\alpha$ ,  $\beta$ , and  $g \in \mathcal{M}$  such that  $E g(B) \cdot (\beta - \alpha)$  is minimized.

(4.3) PROPOSITION. *Suppose  $g \in \mathcal{M}$ . Then, for a  $100(1 - \delta)\%$  confidence interval,  $\beta - \alpha$  is minimized by choosing*

$$\beta = z(g; 1 - \delta/2), \quad \alpha = -\beta,$$

where  $z(g; x)$  solves the equation  $H(z(g; x)) = P\{B(1)/g(B) \leq z(g; x)\} = x$  (in other words, the confidence interval should be centered at  $\bar{Y}_n(1)$ ).

PROOF. We proceed in two steps. First, for any  $a \in \mathbb{R}$  and  $b, y \geq 0$ , it is easily verified that

$$(4.4) \quad \Phi((a + 2b)y) - \Phi(ay) \leq \Phi(by) - \Phi(-by).$$

Integrating both sides of (5.4) with respect to  $G(dy)$  and using (3.14), we get

$$(4.5) \quad H(a + 2b) - H(a) \leq H(b) - H(-b).$$

Furthermore, the symmetry of  $\Phi$  and (3.14) implies that  $H$  is also symmetric, in the sense that  $H(b) - H(0) = H(0) - H(-b)$  for  $b \geq 0$ .

For the second step, observe that the symmetry of  $H$  proves that  $H(z(g; 1 - \delta/2)) - H(-z(g; 1 - \delta/2)) = 1 - \delta$ . Set  $b = z(g; 1 - \delta/2)$ . Then, for any  $\alpha \in \mathbb{R}$ , (4.5) yields

$$H(\alpha + 2b) - H(\alpha) \leq 1 - \delta.$$

Thus, in order that  $H(\beta) - H(\alpha) = 1 - \delta$ , it must be that  $\beta - \alpha \geq 2b$ ; proving our assertion.

We turn now to the choice of  $g \in \mathcal{M}$ . Our goal is to find  $g$  minimizing

$$(4.6) \quad \psi(g) = Eg(B) \cdot z(g; 1 - \delta/2).$$

Next we show that the criterion (4.6) is scale-invariant.

$$(4.7) \text{ LEMMA. } \textit{For } b > 0, \psi(bg) = \psi(g).$$

PROOF. Note that  $z(g; 1 - \delta/2)$  solves

$$\begin{aligned} 1 - \delta/2 &= P\{B(1) \leq z(g; 1 - \delta/2) \cdot g(B)\} \\ &= P\left\{B(1) \leq \frac{1}{b} z(g; 1 - \delta/2) \cdot b \cdot g(B)\right\} \\ &= P\{B(1) \leq z(bg; 1 - \delta/2) \cdot b \cdot g(B)\} \end{aligned}$$

so that the continuity and strict monotonicity of  $H$  imply that

$$z(bg; 1 - \delta/2) = \frac{1}{b} z(g; 1 - \delta/2).$$

Relation (4.6) then yields the lemma.

(4.8) THEOREM. *Suppose  $g \in \mathcal{M}$ . Then*

$$\psi(g) \geq \Phi^{-1}(1 - \delta/2),$$

where  $\Phi^{-1}$  is the inverse of the normal distribution function  $\Phi$ .

PROOF. By Lemma 4.7, we may scale  $g$  so that

$$(4.9) \quad z(g; 1 - \delta/2) = 1.$$

Now, (4.9) implies that

$$H(1) = 1 - \delta/2, \quad \text{or}$$

$$\int_0^\infty \Phi(y) G_g(dy) = 1 - \delta/2,$$

where  $G_g(dy) = P\{g(B) \in dy\}$  (see (2.9)). Thus, we are to show that

$$(4.10) \quad \begin{aligned} \psi(g) &= Eg(B) \cdot z(g; 1 - \delta/2) \\ &= \int_0^\infty (1 - G_g(y)) dy \geq \Phi^{-1}(1 - \delta/2), \end{aligned}$$

subject to

$$(4.11) \quad \int_0^\infty \Phi(y) G_g(dy) = 1 - \delta/2.$$

Integrating by parts, we find that

$$\begin{aligned} \int_0^\infty \Phi(y) G_g(dy) &= -[\Phi(y) \bar{G}_g(y)]_0^\infty + \int_0^\infty \bar{G}_g(y) \phi(y) dy \\ &= \int_0^\infty [\bar{G}_g(y) + 1] \phi(y) dy, \end{aligned}$$

where  $\bar{G}_g(y) = 1 - G_g(y)$  and  $\phi(y)$  is the normal density function.

