# On Worst-Case Regret of Linear Thompson Sampling

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# Overview

1. Problem Definition
2. Confidence-based Policies
3. Failure of LinTS ☹
4. Positive Results ☺

# Stochastic Linear Bandit Problem

- Let $\Theta^\star \in \mathbb{R}^d$ be fixed (and unknown).
- At time $t$, the action set $\mathcal{A}_t \subseteq \mathbb{R}^d$ is revealed to a policy $\pi$.
- The policy chooses $\widetilde{A}_t \in \mathcal{A}_t$.
- It observes a reward $r_t = \langle \Theta^\star, \widetilde{A}_t \rangle + \varepsilon_t$.
- Conditional on the history, $\varepsilon_t$ has zero mean.

# Evaluation Metric

- The objective is to **improve using past experiences**.
- The **cumulative regret** is defined as

$$\mathrm{Regret}(T, \Theta^\star, \pi) := \mathbb{E}\left[\sum_{t=1}^{T} \sup_{A \in \mathcal{A}_t} \langle \Theta^\star, A \rangle - \langle \Theta^\star, \widetilde{A}_t \rangle \,\middle|\, \Theta^\star \right].$$

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- In the Bayesian setting, the **Bayesian regret** is given by

$$\mathrm{BayesRegret}(T, \pi) := \mathbb{E}_{\Theta^\star \sim \mathcal{P}}[\mathrm{Regret}(T, \Theta^\star, \pi)].$$

# Algorithms

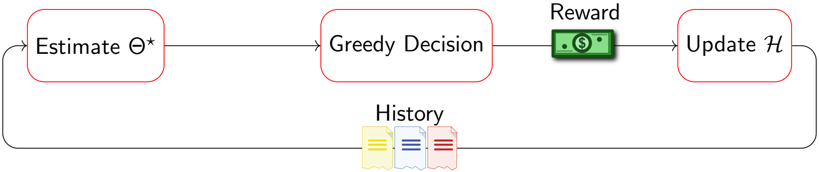
# Greedy

At time $t = 1, 2, \cdots, T$:

- Using the set of observations

$$\mathcal{H}_{t-1} := \{(\widetilde{A}_1, r_1), \cdots, (\widetilde{A}_{t-1}, r_{t-1})\},$$

- Construct an **estimate** $\widehat{\Theta}_{t-1}$ for $\Theta^\star$,
- Choose the action $A \in \mathcal{A}_t$ with **largest** $\langle A, \widehat{\Theta}_{t-1} \rangle$.



# Greedy

The **ridge estimator** is used to obtain $\widehat{\Theta}_t$ (for a fixed $\lambda$):

$$\mathbf{V}_t := \lambda \mathbb{I} + \sum_{i=1}^{t} \widetilde{A}_i \widetilde{A}_i^\top \in \mathbb{R}^{d \times d}, \tag{1}$$

and

$$\widehat{\Theta}_t := \mathbf{V}_t^{-1} \left( \sum_{i=1}^{t} \widetilde{A}_i r_i \right) \in \mathbb{R}^d. \tag{2}$$

# Greedy

**Algorithm 1** Greedy algorithm

**for** $t = 1$ to $T$ **do**
    Pull $\widetilde{A}_t := \arg\max_{A \in \mathcal{A}_t} \langle A, \widehat{\Theta}_{t-1} \rangle$
    Observe the reward $r_t$
    Compute $\mathbf{V}_t = \lambda \mathbb{I} + \sum_{i=1}^{t} \widetilde{A}_i \widetilde{A}_i^\top$
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**end for**

