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# On the rate of convergence to equilibrium for two-sided reflected Brownian motion and for the Ornstein–Uhlenbeck process

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**Abstract**
This paper studies the rate of convergence to equilibrium for two diffusion models that arise naturally in the queueing context: two-sided reflected Brownian motion and the Ornstein–Uhlenbeck process. Specifically, we develop exact asymptotics and upper bounds on total variation distance to equilibrium, which can be used to assess the quality of the steady state as an approximation to finite-horizon performance quantities. Our analysis relies upon the simple spectral structure that these two processes possess, thereby explaining why the convergence rate is "pure exponential," in contrast to the more complex convergence exhibited by one-sided reflected Brownian motion.

**Keywords** Two-sided reflected Brownian motion · Ornstein–Uhlenbeck process · Queueing theory · Total variation distance · Rate of convergence to equilibrium

**Mathematics Subject Classification** 60F05 · 60F10 · 60G05 · 60J60 · 60K25

## 1 Introduction

In this paper, we study the rate of convergence to equilibrium for two diffusion processes that arise naturally as weak limits of queueing systems: *two-sided reflected Brownian motion* (RBM) and the *Ornstein–Uhlenbeck* (O–U) *process*. Our work is a companion to Glynn and Wang [7], which provides a comprehensive analysis of the rate of convergence to equilibrium for one-sided RBM (with lower reflecting boundary

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at the origin and negative drift). As in Glynn and Wang [7], our primary contribution is to develop exact asymptotics and explicit upper bounds on the total variation (TV) distance from equilibrium that can be used to assess the quality of the steady state as an approximation to finite-horizon performance quantities. In addition, these bounds and exact asymptotics can be used to provide guidance to help plan the length of the "warm-up" period required to accurately compute steady-state quantities via simulation for processes that can be approximated by these models (for additional discussion of the *initial transient problem*, see Wang and Glynn [16,17]).

Let $X = (X(t) : t \geq 0)$ be a positive recurrent Markov process for which $X(t) \Rightarrow X(\infty)$ as $t \to \infty$, where $\Rightarrow$ denotes *weak convergence*. As in Glynn and Wang [7], we denote the TV distance of $X(t)$ to equilibrium (conditional on $X(0) = x$) by

$$\begin{aligned} d(t,\, x) &= \sup_A |P_x(X(t) \in A) - P(X(\infty) \in A)| \\ &= \frac{1}{2} \sup_{|f| \leq 1} |E_x f(X(t)) - Ef(X(\infty))|, \end{aligned} \tag{1.1}$$

where $P_x(\cdot)$ denotes the probability law associated with $X$ conditional on $X(0) = x$ and $E_x(\cdot)$ is the corresponding expectation operator, the first supremum is taken over all measurable subsets $A$, and the second supremum is taken over all (measurable) real-valued functions $f$ bounded by 1 in absolute value. The total variation distance is the most widely used metric for assessing rates of convergence of the transient distribution to equilibrium; see, for example, Roberts and Rosenthal [15], Meyn and Tweedie [14], and Diaconis [5]. One nice property of TV distance is that it is guaranteed to be monotone in $t$; see, for example, Roberts and Rosenthal [15].

Let $B = (B(t) : t \geq 0)$ be a one-dimensional standard Brownian motion, so that $EB(t) = 0$ and $\text{var}\, B(t) = t$ for all $t \geq 0$. In Glynn and Wang [7], we consider the rate of convergence to equilibrium for a *one-sided RBM* $X = (X(t) : t \geq 0)$ with drift $-r$ and volatility $\sigma > 0$, so that $X$ satisfies the stochastic differential equation (SDE)

$$\mathrm{d}X(t) = -r\mathrm{d}t + \sigma \mathrm{d}B(t) + L(t).$$

Here, $L = (L(t) : t \geq 0)$ is the continuous non-decreasing process for which $\mathbb{I}(X(t) > 0)\mathrm{d}L(t) = 0$ for $t \geq 0$. This process is known as the *local time* of $X$ at the origin. The process $X$ has an equilibrium distribution if and only if $r > 0$; see Harrison [8, p. 102]. Set $\eta = r/\sigma^2$ and $\nu = r^2/\sigma^2$. Glynn and Wang [7] prove that, for one-sided RBM,

$$d(t,\, x) \sim \sqrt{\frac{2}{\pi}} (\nu t)^{-\frac{3}{2}} \exp\left(-\frac{\nu t}{2}\right) |1 - \eta x| e^{\eta x - 1}$$

as $t \to \infty$, when $x \neq \eta^{-1}$, where we use the notation $a(t) \sim b(t)$ as $t \to \infty$ whenever $a(t)/b(t) \to 1$ as $t \to \infty$. An interesting (and important) feature of this result is the presence of the algebraically decaying pre-factor of order $t^{-\frac{3}{2}}$ that appears in this setting.

