Documentation of model used to produce simulation results in *Efficiency Costs of Meeting Industry-Distributional Constraints under Environmental Permits and Taxes*

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This document presents the steps in the algorithms that compute the equilibrium and welfare levels of an economy consistent with the one presented in *Efficiency Costs of Meeting Industry-Distributional Constraints under Environmental Permits and Taxes*. As documentation for an algorithm, it presents equations in a different order than they might be in documentation designed to present a model.

The documentation is in 2 parts: Part I summarizes the equations used to compute the equilibrium while Part II presents the steps necessary to compute the welfare change due to a policy. Within each part, a section documents a step. Within a section, the **Background** presents the derivation of the equations used by the algorithm. **Computed Equations** presents these equations (look for the **bold** text) while **Implementation in Computer Code** provides a guide to reading the equations in the computer code.

Part I

Equilibrium

0 Set equilibrating prices based on algorithm variables

Background Convergence in the economy is achieved by equilibrating supply and demand for capital used in the production of C, X and Y and equilibrating transfers to households with government revenues. By Walras' Law, supply and demand for labor is equal when the other excess demands are zero. Labor can move instantaneously among uses, so there is one price, regardless of sector, for labor, which is also the numeraire. Benchmark prices for other goods are also set to one.

Some policies require further equilibrating of “supply” and “demand” variables. The relationship between the algorithm variables, which are changed by the equilibrium solver, and the “prices” used to bring about equilibrium in the model is set out below.

Computed Equations The algorithm variables, λ, are initialized to 1 before the beginning of the Calculate Excess Demand loop. The variables are optionally re-initialized, in policy cases, to the values resulting from the previous run of the model (to speed up the solution of the model).

Capital supply and demand

Adjusting the price of capital in each sector equalizes the supply and demand for capital.

$$P_{k_y} = \lambda_{k_y} \tag{1}$$

$$P_{k_x} = \lambda_{k_x} \tag{2}$$

$$P_{k_c} = \lambda_{k_c} \tag{3}$$

Government income neutrality

Government revenue is set equal to government transfers through tax cuts, either through lump sum tax rebates or cuts to tax rates. If neutrality is achieved through lump sum taxes, the equilibrating variables in the respective scenarios are set as the following:

$$\begin{aligned} \tau^m &= 1 \\ \eta &= \lambda_{GIN} \end{aligned} \tag{4$'$}$$

where τ^m adjusts the base tax rate.

Otherwise, if government income neutrality is achieved through reduction in marginal taxes,

$$\begin{aligned} \tau^m &= \lambda_{GIN} \\ \eta &= 1 \end{aligned} \tag{4}$$

Abatement target

Achieving the abatement target requires that actual emissions equal the emissions quota. If the abatement target is to be achieved by an emissions permit price, τ_e:

$$\tau_e = \lambda_{AT} - 1 \tag{5}$$

while if the abatement target is to be achieved through a fuel permit price, τ_x:

$$\tau_x = \lambda_{AT} - 1 \tag{5$'$}$$

Otherwise, λ_{AT} does not impact calculations in the simulation.

Profit neutrality for Y

Profit neutrality–making profit in a policy case equal to the benchmark profit–can be achieved in three ways for the Y sector:

1. **By grandfathering permit revenue:**

$$\alpha_y^{gf} = \lambda_{PN_y} \tag{6}$$

2. **By subsidy to abatement:**

$$s_{ga} = \lambda_{PN_y} \tag{6'}$$

3. **By sector-specific tax cut:**

$$\tau_{k_y}^{cut} = \lambda_{PN_y} \tag{6''}$$

Otherwise, λ_{PN_y} does not impact calculations in the simulation.

Profit neutrality for X

Profit neutrality in the X sector can be achieved in two ways:

1. **By grandfathering permit revenue:**

$$\alpha_x^{gf} = \lambda_{PN_y} \tag{7}$$

2. **By sector-specific tax cut:**

$$\tau_{k_y}^{cut} = \lambda_{PN_y} \tag{7'}$$

Otherwise, λ_{PN_x} does not impact calculations in the simulation.

Environmental tax revenue neutrality

Environmental tax revenue neutrality proxies for a

- **technical mandate for emission reduction, achieved by setting the value of a subsidy to X equal to the environmental tax revenue**
- **performance standard for emission reduction, achieved by setting the value of a subsidy to Y equal to the environmental tax revenue**

If environmental tax revenue neutrality is achieved through subsidy to X:

$$s_x = \lambda_{TRN} - 1 \tag{8}$$

If environmental tax revenue neutrality is achieved through subsidy to Y:

$$s_y = \lambda_{TRN} - 1 \tag{8'}$$

Otherwise, λ_{TRN} does not impact calculations in the simulation.

Implementation in Computer Code Inside the `Calculate Excess Demand` function, the λ vector is referred to as `x[]`, while in the main body of the program, it is `guess_p[]`. Other differences include

- τ^m = `tax_mult`
- τ^{bm} = `K_taxbase[]`

1 Relative supply of capital for use in production of X, Y, and C

Background In contrast to labor, capital cannot flow freely among sectors. Instead, capital is constrained by the following transformation function, which determines the relative allocation of capital (relative because total capital cannot be determined until household income is determined):

$$\overline{k} = \gamma_k \left[\alpha_k k_x^{\frac{\sigma_k - 1}{\sigma_k}} + \beta_k k_y^{\frac{\sigma_k - 1}{\sigma_k}} + (1 - \alpha_k - \beta_k) k_c^{\frac{\sigma_k - 1}{\sigma_k}} \right]^{\frac{\sigma_k}{\sigma_k - 1}} \tag{9}$$

Note: because k cannot be deAs a result, differentials in returns to capital arise among sectors, and owners of capital maximize their profits by allocating capital among the sectors according to the price it receives in each sector and subject to the constraint:

$$\begin{aligned} \max_{k_x, k_y, k_c} \; & \pi(k_x, k_y, k_c) = P_{k_x} k_x + P_{k_y} k_y + P_{k_c} k_c \\ & + \lambda_k \left\{ \overline{k} - \gamma_k \left[\alpha_k k_x^{\frac{\sigma_k - 1}{\sigma_k}} - \beta_k k_y^{\frac{\sigma_k - 1}{\sigma_k}} - (1 - \alpha_k - \beta_k) k_c^{\frac{\sigma_k - 1}{\sigma_k}} \right]^{\frac{\sigma_k}{\sigma_k - 1}} \right\} \end{aligned} \tag{10}$$

where P_{k_i} is price of capital used in i.

The first-order conditions for this problem are

$$\frac{\partial \pi}{\partial k_x} \equiv P_{k_x} - \lambda_k \gamma_k \left[\alpha_k k_x^{\frac{\sigma_k - 1}{\sigma_k}} + \beta_k k_y^{\frac{\sigma_k - 1}{\sigma_k}} + (1 - \alpha_k - \beta_k) k_c^{\frac{\sigma_k - 1}{\sigma_k}} \right]^{\frac{1}{\sigma_k - 1}} \frac{\sigma_k}{\sigma_k - 1} \frac{\sigma_k - 1}{\sigma_k} \alpha_k k_x^{-\frac{1}{\sigma_k}} \equiv 0 \tag{11}$$

$$\frac{\partial \pi}{\partial k_y} \equiv P_{k_y} - \lambda_k \gamma_k \left[\alpha_k k_x^{\frac{\sigma_k - 1}{\sigma_k}} + \beta_k k_y^{\frac{\sigma_k - 1}{\sigma_k}} + (1 - \alpha_k - \beta_k) k_c^{\frac{\sigma_k - 1}{\sigma_k}} \right]^{\frac{1}{\sigma_k - 1}} \frac{\sigma_k}{\sigma_k - 1} \frac{\sigma_k - 1}{\sigma_k} \beta_k k_y^{-\frac{1}{\sigma_k}} \equiv 0 \tag{12}$$

