History as a Widespread Externality:  
Constrained Efficiency and Remedial Policy

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Abstract

In the Arrow-Debreu model of an intertemporal economy with a continuum of agents, suppose that the auctioneer sets prices while the government institutes optimal lump-sum transfers period by period. An earlier paper showed how subgame imperfections arise because agents understand how current decisions such as those determining investment will influence future lump-sum transfers. This observation undermines the second efficiency theorem of welfare economics and makes “history” become a widespread externality. A two-period model is used to show that the standard efficiency theorems are only true for a much weaker concept of efficiency. Possibilities for remedial policy are also discussed.

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History as an Externality

The experience of the old Poor Law has made people very much afraid that any expectation of assistance from public funds will tempt the poor into idleness and thriftlessness. . . . In reality different types of transference act in different ways . . . The main lines of division are between transferences which differentiate against idleness and thriftlessness, transferences which are neutral, and transferences which differentiate in favour of idleness and thriftlessness.

— A.C. Pigou (1932, p. 720)

1. Introduction: Intertemporal Pareto Efficiency

1.1. The Efficiency Theorems of Intertemporal Welfare Economics

The two fundamental efficiency theorems of welfare economics form the basis of most economists’ theoretical presumption that market forces should generally be encouraged wherever possible. The first of these theorems says that perfectly competitive and complete markets produce Pareto efficient allocations. This is true without any additional assumptions, provided that one uses a weak definition of Pareto efficiency — namely, that not all individuals in the economy can be made better off simultaneously by moving to some new feasible allocation. For the more usual definition of Pareto efficiency — namely, that no set of individuals in the economy can be made better off without making some others worse off — the theorem relies on the rather mild assumption that all individuals have locally non-satiated preferences.

By itself, this first efficiency theorem is not at all satisfying from an ethical point of view. It says only that “perfect” markets produce Pareto efficient outcomes, without any mention at all of distributive justice. Indeed, dictatorships and extreme inequality — even slavery (Bergstrom, 1971) or starvation (Coles and Hammond, 1991) — can all be Pareto efficient. For this reason the second efficiency theorem is ethically much more satisfying, since it characterizes (virtually) all Pareto efficient allocations, both just and unjust. The only exceptions, typically, are those which violate a boundary assumption and so may give rise to Arrow’s (1951) famous “exceptional case.” One therefore expects that a truly optimal allocation, fully reflecting ethical views regarding distributive justice, should be among those allocations which are characterized by this second efficiency theorem.
Indeed, the theorem tells us that virtually every Pareto efficient allocation could result from perfectly competitive and complete markets in general equilibrium, provided that purchasing power is redistributed by means of suitable “lump-sum” or “distortion free” taxes and transfers before markets are opened to trade. In addition, various well known convexity and continuity assumptions do have to be satisfied.

These two efficiency theorems were stated and proved, with increasing degrees of generality and mathematical rigour, by a series of writers — including Pareto (1906, 1909), Barone (1908), Lange (1942), Allais (1943), Lerner (1947), and Samuelson (1947) — before Arrow (1951) gave the first complete and definitive treatment. Koopmans (1957) and Debreu (1959) also provide very elegant presentations of these two theorems.1 Earlier Debreu (1954) had extended them significantly to the case of a commodity space which may not be finite dimensional.

Most of this work, however, considered only static or one period economies. Yet Fisher (1907, 1930) and Hicks (1939) were able to describe intertemporal allocations of resources by means of bundles of dated commodities. By using this framework, and extending it to allow uncertainty through the apparatus of state contingent commodities, Allais (1947, 1953), Arrow (1953), Malinvaud (1953), and Debreu (1959) were able simply to reinterpret the efficiency theorems in order to treat intertemporally Pareto efficient allocations and the efficiency properties of those allocations which are achieved by complete and perfectly competitive markets for dated (and state contingent) commodities.

1.2. Time Consistency and Subgame Perfection

As this work on static and then intertemporal efficiency theorems was being completed, however, Strotz (1956) started to investigate serious dynamic inconsistency issues which arise in intertemporal consumer theory. Strotz noticed how, after starting off down any optimal path, the relevant continuation of that path might become suboptimal later on, unless individual preferences met a specific consistency condition which he called “harmonious preferences.” Later, this work was taken up most directly by Pollak (1968).

Strotz’s paper was concerned with particular single person decision problems. More generally, in any \( n \)-person game, a solution or equilibrium path is said to be “time consis-

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1 For an attempt to synthesize and extend these results, see Hammond (1998).
tent” or “dynamically consistent” if the continuation of that solution within any subgame is actually a solution to the subgame. This property will be satisfied by any Nash equilibrium, provided that the players in the game all have dynamically consistent preferences.

Somewhat later, Schelling (1960), Farquharson (1969), Friedman (1971), Hammond (1975), and especially Selten (1965, 1973, 1975) noticed how related but slightly different problems could arise in game theory, even for Nash equilibrium in games whose players all have dynamically consistent preferences. Indeed, even if one has a time consistent Nash equilibrium, deviations from the equilibrium path could still benefit some players if the reactions by the other players have to be best responses in any subgame which is reached even off the equilibrium path. “Subgame perfect” equilibria are those in which no such deviations are worthwhile. Time consistency, on the other hand, merely requires the players’ equilibrium strategies to be best responses within those subgames which are reached on the equilibrium path. This still allows Nash equilibrium strategies not to be best responses to deviations from the equilibrium path.

Considerations of this kind led Phelps and Pollak (1968) and Phelps (1975), followed eventually by Kydland and Prescott (1977), to begin considering subgame perfection issues in connection with first macroeconomic and later microeconomic policy. Just a few writings in a large and expanding field include Kotlikoff, Persson and Svensson (1987), Staiger and Tabellini (1987), Chari (1988), Maskin and Newbery (1990), Karp and Newbery (1993). In addition, Rogers (1986, 1987, 1991) and Judd (1985, 1987) have some especially clear discussions of how, although ex ante it is optimal to tax only wage income (in their models with inelastic labour supply), ex post it is optimal to impose redistributive taxation on capital as well. Finally, Bliss (1991) discusses how such concerns arise in practical policy issues connected with compensating workers who have been adversely affected by the abolition of obstacles to free trade within the European Community. As is clear even from the title of some of these papers, it has been all too common for economists to talk of the weak concept of “time consistency” when they really mean the stronger concept which game theorists call “subgame perfection.” I shall use the latter term throughout the rest of this paper.

Almost all existing work on this subgame perfection problem has typically considered one or more forms of distortionary taxation or other departures from standard first best optimal government policy. Tesfatsion (1986), however, was able to show how exactly
the same kind of subgame imperfection could also arise even in standard microeconomic models of intertemporal general equilibrium with first best optimal lump-sum redistribution. Unfortunately, however, her paper lacks a simple example, and this was partly remedied in Hammond (1993). In fact, imperfections of this kind appear seriously to undermine the relevance of the usual efficiency theorems (especially the second, which is in any case really the only ethically interesting one) if the government is unable to commit itself irrevocably to some “benevolent” policy of welfare maximizing income redistribution.