In this paper, we show that for two-sided RBM and the O–U process, the TV distance has much simpler "pure exponential" asymptotics, so that

$$d(t,\ x) \sim c(x)e^{-\lambda t} \tag{1.2}$$

as $t \to \infty$, for some function $c(\cdot)$, and we identify $c(\cdot)$ and $\lambda$ for both of the models. We also provide upper bounds on $d(t,\ x)$ that can be potentially exploited computationally.

Section 2 develops some general theory, starting with a Hilbert space perspective on rates of convergence and concluding with results focused on TV asymptotics and upper bounds. In Sect. 3, we apply these results to two-sided RBM and the O–U process, thereby verifying (1.2) for these models; see Theorems 2 and 3, respectively.

## 2 General theory

Let $X = (X(t) : t \geq 0)$ be an $\mathbb{S}$-valued Markov process with a unique stationary distribution $\pi$. For $p \geq 1$, let $L^p(\pi)$ be the vector space of real-valued functions $f$ on $\mathbb{S}$ such that

$$E_\pi |f(X(0))|^p < \infty,$$

where $E_\pi(\cdot)$ is the expectation operator conditional on $X(0)$ having distribution $\pi$. Then, $L^2(\pi)$ is a Hilbert space with inner product

$$\langle f,\ g\rangle = E_\pi f(X(0))g(X(0))$$

for $f,\ g \in L^2(\pi)$, having associated norm

$$\|f\| = \sqrt{\langle f,\ f\rangle}.$$

We assume that there exists a countably infinite orthonormal basis

$$(u_i \in L^2(\pi) : i \geq 0)$$

for $L^2(\pi)$ for which $u_0(x) \equiv 1$ for $x \in \mathbb{S}$. Furthermore, we require that there exist

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$$

such that

$$E_x u_i(X(t)) = e^{-\lambda_i t} u_i(x)$$

for $t \geq 0$, so that $u_i$ is an *eigenfunction* of $X$ associated with *eigenvalue* $-\lambda_i$. Note that

$$E_\pi u_i(X(1)) = E_\pi u_i(X(0)) = e^{-\lambda_i} E_\pi u_i(X(0)),$$

for $i \geq 0$, so that $E_\pi u_i(X(0)) = 0$ for $i \geq 1$.

In this case, $f \in L^2(\pi)$ can be written as

$$f = \sum_{i=0}^{\infty} \langle f, u_i \rangle u_i \quad \pi \text{ a.e.}$$

and

$$\|f\| = \left( \sum_{i=0}^{\infty} \langle f, u_i \rangle^2 \right)^{\frac{1}{2}};$$

see, for example, Folland [6, pp. 171–177].

For $f \in L^2(\pi)$, the Cauchy–Schwarz inequality implies that $E_x|f(X(t))| < \infty$ for $\pi$ a.e. $x$. Set

$$(P(t)f)(x) \stackrel{\Delta}{=} E_x f(X(t)).$$

Also, put

$$\begin{aligned} f_n(x) &= \sum_{i=0}^{n} \langle f, u_i \rangle u_i(x), \\ w_n(x) &= \sum_{i=0}^{n} \langle f, u_i \rangle e^{-\lambda_i t} u_i(x). \end{aligned}$$

Clearly, $w_n = P(t) f_n$ $\pi$ a.e. and $w_n \to w_\infty$ in $L^2(\pi)$. Since the Cauchy–Schwarz inequality implies that

$$(P(t)(f_n - f))^2(x) \leq P(t)(f_n - f)^2(x)$$

for $x \in \mathbb{S}$, it follows that

$$\begin{aligned} E_\pi(w_n(X(0)) - (P(t)f)(X(0)))^2 &= E_\pi((P(t)f_n)(X(0)) - (P(t)f)(X(0)))^2 \\ &\leq E_\pi(P(t)(f_n - f))^2(X(0)) \\ &= E_\pi(f_n(X(0)) - f(X(0)))^2 \\ &\to 0 \end{aligned}$$

as $n \to \infty$, since $f_n \to f$ in $L^2(\pi)$. Consequently, $P(t)f = w_\infty$ $\pi$ a.e.

Set $f_c(x) = f(x) - E_\pi f(X(0)) = f(x) - \langle f, u_0 \rangle u_0(x)$, and note that

$$\begin{aligned} \left\| e^{\lambda_1 t} P(t) f_c - \langle f, u_1 \rangle u_1 \right\|^2 &= \left\| e^{\lambda_1 t} (w_\infty - \langle f, u_0 \rangle u_0) - \langle f, u_1 \rangle u_1 \right\|^2 \\ &= \left\| e^{\lambda_1 t} \sum_{i=1}^{\infty} e^{-\lambda_i t} \langle f, u_i \rangle u_i - \langle f, u_1 \rangle u_1 \right\|^2 \end{aligned}$$

$$= \left\| \sum_{i=2}^{\infty} e^{-(\lambda_i - \lambda_1)t} \langle f, u_i \rangle u_i \right\|^2$$
$$= \sum_{i=2}^{\infty} e^{-2(\lambda_i - \lambda_1)t} \langle f, u_i \rangle^2$$
$$\to 0$$

as $t \to \infty$, since $\lambda_i > \lambda_1$ for $i \geq 2$ and $\sum_{i=0}^{\infty} \langle f, u_i \rangle^2 < \infty$ (because $f \in L^2(\pi)$). We have therefore proved Proposition 1.