$$\frac{\partial \pi}{\partial k_c} \equiv P_{k_c} - \lambda_k \gamma_k \left[\alpha_k k_x^{\frac{\sigma_k - 1}{\sigma_k}} + \beta_k k_y^{\frac{\sigma_k - 1}{\sigma_k}} + (1 - \alpha_k - \beta_k) k_c^{\frac{\sigma_k - 1}{\sigma_k}} \right]^{\frac{1}{\sigma_k - 1}} \frac{\sigma_k}{\sigma_k - 1} \frac{\sigma_k - 1}{\sigma_k} (1 - \alpha_k - \beta_k) k_c^{-\frac{1}{\sigma_k}} \equiv 0 \tag{13}$$

$$\frac{\partial \pi}{\lambda_k} \equiv \overline{k} - \gamma_k \left[\alpha_k k_x^{\frac{\sigma_k - 1}{\sigma_k}} - \beta_k k_y^{\frac{\sigma_k - 1}{\sigma_k}} - (1 - \alpha_k - \beta_k) k_c^{\frac{\sigma_k - 1}{\sigma_k}} \right]^{\frac{\sigma_k}{\sigma_k - 1}} \equiv 0 \tag{14}$$

Stacking (11) and (12) eliminates $\lambda_k \gamma_k \left[\alpha_k k_x^{\frac{\sigma_k - 1}{\sigma_k}} + \beta_k k_y^{\frac{\sigma_k - 1}{\sigma_k}} + (1 - \alpha_k - \beta_k) k_c^{\frac{\sigma_k - 1}{\sigma_k}} \right]^{\frac{1}{\sigma_k - 1}}$ from both equations to produce (with subsequent rearrangements)

$$\begin{aligned} \frac{P_{k_x}}{P_{k_y}} &= \frac{\alpha_k}{\beta_k} \left(\frac{k_x}{k_y} \right)^{-\frac{1}{\sigma_k}} \\ \left(\frac{P_{k_x}}{P_{k_y}} \frac{\beta_k}{\alpha_k} \right)^{-\sigma_k} &= \frac{k_x}{k_y} \\ k_x &= \left(\frac{P_{k_x}}{P_{k_y}} \frac{\beta_k}{\alpha_k} \right)^{-\sigma_k} k_y \end{aligned} \tag{15}$$

Similarly, stacking (13) and (12) yields

$$\begin{aligned} \frac{P_{k_c}}{P_{k_y}} &= \frac{1 - \alpha_k - \beta_k}{\beta_k} \left(\frac{k_c}{k_y} \right)^{-\frac{1}{\sigma_k}} \\ \left(\frac{P_{k_c}}{P_{k_y}} \frac{\beta_k}{1 - \alpha_k - \beta_k} \right)^{-\sigma_k} &= \frac{k_c}{k_y} \\ k_c &= \left(\frac{P_{k_c}}{P_{k_y}} \frac{\beta_k}{1 - \alpha_k - \beta_k} \right)^{-\sigma_k} k_y \end{aligned} \tag{16}$$

Substituting (15) and (16) into (14) and rearranging yields

$$\overline{k} - \gamma_k \left[\alpha_k \left(\frac{P_{k_x}}{P_{k_y}} \frac{\beta_k}{\alpha_k} \right)^{1-\sigma_k} k_y^{\frac{\sigma_k - 1}{\sigma_k}} - \beta_k k_y^{\frac{\sigma_k - 1}{\sigma_k}} - (1 - \alpha_k - \beta_k) \left(\frac{P_{k_c}}{P_{k_y}} \frac{\beta_k}{1 - \alpha_k - \beta_k} \right)^{1-\sigma_k} k_y^{\frac{\sigma_k - 1}{\sigma_k}} \right]^{\frac{\sigma_k}{\sigma_k - 1}} \equiv 0$$

$$\frac{\overline{k}}{\gamma_k} = k_y \left[\alpha_k \left(\frac{P_{k_x}}{P_{k_y}} \frac{\beta_k}{\alpha_k} \right)^{1-\sigma_k} - \beta_k - (1 - \alpha_k - \beta_k) \left(\frac{P_{k_c}}{P_{k_y}} \frac{\beta_k}{1 - \alpha_k - \beta_k} \right)^{1-\sigma_k} \right]^{\frac{\sigma_k}{\sigma_k - 1}} \tag{17}$$

Computed Equations Since total capital supply cannot be determined In the code, P_k is before tax, so it needs to be adjusted for taxes to determine capital supply.

Capital taxes are adjusted for general tax rate cuts and sector-specific tax cuts:

$$\tau_{k_y} = \tau_{k_y}^{bm} - (\tau^m - 1) - \tau_y^{cut} \tag{18}$$

$$\tau_{k_x} = \tau_{k_x}^{bm} - (\tau^m - 1) - \tau_x^{cut} \tag{19}$$

$$\tau_{k_c} = \tau_{k_c}^{bm} - (\tau^m - 1) \tag{20}$$

From (17)**,** (15)**, and** (16)

$$k_y^s = \frac{\overline{k}}{\gamma_k \left[\alpha_k \left(\frac{\tau_{k_x} P_{k_x}}{\tau_{k_y} P_{k_y}} \frac{\beta_k}{\alpha_k} \right)^{1-\sigma_k} - \beta_k - (1 - \alpha_k - \beta_k) \left(\frac{\tau_{k_c} P_{k_c}}{\tau_{k_y} P_{k_y}} \frac{\beta_k}{1 - \alpha_k - \beta_k} \right)^{1-\sigma_k} \right]^{\frac{\sigma_k}{\sigma_k - 1}}} \tag{21}$$

$$k_x^s = \left(\frac{\tau_{k_x} P_{k_x}}{\tau_{k_y} P_{k_y}} \frac{\beta_k}{\alpha_k} \right)^{-\sigma_k} k_y \tag{22}$$

$$k_c^s = \left(\frac{\tau_{k_c} P_{k_c}}{\tau_{k_y} P_{k_y}} \frac{\beta_k}{1 - \alpha_k - \beta_k} \right)^{-\sigma_k} k_y \tag{23}$$

The program incidentally calculates, for use in determining convergence, the amount and value of capital lost to "friction"

$$k_{miss} = \overline{k} - k_c^s - k_v^s - k_x^s \tag{24}$$

$$k_{miss}^{value} = P_{k_c} k_{miss} \tag{25}$$

$$k_{miss\_ratio} = \frac{abs(k_c^s - k_c^{bm})}{abs(k_y^s - k_y^{bm}) + abs(k_x^s - k_x^{bm})} \tag{26}$$

where ^{bm} denotes the benchmark value.

Note that the total supply of capital is a function of household income, which cannot yet be calculated, so the capital supplies calculated here indicate not their totals but their relative amounts.

Implementation in Computer Code The computer code differs from the notation used here. To make the math easier, the algorithm combines the capital and labor used in Y into a virtual input V, which in turn is combined with X to produce Y. Thus, the computer code refers to capital used in V to mean the capital used in Y

Further, to facilitate reuse of code, inputs are generically represented by the variable Z and price of inputs by W. As a result, the supply of capital for use in Y is denoted in the code by the variable `Z_s[V][K]` while the counterpart for X is `Z_s[X][K]`. Similarly, the prices of these inputs are denoted `W_s[V][K]` and `W_s[X][K]`

Finally, the code substitutes ρ_k for $\frac{\sigma_k-1}{\sigma_k}$ and, accordingly, $\frac{1}{\rho_k-1}$ for $-\sigma_k$ and $\frac{\rho_k}{\rho_k-1}$ for $1-\sigma_k$. Since lowering σ_k towards negative infinity means that there is less friction in capital movement between sectors (at negative infinity, the iso-plane relating the capital in each sector has no curvature, just as the iso-quant of a production function with infinite elasticity is straight), a decrease in ρ_k, accordingly, also means less friction.