1.3. Outline

This paper will therefore analyse some important features of the widespread externalities that private decisions can create in a sequence economy. In order to do so, Section 2 presents the simplest possible general model of a sequence economy with a continuum of agents and an arbitrary finite set of commodities. One reason for working with a continuum of agents are that aggregate preference sets and demand correspondences acquire suitable convexity properties. But a more important reason is that, as discussed in Hammond (1993), no agent in a continuum economy has the power to influence equilibrium prices even in the future. Otherwise even the usual Walrasian model of perfect competition violates subgame perfection.

For simplicity, discussion is limited to an economy which lasts only two periods. There is also complete certainty, as well as perfect and complete information. Each agent is assumed to have a feasible set of net trades which is separable into two history dependent feasible sets, one for each period. And to have preferences which can be represented by the sum of two separate history dependent single period utility functions, each defined on the relevant set of feasible net trades. Moreover, a Bergson social welfare function in the form of an integral over all agents’ utilities is postulated.

Next, Section 3 briefly reviews work which shows how, under fairly standard assumptions, any intertemporal optimum can be decentralized through intertemporal competitive markets for dated commodities. Equivalently, it can be decentralized through competitive spot markets in each period together with a market for a single Arrow security in the form of a riskless bond.
Within this model, Section 4 finally brings to the fore the subgame perfection issues discussed by Tesfatsion (1986) and Hammond (1993). Of course, an intertemporal optimum prescribes an allocation for both periods together. Moreover, the standard decentralization through complete markets discussed in Section 2 involves lump-sum transfers in the first period only. Nevertheless, subgame perfection implies that lump-sum transfers cannot be prevented from occurring in the second period as well, if individuals choose to depart from the choice of history prescribed by the intertemporal optimum. Indeed, a subgame perfect optimal policy in the second period will make the current allocation to each agent depend on that agent’s personal history, as well as on the entire distribution of all agents’ personal histories. Subgame perfect optimal lump-sum transfers in the second period exhibit exactly the same dependence. Section 4 shows how to describe the relevant history in formal terms. It also shows how policy feeds back from history created in the first period to optimal redistribution in the second period. This is precisely what makes history become a widespread externality in the first period.

Thereafter, Section 5 introduces the notion of a history constraint. It also gives appropriate “history constrained sequential” versions of the usual concepts involved in stating the fundamental efficiency theorems of welfare economics. These include, of course, feasible allocation rules, Pareto efficiency, systems of lump-sum transfers, and both Walrasian and compensated equilibrium relative to such transfer systems. In fact history constrained sequential Pareto efficiency is very similar to the notion of “social Nash optimality” used by Grossman (1977) and Repullo (1988) in particular. It also relates very closely to the notion of externality constrained Pareto efficiency that was investigated in Hammond (1995).

After all this introductory material, Section 6 presents history constrained sequential versions of the efficiency theorems that apply to the model introduced in Section 2. These theorems are based on appropriate transfer systems which allow agents in the first period to be taxed or subsidized in a general way depending on what they contribute to history. The determination of an appropriate history, however, is a public good problem, which cannot easily be solved through a price mechanism.

The notion of history that is used in Sections 4–6 is extremely broad, because it changes if any non-null set of agents alter their choices of personal history. The corresponding history constraints are therefore very severe. In fact, the choice of an appropriate history in these
models is really a public good problem, which has to be made centrally in virtually any realistic economic environment. So it might be highly desirable to have more features of history determined through decentralized markets, if possible. For this reason, Section 7 considers the implications of assuming that history can be described adequately by a finite collection of real-valued sufficient statistics, each of which is the population mean of a suitably defined personal historical variable. An obvious example is when the only historical variables that matter arise from individual agents’ decisions to accumulate a finite collection of different physical capital goods, and specifically, it is only aggregate stocks per head which affect what level of social welfare is possible in the second period. It is then shown that Pigovian taxes or subsidies on each separate historical variable allow the constrained efficiency theorems of Section 6 to be derived once more, subject to the obvious weaker history constraints.

Once the optimal choice of history is known, the form of remedial policy is easy to determine, as explained in Section 8. The widespread externality comes about because each agent’s transfer is a predictable function of the choice of personal history in the first period, as well as of the distribution of personal histories. This is easily remedied by requiring all agents in the first period to pay the respective present values of any extra transfer which will be due to them individually in the second period because of their choices of personal history in the first period. Obviously, this internalizes all external effects. Putting such a scheme into effect, however, does require knowledge of what the second period transfer system will be, and how to convert the transfers into present values. Also, even in the special framework of Section 7, in general remedial policy cannot be based on linear Pigovian prices.

Section 9 contains a few brief concluding remarks.

2. A Simple Sequence Economy with a Continuum of Agents

2.1. A Continuum of Agents

Let \((A, \mathcal{B}, \alpha)\) be a measure space of economic agents, where \(A\) is a compact metric space, \(\mathcal{B}\) is the Borel \(\sigma\)-field, and \(\alpha\) is a non-atomic Borel measure with \(\alpha(A) = 1\). As explained in Hildenbrand (1974), it loses no generality to take \(A\) as the subset \([0, 1]\) of the real line \(\mathbb{R}\), and \(\alpha\) as Lebesgue measure.
2.2. A Two-Period Commodity Space

The commodity space is assumed to be the finite-dimensional Euclidean product space $X \times Y$, where $X := \mathbb{R}^{G_1}$ and $Y := \mathbb{R}^{G_2}$. Each member $(x, y) \in X \times Y$ represents a pair of net trade vectors, with $x$ as the net trade vector for period one, and $y$ as the net trade vector for period two.

2.3. Personal Histories and Individually Feasible Net Trades

It is assumed that there is a set $H$ of possible personal histories at the end of period one. For most of the paper $H$ can be an entirely abstract metric space equipped with its Borel $\sigma$-field; only in Section 8 will it need a vector space structure.

Each agent $a \in A$ is assumed to have a set $F_a$ of feasible net trades which can be decomposed into the form

$$F_a = \{ (x, y) \in X \times Y \mid \exists h \in H : (x, h) \in X_a, y \in Y_a(h) \}.$$

Thus, the set $X_a$ represents the feasible choices of combinations $(x, h)$ of a net trade vector with a personal history in period one. Also, for each $h \in H$, the set $Y_a(h)$ represents the feasible choices of net trade vector in period two, conditional on the choice of $h$ as a personal history in period one. The implicit assumption is that personal history is always richly enough described so that it completely determines what is feasible for the agent in period two.

For the usual technical reasons, it will be assumed that the feasible set correspondences defined by $a \mapsto X_a$ and by $(a, h) \mapsto Y_a(h)$ both have measurable graphs.

Finally, to avoid some unimportant technical difficulties, I will assume free disposal of traded goods at the individual level. Hence, in the first period, if $(x, h) \in X_a$ and $x' \succeq x$, then $(x', h) \in X_a$. Similarly, in the second period, if $y \in Y_a(h)$ and $y' \succeq y$, then $y' \in Y_a(h)$.
2.4. Individually Feasible Allocations and the Attainable Set

An *individually feasible allocation* \((x, h)\) in the first period economy will be a measurable mapping \(a \mapsto (x_a, h_a)\) with the property that \((x_a, h_a) \in X_a\) (\(\alpha\)-a.e. in \(A\)).