**Proposition 1** *For $f \in L^2(\pi)$, we have*

$$e^{\lambda_1 t}((P(t)f)(\cdot) - E_\pi f(X(0))) \to \langle f, u_1 \rangle u_1(\cdot)$$

*as $t \to \infty$ in $L^2(\pi)$.*

Proposition 1 asserts that, in the $L^2(\pi)$ norm,

$$(P(t)f)(\cdot) - E_\pi f(X(0)) \approx e^{-\lambda_1 t} \langle f, u_1 \rangle u_1(\cdot)$$

for $t$ large. However, given our interest in the TV norm [which is more closely related to $L^1(\pi)$], we will now modify our setup somewhat. Note that the equality $P(t)f = w_\infty$ implies that

$$E_x f(X(t)) = \int_{\mathbb{S}} p(t, x, y) f(y) \pi(\mathrm{d}y)$$

for $\pi$ a.e. $x$ and $f \in L^2(\pi)$, where

$$p(t, x, y) = \sum_{i=0}^{\infty} e^{-\lambda_i t} u_i(x) u_i(y).$$

Such a representation for the transition density (with respect to $\pi$) in terms of decaying exponentials is often called a *spectral representation*. This motivates our next assumption.

(A1) $X = (X(t) : t \geq 0)$ has a stationary distribution $\pi$ for which

$$P_x(X(t) \in \mathrm{d}y) = p(t, x, y) \pi(\mathrm{d}y)$$

for $x, y \in \mathbb{S}$. Furthermore,

$$p(t, x, y) = \sum_{i=0}^{\infty} e^{-\lambda_i t} u_i(x) u_i(y)$$

for $x,\ y \in \mathbb{S}$, where $E_\pi u_i^2(X(0)) = 1$ for $i \geq 0$, $0 = \lambda_0 < \lambda_1 < \cdots$, $u_0(x) \equiv 1$ for $x \in \mathbb{S}$, and there exists $t_0 > 0$ such that

$$\sum_{i=0}^{\infty} e^{-\lambda_i t_0} |u_i(x)| < \infty \tag{2.1}$$

for $x \in \mathbb{S}$.

Equipped with (A1), we arrive at Theorem 1.

**Theorem 1** *If X satisfies (A1), then*

$$d(t,\ x) \sim \frac{1}{2} e^{-\lambda_1 t} |u_1(x)| \int_{\mathbb{S}} |u_1(y)| \pi(\mathrm{d}y) \tag{2.2}$$

*as* $t \to \infty$*, provided that* $u_1(x) \neq 0$*. Also,*

$$d(t,\ x) \leq \frac{1}{2} \sum_{i=1}^{\infty} e^{-\lambda_i t} |u_i(x)| \tag{2.3}$$

*for* $t \geq t_0$ *and* $x \in \mathbb{S}$*.*

***Proof*** It is well known that, because $X(t)$ has a density with respect to $\pi$,

$$d(t,\ x) = \frac{1}{2} \int_{\mathbb{S}} |p(t,\ x,\ y) - 1| \pi(\mathrm{d}y).$$

(See, for example, Proposition 3 of Roberts and Rosenthal [15].) So, in view of the fact that $\lambda_0 = 0$ and $u_0(y) \equiv 1$ for $y \in \mathbb{S}$,

$$d(t,\ x) = \frac{1}{2} \int_{\mathbb{S}} \left| \sum_{i=1}^{\infty} e^{-\lambda_i t} u_i(x) u_i(y) \right| \pi(\mathrm{d}y).$$

For $t \geq t_0$, the Cauchy–Schwarz inequality implies that

$$\begin{aligned} d(t,\ x) &\leq \frac{1}{2} \sum_{i=1}^{\infty} e^{-\lambda_i t} |u_i(x)| \int_{\mathbb{S}} |u_i(y)| \pi(\mathrm{d}y) \\ &\leq \frac{1}{2} \sum_{i=1}^{\infty} e^{-\lambda_i t} |u_i(x)| \sqrt{\|u_i\|^2} \\ &\leq \frac{1}{2} \sum_{i=1}^{\infty} e^{-\lambda_i t} |u_i(x)|, \end{aligned} \tag{2.4}$$

yielding result (2.3). To obtain result (2.2), note that

$$e^{\lambda_1 t} d(t, x) = \frac{1}{2} \int_{\mathbb{S}} \left| \sum_{i=1}^{\infty} e^{-(\lambda_i - \lambda_1)t} u_i(x) u_i(y) \right| \pi(\mathrm{d}y).$$