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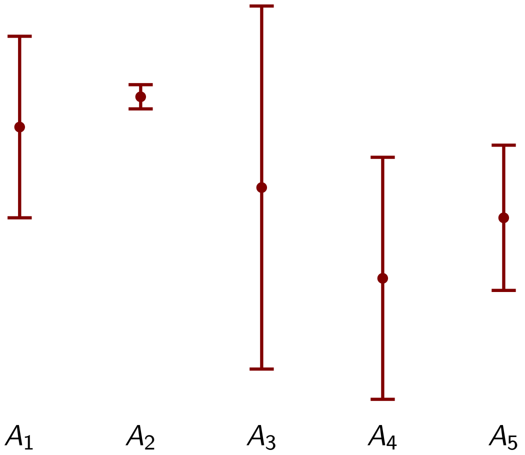
6: **end for**

Greedy makes wrong decisions due to **over**- or **under-estimating** the true rewards.

- The over-estimation is **automatically** corrected.
- The under-estimation can cause **linear regret**.

# Greedy



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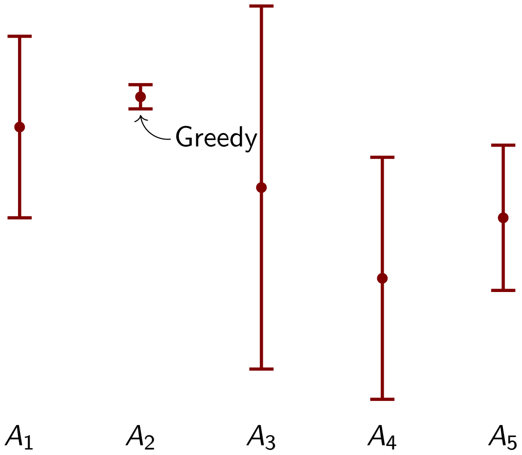

# Optimism in Face of Uncertainty (OFU) Algorithm

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- For $\rho > 0$, define the **confidence set** $\mathcal{C}_t(\rho)$ to be

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  where

$$\|X\|_{\mathbf{V}_t}^2 = X^\top \mathbf{V}_t X \in \mathbb{R}^+.$$

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**Theorem (Informal, Abbasi-Yadkori, Pál, and Szepesvári 2011)**

*Letting $\rho := \widetilde{\mathcal{O}}(\sqrt{d})$, we have $\Theta^\star \in \mathcal{C}_t(\rho)$ with high probability.*

# Optimism in Face of Uncertainty (OFU) Algorithm

**Algorithm 2** OFUL algorithm

1: **for** $t = 1$ to $T$ **do**
2: Pull $\widetilde{A}_t := \arg\max_{A \in \mathcal{A}_t} \sup_{\Theta \in \mathcal{C}_{t-1}(\rho)} \langle A, \Theta \rangle$
3: Observe the reward $r_t$
4: Compute $\mathbf{V}_t = \lambda \mathbb{I} + \sum_{i=1}^{t} \widetilde{A}_i \widetilde{A}_i^\top$
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It can be shown that

$$\sup_{\Theta \in \mathcal{C}_t(\rho)} \langle A, \Theta \rangle = \langle A, \widehat{\Theta}_t \rangle + \rho \|A\|_{\mathbf{V}_{t-1}^{-1}}.$$

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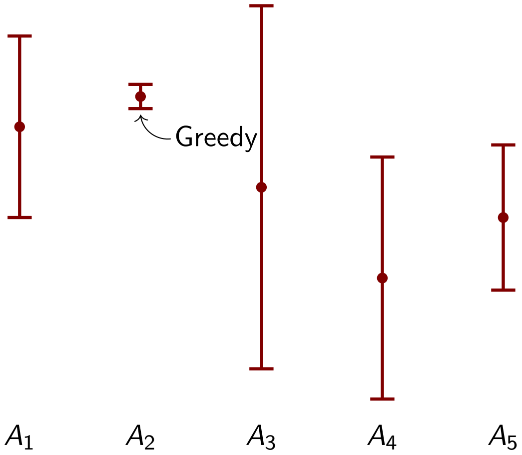