Let  $K(y)$  be the distribution function defined by

$$K(y) = \begin{cases} 0, & y < p, \\ 1, & y \geq p, \end{cases}$$

where  $p = \Phi^{-1}(1 - \delta/2)$ . Note that

$$\int_0^\infty [\bar{K}(y) + 1] \phi(y) dy = 1 - \delta/2 \quad \text{and}$$

$$\int_0^\infty \bar{K}(y) dy = p,$$

where  $\bar{K}(y) = 1 - K(y)$ . Thus, we can reformulate (4.10) and (4.11) as follows:

$$(4.12) \quad \int_0^\infty (\bar{G}_g(y) - \bar{K}(y)) dy \geq 0$$

subject to

$$(4.13) \quad \int_0^\infty (\bar{G}_g(y) - \bar{K}(y)) \phi(y) dy = 0.$$

Since  $\phi$  is strictly decreasing on  $[0, \infty)$ , and because

$$\begin{aligned}\bar{G}_g(y) - \bar{K}(y) &\leq 0 \quad \text{for } y < p, \\ \bar{G}_g(y) - \bar{K}(y) &\geq 0 \quad \text{for } y \geq p,\end{aligned}$$

it follows that

$$(4.14) \quad \int_0^p (\bar{G}_g(y) - \bar{K}(y)) dy \cdot \phi(p) \geq \int_0^p (\bar{G}_g(y) - \bar{K}(y)) \phi(y) dy \quad \text{and}$$

$$(4.15) \quad \int_p^\infty (\bar{G}_g(y) - \bar{K}(y)) dy \cdot \phi(p) \geq \int_p^\infty (\bar{G}_g(y) - \bar{K}(y)) \phi(y) dy.$$

Adding (4.14) and (4.15) together, we get

$$\int_0^\infty (\bar{G}_g(y) - \bar{K}(y)) dy \cdot \phi(p) \geq \int_0^\infty (\bar{G}_g(y) - \bar{K}(y)) \phi(y) dy.$$

Relation (4.13) then yields (4.12).

(4.16) COROLLARY. *Suppose  $g \in \mathcal{M}$  is nonnegative. Under Assumption (2.1),*

$$\lim_{n \rightarrow \infty} n^{1/2} EL_n \geq 2\sigma \Phi^{-1}(1 - \delta/2).$$

This corollary follows immediately from Propositions 4.1 and 4.3, and Theorem 4.8. The lower bound of Corollary 4.16 has an important interpretation. Consider a steady-state simulation output analysis algorithm which is based on constructing an estimator  $s_n$  which consistently estimates  $\sigma$ :

$$(4.17) \quad s_n \Rightarrow \sigma,$$

as  $n \rightarrow \infty$ . Among the algorithms of this type are the regenerative method of simulation, spectral methods, and autoregressive procedures (see Chapter 3 of Bratley, Fox and Schrage [3] for a description of these techniques). The following proposition is a straightforward application of the converging-together lemma (see p. 25 of [2]).

(4.18) PROPOSITION. *If  $s_n$  is an estimator satisfying (4.17), then (2.1) implies that*

$$(4.19) \quad n^{1/2}(\bar{Y}_n(1) - \mu)/s_n \Rightarrow B(1),$$

as  $n \rightarrow \infty$ .

The weak convergence result (4.19) permits construction of asymptotic  $100(1 - \delta)\%$  confidence intervals for  $\mu$ :

$$(4.20) \quad \left[ \bar{Y}_n(1) - z(\delta) \frac{s_n}{n^{1/2}}, \bar{Y}_n(1) + z(\delta) \frac{s_n}{n^{1/2}} \right],$$

where  $z(\delta) = \Phi^{-1}(1 - \delta/2)$ . If  $L_n$  is the length of the interval (4.20), it is clear that as  $n \rightarrow \infty$ ,

$$(4.21) \quad n^{1/2}L_n \Rightarrow 2\sigma \Phi^{-1}(1 - \delta/2),$$

which is precisely the lower bound of Corollary 4.16. If  $\{s_n; n \geq 1\}$  is uniformly integrable (conditions guaranteeing this appear in Glynn and Iglehart [11, Section 6]), then we further have that

$$(4.22) \quad \lim_{n \rightarrow \infty} n^{1/2}EL_n = 2\sigma\Phi^{-1}(1 - \delta/2).$$

Corollary 4.16 and the limit theorems (4.21) and (4.22) suggest that, from the viewpoint of expected confidence interval length, output analysis methods which consistently estimate  $\sigma$  dominate standardized time series procedures asymptotically.