2 Per-unit demand for inputs: K and L in C and X

Background Demand for capital and labor for use in C, X,and Y will obviously be a function of the demand for these outputs, which in the case of X is, in turn, a function of the demand for Y. However, because firms use constant-returns-to-scale technology in producing each output, they can determine the optimal per-unit-of-output use of inputs independently of each output's demand.

Firms' Optimization Problem: C

Accordingly in producing C, firms face the following optimization problem:

$$\min_{\frac{K}{C},\frac{L}{C}} cost_c\left(\frac{K}{C},\frac{L}{C}\right) = P_{k_c}\frac{K}{C} + P_l\frac{L}{C} \tag{27}$$

$$s.t.\ f^c\left(\frac{K}{C},\frac{L}{C}\right) \geq 1 \tag{28}$$

$$\frac{K}{C} \geq 0 \tag{29}$$

$$\frac{L}{C} \geq 0 \tag{30}$$

where $P_l(=1)$ is price of labor.

Assuming further that the production technology exhibits constant elasticity-of-substitution (CES), the minimization problem becomes

$$\min_{\frac{K}{C},\frac{L}{C}} cost_c\left(\frac{K}{C},\frac{L}{C}\right) = P_{k_c}\frac{K}{C} + P_l\frac{L}{C} \tag{27$'$}$$

$$s.t.\ \frac{\gamma_c\left[\alpha_{c_k}K^{\frac{\sigma_c-1}{\sigma_c}} + \alpha_{c_l}L^{\frac{\sigma_c-1}{\sigma_c}}\right]^{\frac{\sigma_c}{\sigma_c-1}}}{C} \geq 1 \tag{28$'$}$$

$$\frac{K}{C} \geq 0 \tag{29$'$}$$

$$\frac{L}{C} \geq 0 \tag{30$'$}$$

where $\alpha_{c_k} + \alpha_{c_l} = 1$.

Combining (27′) and (28′), expressing every input as a fraction of C, and assuming an interior solution (so that (29′) and (30′) can be ignored) yields

$$\min_{\frac{K}{C},\frac{L}{C}} cost_c\left(\frac{K}{C},\frac{L}{C}\right) = P_{k_c}\frac{K}{C} + P_l\frac{L}{C}$$
$$+\lambda_c\left\{1-\gamma_c\left[\alpha_{c_k}\left(\frac{K}{C}\right)^{\frac{\sigma_c-1}{\sigma_c}} + \alpha_{c_l}\left(\frac{L}{C}\right)^{\frac{\sigma_c-1}{\sigma_c}}\right]^{\frac{\sigma_c}{\sigma_c-1}}\right\} \tag{31}$$

Substituting k_c and l_c for $\frac{K}{C}$ and $\frac{L}{C}$ reduces the notation to

$$\min_{k_c,l_c} cost_c\left(k_c,l_c\right) = P_{k_c}k_c + P_l l_c$$
$$+\lambda_c\left\{1-\gamma_c\left[\alpha_{c_k}k_c{}^{\frac{\sigma_c-1}{\sigma_c}} + \alpha_{c_l}l_c{}^{\frac{\sigma_c-1}{\sigma_c}}\right]^{\frac{\sigma_c}{\sigma_c-1}}\right\} \tag{31$'$}$$

First-order Conditions: C

Taking the derivative of (31′) with respect to λ_c yields the constraint:

$$1-\gamma_c\left[\alpha_{c_k}k_c{}^{\frac{\sigma_c-1}{\sigma_c}} + \alpha_{c_l}l_c{}^{\frac{\sigma_c-1}{\sigma_c}}\right]^{\frac{\sigma_c}{\sigma_c-1}} \equiv 0 \tag{32}$$

Taking the derivative of (31′) with respect to k_c yields the usual first-order condition for cost minimization of a CES production function:

$$P_{k_c} - \lambda_c\gamma_c\frac{\sigma_c}{\sigma_c-1}\left[\alpha_{c_k}k_c{}^{\frac{\sigma_c-1}{\sigma_c}} + \alpha_{c_l}l_c{}^{\frac{\sigma_c-1}{\sigma_c}}\right]^{\frac{1}{\sigma_c-1}}\frac{\sigma_c-1}{\sigma_c}\alpha_{c_k}k_c{}^{\frac{-1}{\sigma_c}} \equiv 0 \tag{33}$$

Rearranging yields

$$P_{k_c} \equiv \lambda_c\gamma_c\left[\alpha_{c_k}k_c{}^{\frac{\sigma_c-1}{\sigma_c}} + \alpha_{c_l}l_c{}^{\frac{\sigma_c-1}{\sigma_c}}\right]^{\frac{1}{\sigma_c-1}}\alpha_{c_k}k_c{}^{\frac{-1}{\sigma_c}} \tag{33$'$}$$

Taking the derivative of (31′) with respect to l_c yields the analogous equations:

$$P_l - \lambda_c\gamma_c\frac{\sigma_c}{\sigma_c-1}\left[\alpha_{c_k}k_c^{\ \frac{\sigma_c-1}{\sigma_c}} + \alpha_{c_l}l_c^{\ \frac{\sigma_c-1}{\sigma_c}}\right]^{\frac{1}{\sigma_c-1}}\frac{\sigma_c-1}{\sigma_c}\alpha_{c_l}l_c^{\ \frac{-1}{\sigma_c}} \equiv 0 \tag{34}$$

$$P_l \equiv \lambda_c\gamma_c\left[\alpha_{c_k}k_c^{\ \frac{\sigma_c-1}{\sigma_c}} + \alpha_{c_l}l_c^{\ \frac{\sigma_c-1}{\sigma_c}}\right]^{\frac{1}{\sigma_c-1}}\alpha_{c_l}l_c^{\ \frac{-1}{\sigma_c}} \tag{34′}$$

Stacking (33′) and (34′) eliminates $\lambda_c\gamma_c\left[\alpha_{c_k}l_k^{\ \frac{\sigma_c-1}{\sigma_c}} + \alpha_{c_l}l_c^{\ \frac{\sigma_c-1}{\sigma_c}}\right]^{\frac{1}{\sigma_c-1}}$ from both equations to produce (with subsequent rearrangements)

$$\frac{P_{k_c}}{P_l} = \frac{\alpha_{c_k}}{\alpha_{c_l}}\left(\frac{k_c}{l_c}\right)^{\frac{-1}{\sigma_c}}$$

$$\frac{P_l}{P_{k_c}} = \frac{\alpha_{c_l}}{\alpha_{c_k}}\left(\frac{k_c}{l_c}\right)^{\frac{1}{\sigma_c}}$$

$$k_c^{\frac{1}{\sigma_c}} = \frac{\alpha_{c_k}}{\alpha_{c_l}}\frac{P_l}{P_{k_c}}l_c^{\frac{1}{\sigma_c}}$$

$$k_c = \left(\frac{\alpha_{c_k}}{\alpha_{c_l}}\frac{P_l}{P_{k_c}}\right)^{\sigma_c} l_c \tag{35}$$