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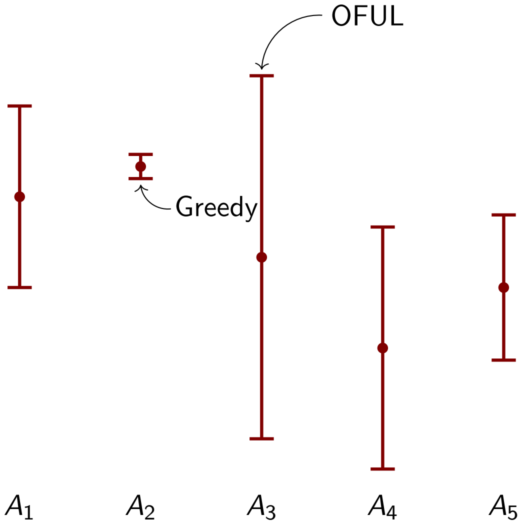

# Linear Thompson Sampling (LinTS) Algorithm

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- Key idea: use **randomization** to address under-estimation.
- LinTS samples from the **posterior** distribution of $\Theta^\star$.

**Algorithm 3** LinTS algorithm

1: **for** $t = 1$ to $T$ **do**
2: Sample $\widetilde{\Theta}_t \sim \mathbb{P}(\Theta^\star \mid \mathcal{H}_{t-1})$
3: Pull $A_t := \arg\max_{A \in \mathcal{A}_t} \langle A, \widetilde{\Theta}_t \rangle$
4: Observe the reward $r_t$
5: Update $\mathcal{H}_t \leftarrow \mathcal{H}_{t-1} \bigcup \{(A_t, r_t)\}$
6: **end for**

# Linear Thompson Sampling (LinTS) Algorithm

- Under **normality**, LinTS becomes:

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**Algorithm 4** LinTS algorithm under normality

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1: **for** $t = 1$ to $T$ **do**
2: Sample $\widetilde{\Theta}_t \sim \mathcal{N}(\widehat{\Theta}_{t-1}, \mathbf{V}_{t-1}^{-1})$
3: Pull $A_t := \arg\max_{A \in \mathcal{A}_t} \langle A, \widetilde{\Theta}_t \rangle$
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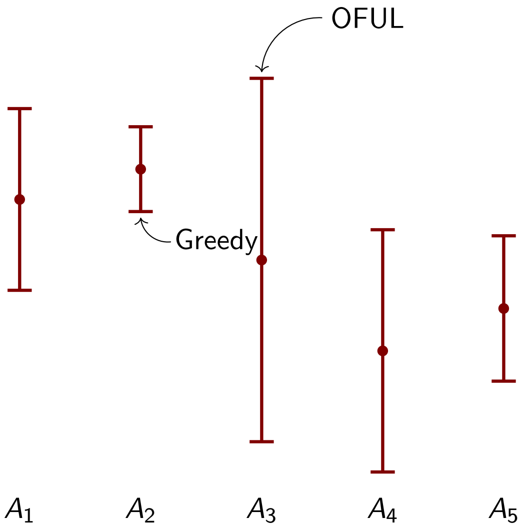

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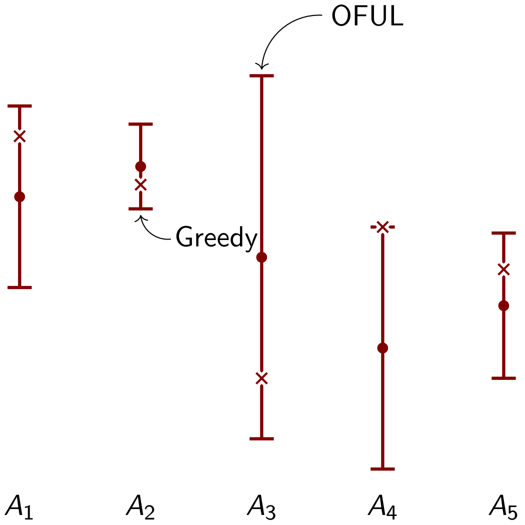