In the second period, even the definition of an individually feasible allocation will depend on the distribution of personal histories at the end of the first period. Given the space \(A \times H\) equipped with its product topology, this distribution will be described by a member \(\nu\) of the space \(\Delta(A \times H)\) of Borel measures over \(A \times H\). Then, an *individually feasible allocation* \(y\) in the second period economy will be a measurable mapping \((a, h) \mapsto y_a(h)\) with the property that \(y_a(h) \in Y_a(h)\) (\(\nu\)-a.e. in \(A \times H\)).

2.5. The Attainable Set

Free disposal at the aggregate level will not be assumed. It seems better to postulate instead that disposal is only possible if some individual agent can undertake it. It is also assumed that there is no production except by individual agents. Accordingly, an *attainable allocation* in the first (resp. second) period economy will be one that is individually feasible and also satisfies the first (resp. second) period aggregate feasibility constraint \(\int_A x_a \alpha(da) = 0\) (resp. \(\int_{A \times H} y_a(h) \nu(da \times dh) = 0\)).

2.6. Additive Social Welfare

As has been usual in models with a continuum of agents, it is assumed that the Bergson social welfare function at the start of the first period takes the integral form

\[
\Omega_1(x, h, y) \equiv \int_{A \times H} U_a(x_a, h, y_a(h)) \nu(da \times dh)
\]

for some suitable family of continuous cardinal utility functions \(U_a\) defined on agents’ feasible sets \(F_a\). In fact, for obvious technical reasons, it is also assumed that the mapping \((a, x, h, y) \mapsto U_a(x, h, y)\) is jointly measurable in all four variables.

In the sequence economy, subgame perfection forces us to consider optimal policies in each possible second period subeconomy, starting with a given history described by a distribution \(\nu \in \Delta(A \times H)\). It will be assumed that this history is sufficient to determine the second period social welfare function, and that this takes the integral form

\[
\Omega_{2,\nu}(y) \equiv \int_{A \times H} v_a(y_a(h); h) \nu(da \times dh)
\]
for some suitable family of continuous cardinal utility functions \( v_a(\cdot; h) \) defined on agents’ conditionally feasible sets \( Y_a(h) \). The notation is deliberately chosen to suggest that personal history \( h \) should be regarded as a parameter of the history-dependent utility function for the second period, rather than as an argument of that utility function.

2.7. Additive Utility

In order that both these additive separability assumptions be satisfied simultaneously, the most reasonable assumption is that in fact

\[
\Omega_1(x, h, y) \equiv \int_A [u_a(x, h) + v_a(y; h)] \alpha(da)
\]

for a suitable family of continuous single period utility functions \((u_a, v_a)_{a \in A}\) with the indicated arguments. Note that \( h_a \) is regarded as an argument of \( u_a(x, h) \), because this “myopic” utility function is intended to represent \( a \)'s preferences for combinations of net trades and personal histories, while disregarding the effect of personal history on future utility. The obvious implication of this assumption is that, for almost all agents \( a \in A \), the preferences of agent \( a \) can be represented by the additively separable utility function

\[
U_a(x, h, y) \equiv u_a(x, h) + v_a(y; h).
\]

In fact, I shall assume that this is true for every agent \( a \in A \), not just almost every agent. Moreover, for the usual technical reasons, it is now assumed that the mappings \((a, x, h) \mapsto u_a(x, h)\) and \((a, h, y) \mapsto v_a(y; h)\) are jointly measurable with respect to their respective sets of three variables.

2.8. Sequential Monotonicity

Local non-satiation, or a stronger sufficient condition for local non-satiation such as monotone preferences, plays a significant role in the usual fundamental efficiency theorems of welfare economics. The corresponding assumption here is sequential monotonicity requiring that, for each agent \( a \in A \), both single period utility functions are monotone in that period’s net trade vector — i.e., for all \( h \in H \), the function \( u_a(x, h) \) is monotone in \( x \), whereas \( v_a(y; h) \) is monotone as a function of \( y \). In symbols, this means that in the first period, whenever \((x, h) \in X_a\) with \( x' \succeq x \), then \( u_a(x', h) \geq u_a(x, h) \) with strict inequality if \( x' \gg x \). Similarly, in the second period, whenever \( y \in Y_a(h) \) and \( y' \succeq y \), then \( v_a(y'; h) \geq v_a(y; h) \) with strict inequality if \( y' \gg y \).
2.9. Example

A simple example of such a two period economy is the following, whose properties will be discussed further in several of the later sections. Each agent $a \in A = [0, 1]$ has fixed endowments $e_a, f_a \in \mathbb{R}^{G+}$ of all commodities in the set $G$ within each of the two periods. Obviously, it is assumed that the two functions $e, f : A \to \mathbb{R}^{G+}$ are both measurable. Then the mean endowment vectors $\bar{e} := \int_A e_a \alpha(da)$ and $\bar{f} := \int_A f_a \alpha(da)$ for both periods are well defined. For reasons to be explained in Section 3.1, it will be assumed that $\bar{e} \geq -\bar{f}$.

A personal history $h_a \in \mathbb{R}^{G+}$ will be a non-negative commodity vector that is stored for private use in the second period, so that the agent begins period two with the new endowment vector $f_a + h_a \in \mathbb{R}^{G+}$. If agent $a$ has the two-period net trade vector $(x_a, y_a)$ and the personal history $h_a$, the resulting two-period consumption stream is

$$(c_a, d_a) := (x_a + e_a - h_a, y_a + f_a + h_a).$$

It is then assumed that individual feasibility requires $(c_a, d_a) \geq (0, 0)$ to be satisfied. This implies that the individual feasible sets take the form:

$$X_a = \{ (x_a, h_a) \in \mathbb{R}^G \times \mathbb{R}^G \mid x_a - h_a \geq -e_a \}; \quad Y_a(h_a) = \{ y_a \in \mathbb{R}^G \mid y_a \geq -f_a - h_a \}.$$ 

Finally, it is assumed that each agent in each period has the same utility function $\bar{u}(c)$ of consumption $c$, where $\bar{u}$ is twice continuously differentiable with gradient vector $\bar{u}'(c) \gg 0$ and with its Hessian matrix $\bar{u}''(c)$ negative definite for all $c \geq 0$.