Substituting (35) into (32) yields, after rearrangement, familiar CES demand functions, expressed as per-unit-of-output:

$$k_c = \frac{1}{\gamma_c}\left[\alpha_{c_k} + \alpha_{c_l}\left(\frac{\alpha_{c_k}}{\alpha_{c_l}}\frac{P_l}{P_{k_c}}\right)^{\sigma_c-1}\right]^{\frac{\sigma_c}{\sigma_c-1}} \tag{36}$$

$$l_c = \frac{1}{\gamma_c}\left[\alpha_{c_k}\left(\frac{\alpha_{c_k}}{\alpha_{c_l}}\frac{P_l}{P_{k_c}}\right)^{\sigma_c-1} + \alpha_{c_l}\right]^{\frac{\sigma_c}{\sigma_c-1}} \tag{37}$$

Firms' Optimization Problem: X

Firms produce X with the same technology as they produce C, so firms face an optimization problem equivalent to the one in C:

$$\min_{\frac{K}{X},\frac{L}{X}} cost_x\left(\frac{K}{X},\frac{L}{X}\right) = P_{k_x}\frac{K}{X} + P_l\frac{L}{X} \tag{38}$$

$$s.t.\ f^x\left(\frac{K}{X},\frac{L}{X}\right) \geq 1 \tag{39}$$

$$\frac{K}{X} \geq 0 \tag{40}$$

$$\frac{L}{X} \geq 0 \tag{41}$$

As with C, assuming a CES production function, substituting k_x and l_x for $\frac{K}{X}$ and $\frac{L}{X}$, and assuming an interior solution reduces the problem to

$$\begin{gathered}\min_{k_x, l_x} cost_x\left(k_x, l_x\right) = P_x k_x + P_x l_x \\ +\lambda_x \left\{1 - \gamma_x \left[\alpha_{x_k} k_x^{\ \frac{\sigma_x - 1}{\sigma_x}} + \alpha_{x_l} l_x^{\ \frac{\sigma_x - 1}{\sigma_x}}\right]^{\frac{\sigma_x}{\sigma_x - 1}}\right\}\end{gathered} \tag{42}$$

where $\alpha_{x_k} + \alpha_{x_l} = 1$.

Solving this optimization problem yields demand functions equivalent to those of C:

$$k_x = \frac{1}{\gamma_x}\left[\alpha_{x_k} + \alpha_{x_l}\left(\frac{\alpha_{x_k}}{\alpha_{x_l}}\frac{P_l}{P_{k_x}}\right)^{\sigma_x - 1}\right]^{\frac{\sigma_x}{\sigma_x - 1}} \tag{43}$$

$$l_x = \frac{1}{\gamma_x}\left[\alpha_{x_k}\left(\frac{\alpha_{x_k}}{\alpha_{x_l}}\frac{P_l}{P_{k_x}}\right)^{\sigma_x - 1} + \alpha_{x_l}\right]^{\frac{\sigma_x}{\sigma_x - 1}} \tag{44}$$

Computed Equations In the computer code, the demand functions are stated in an alternative but equivalent form:

C

$$k_c^d = \frac{1}{\gamma_c}\left(\frac{\alpha_{c_k}\gamma_c cost_c}{P_{k_c}}\right)^{\sigma_c} \tag{45}$$

$$l_c^d = \frac{1}{\gamma_c}\left(\frac{\alpha_{c_l}\gamma_c cost_c}{P_l}\right)^{\sigma_c} \tag{46}$$

where

$$cost_c = \frac{1}{\gamma_c}\left(\alpha_{c_k}^{\sigma_c} P_{k_c}^{1-\sigma_c} + \alpha_{c_l}^{\sigma_c} P_{l_c}^{1-\sigma_c}\right)^{\frac{1}{1-\sigma_c}} \tag{47}$$

X

$$k_x^d = \frac{1}{\gamma_x}\left(\frac{\alpha_{x_k}\gamma_x cost_x}{P_{k_x}}\right)^{\sigma_x} \tag{48}$$

$$l_x^d = \frac{1}{\gamma_x}\left(\frac{\alpha_{x_l}\gamma_x cost_x}{P_l}\right)^{\sigma_x} \tag{49}$$

where

$$cost_x = \frac{1}{\gamma_x}\left(\alpha_{x_k}^{\sigma_x} P_{k_x}^{1-\sigma_x} + \alpha_{x_l}^{\sigma_x} P_{l_x}^{1-\sigma_x}\right)^{\frac{1}{1-\sigma_x}} \tag{50}$$

By imposing the zero-profit condition, the prices of C and X can also be calculated:

$$P_c = P_{k_c} k_c + P_l l_c \tag{51}$$
$$P_x = P_{k_x} k_x + P_l l_x \tag{52}$$
$$P_x = P_x + \tau_x - s_x \tag{53}$$

Note that these prices could also have been expressed as the unit cost, calculated above. Calculating them this way provides a check.

Implementation in Computer Code k_c is stated as `Z_u[C][K]` (`_u` indicates unit) and l_c as `Z_u[C][L]`. The other unit demands are represented analogously.

α_{c_k}, α_{c_l}, and γ_c are stated as `alpha_tild[C][K]`, `alpha_tild[C][L]`, and `gamma_tild[C]`. The `_tild` stands for ˜ , for consistency with the calibration program.

3 Per-unit demand for inputs: K, L & X in Y

Background Y is produced by combining K, L, and X in a constant returns-to-scale technology. To make this problem more tractable, K and X are nested together (in a product called V) and then combined with L. As with C and X, because firms produce V and Y with constant returns-to-scale technology and because emissions exhibits constant returns-to-scale with respect to abatement effort and X (so emissions per unit of intermediate input $\frac{E}{X}$ are a function of the ratio of abatement labor to intermediate input, $\frac{E}{X}$, only), their optimization problem can be solved in terms of unit costs without loss of generality.

Firms' Optimization Problem: V

$$\min_{\frac{K}{V}, \frac{X}{V}, \frac{G_a}{V}} cost_v \left(\frac{K}{V}, \frac{X}{V}, \frac{G_a}{V} \right) = P_k \frac{K}{V} + P_x \frac{X}{V} + P_V \frac{G_a}{V} + \tau_e \frac{E}{V} \tag{54}$$
$$s.t. \; f^V \left(\frac{K}{V}, \frac{X}{V} \right) \geq 1 \tag{55}$$
$$\frac{E}{V} = f^e \left(X, V, G_a \right) \tag{56}$$
$$\frac{K}{V} \geq 0 \tag{57}$$
$$\frac{X}{V} \geq 0 \tag{58}$$
$$\frac{V_a}{V} \geq 0 \tag{59}$$

where

- G_a is G used in emission reduction
- τ_e is the tax on emmissions/price of emissions quota
- E is emissions

Parameterizing the production function and the emission function yields

$$\min_{\frac{K}{V},\frac{X}{V},\frac{G_a}{V}} cost_v\left(\frac{K}{V},\frac{X}{V},\frac{G_a}{V}\right) = P_k\frac{K}{V} + P_x\frac{X}{V} + P_g\frac{G_a}{V} + \tau\frac{E}{V} \tag{60}$$

$$s.t.\ \frac{\gamma_v}{V}\left[\alpha_{v_k}K^{\frac{\sigma_v-1}{\sigma_v}} + \alpha_{v_x}X^{\frac{\sigma_v-1}{\sigma_v}}\right]^{\frac{\sigma_v}{\sigma_v-1}} \geq 1 \tag{61}$$

$$\frac{E}{V} = \alpha_e\left[1+\beta\left(\frac{G_a}{X}\right)^{\rho_e}\right]^{\frac{-1}{\rho_e}}\frac{X}{V} \tag{62}$$

$$\frac{K}{V} \geq 0 \tag{63}$$

$$\frac{X}{V} \geq 0 \tag{64}$$

$$\frac{G_a}{V} \geq 0 \tag{65}$$

where $\alpha_{v_k} + \alpha_{v_x} = 1$

Note: if we write $ex \equiv E/X = f(a)$ where $a \equiv G_a/X$, the function $f(a)$, under certain parameterization, meets meets the following criteria:

- $f'(0) \Rightarrow -\infty$. This implies that there will be abatement if the emission tax is positive.
- $f(\infty) = 0$ If abatement is very large, no pollution.
- $f(0) = 1$ If no abatement, pollution does not become infinite.