Condition (2.1) and inequality (2.4) allow us to apply the dominated convergence theorem to conclude that

$$\int_{\mathbb{S}} \left| \sum_{i=2}^{\infty} e^{-(\lambda_i - \lambda_1)t} u_i(x) u_i(y) \right| \pi(\mathrm{d}y) \to 0$$

as $t \to \infty$, thereby yielding result (2.2). □

We note that Theorem 1 implies that the TV distance converges to 0 according to "pure exponential" asymptotics (with no algebraically decaying pre-factor). To understand why one-sided RBM has more complex asymptotics, it is instructive to note that its spectral representation (with respect to its stationary distribution) has the form

$$p(t, x, y) = 1 + \int_{\frac{\nu}{2}}^{\infty} e^{-\lambda t} u_\lambda(x) u_\lambda(y) \left( \frac{s(\lambda)}{2\pi \lambda} \right) \mathrm{d}\lambda,$$

where $s(\lambda) = \sqrt{2\lambda - \nu}/\sigma$ for $\lambda \geq \nu/2$ and

$$u_\lambda(x) = e^{\eta x} \left( \cos(s(\lambda)x) - \frac{\eta}{s(\lambda)} \sin(s(\lambda)x) \right)$$

for $\lambda > \nu/2$. In particular, one-sided RBM's spectral representation has no positive "gap" between $\lambda = \nu/2$ and larger values in the spectrum, as required by (A1). This lack of a gap is what contributes to the algebraic pre-factor for one-sided RBM; see Glynn and Wang [7].

## 3 Rates of convergence for two-sided RBM and the O–U process

We shall now apply Theorem 1 to our two models. We say that $X = (X(t) : t \geq 0)$ is a two-sided RBM (with reflecting boundaries at 0 and $\ell > 0$) having drift $-r$ and volatility $\sigma > 0$ if it satisfies the SDE

$$\mathrm{d}X(t) = -r\mathrm{d}t + \sigma \mathrm{d}B(t) + \mathrm{d}L(t) - \mathrm{d}U(t), \tag{3.1}$$

where $B$ and $L$ are as in the one-sided setting, and $U = (U(t) : t \geq 0)$ is a non-decreasing continuous process satisfying $\mathbb{I}(X(t) < \ell)\mathrm{d}U(t) = 0$ for $t \geq 0$. The process $U$ is called the local time for $X$ at $\ell$. Because of the reflecting barriers at 0 and $\ell$, $X$ is always positive recurrent, regardless of the value of $r \in \mathbb{R}$ (unlike the one-sided case where $r > 0$ is required). In particular, the density $p(\cdot)$ (with respect to Lebesgue measure) of the stationary distribution $\pi$ for $X$ is given by

$$p(y) = \begin{cases} \frac{2\eta e^{-2\eta y}}{1-e^{-2\eta\ell}}, & 0 \le y \le \ell, \ \eta \ne 0 \\ \frac{1}{\ell}, & 0 \le y \le \ell, \ \eta = 0, \end{cases} \tag{3.2}$$

where $\eta$ is as in the one-sided setting; see Harrison [8, p. 99].

Two-sided RBM arises as the weak limit of the number-in-system process for queues having a buffer with finite capacity $\ell$; see Whitt [18, p. 154]. In Berger and Whitt [1], the authors study the quality of Brownian approximations for both the number-in-system process and the overflow process in $G/G/1/C$ systems. They further compute asymptotic variance parameters associated with $L(\cdot)$ and $U(\cdot)$; see also Williams [19] for a direct proof. More recently, Zhang and Glynn [20] study large deviations asymptotics for $U(\cdot)$ as well as associated conditional queue dynamics, and D'Auria and Kella [4] compute the stationary distribution for two-sided RBM in the presence of Markov modulation.

In Linetsky [13], it is shown that, for $0 \le x, \ y \le \ell$, the transition density of $X$ with respect to $\pi$ is given by

$$p(t, x, y) = \sum_{i=0}^{\infty} e^{-\lambda_i t} u_i(x) u_i(y)$$

with $\lambda_0 = 0$, $u_0(x) \equiv 1$, and, when $r \ne 0$,

$$\lambda_i = \frac{\nu}{2} + \frac{\pi^2 i^2 \sigma^2}{2\ell^2}$$

and

$$u_i(x) = \sqrt{\frac{1 - e^{-2\eta\ell}}{\eta^3 \ell + \frac{\pi^2 i^2 \eta}{\ell}}} \left\{ \frac{\pi i}{\ell} \cos\left(\frac{x\pi i}{\ell}\right) - \eta \sin\left(\frac{x\pi i}{\ell}\right) \right\} e^{\eta x},$$

for $i \ge 1$ (with $\nu$ and $\eta$ as in the one-sided case). On the other hand, if $r = 0$,

$$\lambda_i = \frac{\pi^2 i^2 \sigma^2}{2\ell^2}$$

and

$$u_i(x) = \sqrt{2} \cos\left(\frac{x\pi i}{\ell}\right)$$

for $i \ge 1$. Furthermore, $E_\pi u_i(X(0))^2 = 1$ for $i \ge 0$.