One further point, pertinent to expected confidence interval length, remains to be investigated. The examples of §3 show that for any  $k \geq 1$ , there exists  $g_k \in \mathcal{M}$  such that  $B(1)/g_k(B)$  has a Student's  $t$  distribution with  $k$  degrees of freedom. If  $g_k(X_n)$  is uniformly integrable, then it follows that if  $L_n(k)$  is the length of such a confidence interval,

$$\lim_{n \rightarrow \infty} n^{1/2}EL_n(k) = 2\sigma H_k^{-1}(1 - \delta/2),$$

where  $H_k^{-1}(p)$  is the  $p$ th quantile of a Student's  $t$  with  $k$  degrees of freedom. Since

$$\lim_{k \rightarrow \infty} H_k^{-1}(1 - \delta/2) = \Phi^{-1}(1 - \delta/2),$$

this discussion proves that

$$(4.23) \quad \inf_{g \in \mathcal{M}} \lim_{n \rightarrow \infty} n^{1/2}EL_n = 2\sigma\Phi^{-1}(1 - \delta/2);$$

thus, the lower bound of Corollary 4.16 is tight. Relation (4.23) raises the question of whether there exists  $g \in \mathcal{M}$  such that

$$(4.24) \quad \lim_{n \rightarrow \infty} n^{1/2}EL_n = 2\sigma\Phi^{-1}(1 - \delta/2);$$

in other words, is the lower bound attained with  $\mathcal{M}$ ?

A glance at the proof of Theorem 4.9 shows that  $\phi(g) > \Phi^{-1}(1 - \delta/2)$  unless  $G_g(dy)$  is a point-mass distribution. Thus, in order to find  $g \in \mathcal{M}$  satisfying (4.24), it must be that

$$(4.25) \quad P\{g(\sigma B) = \alpha\sigma\} = 1$$

for some  $\alpha > 0$ . Our next result shows that such a  $g$  cannot exist.

(4.26) PROPOSITION. *There exists no  $g \in \mathcal{M}$  such that (4.25) holds.*

PROOF. We will prove something stronger: the requirements  $P\{B \in D(g)\} = 0$  and (4.25) are incompatible. We start by showing that for every  $x \in C_0[0, 1] \equiv \{x \in C[0, 1]: x(0) = 0\}$  and  $\epsilon > 0$ ,

$$(4.27) \quad P\{\rho(\sigma B, x) < \epsilon\} > 0,$$

where  $\rho$  is the uniform metric on  $C[0, 1]$ . To see this, fix  $x \in C_0[0, 1]$  and  $\delta > 0$ . Since  $[0, 1]$  is compact,  $x$  is uniformly continuous on  $[0, 1]$ , so there exists  $N = N(\epsilon)$  such that

$$(4.28) \quad |x(t) - x(k/N)| < \epsilon/4$$

for  $kN \leq t \leq (k+1)/N$ , where  $0 \leq k < N$ . Now, the independent increments of Brownian motion imply that if  $\Delta z(k/N) \equiv z((k+1)/N) - z(k/N)$ , then

(4.29)

$$P(A(\epsilon)) \equiv P\left\{|\sigma\Delta B(k/N) - \Delta x(k/N)| < \epsilon/4N, \right. \\ \left. \sigma \max_{k/N \leq t \leq (k+1)/N} |B(t) - B(k/N)| < \epsilon/2, \quad 0 \leq k < N\right\} \\ = \prod_{k=0}^{N-1} P\left\{|\sigma\Delta B(k/N) - \Delta x(k/N)| < \epsilon/4N, \right. \\ \left. \sigma \max_{k/N \leq t \leq (k+1)/N} |B(t) - B(k/N)| < \epsilon/2\right\} > 0,$$

by virtue of the fact that for any  $z$  with  $|z| < \eta$ ,  $P\{|B(t) - z| < \eta, \max_{0 \leq s \leq t} |B(s)| < 2\eta\} > 0$ . Now, on the event  $A(\epsilon)$ , a simple triangle inequality argument shows that  $|\sigma B(t) - x(t)| < \epsilon$  for  $0 \leq t \leq 1$  (use (4.28)), proving (4.27).

From (4.27) and (4.25), it follows that for some  $\alpha > 0$

$$P\{\rho(\sigma B, x) < \epsilon, g(\sigma B) = \alpha\sigma\} > 0,$$

so that there necessarily exists  $y = y(x, \epsilon)$  such that  $\rho(y, x) < \epsilon$  with  $g(y) = \alpha\sigma$ . Thus, the range of  $g$  over any  $\epsilon$ -neighborhood of  $x$  contains the set  $\{\alpha\sigma: \sigma > 0\}$ ; clearly, then  $g$  cannot be continuous at  $x$ .

Hence,  $x \in D(g)$ . Since  $x$  was arbitrary, this implies that  $D(g) = C_0[0, 1]$ , violating the assumption  $P\{B \in D(g)\} = 0$ .