Substituting (62) into the cost function and expressing it in terms of unit production, the optimization problem becomes

$$\begin{aligned}\min_{k,x,g_a} cost_v(k,x,g_a) = {} & P_k k + P_x x + P_g g_a + \tau\alpha_e\left[1+\beta\left(\frac{g_a}{x}\right)^{\rho_e}\right]^{\frac{-1}{\rho_e}} x \\ & +\lambda_v\left\{1-\gamma_v\left[\alpha_{v_k}k^{\frac{\sigma_v-1}{\sigma_v}} + \alpha_{v_x}x^{\frac{\sigma_v-1}{\sigma_v}}\right]^{\frac{\sigma_v}{\sigma_v-1}}\right\}\end{aligned} \tag{66}$$

First-order conditions Taking the derivative of (66) with respect to k yields

$$P_k - \lambda_v\gamma_v\frac{\sigma_v}{\sigma_v-1}\left[\alpha_{v_k}k^{\frac{\sigma_v-1}{\sigma_v}} + \alpha_{v_x}x^{\frac{\sigma_v-1}{\sigma_v}}\right]^{\frac{1}{\sigma_v-1}}\frac{\sigma_v-1}{\sigma_v}\alpha_{v_k}k^{\frac{-1}{\sigma_v}} \equiv 0 \tag{67}$$

Reducing,

$$P_k - \lambda_v \gamma_v \left[\alpha_{v_k} k^{\,\frac{\sigma_v - 1}{\sigma_v}} + \alpha_{v_x} x^{\,\frac{\sigma_v - 1}{\sigma_v}}\right]^{\frac{1}{\sigma_v - 1}} \alpha_{v_k} k^{-\frac{1}{\sigma_v}} \equiv 0 \tag{67$'$}$$

Taking the derivative of (66) with respect to x yields

$$\begin{aligned} P_x + \tau\alpha_e \left[1 + \beta\left(\frac{g_a}{x}\right)^{\rho_e}\right]^{\frac{-1}{\rho_e}} + \tau\alpha_e \frac{-1}{\rho_e}\left[1 + \beta\left(\frac{g_a}{x}\right)^{\rho_e}\right]^{\frac{-(1+\rho_e)}{\rho_e}} \beta\left(\frac{g_a}{x}\right)^{\rho_e - 1} x\rho_e \frac{-g_a}{x^2} \\ - \lambda_v \gamma_v \frac{\sigma_v}{\sigma_v - 1}\left[\alpha_{v_k} k^{\,\frac{\sigma_v - 1}{\sigma_v}} + \alpha_{v_x} x^{\,\frac{\sigma_v - 1}{\sigma_v}}\right]^{\frac{1}{\sigma_v - 1}} \frac{\sigma_v - 1}{\sigma_v} \alpha_{v_x} x^{\,\frac{-1}{\sigma_v}} \equiv 0 \end{aligned} \tag{68}$$

Rearranging,

$$\begin{aligned} \overbrace{P_x + \left\{\tau_e\alpha_e\left[1 + \beta\left(\frac{g_a}{x}\right)^{\rho_e}\right]^{\frac{-1}{\rho_e}} + \tau_e\alpha_e\beta\left(\frac{g_a}{x}\right)^{\rho_e}\left[1 + \beta\left(\frac{g_a}{x}\right)^{\rho_e}\right]^{\frac{-(1+\rho_e)}{\rho_e}}\right\}}^{\hat{P_x}} \\ \equiv \lambda_v\gamma_v\left[\alpha_{v_k} k^{\,\frac{\sigma_v - 1}{\sigma_v}} + \alpha_{v_x} x^{\,\frac{\sigma_v - 1}{\sigma_v}}\right]^{\frac{1}{\sigma_v - 1}} \alpha_{v_x} x^{\,\frac{-1}{\sigma_v}} \end{aligned} \tag{68$'$}$$

$$\hat{P_x} = \overbrace{P_x}^{1} + \left\{\overbrace{\tau_e\alpha_e\left[1 + \beta\left(\frac{g_a}{x_v^d v}\right)^{\rho_e}\right]^{\frac{-1}{\rho_e}}}^{2} + \overbrace{\tau_e\alpha_e\beta\left(\frac{g_a}{x_v^d v}\right)^{\rho_e}\left[1 + \beta\left(\frac{g_a}{x_v^d v}\right)^{\rho_e}\right]^{\frac{-(1+\rho_e)}{\rho_e}}}^{3}\right\} \tag{69}$$

The terms of (69) can be interpreted as the marginal cost due to

1. purchasing x
2. increasing emissions due to increasing x (holding $\frac{g_a}{x}$ constant)
3. increasing emissions due to decreasing the ratio $\frac{g_a}{x}$

Taking the derivative of (66) with respect to g_a yields

$$P_g + \tau\alpha_e \frac{-1}{\rho_e}\left[1 + \beta\left(\frac{g_a}{x}\right)^{\rho_e}\right]^{\frac{-(1+\rho_e)}{\rho_e}} \beta\left(\frac{g_a}{x}\right)^{\rho_e - 1} x\rho_e \frac{1}{x} \equiv 0 \tag{70}$$

Rearranging,

$$P_g \equiv \tau\alpha_e\beta\left(\frac{g_a}{x}\right)^{\rho_e-1}\left[1+\beta\left(\frac{g_a}{x}\right)^{\rho_e}\right]^{\frac{-(1+\rho_e)}{\rho_e}} \tag{70$'$}$$

$$\frac{P_g}{\tau\alpha_e\beta} \equiv \left(\frac{g_a}{x}\right)^{\rho_e-1}\left[1+\beta\left(\frac{g_a}{x}\right)^{\rho_e}\right]^{\frac{-(1+\rho_e)}{\rho_e}} \tag{70$''$}$$

$$\frac{g_a}{x} \equiv \left(\frac{P_g}{\tau\alpha_e\beta}\right)^{\frac{1}{\rho_e-1}}\left[1+\beta\left(\frac{g_a}{x}\right)^{\rho_e}\right]^{\frac{1+\rho_e}{\rho_e(\rho_e-1)}} \tag{70$'''$}$$

If $\hat{P}_x$ were known, x^d and k^d could be solved as k^d and l^d were solved for C and X:

$$k_v^d = \frac{1}{\gamma_y}\left(\frac{\alpha_{y_k}\gamma_y cost_v}{P_k}\right)^{\sigma_y} \tag{71}$$

$$x_v^d = \frac{1}{\gamma_y}\left(\frac{\alpha_{y_x}\gamma_y cost_v}{\hat{P}_x}\right)^{\sigma_y} \tag{72}$$

where

$$cost_v = \frac{1}{\gamma_y}\left(\alpha_{y_x}^{\sigma_y}\hat{P_x}^{1-\sigma_y}+\alpha_{y_k}^{\sigma_y}P_k^{1-\sigma_y}\right)^{\frac{1}{1-\sigma_y}} \tag{73}$$

Since $cost_v = P_v$ (by the zero-profit condition), it could be used to calculate the price of Y:

$$P_y = cost_y + \tau_y - s_y = \frac{1}{\gamma_y}\left(\alpha_{y_v}^{\sigma_y}P_v^{1-\sigma_y}+\alpha_{y_l}^{\sigma_y}P_l^{1-\sigma_y}\right)^{\frac{1}{1-\sigma_y}} + \tau_y - s_y \tag{74}$$