When $r \neq 0$, the coefficient for the cosine term appearing in $u_i(\cdot)$ can be bounded by $(\pi i/\ell)(\pi^2 i^2 \eta/\ell)^{-\frac{1}{2}} = (\eta\ell)^{-\frac{1}{2}}$, whereas the coefficient for the sine function is bounded by $\eta(\eta^3\ell)^{-\frac{1}{2}} = (\eta\ell)^{-\frac{1}{2}}$. Consequently,

$$|u_i(x)| \leq \frac{2}{\sqrt{\eta\ell}} e^{\eta x}$$

for $0 \leq x \leq \ell$ when $r \neq 0$ and $i \geq 1$. On the other hand,

$$|u_i(x)| \leq \sqrt{2}$$

for $i \geq 1$ and $r = 0$. Since

$$\begin{aligned}
\sum_{i=1}^{\infty} e^{-\lambda_i t} &= e^{-\frac{\nu t}{2}} \sum_{i=1}^{\infty} e^{-\frac{\pi^2 i^2 \sigma^2}{2\ell^2} t} \\
&\leq e^{-\frac{\nu t}{2}} \sum_{i=1}^{\infty} e^{-\frac{\pi^2 i \sigma^2}{2\ell^2} t} \\
&= e^{-\left(\frac{\nu}{2} + \frac{\pi^2\sigma^2}{2\ell^2}\right)t} \left(1 - e^{-\frac{\pi^2\sigma^2}{2\ell^2} t}\right)^{-1} \\
&= e^{-\lambda_1 t} \left(1 - e^{-\frac{\pi^2\sigma^2}{2\ell^2} t}\right)^{-1},
\end{aligned}$$

evidently (A1) is satisfied. We therefore obtain Theorem 2.

**Theorem 2** *When $X = (X(t) : t \geq 0)$ is a two-sided RBM satisfying (3.1),*

$$d(t, x) \sim \frac{1}{2} e^{-\left(\frac{\nu}{2} + \frac{\pi^2\sigma^2}{2\ell^2}\right)t} |u_1(x)| \int_0^{\ell} |u_1(y)| \pi(\mathrm{d}y)$$

*as $t \to \infty$ when $u_1(x) \neq 0$. Also,*

$$d(t, x) \leq e^{-\left(\frac{\nu}{2} + \frac{\pi^2\sigma^2}{2\ell^2}\right)t} \frac{1}{\sqrt{\eta\ell}} e^{\eta x} \left(1 - e^{-\frac{\pi^2\sigma^2}{2\ell^2} t}\right)^{-1}$$

*for $t > 0$ and $x \in [0, \ell]$ when $r \neq 0$, whereas*

$$d(t, x) \leq \frac{1}{\sqrt{2}} e^{-\frac{\pi^2\sigma^2}{2\ell^2} t} \left(1 - e^{-\frac{\pi^2\sigma^2}{2\ell^2} t}\right)^{-1}$$

*for $t > 0$ and $x \in [0, \ell]$ when $r = 0$.*

We note that the exponential rate parameter associated with the rate of convergence to equilibrium is always larger than in the one-sided case with the same drift $r$ and volatility $\sigma$ (by an amount $\pi^2\sigma^2/(2\ell^2)$), and (not surprisingly) the rate is faster when

$\ell$ is small relative to $\sigma$. We further point out that our TV distance upper bound has the desirable feature that it is within a constant factor of the exact asymptotic for $d(t, x)$.

We turn next to the O–U process. We say that $X = (X(t) : t \geq 0)$ is an O–U process with mean reversion rate $\mu$ and volatility $\sigma > 0$ if it satisfies the SDE

$$\mathrm{d}X(t) = -\mu X(t)\mathrm{d}t + \sigma \mathrm{d}B(t) \tag{3.3}$$

for $t \geq 0$. In order to guarantee positive recurrence, we require that $\mu$ be positive. Then, the density $p(\cdot)$ (with respect to Lebesgue measure) of the stationary distribution $\pi(\cdot)$ is given by

$$p(y) = \sqrt{\frac{\mu}{\pi\sigma^2}} e^{-\frac{\mu y^2}{\sigma^2}},$$

$-\infty < y < \infty$.