3. Intertemporal Optimality and Market Decentralization

3.1. Intertemporal Optimality

The usual concept of an intertemporal welfare optimum is evidently an attainable allocation $(\bar{x}, \bar{h}, \bar{y})$ of net trade vectors in each period, and of personal histories at the end of the first period, which together maximize the welfare integral $\int_{A \times H} U_a(x_a, h_a, y_a)\alpha(da)$ subject to individual and aggregate feasibility constraints. Any such welfare optimum is evidently intertemporally Pareto efficient in the sense that any alternative attainable allocation $(x, h, y)$ for which

$$U_a(x_a, h_a, y_a) \geq U_a(\bar{x}_a, \bar{h}_a, \bar{y}_a) \text{ (}\alpha\text{-a.e. in } A)$$

implies

$$U_a(x_a, h_a, y_a) \geq U_a(\bar{x}_a, \bar{h}_a, \bar{y}_a) \text{ (}\alpha\text{-a.e. in } A).$$
must in fact have

\[ U_a(x_a, h_a, y_a) = U_a(\hat{x}_a, \hat{h}_a, \hat{y}_a) \quad (\omega\text{-a.e. in } A). \]

In fact, however, this paper will use a stronger concept of optimality, which pays attention even to null sets of agents. An \( f \)-optimum\(^2\) is an intertemporal welfare optimum with the additional property that, for all finite sets of agents \( C \subset A \), the allocation \( (\hat{x}_a, \hat{h}_a, \hat{y}_a) \) \((\epsilon \in C)\) to the members of \( C \) maximizes the utility sum \( \sum_{a \in C} U_a(x_a, h_a, y_a) \) subject to the individual feasibility constraints \( (x_a, h_a, y_a) \in F_\epsilon \) \((\epsilon \in C)\), as well as the aggregate feasibility constraints \( \sum_{\epsilon \in C} x_a = \sum_{\epsilon \in C} \hat{x}_a \) and \( \sum_{\epsilon \in C} y_a = \sum_{\epsilon \in C} \hat{y}_a \). That is, the resources that are made available to any finite set of agents \( C \) must also be distributed optimally among the members of \( C \).

For the special example described in Section 2.9, the intertemporal \( f \)-welfare optimum not only exists, but is also unique and easy to describe. It is simply the consumption allocation satisfying \( c_a = d_a = \frac{1}{2}(\bar{e} + \bar{f}) \) for all \( \epsilon \in A \), and not just for almost all \( \epsilon \in A \). The reason for assuming that \( \bar{e} \geq \bar{f} \) was precisely to allow the \( f \)-optimum to be characterized this simply. Because no agent in the first period consumes more of any commodity than the existing mean endowment of that commodity, there is no need to consider any possible corner solution.

This example also illustrates how, when marginal utilities of income are well defined, an \( f \)-optimum involves choosing income levels which equate those marginal utilities for all individuals, not just almost all individuals. Usually there is no problem in doing this. Indeed, in the general case, we are looking for an allocation in the non-empty \( f \)-core of a particular transferable utility game.\(^3\)

\(^2\) A term evoking the “\( f \)-core” of Kaneko and Wooders (1986, 1989) and of Hammond, Kaneko and Wooders (1989) has deliberately been chosen, since the \( f \)-core is also sensitive to the fates of finite sets of agents in a continuum economy.

\(^3\) It seems; this should be checked with the experts.
3.2. Compensated Equilibrium

In order to show that an \( f \)-optimum can be decentralized in the usual way as a Walrasian equilibrium relative to a system of lump-sum transfers, the following constructions are useful. First, define the new feasible set

\[
F_a^* := \text{proj}_{X \times Y} F_a := \{ (x, y) \in X \times Y \mid \exists h \in H : (x, h, y) \in F_a \}
\]

of possible net trade vectors, and the new utility function

\[
u_a^*(x, y) := \max_{h \in H} \{ u_a(x, h) + v_a(y, h) \mid (x, h, y) \in F_a \}
\]
on this domain. It is assumed, of course, that the relevant maximum is always achieved; a sufficient condition for this is that the set

\[
H(x, y) := \text{proj}_H F_a := \{ h \in H \mid (x, h, y) \in F_a \}
\]
of histories that are compatible with \((x, y)\) is compact. By Berge’s maximum theorem, the function \(u_a^*(x, y)\) must be continuous. Denote the corresponding weak preference relation on \(F_a^*\) by \(R_a\), and the corresponding strict preference relation by \(P_a\). Let

\[
R_a(x, y) := \{ (x', y') \in F_a^* \mid (x', y') R_a (x, y) \}
\]
\[
P_a(x, y) := \{ (x', y') \in F_a^* \mid (x', y') P_a (x, y) \}
\]
denote the corresponding upper contour and strict preference sets, respectively.

Now, arguing as in Hildenbrand (1974), given the optimal allocation \((\hat{x}, \hat{y})\) of net trade vectors, which must also be Pareto efficient, the set \(P := \int_A P_a(\hat{x}_a, \hat{y}_a) \alpha(da)\) cannot contain the origin of the product space \(X \times Y\). Moreover, because the measure \(\alpha\) is non-atomic, the set \(P\) is convex. By local non-satiation, however, the origin must be a boundary point of \(P\). Hence there exists a hyperplane \(px + qy = 0\) through the origin of \(X \times Y\) which has the set \(P\) all on one side. That is, there exists a price vector \((p, q) \neq (0, 0)\) such that \((x, y) \in P \implies px + qy \geq 0\). Using local non-satiation once again, it follows that for almost all \(a \in A\) one has \((x_a, y_a) \in P_a(\hat{x}_a, \hat{y}_a)\) implies \(px_a + qy_a \geq p\hat{x}_a + q\hat{y}_a\). But then, because each agent’s utility function is continuous, it also follows that for almost all \(a \in A\) one has

\[(x_a, y_a) \in R_a(\hat{x}_a, \hat{y}_a) \implies px_a + qy_a \geq p\hat{x}_a + q\hat{y}_a.\]
In this sense, the optimal allocation \((\hat{x}, \hat{y})\) of net trade vectors is a *compensated equilibrium* at prices \((p, q)\) relative to the wealth distribution given by \(w_a(p, q) := p \hat{x}_a + q \hat{y}_a\) for all \(a \in A\). Moreover, it is a compensated equilibrium in which individuals are free to choose their own personal histories.

### 3.3. Uncompensated Equilibrium

It remains to show that \((\hat{x}, \hat{y})\) is a *uncompensated* or Walrasian equilibrium at prices \((p, q)\) relative to the wealth distribution \(w_a(p, q)\), in the sense that, for almost all \(a \in A\), one has

\[(x_a, y_a) \in P_a(\hat{x}_a, \hat{y}_a) \implies px_a + qy_a > p \hat{x}_a + q \hat{y}_a.\]

Showing this requires three additional assumptions.

First, assume for simplicity that each agent’s feasible set \(F_a^*\) is convex. Now, \(F_a^*\) is the union \(\cup_{h \in H} F_a(h)\) of the sections \(F_a(h) := \{ (x, y) \in X \times Y \mid (x, h, y) \in F_a \}\). This is usually not convex even if each section \(F_a(h)\) is convex. Nevertheless, it will be provided that, for example, each section satisfies \(F_a(h) \subseteq F_a(\bar{h})\) for some \(\bar{h} \in H\) such that \(F_a(\bar{h})\) is convex. In any case, a significant weakening of the assumption that each \(F_a^*\) is convex is presented in Hammond (1993a), which also discusses the relationship to the assumption due to Broome (1972) and Mas-Colell (1977) that divisible goods are “overridingly desirable” in the preference ordering — i.e., that any reduction in the quantities of indivisible goods consumed can always be overcompensated by an increase in the quantities of divisible goods.