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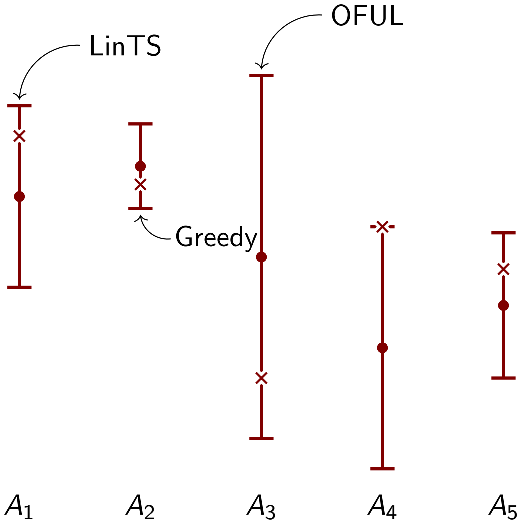
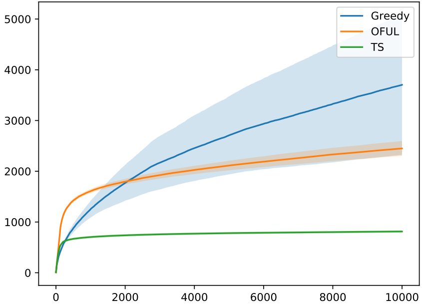


# Why Is LinTS Popular?

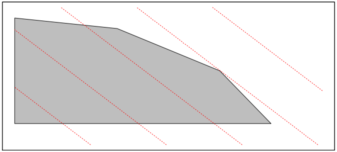
- **Empirical superiority**:
  - $d = 120$, $\Theta^\star \sim \mathcal{N}(0, \mathbb{I}_d)$,
  - $k = 10$, $X \sim \mathcal{N}(0, \mathbb{I}_{12})$,
  - Each $A_t$ contains $X$ as a block[1].




[1] This is the 10-armed contextual bandit with 12 dimensional covariates.

# Why is LinTS Popular?

- **Computation efficiency**: when $\mathcal{A}_t$ is a polytope $\cdots$
  - LinTS solves an LP problem,



  - OFUL becomes an NP-hard problem!

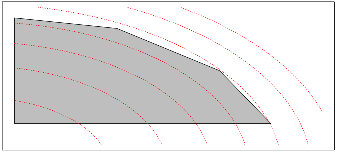


Photo credit: Russo and Van Roy 2014

# Comparison of Regret Bounds

**Theorem (Abbasi-Yadkori, Pál, and Szepesvári 2011)**

*Under some conditions, the regret of OFUL is bounded by*

$$\text{Regret}(T, \Theta^\star, \pi^{OFUL}) \leq \widetilde{\mathcal{O}}(d\sqrt{T}).$$

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*Under minor assumptions, the Bayesian regret of LinTS is bounded by*

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**Theorem (Dani, Hayes, and Kakade 2008)**

*There is a Bayesian linear bandit problem that satisfies*

$$\inf_{\pi} \mathrm{BayesRegret}(T, \pi) \ge \Omega(d\sqrt{T}).$$

# A Worst-Case Regret Bound for LinTS

- Question: can one prove a similar worst-case regret bound for LinTS?
- The only known results require **inflating** the posterior variance.