The O–U process arises as a weak approximation to the $M/M/\infty$ queue; see Iglehart [9] for the theoretical support for this approximation. In the finance literature, the O–U process is often called the *Vasicek SDE* and can be used to model the evolution of interest rates; see, for example, Brigo and Mercurio [3], Chapter 3. In these settings, $X$ is shifted by a *mean parameter* $a > 0$, so that $\mathrm{d}X(t) = \mu(a - X(t))\mathrm{d}t + \sigma \mathrm{d}B(t)$ for all $t \geq 0$. The process $X' = (X'(t) : t \geq 0)$ for which $X'(t) = X(t) - a$ then satisfies (3.3).

For the O–U process (3.3), the transition density with respect to $\pi$ takes the form

$$p(t, x, y) = \sum_{i=0}^{\infty} e^{-\lambda_i t} u_i(x) u_i(y) \tag{3.4}$$

with $\lambda_0 = 0$, $u_0(\cdot) \equiv 1$, and

$$\begin{aligned} \lambda_i &= i\mu, \\ u_i(x) &= (2^i i!)^{-\frac{1}{2}} H_i\left(\frac{\sqrt{\mu}}{\sigma} x\right) \end{aligned}$$

for $i \geq 1$, where $H_i(\cdot)$ is the $i$th-order *Hermite polynomial* defined by

$$H_i(x) = (-1)^i e^{x^2} \frac{\mathrm{d}^i}{\mathrm{d}x^i}\left(e^{-x^2}\right)$$

for $i \geq 0$. Again, the $u_i$ have the property that

$$E_\pi u_i(X(0))^2 = 1$$

for all $i \geq 0$. The representation (3.4) follows from pp. 332–333 of Karlin and Taylor [10] for the canonical case where $\mu = \sigma = 1$, and the observation that

$$p(t,\ x,\ y) = \frac{\sqrt{\mu}}{\sigma}\tilde{p}\left(\mu t,\ \frac{\sqrt{\mu}}{\sigma}x,\ \frac{\sqrt{\mu}}{\sigma}y\right),$$

where $\tilde{p} = (\tilde{p}(t,\ x,\ y) : t > 0,\ x,\ y \in \mathbb{R})$ is the transition density of the canonical O–U process.

We can now appeal to Theorem 1 of Bonan and Clark [2] to establish the existence of $\tilde{c} < \infty$ such that

$$|u_i(x)| \leq \tilde{c} \exp\left(\frac{\mu x^2}{2\sigma^2}\right) i^{-\frac{1}{12}} \tag{3.5}$$

for $i \geq 1$. It follows that (2.1) holds, so that (A1) is satisfied, and hence Theorem 1 can be applied. Note that

$$\sum_{i=1}^{\infty} e^{-\lambda_i t}|u_i(x)| \leq \tilde{c}e^{-\mu t}(1 - e^{-\mu t})^{-1} \exp\left(\frac{\mu x^2}{2\sigma^2}\right)$$

for $t > 0$ and $x \in \mathbb{R}$. Since $u_1(x) = \frac{\sqrt{2\mu}}{\sigma}x$, we arrive at Theorem 3.

**Theorem 3** *When $X = (X(t) : t \geq 0)$ is an O–U process satisfying (3.3),*

$$d(t,\ x) \sim e^{-\mu t}|x|\sqrt{\frac{\mu}{\pi\sigma^2}}$$

*as $t \to \infty$ when $x \neq 0$. Also,*

$$d(t,\ x) \leq e^{-\mu t}(1 - e^{-\mu t})^{-1}\frac{\tilde{c}}{2}\exp\left(\frac{\mu x^2}{2\sigma^2}\right)$$

*for $t > 0$ and $x \in \mathbb{R}$.*

***Remark 1*** In Lachaud [12], an exact asymptotic for $d(t,\ x)$ is derived via a non-rigorous Taylor series argument. The first part of Theorem 3 makes this rigorous.

Again, our upper bound on the TV distance has the desirable property that the upper bound is within a constant factor of our TV distance asymptotic. However, our upper bound involves the constant $\tilde{c}$.

In order to deal with this problem, we can potentially use Theorem 1 of Krasikov [11] to quantify $\tilde{c}$. However, the bounds given there on $H_i(\cdot)$ only apply for $i \geq 6$. We therefore seek an alternative approach to upper bounding $d(t,\ x)$ for the O–U process. Fortunately, since the O–U process is Gaussian, this is feasible.

For $z \in \mathbb{R}$, let

$$\phi(z) = \frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}$$

be the density of a standard Gaussian random variable $Z$ (with mean zero and unit variance).