We now turn to the question of end-point variability. To be precise, observe that if  $g \in \mathcal{M}$ , then (2.1) implies that

$$(4.30) \quad n^{1/2}L_n \Rightarrow \sigma g(B) \cdot (\beta - \alpha)$$

as  $n \rightarrow \infty$  (see (4.2) for a more complete argument). The limit distribution of the confidence interval length is, of course, degenerate if (and only if)  $g(B)$  is degenerate. Suppose that, in fact,  $g(B)$  is degenerate so that there exists  $\alpha$  such that  $P\{g(B) = \alpha\} = 1$ . Note that  $\alpha > 0$  by (2.3iii). On the other hand, it follows from (2.3i) that

$$(4.31) \quad P\{g(\sigma B) = \alpha\sigma\} = 1$$

for all  $\sigma > 0$ . But (4.31) is, of course, just (4.25); Proposition 4.26 therefore proves that no such  $g$  can exist. Consequently, we may conclude that  $g(B)$  must be nondegenerate. The limit theorem (4.30) therefore states that  $L_n$  exhibits nondegenerate random fluctuations of order  $n^{-1/2}$ .

Another way to quantify the above phenomenon is to examine the quantity  $E(L_n - EL_n)^2$ .

(4.32) PROPOSITION. *If  $\{g^2(X_n): n \geq 1\}$  is uniformly integrable, then under (2.1),*

$$(4.33) \quad \lim_{n \rightarrow \infty} nE(L_n - EL_n)^2 = \sigma^2 E(g(B) - Eg(B))^2 (\beta - \alpha)^2$$

*provided  $g \in \mathcal{M}$ . Furthermore, the right-hand side of (4.33) is positive.*

PROOF. The uniform integrability of  $\{g^2(X_n): n \geq 1\}$  implies that  $\{g(X_n): n \geq 1\}$  (see p. 100 of [6]), so

$$\lim_{n \rightarrow \infty} n^{1/2} E g(\bar{Y}_n) = \lim_{n \rightarrow \infty} E g(X_n) = E g(B) \quad \text{and}$$

$$\lim_{n \rightarrow \infty} n E g^2(\bar{Y}_n) = \lim_{n \rightarrow \infty} E g^2(X_n) = E g^2(B);$$

combining the above two limit relations yields (4.33). As for the positivity, this follows from the nondegeneracy of  $g(B)$  for  $g \in \mathcal{M}$ .

We now wish to compare the end-point variability of standardized time series procedures to that obtained via methods which consistently estimate  $\sigma$ . Our analysis will be restricted to the regenerative method of simulation; we do this only because the required limit theorems are available in this context.

As (4.21) indicates,  $n^{1/2}L_n$  converges to a degenerate RV. Thus,  $L_n$  asymptotically exhibits no random fluctuations of order  $n^{-1/2}$ . We can, in fact, be more precise.

(4.34) PROPOSITION. *Let  $Y$  be a regenerative process and  $f$  a real-valued function on the state space of  $Y$ . Assume that 0 and  $\tau_1$  are the first two regeneration times,  $Y_1(f) = \int_0^{\tau_1} f(Y(s)) ds$ , and  $E(Y_1(|f|)^8 + \tau_1^8) < \infty$ . Then, if  $s_n$  is the regenerative estimator for  $\sigma$  (see, for example, [10]), there exists  $\eta$  such that*

$$(4.35) \quad \begin{aligned} \text{(i)} \quad & n(L_n - EL_n) \Rightarrow \eta N(0, 1) \quad \text{as } n \rightarrow \infty; \\ \text{(ii)} \quad & n^2 E(L_n - EL_n)^2 \rightarrow \eta^2 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

PROOF. Under the above moment hypothesis, there exists  $\kappa$  such that

$$(4.36) \quad n^{1/2}(s_n - \sigma) \Rightarrow \kappa N(0, 1),$$

as  $n \rightarrow \infty$ ; furthermore, the sequence  $\{n(s_n - \sigma)^2: n \geq 1\}$  is uniformly integrable (see Sections 5 and 6 of [10]). Thus,

$$(4.37) \quad n E \left( L_n - \frac{2z(\delta)\sigma}{n^{1/2}} \right) \rightarrow 0,$$

as  $n \rightarrow \infty$ ; combining (4.36) and (4.37), we get (4.35i). For (ii), we use the uniform integrability to obtain  $nE(s_n - \sigma)^2 \rightarrow \kappa^2$ ; this evidently implies that

$$(4.38) \quad n^2 E(L_n - EL_n)^2 \rightarrow 4z^2(\delta)\kappa^2,$$

proving (ii).

We conclude that the end-point variability of the regenerative confidence interval is of order  $n^{-1}$ , as opposed to  $n^{-1/2}$  for standardized time series.

**Acknowledgements.** The authors wish to thank Thomas G. Kurtz for the idea behind the proof of Proposition 4.26. Also we are grateful to several referees and an associate editor for suggestions which improved the paper.

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