However, $\hat{P}_x$ depends on $\frac{g_a}{x}$ and $\frac{g_a}{x}$, even if it had a closed form solution, would depend on P_g. P_g depends on P_y and P_y, in turn, depends on $\hat{P}_x$. Obviously, several variables are determined simultaneously and, as a result, must be solved via iteration, which is outlined in the **Computed Equations**. After the iteration, the per-unit demands for V and L in Y can be solved, based on the optimization below:

Firm's Optimization Problem: Y $cost_v$ could also be used to calculate the price of V, which is used in Y's optimization problem:

$$\min_{v_y,l_y} cost_y\left(v_y,l_y\right) = P_v v_y + P_l l_y$$
$$+\lambda_y\left\{1-\gamma_y\left[\alpha_{y_v}v_y^{\frac{\sigma_y-1}{\sigma_y}}+\alpha_{y_l}l_y^{\frac{\sigma_y-1}{\sigma_y}}\right]^{\frac{\sigma_y}{\sigma_y-1}}\right\} \tag{75}$$

Since Y's optimization problem is completely analogous to C's, it has the analogous per-unit demands, which are presented below:

Computed Equations $\frac{g_a}{x}$ is determined by the following iteration, which lies, in turn, with an iteration on P_y. Obviously, these iterations start with guesses for $\frac{g_a}{x}$ and P_y.

Specifically, the derivative of V's optimization problem with respect to g_a,

$$\frac{g_a}{x} \equiv \left(\frac{P_g}{\tau \alpha_e \beta}\right)^{\frac{1}{\rho_e - 1}} \left[1 + \beta \left(\frac{g_a}{x}\right)^{\rho_e}\right]^{\frac{1+\rho_e}{\rho_e(\rho_e - 1)}} \tag{70'''}$$

is iterated until convergence, where P_g is a price index based on P_c and P_y:

$$P_g = P_g^{avg} = \frac{P_y(\alpha^u_{g_y} P_c)^{\sigma^u_g} + P_c(\alpha^u_{g_c} P_y)^{\sigma^u_g}}{(\alpha^u_{g_y} P_c)^{\sigma^u_g} + (\alpha^u_{g_c} P_y)^{\sigma^u_g}} \tag{76}$$

By the zero-profit condition, price equals unit cost, so

$$P_v = cost_v \tag{77}$$

where

$$cost_v = \frac{1}{\gamma_y} \left(\alpha^{\sigma_y}_{y_x} \hat{P}_x^{\,1-\sigma_y} + \alpha^{\sigma_y}_{y_k} P_k^{1-\sigma_y}\right)^{\frac{1}{1-\sigma_y}} \tag{73}$$

$$\hat{P}_x = P_x + \left\{\tau \alpha_e \left[1 + \beta \left(\frac{g_a}{x}\right)^{\rho_e}\right]^{\frac{-1}{\rho_e}} + \tau \alpha_e \beta \left(\frac{g_a}{x}\right)^{\rho_e} \left[1 + \beta \left(\frac{g_a}{x}\right)^{\rho_e}\right]^{\frac{-(1+\rho_e)}{\rho_e}}\right\} \tag{69}$$

P_v can in turn be used within

$$\begin{aligned} P_y &= cost_y + \tau_y - s_y & (78) \\ &= \frac{1}{\gamma_y} \left(\alpha^{\sigma_y}_{y_v} P_v^{1-\sigma_y} + \alpha^{\sigma_y}_{y_l} P_l^{1-\sigma_y}\right)^{\frac{1}{1-\sigma_y}} + \tau_y - s_y & (74) \end{aligned}$$

to calculate P_y.

Iteration on (73) and 74), and (70''') within that iteration, is repeated until convergence of P_y

After convergence, per-unit inputs to V and Y can be calculated. From (71) and (72), k^d and x^d are calculated with the equivalent formulae as with other inputs, with the exception that the effective price of X includes the emissions tax via (69).

$$\begin{aligned} k^d_v &= \frac{1}{\gamma_v} \left(\frac{\alpha_{y_v} \gamma_v cost_v}{P_k}\right)^{\sigma_v} & (71) \\ x^d_v &= \frac{1}{\gamma_v} \left(\frac{\alpha_{y_l} \gamma_v cost_v}{\hat{P}_x}\right)^{\sigma_v} & (72) \end{aligned}$$

Demand for g_a PER UNIT OF V is calculated as

$$g_a = \frac{g_a}{x} x_v^d \tag{79}$$

Per-unit-of-v emissions are calculated as

$$e_v = \alpha_e \left[1 + \beta \left(\frac{g_a}{x_v^d} \right)^{\rho_e} \right]^{\frac{-1}{\rho_e}} x_v^d \tag{80}$$

Per-unit demands for inputs in Y are

$$v_y^d = \frac{1}{\gamma_y} \left(\frac{\alpha_{y_v} \gamma_y cost_y}{P_{v_y}} \right)^{\sigma_y} \tag{81}$$

$$l_y^d = \frac{1}{\gamma_y} \left(\frac{\alpha_{y_l} \gamma_y cost_y}{P_l} \right)^{\sigma_y} \tag{82}$$

To confirm the calculation of P_v and P_y, consumer and producer prices for Y, using the zero-profit condition, are calculated as

$$P_v = \hat{P}_v v + P_x x \tag{83}$$

$$P_y = P_v v + P_l l + \tau e + P_g g_a + \tau_y - s_y \tag{84}$$

$$P_y^{net} = P_v v + P_l l + P_g g_a \tag{85}$$

Implementation in Code For consistency with steps 1 and 2, when V and X are used as inputs, their prices are presented like other input prices (i.e. as `W`, so that P_v is `W[Y][Vi]`).

Prices, wages, and input quantities are stored in matrices of dimension 5x2 (5 "outputs" and 2 "inputs"). Since `V` is the second output (in alphabetical order) but a prior input to `X`, `Vi` and `Xi` must be used in denoting the use of V and X as inputs. Because the matrices are only 2 columns wide, the use of g_a is stored in `Z_u[E][Yi]` instead of in `Z_u[Y][Ya]`

4 Demand for Final Goods: C, Y & leisure

Background Consumers gain utility from goods (G) and from resources (F) that, instead of supplying to the market, they keep for their own use (i.e. potential labor used as leisure and potential capital not supplied as a production input). Utility is produced according to a CES utility function:

$$U = \left(\alpha_{u_g} G^{\frac{\sigma_u - 1}{\sigma_u}} + \alpha_{u_f} F^{\frac{\sigma_u - 1}{\sigma_u}} \right)^{\frac{\sigma_u}{\sigma_u - 1}} \tag{86}$$

In turn, F and G are constructed from CES functions:

$$F = \left(\alpha_{f_l} \ell_l^{\frac{\sigma_f - 1}{\sigma_f}} + \alpha_{f_k} \ell_k^{\frac{\sigma_f - 1}{\sigma_f}} \right)^{\frac{\sigma_f}{\sigma_f - 1}} \tag{87}$$

$$G = \left(\alpha_{g_c} C^{\frac{\sigma_g - 1}{\sigma_g}} + \alpha_{g_y} Y^{\frac{\sigma_g - 1}{\sigma_g}} \right)^{\frac{\sigma_g}{\sigma_g - 1}} \tag{88}$$

and ℓ_l is potential labor not supplied to the market (i.e. leisure) and ℓ_k is potential capital not supplied to the market.