In the presence of this first convexity assumption, or the generalization considered in Hammond (1993a), the usual sufficient condition that each agent \(a \in A\) has a “cheaper point” \((x_a, y_a)\) within the feasible set \(F_a^*\) for which \(px_a + qy_a < p \hat{x}_a + q \hat{y}_a\) is enough to guarantee that the compensated equilibrium \((\hat{x}_a, \hat{y}_a)\) is an uncompensated equilibrium. The second and third assumptions about to be presented ensure that each agent does have a cheaper point in this sense.

The second assumption is that the origin belongs to the interior of the aggregate feasible set \(\int_{A} F_a^* \alpha(da)\). This ensures that a non-null set of agents \(a \in A\) must have cheaper points and so are in uncompensated equilibrium, with \((x_a, y_a) \in P_a(\hat{x}_a, \hat{y}_a)\) implying that \(px_a + qy_a > p \hat{x}_a + q \hat{y}_a\).
The third assumption is that the optimal allocation \((\hat{x}, \hat{y})\) should be “non-oligarchic” in the sense that, whenever \(C\) is a measurable subset of \(A\) for which both \(\alpha(C)\) and \(\alpha(A \setminus C)\) are positive, there should be a measurable subset \(A^*\) of \(C\) for which \(\alpha(A^*)\) is positive and also

\[
0 \in \int_{A^*} P_a(\hat{x}_a, \hat{y}_a) \alpha(da) + \int_{C \setminus A^*} R_a(\hat{x}_a, \hat{y}_a) \alpha(da) + \int_{A \setminus C} F^*_a \alpha(da).
\]

That is, there can be no set \(C\) which is “oligarchic” in the sense of being so well off that it is unable to generate a Pareto improvement for its own members from the resources of the complementary set \(A \setminus C\).

Together these three assumptions suffice to ensure that the compensated equilibrium \((\hat{x}, \hat{y}, p, q)\) relative to the wealth distribution \(w_a(p, q)\) is actually a Walrasian equilibrium relative to this distribution.

Note in particular that, because of sequential monotonicity, this Walrasian equilibrium property implies both \(p > 0\) and \(q > 0\).

3.4. Walrasian \(f\)-Equilibrium

So far, it has been established that an intertemporal welfare optimum \((\hat{x}, \hat{y})\) is a Walrasian equilibrium at prices \((p, q)\) relative to the wealth distribution \(w_a(p, q)\), in the sense that, for almost all \(a \in A\), one has

\[
(x_a, y_a) \in P_a(\hat{x}_a, \hat{y}_a) \Longrightarrow px_a + qy_a > p\hat{x}_a + q\hat{y}_a = w_a(p, q).
\]

The question that naturally arises is whether an \(f\)-optimum \((\hat{x}, \hat{y})\) might also be a Walrasian \(f\)-equilibrium in the sense that, for all \(a \in A\), one has

\[
(x_a, y_a) \in P_a(\hat{x}_a, \hat{y}_a) \Longrightarrow px_a + qy_a > p\hat{x}_a + q\hat{y}_a = w_a(p, q).
\]

In fact, it is rather evident that this need not be true in general economies without smooth preferences, especially if there may be individual non-convexities. Nevertheless, if there is some (null) set of agents \(C\) who are not in Walrasian equilibrium at the optimum \((\hat{x}, \hat{y})\), it is obviously possible to find a new allocation \((x^*, y^*)\) which coincides with \((\hat{x}, \hat{y})\) for all \(a \in A \setminus C\), and which is a Walrasian equilibrium for all agents because, even for \(a \in C\), it is true that \((x^*_a, y^*_a)\) together maximize \(u^*_a(x_a, y_a)\) subject to \((x_a, y_a) \in F^*_a\) and
\[ px_a + q y_a \leq p \hat{x}_a + q \hat{y}_a = w_a(p, q). \] Since the allocation \((\hat{x}, \hat{y})\) has been changed for only a null set of agents, \((x^*_a, y^*_a)\) is obviously feasible, so it is a Walrasian \(f\)-equilibrium at prices \((p, q)\) given the transfer system \(w_a(p, q)\). Moreover, the welfare integral cannot have been changed. Therefore, \((x^*_a, y^*_a)\) is also a welfare optimum. Furthermore, since it is a Walrasian \(f\)-equilibrium, a standard proof of the first efficiency theorem of welfare economics shows that it must be an \(f\)-optimum.

Hence, under the assumptions set out in Section 3.3, there is no loss of generality in considering \(f\)-optimal allocations that are Walrasian \(f\)-equilibria relative to appropriate rules for lump-sum redistribution.

### 3.5. Sequential Decentralization with an Arrow Security Market

The fundamental idea of Arrow (1953, 1964) is easy to apply to this intertemporal welfare optimum. Since there is no uncertainty, and only two periods, complete markets for Arrow securities in fact require only a market for a single riskless Arrow security that pays one unit of the numéraire for sure in period two. With such a security, the Walrasian equilibrium \((\hat{x}, \hat{y}, p, q)\) relative to the wealth distribution \(w_a(p, q)\) corresponds to an equilibrium \((\hat{x}, \hat{y}, \hat{b}, p, q, \rho)\) in spot markets each period, as well as in the market for the single Arrow security whose price in the first period is \(\rho\), which must be positive because \(q \neq 0\). Also, \(b_a\) denotes agent \(a\)’s net purchase of this security. However, each agent’s single budget constraint

\[ px_a + q y_a \leq w_a(p, q) \]

must be replaced by an equivalent pair of budget constraints, one for each separate period, of the particular form

\[ px_a + \rho b_a \leq w_a(p, q); \quad (q/\rho) y_a \leq b_a. \]

Note that the equilibrium value of \(\rho\) is actually completely indeterminate; if \(\rho\) is multiplied by any positive scalar \(\lambda\), there is still essentially the same Walrasian equilibrium after each agent’s bondholding \(b_a\) has been divided by \(\lambda\). For this reason, it is harmless to normalize and take \(\rho = 1\).
3.6. Example

In the example that was discussed in Sections 2.9 and 3.1, the intertemporal $f$-welfare optimum involves the consumption allocation with $c_a = d_a = \frac{1}{2}(\bar{e} + \bar{f})$ for all $a \in A$. This can be decentralized by the single budget constraint $pc_a + pd_a \leq p(\bar{e} + \bar{f})$, where $p$ is any normalized gradient vector of the utility function $\bar{u}(c)$, evaluated at $\frac{1}{2}(\bar{e} + \bar{f})$. Note that an agent with endowment stream $(e_a, f_a)$ receives the net lump-sum transfer $p(\bar{e} - e_a + \bar{f} - f_a)$, so that all real wealth is equally divided. Alternatively, if a market for Arrow securities is introduced, the $f$-welfare optimum can be decentralized by the pair of budget constraints

$$pc_a + b_a \leq p(\bar{e} + \bar{f}) \quad \text{and} \quad pd_a \leq b_a.$$

4. Subgame Imperfection

4.1. Second Period Optimality

Suppose that the first period of the economy is already over and that the different agents’ allocations of $(x_a, h_a)$ (all $a \in A$) have given rise to the joint distribution $\nu \in \Delta(A \times H)$ of agents’ labels and personal histories. Then there is a well-defined second period objective given by the welfare integral $\int_A v_a(y_a; h_a)\alpha(da)$. This should be maximized subject to the remaining individual feasibility constraints $y_a \in Y_a(h_a)$ (all $a \in A$) and the aggregate feasibility constraint $\int_A y_a\alpha(da) = 0$.