**Proposition 2** *Suppose that* $f \in L^2(\pi)$. *If* $0 < \gamma^2 < 2$ *and* $\tau \in \mathbb{R}$, *then*

$$|Ef(\gamma Z + \tau) - Ef(Z)| \leq \sqrt{Ef^2(Z)}\sqrt{\frac{\exp\left(\frac{\tau^2}{2-\gamma^2}\right)}{\gamma\sqrt{2-\gamma^2}} - 1}.$$

***Proof*** We start by observing that the Cauchy–Schwarz inequality implies that

$$\begin{aligned}
|Ef(\gamma Z + \tau) - Ef(Z)| &= \left|\int_{-\infty}^{\infty} f(y)\left(\frac{\phi\left(\frac{y-\tau}{\gamma}\right)}{\phi(y)\gamma} - 1\right)\phi(y)\mathrm{d}y\right| \\
&= \left|Ef(Z)\left(\frac{\phi\left(\frac{Z-\tau}{\gamma}\right)}{\phi(Z)\gamma} - 1\right)\right| \\
&\leq \sqrt{Ef^2(Z)}\sqrt{E\left(\frac{\phi\left(\frac{Z-\tau}{\gamma}\right)}{\phi(Z)\gamma} - 1\right)^2} \\
&= \sqrt{Ef^2(Z)}\sqrt{E\left(\frac{\phi\left(\frac{Z-\tau}{\gamma}\right)^2}{\phi(Z)^2\gamma^2} - 1\right)},
\end{aligned}$$

where the third equality follows from the fact that

$$E\frac{\phi\left(\frac{Z-\tau}{\gamma}\right)}{\phi(Z)\gamma} = 1.$$

But

$$\begin{aligned}
&\frac{1}{\gamma^2}E\frac{\phi\left(\frac{Z-\tau}{\gamma}\right)^2}{\phi(Z)^2} \\
&= \frac{1}{\gamma^2}\int_{-\infty}^{\infty} e^{-\frac{(y-\tau)^2}{\gamma^2}+y^2}\frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}\mathrm{d}y \\
&= \frac{1}{\gamma^2}\int_{-\infty}^{\infty} e^{-\left(\frac{1}{\gamma^2}-\frac{1}{2}\right)y^2+\frac{2\tau}{\gamma^2}y-\frac{\tau^2}{\gamma^2}}\frac{\mathrm{d}y}{\sqrt{2\pi}} \\
&= \frac{1}{\gamma}\int_{-\infty}^{\infty} e^{-\frac{1}{2\gamma^2}\left(\sqrt{2-\gamma^2}y-\frac{2\tau}{\sqrt{2-\gamma^2}}\right)^2+\frac{2\tau^2}{(2-\gamma^2)}\frac{1}{\gamma^2}-\frac{\tau^2}{\gamma^2}}\sqrt{\frac{2-\gamma^2}{2\pi\gamma^2}}\mathrm{d}y\frac{1}{\sqrt{2-\gamma^2}}
\end{aligned}$$

$$= \frac{1}{\gamma} e^{\frac{\tau^2}{\gamma^2}\left(\frac{2}{2-\gamma^2}-1\right)} \frac{1}{\sqrt{2-\gamma^2}} \int_{-\infty}^{\infty} \phi\left(\frac{2-\gamma^2}{2\gamma^2}\left(y - \frac{2\tau}{2-\gamma^2}\right)^2\right) \sqrt{\frac{2-\gamma^2}{\gamma^2}} \mathrm{d}y$$

$$= \frac{1}{\gamma\sqrt{2-\gamma^2}} e^{\frac{\tau^2}{2-\gamma^2}},$$

from which the result follows. □

Of course, it is well known that, conditional on $X(0) = x$, the O–U process has Gaussian marginals, so that (3.4) sums to a Gaussian density. In particular, $X(t)$ is Gaussian with mean $xe^{-\mu t}$ and variance $\sigma^2/(2\mu)(1 - e^{-2\mu t})$. Note that

$$f(X(t)) \stackrel{\mathscr{D}}{=} \tilde{f}\left(\sqrt{1 - e^{-2\mu t}} Z + \frac{\sqrt{2\mu}}{\sigma} x e^{-\mu t}\right)$$

and $f(X(\infty)) \stackrel{\mathscr{D}}{=} \tilde{f}(Z)$, where $\tilde{f}(x) = f(\sigma/\sqrt{2\mu} x)$. So, for $f \in L^2(\pi)$, Proposition 2 applies to $\tilde{f}$ with $\gamma = \sqrt{1 - e^{-2\mu t}}$ and $\tau = \sqrt{2\mu}/\sigma x e^{-\mu t}$, thereby yielding the identity

$$\begin{aligned} |E_x f(X(t)) - Ef(X(\infty))| &= |E_x f_c(X(t)) - Ef_c(X(\infty))| \\ &\le \sqrt{\operatorname{var} \tilde{f}(Z)} \sqrt{\frac{\exp\left(\frac{2x^2 e^{-2\mu t}\mu}{\sigma^2(1+e^{-2\mu t})}\right)}{(1 - e^{-4\mu t})^{\frac{1}{2}}} - 1} \\ &\le \sqrt{\operatorname{var} \tilde{f}(Z)} \sqrt{\frac{\exp\left(2x^2 e^{-2\mu t}\mu/\sigma^2\right)}{(1 - e^{-4\mu t})^{\frac{1}{2}}} - 1}. \end{aligned}$$

Note that if $|f| \le 1$, then $|\tilde{f}| \le 1$ and $\operatorname{var} \tilde{f}(Z) \le 4$. Hence, we are led to Theorem 4, which provides an explicit upper bound on the total variation distance to equilibrium.