**Algorithm 5** LinTS algorithm under normality

1: **for** $t = 1$ to $T$ **do**
2: Sample $\widetilde{\Theta}_t \sim \mathcal{N}(\widehat{\Theta}_{t-1}, \beta^2 \mathbf{V}_{t-1}^{-1})$
3: Pull $A_t := \arg\max_{A \in \mathcal{A}_t} \langle A, \widetilde{\Theta}_t \rangle$
4: Observe the reward $r_t$
5: Compute $\mathbf{V}_t = \lambda \mathbb{I} + \sum_{i=1}^{t} \widetilde{A}_i \widetilde{A}_i^\top$
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7: **end for**

# A Worst-Case Regret Bound for LinTS

**Theorem (Abeille and Lazaric 2017; Agrawal and Goyal 2013)**

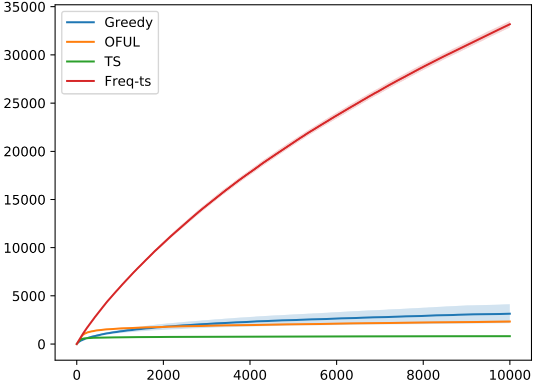
*If* $\beta \propto \sqrt{d}$*, then*

$$\text{Regret}(T, \Theta^\star, \pi^{LinTS}) \leq \widetilde{\mathcal{O}}(d\sqrt{d\,T}).$$

This result is far from optimal by a $\sqrt{d}$ factor.

# Empirical Performance of Inflated LinTS

- Unfortunately, the inflated variant of LinTS performs poorly...



# A General Regret Bound

# Randomized OFUL

- By a **worth function**, we mean a function $\widetilde{\mathrm{M}}_t$ that maps each $A \in \mathcal{A}_t$ to $\mathbb{R}$ such that

$$|\widetilde{\mathrm{M}}_t(A) - \langle A, \widehat{\Theta}_{t-1} \rangle| \leq \rho \|A\|_{\mathbf{V}_{t-1}^{-1}}$$

with probability at least $1 - \frac{1}{T^2}$.

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with probability at least $1 - \frac{1}{T^2}$.

- Next, define **Randomized OFUL (ROFUL)** to be:

**Algorithm 6** ROFUL algorithm

1: **for** $t = 1$ to $T$ **do**
2: Pull $\widetilde{A}_t := \arg\max_{A \in \mathcal{A}_t} \widetilde{\mathrm{M}}_t(A)$
3: Observe the reward $r_t$
4: Compute $\mathbf{V}_t = \lambda \mathbb{I} + \sum_{i=1}^{t} \widetilde{A}_i \widetilde{A}_i^\top$
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6: **end for**

# ROFUL Representations

Examples of worth functions:

- Greedy: $\widetilde{\mathrm{M}}_t(A) = \langle A, \widehat{\Theta}_{t-1} \rangle$
- OFUL: $\widetilde{\mathrm{M}}_t(A) = \langle A, \widehat{\Theta}_{t-1} \rangle + \rho \|A\|_{\mathbf{V}_{t-1}^{-1}}$
- LinTS: $\widetilde{\mathrm{M}}_t(A) = \langle A, \widetilde{\Theta}_{t-1} \rangle$

# A General Regret Bound

## Definition

We say a worth function $\widetilde{\mathrm{M}}_t$ is **optimistic** if

$$\sup_{A \in \mathcal{A}_t} \widetilde{\mathrm{M}}_t(A) \geq \sup_{A \in \mathcal{A}_t} \langle A, \Theta^\star \rangle \tag{3}$$

with probability at least p.