In the particular case when the first period allocations are all equal to those for the intertemporal welfare optimum $(\hat{x}, \hat{h}, \hat{y})$, namely $(\hat{x}_a, \hat{h}_a)$ (all $a \in A$), the resulting history $\hat{\nu}$ is optimal, and a welfare maximizing continuation is indeed to proceed with $y$, as originally planned. There is no subgame imperfection here, of course. Moreover, exactly the same policy appears at first to remain optimal even if a null set of agents deviate from their optimal allocations, so that $(x_a, h_a) = (\hat{x}_a, \hat{h}_a)$ only for almost all $a \in A$, and not for all $a \in A$. After all, such a deviation makes no difference to the distribution $\hat{\nu}$, nor to the maximized value $\int_A[u_a(\hat{x}_a; \hat{h}_a) + v_a(\hat{y}_a; \hat{h}_a)]\alpha(da)$ of the welfare integral that results from continuing with the second-period plan $y$. Indeed, since the overall allocation is effectively the same even if a null set of agents have different allocations, one could well argue that such deviations should be ignored. If they are, the intertemporal welfare optimum $(\hat{x}, \hat{h}, \hat{y})$ is subgame perfect, and there is nothing more to write about.
Neglecting null sets of agents in this way, however, poses fundamental difficulties. For one thing, there can be no corresponding limit result for large finite economies; in any finite economy, no matter how large, if any one agent were to have a different history at the start of period two, the second period welfare optimal allocation to that agent would be affected. Furthermore, if individual feasibility constraints are to be taken seriously, what are we to do about the (null) set of agents for whom $\hat{y}_a \notin Y_a(h_a)$ after a deviation from $\hat{h}_a$ to $h_a$? Such agents, for instance, played an important role in creating a need for additional incentive constraints in the two-period economy considered by Hammond (1992). And the blocking powers of finite sets of agents in a continuum economy were used to prove an $f$-core equivalence theorem in Hammond, Kaneko and Wooders (1989).

Accordingly, as with the $f$-optimum considered in Section 3.1, I am going to require that the second-period allocation be an $f$-optimum in each subeconomy. That is, not only should it maximize $\int_A v_a(y_a; h_a)\alpha(da)$ subject to $y_a \in Y_a(h_a)$ (almost all $a \in A$) and $\int_A y_a\alpha(da) = 0$. In addition, for any finite set of agents $C \subset A$, it should also maximize the sum $\sum_{a \in C} v_a(y_a; h_a)$ subject to $y_a \in Y_a(h_a)$ (all $a \in C$) for the given level of total net demands $\sum_{a \in C} y_a$ made available to the members of $C$. It is important to notice that, as with the intertemporal $f$-optimum considered in Section 3.1, such an optimum does generally exist.

With this extra requirement on the second-period optimum, subgame perfection becomes a serious issue once again, and in fact the intertemporal $f$-optimum is generally subgame imperfect. To show this, let us first notice that the second-period $f$-optimal allocation $y_a$ to each agent $a$ is no longer a function only of the distribution $\nu \in \Delta(A \times H)$, but also a function of the agent’s own history $h_a$. Let $\eta_a(\nu, h_a)$ denote this function. It must solve the problem of maximizing $\int_A v_a(y_a; h_a)\alpha(da)$ subject to $y_a \in Y_a(h_a)$ (for all $a \in A$) and $\int_A y_a\alpha(da) = 0$. And, for any finite set of agents $C \subset A$, it should also maximize the sum $\sum_{a \in C} v_a(y_a; h_a)$ subject to $y_a \in Y_a(h_a)$ (all $a \in C$) and $\sum_{a \in C} y_a = \sum_{a \in C} \eta_a(\nu, h_a)$.

For the example discussed in Sections 2.9, 3.1 and 3.6, history is represented by the mean consumption vector $\bar{c}$, which determines the mean storage vector $\bar{h} = \bar{e} - \bar{c}$. So the mean endowment vector $\bar{f} + \bar{h}$ is available for distribution in the second period. Given this history, the obvious $f$-optimum in the second-period subeconomy satisfies $d_a = \bar{h} + f$ for all $a \in A$. 

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4.2. The First-Period Economy

Consider now the first-period subeconomy. Each agent $a$‘s anticipated utility, given rational expectations about what will happen in period two, is

$$u_a(x_a, h_a) + v_a(\eta_a(\nu, h_a); h_a).$$

The dependence of anticipated utility on the distribution $\nu$ of all personal histories is evidence of a widespread externality. Each agent is directly affected by the distribution of personal histories among all other agents.

Indeed, for the example discussed in Sections 2.9, 3.1, 3.6 and 4.1, each agent $a \in A$ has anticipated utility $\bar{u}(c_a) + \bar{u}(\bar{h} + \bar{f})$. This depends directly on $\bar{h}$, but second-period utility $\bar{u}(\bar{h} + \bar{f})$ is independent of $h_a = e_a - c_a$.

There is no problem here if a central planner can simply choose $(x, h)$ in the first period in order to maximize the social welfare integral $\int_A[u_a(x_a, h_a) + v_a(\eta_a(\nu, h_a); h_a)]\alpha(da)$ subject to $(x_a, h_a) \in X_a$ (all $a \in A$) and $\int_A x_a\alpha(da) = 0$. The usual principle of optimality ensures that the first-period part $(\hat{x}, \hat{h})$ of an intertemporal welfare optimum will be chosen, which will then be extended to the entire path of such an optimum. The trouble comes from trying to decentralize this optimum through competitive markets in the first-period economy.

Indeed, suppose that each agent is faced with the first of the two budget constraints in the previous decentralization through spot markets with a market for an Arrow security. That is, each agent $a \in A$ faces the constraint $p x_a + \rho b_a \leq w_a(p, q)$, where $\rho > 0$. Then, by choosing $b_a = [w_a(p, q) - p x_a]/\rho$, agent $a$ can fund any desired $x_a$; since local non-satiation in $x_a$ evidently implies global non-satiation, there cannot be any utility maximum. A fortiori, there can be no equilibrium. The same argument works for any other price vector, provided that there is a market for Arrow securities.

Closing down the market for Arrow securities by imposing the constraint $b_a \geq 0$ on each agent will generally overcome the non-existence problem. Yet it still does not decentralize the intertemporal optimum, in general, or produce any very satisfactory first period allocation. For suppose an attempt is made to decentralize the first-period part $(\hat{x}, \hat{h})$ of an intertemporal optimum by confronting each agent $a \in A$ with the budget constraint $p x_a \leq p \hat{x}_a$. Then $a$ will maximize $u_a(x_a, h_a) + v_a(\eta_a(\nu, h_a); h_a)$ subject to the individual
feasibility constraint \((x_a, h_a) \in X_a\) and this budget constraint. Whereas agent \(a\) should be maximizing \(u_a(x_a, h_a) + v_a(\hat{y}_a; h_a)\) instead. The dependence of the future net trade vector \(y_a\) on \(h_a\) through the policy feedback function \(\eta_a(\nu, h_a)\) is liable to distort the agent’s choice of personal history \(h_a\).