These functions, along with income, determine the supply of K and L and the demand for C and Y

Computed Equations

Income To calculate income, some preliminary calculations of prices are necessary

$$\tau_l = \tau_l^{bm} + (\tau^m - 1) \tag{89}$$

$$P_{k_{avg}} = (1 - \tau_{k_c}) k_c^s P_{k_c} + (1 - \tau_{k_y}) k_y^s P_{k_y} + (1 - \tau_{k_x}) k_x^s P_{k_x} \tag{90}$$

Note that k^s is, at this point, a per-unit measure

$$P_f = (\alpha_{f_k}^{\sigma_f} P_{k_{avg}}^{1-\sigma_f} + \alpha_{fl}^{\sigma_f} (1 - \tau_l)^{1-\sigma_f})^{\frac{1}{1-\sigma_f}} \tag{91}$$

$$P_g = (\alpha_{g_c}^{\sigma_g} P_c^{1-\sigma_g} + \alpha_{g_y}^{\sigma_g} P_y^{1-\sigma_g})^{\frac{1}{1-\sigma_g}} \tag{92}$$

$$P_u = (\alpha_{u_f}^{\sigma_u} P_f^{1-\sigma_u} + \alpha_{u_g}^{\sigma_u} P_g^{1-\sigma_u})^{\frac{1}{1-\sigma_u}} \tag{93}$$

Household income is made up of transfers, labor income, and capital income. Where applicable, income from grandfathered permits is added.

$$transfers = P_g * transfers_{real} + (1 - \eta) * Scalar[GIN] \tag{94}$$

$$I_0 = \bar{K} P_{k_{avg}} + \bar{L} P_l (1 - \tau_l) + transfers + rev_y^{gf} (1 - \tau_{k_y}) + rev_x^{gf} (1 - \tau_{k_x}) + S_a (1 - \tau_{k_y}) \tag{95}$$

where

- $\bar{K}$ **is the amount of potential capital available**
- $\bar{L}$ **is the amount of potential labor available**
- $transfers_{real}$ **are observed/benchmark transfers**

- **if emissions reduction is achieved through emissions permits**

$$ref_y^{gf} = \tau_e E^q \alpha_y^{gf} \tag{96}$$

$$ref_x^{gf} = \tau_e E^q \alpha_x^{gf} \tag{97}$$

- **if emissions reduction is achieved through fuel permits**

$$ref_y^{gf} = \tau_x \frac{E^q}{e_v} x_v \alpha_y^{gf} \tag{98}$$

$$ref_y^{gf} = \tau_x \frac{E^q}{e_v} x_v \alpha_x^{gf} \tag{99}$$

$$\tag{100}$$

where $\frac{E^q}{e_v} x_v$ is equivalent to X, in equilibrium.

- **if there is an abatement subsidy**

$$S_a = \tau_e E^{bm} \tag{101}$$

Spending on F and G **Income can be decomposed into "spending" on F (i.e. the value of potential labor and capital consumed by the household rather than supplied to the market) and into spending on G (goods) From the utility function, the "demand" for (i.e. spending on) F can be calculated as below and the spending on G as the residual:**

$$I_f = \frac{I_0 (\alpha_{u_f} P_g)^{\sigma_u} P_f}{(\alpha_{u_f} P_g)^{\sigma_u} P_f + (\alpha_{u_g} P_f)^{\sigma_u} P_g} \tag{102}$$

$$I_g = I_0 - F \tag{103}$$

Supply of capital & labor **Similarly, spending on F can be decomposed into spending on leisure and spending on unused capital. The residual between spending on a resource and the total amount available is supplied to the market.**

$$\ell_l = \frac{I_f (\alpha_{f_l} P_{k_{avg}})^{\sigma_f} (1 - \tau_l)}{(\alpha_{f_l} P_{k_{avg}})^{\sigma_f} (1 - \tau_l) + (\alpha_{f_k} (1 - \tau_l))^{\sigma_f} P_{k_{avg}}} \tag{104}$$

$$L^s = \bar{L} - \frac{\ell_l}{1 - \tau_l} \tag{105}$$

$$\ell_k = I_f - \ell_l \tag{106}$$

$$K^s = \bar{K} - \frac{\ell_k}{P_{k_{avg}}} \tag{107}$$

K^s **can be decomposed into sector-specific capital supply:**

$$\begin{aligned} K_c^s &= k_c^s K^s && (108) \\ K_y^s &= k_y^s K^s && (109) \\ K_x^s &= k_x^s K^s && (110) \\ K_{miss} &= k_{miss} K^s && (111) \end{aligned}$$

Consumer Demand for C and Y **Consumer spending on G can be decomposed into spending on C and Y**

$$\begin{aligned} I_c &= \frac{I_g(\alpha_{g_c}P_y)^{\sigma_g}P_c}{(\alpha_{g_c}P_y)^{\sigma_g}P_c + (\alpha_{g_y}P_c)^{\sigma_g}P_y} && (112) \\ C_c^d &= \frac{I_c}{P_c} && (113) \\ Y_c^d &= \frac{I_g - I_c}{P_y} && (114) \end{aligned}$$

Total demand for C and Y **The producer demand for Y will be the portion of G_a that comes from Y. This portion will be calculated based on the percentage of Y in the consumer portion of G: $\frac{Y_c}{G_c} = \frac{Y_c}{C_c+Y_c}$. G_a is calculated based on g_a, the per-unit-of-V use of abatement, multiplied by V, which in turn is calculated as $v_y^d(Y_c^d + Y_p^d)$:**

$$\begin{aligned} G_a &= g_a V \\ &= g_a v_y Y \\ &= g_a v_y (Y_a^d + Y_c^d) \\ &= g_a v_y \left(G_a \frac{Y_c^d}{C_c^d + Y_c^d} + Y_c^d\right) \\ &= \frac{g_a v_y Y_c^d}{1 - \frac{g_a v_y Y_c^d}{C_c^d + Y_c^d}} && (115) \end{aligned}$$

$$\begin{aligned} Y_a^d &= G_a \frac{Y_c}{C_c^d + Y_c^d} && (116) \\ C_a^d &= G_a - Y_a^d && (117) \\ C^d &= C_c^d + C_a^d && (118) \\ Y^d &= Y_c^d + Y_a^d && (119) \end{aligned}$$

5 Demand for Inputs: V, X, & E

Background The production of emissions and demand for V and X are derived from the demand for Y.

Computed Equations

$$V^d = Y^d v_y^d \tag{120}$$

$$X^d = V^d x_v^d \tag{121}$$

$$E = V^d e_v \tag{122}$$

6 Demand for Inputs: K & L & tax revenue

Background The demands for K_c, K_x, and K_y are derived from the demands for C, X, and V

Computed Equations

$$K_c^d = C^d k_c^d \tag{123}$$

$$L_c^d = C^d l_c^d \tag{124}$$

$$K_x^d = X^d k_x^d \tag{125}$$

$$L_x^d = X^d l_x^d \tag{126}$$

$$K_y^d = V^d k_y^d \tag{127}$$

$$L_y^d = Y l_y^d \tag{128}$$

$$G_a = V^d g_a \tag{129}$$

Emissions tax revenue is calculated as

$$T_e = \tau_e E \tag{130}$$

If emissions reductions are achieved through a quota, tax revenue is taken to mean permit revenue and calculated to account for the possibility of grandfathering:

$$T_e = \tau_e E^q \tag{130'}$$

Government income is calculated as the sum of emission tax revenue, labor tax, and capital tax, adjusted for taxes and subsidies to X and Y, grandfathering, and abatement subsidies.

$$\begin{aligned} gov\_income = {} & T_e + L^s \tau_l + \tau_{k_c} K_c^s P_{k_c} + \tau_{k_y} K_y^s P_{k_y} + \tau_{k_x} K_x^s P_{k_x} \\ & + (\tau_x - s_x) X + (\tau_y - s_y) Y - (1 - \tau_{k_y}) rev_y^{gf} - (1 - \tau_{k_x}) rev_x^{gf} - S_a \end{aligned} \tag{131}$$