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## Theorem

*Let $(\widetilde{\mathrm{M}}_t)_{t=1}^{T}$ be a sequence of optimistic worth functions. Then, the regret of ROFUL with this worth function is bounded by*

$$\mathrm{Regret}(T, \pi^{\mathrm{ROFUL}}) \leq \widetilde{\mathcal{O}}\left(\rho \sqrt{\frac{dT}{\mathrm{p}}}\right).$$

# A Sufficient Condition for Optimism

- Recall that the worth function for LinTS is given by

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- Hence, we have

$$\sup_{A \in \mathcal{A}_t} \widetilde{\mathsf{M}}_t(A) - \sup_{A \in \mathcal{A}_t} \langle A, \Theta^\star \rangle \geq \widetilde{\mathsf{M}}_t(A_t^\star) - \langle A_t^\star, \Theta^\star \rangle$$

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&= \langle A_t^\star, \widetilde{\Theta}_t - \widehat{\Theta}_{t-1} \rangle + \underbrace{\langle A_t^\star, \widehat{\Theta}_{t-1} - \Theta^\star \rangle}_{\text{Error term}}.
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# A Sufficient Condition for Optimism

Define

- Error vector $E := \Theta^\star - \widehat{\Theta}_{t-1}$
- Compensator vector $C := \widetilde{\Theta}_t - \widehat{\Theta}_{t-1}$

The optimism assumption holds if, with probability p, the following holds

$$\langle A_t^\star, C \rangle \geq \langle A_t^\star, E \rangle.$$

# Omniscient Adversary and LinTS

- An **adversary** chooses $\mathcal{A}_t$ at time $t$.
- The adversary is **omniscient** if he knows $\widehat{\Theta}_{t-1}$ and $\Theta^\star$.

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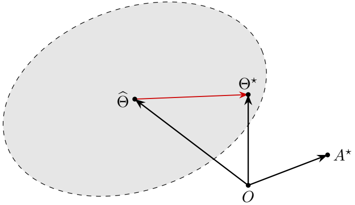
- Then $c > 0$ is chosen so that

$$\langle A, \Theta^\star \rangle > 0 \qquad \text{and} \qquad \langle A, \widehat{\Theta}_{t-1} \rangle < -\frac{1}{2} \cdot \|A\|_{\mathbf{V}_{t-1}^{-1}} \cdot \underbrace{\|E\|_{\mathbf{V}_{t-1}}}_{\approx \sqrt{d}} \ll 0.$$



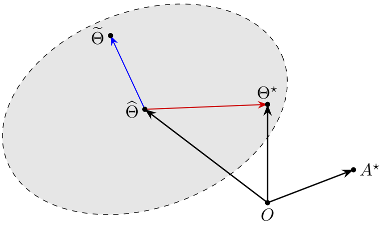
# Omniscient Adversary and LinTS

- The adversary sets $\mathcal{A}_t = \{0, A\}$.
- LinTS chooses $A$ if and only if

$$\langle A, \widetilde{\Theta}_t \rangle = \langle A, \widetilde{\Theta}_t - \widehat{\Theta}_{t-1} \rangle + \langle A, \widehat{\Theta}_{t-1} \rangle > 0.$$

- This requires

$$\langle A, C \rangle \sim \mathcal{N}(0, \mathbf{V}_{t-1}^{-1}) > \frac{1}{2} \cdot \|A\|_{\mathbf{V}_{t-1}^{-1}} \cdot \underbrace{\|E\|_{\mathbf{V}_{t-1}}}_{\approx\sqrt{d}}.$$



# Omniscient Adversary and LinTS

- Next, we have

$$\mathbb{P}(\langle A, \widetilde{\Theta}_t \rangle > 0) \leq \exp\left(-\Omega(d)\right)!$$

- LinTS chooses the optimal arm $A$ w.p. **exponentially small in** $\Omega(d)$.
- When $\widetilde{A}_t = 0$, the reward contains **no new information** about $\Theta^\star$.
- The adversary reveals the same action set in the next rounds.
- The regret will grow **linearly**.

# Bayesian Analyses are Brittle

- The key point was the **adversary's knowledge of** $E$.
- This can be relaxed by **slightly modifying** the noise distribution.
- In this case, we can set up a problem so that $\mathbb{E}[E] \neq 0$.
- **Reducing the noise variance** reveals information about $E$.

# Bayesian Analyses are Brittle

We prove that the inflation is **necessary** for LinTS to work.