Indeed, it is rather easy to come up with examples where the optimal feedback function \(\eta_a(\nu, h_a)\) makes the utility \(v_a(\eta_a(\nu, h_a); h_a)\) of each agent \(a \in A\) entirely independent of \(h_a\). Then each agent will make an entirely myopic choice of \(h_a\), so that \((x_a, h_a)\) together maximize \(u_a(x_a, h_a)\) subject to \(px_a \leq \hat{p}x_a\) and \((x_a, h_a) \in X_a\). In this case nobody will give any thought to the future in choosing their personal history, because all incentives to do so have been completely destroyed. Only first-period utility remains under the agent’s personal control.

In the specific example discussed in Sections 2.9, 3.1, 3.6 and 4.1, this implies that each agent \(a \in A\) will choose \(c_a = e_a\), unless restrictions are imposed to prevent this. Instead of the intertemporal optimum with an equal distribution of the total endowment over both periods, the result will be autarky in the first period, followed by equal distribution in the second.

5. History Constrained Sequential Allocations

Let \(\hat{h} = (\hat{h}_a)_{a \in A}\) denote any fixed allocation of personal histories, with \(\hat{\nu} \in \Delta(A \times H)\) as the induced “history”. Given \(\hat{h}\) and \(\hat{\nu}\), a history-constrained sequential allocation rule is a pair \((\mathbf{x}, \mathbf{y}(\cdot))\) of measurable functions \(\mathbf{x} : A \to X\) and \(\mathbf{y} : A \times H \to Y\) satisfying the restrictions that:

(i) \((x_a, \hat{h}_a) \in X_a\) \((\alpha\text{-a.e. in } A)\);
(ii) \(\int_A x_a \alpha(da) = 0\);
(iii) \(y_a(h) \in Y_a(h)\) \((\nu\text{-a.e. in } A \times H)\);
(iv) \(\int_{A \times H} y_a(h) \nu(da \times dh) = 0\).
A history-constrained sequential allocation rule \((\hat{x}, \hat{y}(\cdot))\) is said to be \textit{history-constrained sequentially Pareto efficient} (or HCSP-efficient) if, given any alternative (feasible) history-constrained sequential allocation rule \((x, y(\cdot))\), one has:

(i) \(u_a(x_a, \hat{h}_a) \geq u_a(\hat{x}_a, \hat{h}_a)\) for all \(a \in A\) only if \(u_a(x_a, \hat{h}_a) = u_a(\hat{x}_a, \hat{h}_a)\) (\(\alpha\)-a.e. in \(A\));

(ii) \(v_a(y_a(h); h) \geq v_a(\hat{y}_a(h); h)\) for all \(a \in A\) and \(h \in H\) only if \(v_a(y_a(h); h) = v_a(\hat{y}_a(h); h)\) (\(\nu\)-a.e. in \(A \times H\)).

A \textit{sequential transfer system} is a pair of measurable functions \((m_a(p), n_a(q, h))\) which satisfy the restrictions that, for every pair of price vectors \(p \neq 0\) and \(q \neq 0\), one has

\[
\int_A m_a(p) \alpha(da) = 0 \quad \text{and} \quad \int_{A \times H} n_a(q, h) \nu(da \times dh) = 0.
\]

Given \(\hat{h}\) and \(\hat{\nu}\), a \textit{history-constrained sequential Walrasian equilibrium relative to a sequential transfer system} \((m_a(p), n_a(q, h))\) is a history-constrained sequential allocation rule \((\hat{x}, \hat{y}(\cdot))\) and a pair of price vectors \(p \neq 0\) and \(q \neq 0\) such that:

(i) for all \(a \in A\) one has \(p \hat{x}_a \leq m_a(p)\) and \(p x \geq m_a(p)\) whenever \((x, \hat{h}_a) \in X_a\) with \(u_a(x, \hat{h}_a) > u_a(\hat{x}_a, \hat{h}_a)\);

(ii) for all \(a \in A\) and \(h \in H\) one has \(q \hat{y}_a(h) \leq n_a(q, h)\) and \(q y \geq n_a(q, h)\) whenever \(y \in Y_a(h)\) with \(v_a(y; h) > v_a(\hat{y}_a(h); h)\).

By contrast, a \textit{history-constrained sequential compensated equilibrium relative to a transfer system} \((m_a(p), n_a(q, h))\) is defined with weak inequalities in (i) and (ii) above. That is, it consists of a history-constrained sequential allocation rule \((\hat{x}, \hat{y}(\cdot))\) and a pair of price vectors \(p \neq 0\) and \(q \neq 0\) such that:

(i’) for all \(a \in A\) one has \(p \hat{x}_a \leq m_a(p)\) and \(p x \geq m_a(p)\) whenever \((x, \hat{h}_a) \in X_a\) with \(u_a(x, \hat{h}_a) \geq u_a(\hat{x}_a, \hat{h}_a)\);

(ii’) for all \(a \in A\) and \(h \in H\) one has \(q \hat{y}_a(h) \leq n_a(q, h)\) and \(q y \geq n_a(q, h)\) whenever \(y \in Y_a(h)\) with \(v_a(y; h) \geq v_a(\hat{y}_a(h); h)\).
6. History Constrained Sequential Efficiency Theorems

Here are history constrained sequential versions of the efficiency theorems that apply to the model introduced in Section 2.

**First Sequential Efficiency Theorem.** Under the assumption of sequential local non-satiation, any history-constrained sequential Walrasian equilibrium relative to a transfer system must be history-constrained sequentially Pareto efficient.

**Proof:** Given \( \hat{h} \) and \( \hat{\nu} \), let \((\hat{x}, \hat{y}(\cdot), p, q)\) be a history-constrained sequential Walrasian equilibrium relative to the transfer system \((m_a(p), n_a(q, h))\). Because of sequential local non-satiation, this must be a history-constrained sequential compensated equilibrium as well, and budget exhaustion must also be satisfied. Let \((x, y(\cdot))\) denote any other (feasible) history-constrained sequential allocation rule.

Suppose that in the first period one has \( u_a(x_a, \hat{h}_a) \geq u_a(\hat{x}_a, \hat{h}_a) \) for all \( a \in A \). Then the compensated equilibrium and budget exhaustion properties ensure that \( p \hat{x}_a \geq m_a(p) = p \hat{x}_a \) for all \( a \in A \). Integrating over \( A \) and using the aggregate feasibility conditions \( \int_A x_a \alpha(da) = \int_A \hat{x}_a \alpha(da) = 0 \) implies that

\[
0 = \int_A p x_a \alpha(da) \geq \int_A m_a(p) \alpha(da) = \int_A p \hat{x}_a \alpha(da) = 0.
\]

This can only be true if \( \alpha \)-a.e. in \( A \) one has \( p x_a = m_a(p) \). Because a Walrasian equilibrium implies preference maximization subject to the budget constraints, this implies that actually \( \alpha \)-a.e. in \( A \) one has \( u_a(x_a, h_a) \leq u_a(\hat{x}_a, \hat{h}_a) \), and so \( u_a(x_a, h_a) = u_a(\hat{x}_a, \hat{h}_a) \).