**Theorem 4** *Suppose that $X$ is an O–U process obeying the SDE* (3.3). *Then, for each $t > 0$ and $x \in \mathbb{R}$,*

$$d(t, x) \le \sqrt{\frac{\exp\left(2x^2 e^{-2\mu t}\mu/\sigma^2\right)}{(1 - e^{-4\mu t})^{\frac{1}{2}}} - 1}. \tag{3.6}$$

Note that the bound (3.6) is within a constant factor of Theorem 3's asymptotic, establishing that Theorem 4 provides a reasonably tight bound and can be used effectively in practice.

## 4 Conclusions

In this paper, we derived bounds and asymptotics that can be used to assess when a finite-horizon formulation can be replaced by an equilibrium formulation for both finite-buffer and infinite-server models, and to provide insight into related simulation start-up issues. Because the Hilbert spaces for both processes have countable bases of eigenfunctions in which the "second eigenvalue" is isolated from the rest of the spectrum (i.e., the second eigenvalue is not a limit point of other eigenvalues), simple spectral representations that lead to exact exponential asymptotics for the rate of convergence to equilibrium can be obtained in a relatively straightforward fashion (in sharp contrast with one-sided RBM, for which the analysis is significantly more challenging).

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## References

1. Berger, A.W., Whitt, W.: The Brownian approximation for rate-control throttles and the G/G/1/C queue. Discret. Event Dyn. Syst. **2**, 7–60 (1992)
2. Bonan, S.S., Clark, D.S.: Estimates of the Hermite and Freud polynomials. J. Approx. Theory **63**, 210–224 (1990)
3. Brigo, D., Mercurio, F.: Interest Rate Models–Theory and Practice (With Smile, Inflation, and Credit), 2nd edn. Springer, Berlin (2006)
4. D'Auria, B., Kella, O.: Markov modulation of a two-sided reflected Brownian motion with application to fluid queues. Stoch. Process. Appl. **122**, 1566–1581 (2012)
5. Diaconis, P.: The Markov chain Monte Carlo revolution. Bull. Am. Math. Soc. **46**(2), 179–1205 (2009)
6. Folland, G .B.: Real Analysis: Modern Techniques and Their Applications, 2nd edn. Wiley, New York (1999)
7. Glynn, P.W., Wang, R.J.: On the rate of convergence to equilibrium for reflected Brownian motion. Queueing Syst. **89**, 165–197 (2018)
8. Harrison, J.M.: Brownian Models of Performance and Control. Cambridge University Press, Cambridge (2013)
9. Iglehart, D.L.: Limiting diffusion approximations for the many-server queue and the repairman problem. J. Appl. Prob. **2**(2), 429–441 (1965)
10. Karlin, S., Taylor, H.M.: A Second Course in Stochastic Processes. Academic Press, New York (1981)
11. Krasikov, I.: New bounds on the Hermite polynomials. arXiv:math/0401310 pp. 1–6 (2004)
12. Lachaud, B.: Cut-off and hitting times of a sample of Ornstein–Uhlenbeck processes and its average. J. Appl. Prob. **42**(4), 1069–1080 (2005)
13. Linetsky, V.: On the transition densities for reflected diffusions. Adv. Appl. Prob. **37**, 435–460 (2005)
14. Meyn, S., Tweedie, R.L.: Markov Chains and Stochastic Stability, 2nd edn. Cambridge University Press, Cambridge (2009)
15. Roberts, G.O., Rosenthal, J.S.: General state space Markov chains and MCMC algorithms. Probab. Surv. **1**, 20–71 (2004)
16. Wang, R.J., Glynn, P.W.: Measuring the initial transient: reflected Brownian motion. In: A. Tolk, S.Y. Diallo, I.O. Ryzhov, L. Yilmaz, S. Buckley, and J.A. Miller (eds) Proceedings of the 2014 Winter Simulation Conference, pp. 652–661 (2014)
17. Wang, R.J., Glynn, P.W.: On the marginal standard error rule and the testing of initial transient deletion methods. ACM Trans. Model. Comput. Simul. **27**(1), 1–30 (2016)
18. Whitt, W.: Stochastic-Process Limits. Springer, New York (2002)
19. Williams, R.J.: Asymptotic variance parameters for the boundary local times of reflected Brownian motion on a compact interval. J. Appl. Prob. **29**, 996–1002 (1992)
20. Zhang, X., Glynn, P.W.: On the dynamics of a finite buffer queue conditioned on the amount of loss. Queueing Syst. **67**(2), 91–110 (2011)