**Theorem**

*There exists a linear bandit problem such that for $T \leq \exp(\Omega(d))$, we have*

$$\text{BayesRegret}(T, \pi^{LinTS}) = \Omega(T).$$

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The counter-example satisfies the following properties:

- $\Theta^\star \sim \mathcal{N}(0, \mathbb{I}_d)$,
- LinTS uses the right prior,
- LinTS assumes noises are standard normal,
- $r_t = \langle \Theta^\star, A_t \rangle$. (i.e., **noiseless** data!)

# Reducing the Inflation Parameter

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- What if we assume that $A_t^\star$ is in a **random** direction?

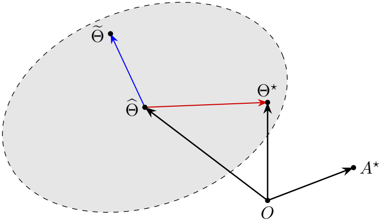
# Diversity Assumption

## Assumption (Optimal arm diversity)

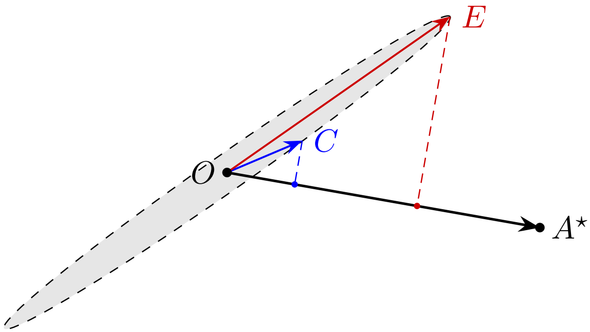
Assume that for any $V \in \mathbb{R}^d$ with $\|V\|_2 = 1$, we have

$$\mathbb{P}\left( \langle A_t^\star, V \rangle > \frac{\nu}{\sqrt{d}} \|A_t^\star\|_2 \right) \leq \frac{1}{t^3},$$

for some fixed $\nu \in [1, \sqrt{d}]$.



# Diversity is not Sufficient



# Improved Worst-Case Regret Bound for LinTS

Define **thinness** of a matrix $\boldsymbol{\Sigma}$ to be

$$\psi(\boldsymbol{\Sigma}) := \sqrt{\frac{d \cdot \|\boldsymbol{\Sigma}\|_{\mathrm{op}}}{\|\boldsymbol{\Sigma}\|_{*}}}.$$

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## Assumption

For $\Psi, \omega > 0$, we have

$$\mathbb{P}\left( \|A^\star\|_{\mathbf{V}_t^{-1}} < \omega \sqrt{\frac{\|\mathbf{V}_t^{-1}\|_*}{d}} \cdot \|A^\star\|_2 \right) \leq \frac{1}{t^3}$$

for any positive definite $\mathbf{V}_t^{-1}$ with $\psi(\mathbf{V}_t^{-1}) \leq \Psi$.

# Main Results

For $\beta := \frac{\nu\Psi}{\omega} \cdot \frac{\rho}{\sqrt{d}}$, optimism holds. So, we have the following result:

**Theorem**

*If* $\sum_{t=1}^{T} \mathbb{P}\big(\psi(\mathbf{V}_t^{-1}) > \Psi\big) \leq C$, *we have*

$$\text{Regret}(T, \Theta^\star, \pi^{TS}) \leq \mathcal{O}\Big(\rho\beta\sqrt{dT\log(T)} + C\Big).$$

# Empirical Scrutiny on Thinness

Thinness in the simulations in Russo and Van Roy (2014):