Government expenditures are

$$gov\_expend = transfers + s_{ya} Y_p P_y \tag{130.3}$$

$$K^d_{total} = K^d_c + K^d_x + K^d_y \tag{132}$$

$$L^d_{total} = L^d_c + L^d_x + L^d_y \tag{133}$$

Calculate Profit Profit in a sector is return to capital in that sector: price of capital multiplied by the amount of capital. Because the amount of capital in a sector may differ between the benchmark and a policy case, two definitions of profit are calculated: one based on the current level of capital and another based on the original. The latter is the definition of interest.

$$P^{defl} = P^{bm}_{index} / P_{index} \tag{134}$$

$$\pi_c = P^{defl}(1 - \tau_{k_c}) P_{k_c} K^d_c \tag{135}$$

$$\pi^0_c = P^{defl}(1 - \tau_{k_c}) P_{k_c} K^{d^{bm}}_c \tag{136}$$

$$\pi_x = P^{defl}(1 - \tau_{k_x}) P_{k_x} K^d_x \tag{137}$$

$$\pi^0_x = P^{defl}(1 - \tau_{k_x}) P_{k_x} K^{d^{bm}}_x \tag{138}$$

$$\pi^{gf}_x = \pi_x + P^{defl}(1 - \tau_{k_x}) rev^{gf}_x \tag{139}$$

$$\pi^{0^{gf}}_x = \pi^0_x + P^{defl}(1 - \tau_{k_x}) rev^{gf}_x \tag{140}$$

$$\pi_y = P^{defl}(1 - \tau_{k_y}) P_{k_y} K^d_y \tag{141}$$

$$\pi^0_y = P^{defl}(1 - \tau_{k_y}) P_{k_y} K^{d^{bm}}_y \tag{142}$$

$$\pi^{gf}_y = \pi_y + P^{defl}(1 - \tau_{k_y}) rev^{gf}_y \tag{143}$$

$$\pi^{0^{gf}}_x = \pi^0_y + P^{defl}(1 - \tau_{k_y}) rev^{gf}_y \tag{144}$$

$$\tag{145}$$

7 Calculate Excess Demands

$$K_c^{ed} = K_c^d - K_c^s \tag{146}$$

$$K_x^{ed} = K_x^d - K_x^s \tag{147}$$

$$K_y^{ed} = K_y^d - K_y^s \tag{148}$$

$$Budget\_surplus = gov\_income - gov\_expend \tag{149}$$

$$E^{ed} = E - E^q \tag{150}$$

$$\pi_y^{ed} = \pi_y - pi_y^{bm} \tag{151}$$

$$\pi_x^{ed} = \pi_x - pi_x^{bm} \tag{152}$$

$$T_e^{ed} = \tau_e E - s_x X \text{ or } \tau_e E - s_y Y \tag{153}$$

The excess demand equations sent to the Newton algorithm are normalized:

$$fvec[1] = \frac{K_c^{ed}}{Exdem\_Denom[C]} \tag{154}$$

$$fvec[2] = \frac{K_x^{ed}}{Exdem\_Denom[X]} \tag{155}$$

$$fvec[3] = \frac{K_y^{ed}}{Exdem\_Denom[Y]} \tag{156}$$

$$fvec[4] = \frac{Budget\_surplus}{Exdem\_Denom[BS]} \tag{157}$$

$$fvec[5] = \frac{E^{ed}}{Exdem\_Denom[E]} \tag{158}$$

$$fvec[6] = \frac{\pi_y^{bm}}{Exdem\_Denom[\pi_y]} \tag{159}$$

$$fvec[7] = \frac{\pi_x^{bm}}{Exdem\_Denom[\pi_x]} \tag{160}$$

$$fvec[8] = \frac{T_e^d}{Exdem\_Denom[T_e]} \tag{161}$$

$$\tag{162}$$

where *Exdem_Denom* is set to 1 in the benchmark case and, in policy cases, the value from the benchmark:

$$Exdem\_Denom[C] = K_C^{bm} \tag{163}$$

$$Exdem\_Denom[V] = K_V^{bm} \tag{164}$$

$$Exdem\_Denom[X] = K_X^{bm} \tag{165}$$

$$Exdem\_Denom[BS] = gov^{bm}\_inc + 1 \tag{166}$$

$$Exdem\_Denom[E] = E^{bm} \tag{167}$$

$$Exdem\_Denom[\pi_y] = \pi_Y^{bm} \tag{168}$$

$$Exdem\_Denom[\pi_x] = \pi_X^{bm} \tag{169}$$

$$Exdem\_Denom[T_e] = E^{bm} \tag{170}$$

The program computes the following as convergence checks:

$$Excess\_demand^{total} = K_c^{ed} + K_x^{ed} + K_y^{ed} + Budget\_surplus + E^{ed} + (L_{total}^{d} - \overline{L}) \tag{171}$$

$$Value\_of\_Excess\_demand^{total} = P_{k_c}K_c^{ed} + P_{k_x}K_x^{ed} + P_{k_y}K_y^{ed} + Budget\_surplus + \tau_e E^{ed} + (L_{total}^{d} - \overline{L}) \tag{172}$$

Part II
Welfare

Background EV is calculated by subtracting benchmark income from the money-metric utility produced in the policy. The money-metric utility is produced by calculating the utility resulting from the policy and adjusting by the per-unit value of utility at old prices.

Computed Equations **Utility is calculated by the equation**

$$U = \left(\alpha_{u_g} G^{\frac{\sigma_u - 1}{\sigma_u}} + \alpha_{u_f} F^{\frac{\sigma_u - 1}{\sigma_u}}\right)^{\frac{\sigma_u}{\sigma_u - 1}} \tag{173}$$

where

$$F = \left(\alpha_{f_l} \ell_l^{\frac{\sigma_f - 1}{\sigma_f}} + \alpha_{f_k} \ell_k^{\frac{\sigma_f - 1}{\sigma_f}}\right)^{\frac{\sigma_f}{\sigma_f - 1}} \tag{174}$$

$$G = \left(\alpha_{g_c} C^{\frac{\sigma_g - 1}{\sigma_g}} + \alpha_{g_y} Y^{\frac{\sigma_g - 1}{\sigma_g}}\right)^{\frac{\sigma_g}{\sigma_g - 1}} \tag{175}$$

and ℓ_l is potential labor not supplied to the market (i.e. leisure) and ℓ_k is potential capital not supplied to the market.

The per-unit value of utility is calculated as

$$P_u^{bm} = (\alpha_{u_f}^{\sigma_u} P_f^{bm^{1-\sigma_u}} + \alpha_{u_g}^{\sigma_u} P_g^{bm^{1-\sigma_u}})^{\frac{1}{1-\sigma_u}} \tag{176}$$

where

$$P_f^{bm} = (\alpha_{f_k}^{\sigma_f} P_{k_{avg}}^{bm^{1-\sigma_f}} + \alpha_{f_l}^{\sigma_f} (1 - \tau_l)^{bm^{1-\sigma_f}})^{\frac{1}{1-\sigma_f}} \tag{177}$$

$$P_g^{bm} = (\alpha_{g_c}^{\sigma_g} P_c^{bm^{1-\sigma_g}} + \alpha_{g_y}^{\sigma_g} P_y^{bm^{1-\sigma_f}})^{\frac{1}{1-\sigma_g}} \tag{178}$$

So, EV is calculated as

$$EV = P_u^{bm} U - I_0^{bm} \tag{179}$$