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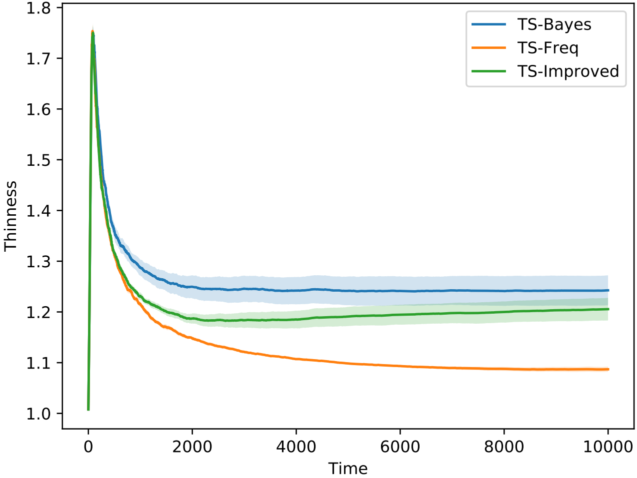


# Conclusion

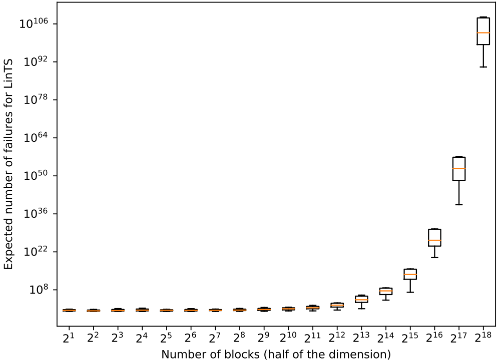
- Proved that LinTS without inflation can incur linear regret.
- Provided a general regret bound for confidence-based policies.
- Introduced sufficient conditions for reducing the inflation parameter.

*Thank you!*

*Any questions?*

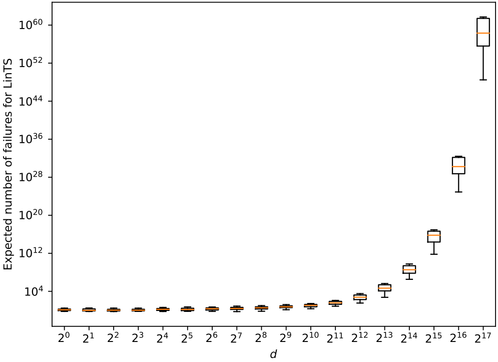
# Failure of LinTS: Example 1

| | Environment | LinTS |
|---|---|---|
| Prior | $\mathcal{N}(0, \mathbb{I}_d)$ | $\mathcal{N}(0, \mathbb{I}_d)$ |
| Noise | $\mathcal{N}(0, 0)$ | $\mathcal{N}(0, 1)$ |



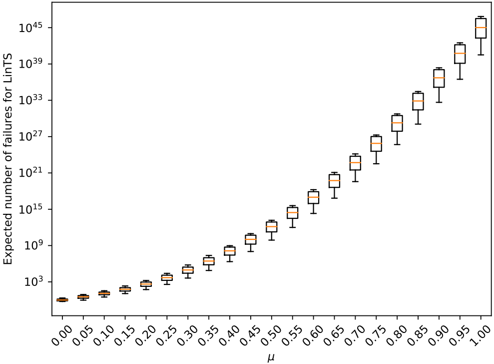
# Failure of LinTS: Example 2

| | Environment | LinTS |
|---|---|---|
| Prior | $\mathcal{N}(0.1 \cdot \mathbf{1}_d, \mathbb{I}_d)$ | $\mathcal{N}(0, \mathbb{I}_d)$ |
| Noise | $\mathcal{N}(0, 1)$ | $\mathcal{N}(0, 1)$ |



# Failure of LinTS: Example 2

| | Environment | LinTS |
|---|---|---|
| Prior | $\mathcal{N}(\mu \cdot \mathbf{1}_{2000}, \mathbb{I}_{2000})$ | $\mathcal{N}(0, \mathbb{I}_{2000})$ |
| Noise | $\mathcal{N}(0, 1)$ | $\mathcal{N}(0, 1)$ |



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