Similarly, suppose that in the second period one has \( v_a(y_a(h); h) \geq v_a(\hat{y}_a(h); h) \) for all \( a \in A \) and \( h \in H \). Arguing as for the first period, \( q y_a(h) \geq n_a(p, h) = q \hat{y}_a(h) \) for all \( a \in A \) and \( h \in H \). Integrating over \( A \times H \) and using the aggregate feasibility conditions \( \int_{A \times H} y_a(h) \nu(da \times dh) = \int_{A \times H} \hat{y}_a(h) \nu(da \times dh) = 0 \) implies that

\[
0 = \int_{A \times H} q y_a(h) \nu(da \times dh) \geq \int_{A \times H} n_a(q, h) \nu(da \times dh) = \int_{A \times H} q \hat{y}_a(h) \nu(da \times dh) = 0.
\]

This can only be true if \( \nu \)-a.e. in \( A \times H \) one has \( q y_a(h) = n_a(p, h) \). Because a Walrasian equilibrium implies preference maximization subject to the budget constraints, this implies that actually \( \nu \)-a.e. in \( A \times H \) one has \( v_a(y_a(h); h) \leq v_a(\hat{y}_a(h); h) \) and so \( v_a(y_a(h); h) = v_a(\hat{y}_a(h); h) \). \( \blacksquare \)
SECOND SEQUENTIAL EFFICIENCY THEOREM. Under the assumption of sequential local non-satiation, given any history-constrained sequentially Pareto efficient allocation rule, there must exist prices with which the allocation forms a history-constrained sequential compensated equilibrium relative to a suitable sequential transfer system.

PROOF: Given \( \hat{h} \) and \( \hat{\nu} \), let \((\hat{x}, \hat{y}(\cdot))\) be a history-constrained sequentially Pareto efficient allocation rule.

In the first period, and for all \( a \in A \), define the set
\[
P_a := \{ x \in X \mid (x, \hat{h}_a) \in X_a \text{ and } u_a(x, \hat{h}_a) > u_a(\hat{x}_a, \hat{h}_a) \}
\]
of first-period net trade vectors \( x \) that, when accompanied by the personal history \( \hat{h}_a \), are strictly preferred to the pair \((\hat{x}_a, \hat{h}_a)\), given the net trade vector that the agent expects to receive in the second period. Define \( \hat{P} := \int_A P_a \alpha(da) \). Then \( 0 \notin \hat{P} \) because of history-constrained sequential Pareto efficiency. Moreover, \( \hat{P} \) is a convex set because \( \alpha \) is a non-atomic measure. So there exists a first-period price vector \( \hat{p} \neq 0 \) and a corresponding hyperplane \( \hat{p}x = 0 \) through the origin such that \( x \in \hat{P} \) implies \( \hat{p}x \geq 0 \). Because of sequential local non-satiation, \( \hat{x}_a \) is a boundary point of \( P_a \) for all \( a \in A \). Because of the fact that \( \int_A \hat{x}_a \alpha(da) = 0 \), it then follows that \( x_a \in \hat{P}_a \) implies \( \hat{p}x_a \geq \hat{p}\hat{x}_a \) (\( \alpha \)-a.e. in \( A \)).

In the second period, and for all \( a \in A \) and all \( h \in H \), define the set
\[
Q_a(h) := \{ y \in X \mid v_a(y; h) > v_a(\hat{y}_a(h); h) \}
\]
of second-period net trade vectors that are strictly preferred to \( \hat{y}_a(h) \), given the personal history \( h \). Define \( \hat{Q} := \int_{A \times H} \hat{Q}_a(h) \nu(da \times dh) \). Arguing as I did above for the first period, in the second period too one has \( 0 \notin \hat{Q} \) because of history-constrained sequential Pareto efficiency, and \( \hat{Q} \) convex because \( \nu \) is a non-atomic measure. So there exists a second-period price vector \( \hat{q} \neq 0 \) such that \( y \in \hat{Q} \) implies \( \hat{q}y \geq 0 \). Because of sequential local non-satiation and the fact that \( \int_{A \times H} \hat{y}_a(h) \nu(da \times dh) = 0 \), it then follows that \( y_a \in \hat{Q}_a(h) \) implies \( \hat{q}y_a \geq \hat{q}\hat{y}_a(h) \) (\( \nu \)-a.e. in \( A \times H \)).

Now define the sequential transfer system so that, for the specific price vectors \( p \neq 0 \) and \( q \neq 0 \) derived above, one has \( m_a(p) = p \hat{x}_a \) (all \( a \in A \)) and \( n_a(q, h) = q \hat{y}_a(h) \) (all \( (a, h) \in A \times H \)). Then the results of the previous two paragraphs imply that \((\hat{x}, \hat{y}(\cdot), p, q)\) is a history-constrained sequential compensated equilibrium relative to the transfer system \((m_a(p), n_a(q, h))\).
7. Historical Aggregates

In this section, suppose that the set $H$ of personal histories can be taken as a subset of some finite-dimensional Euclidean space, and that the population mean $\bar{h} := \int_A h \alpha(da)$ is a sufficient statistic for the distribution $\nu \in \Delta(A \times H)$ in determining the second-period $f$-optimum. Assume, moreover, that the feasible set $F_a$ of each agent $a \in A$ is a convex subset of $X \times H \times Y$.

Under these assumptions, given the mean $\hat{h}$, let $(\hat{x}, \hat{y}(\cdot))$ be a history-constrained sequentially Pareto efficient allocation rule. In the first period, and for all $a \in A$, define the set

$$P_a := \{ (x, h) \in X \times H \mid u_a(x, h) + v_a(\hat{y}_a(h); h) > u_a(\hat{x}_a, \hat{h}_a) + v_a(\hat{y}_a; \hat{h}_a) \}$$

of first-period net trade vectors $x$ and personal histories $h$ that are jointly preferred to $(\hat{x}_a, \hat{h}_a)$, given the net trade vector that the agent expects to receive in the second period. Define $P := \int_A P_a \alpha(da) - \{(0, \hat{h})\}$. Note that $0 \notin P$ because $(\hat{x}, \hat{y}(\cdot))$ is history-constrained sequentially Pareto efficient. Moreover, $P$ is a convex set because $\alpha$ is a non-atomic measure. So there exists a first-period price vector $(p, t) \neq (0, 0)$ and a corresponding hyperplane $px + tz = 0$ through the origin such that $(x, z) \in P$ implies $px + tz \geq 0$. Because of sequential local non-satiation, $(\hat{x}_a, \hat{h}_a)$ is a boundary point of $P_a$ for all $a \in A$. Because $\int_A (\hat{x}_a, \hat{h}_a) \alpha(da) = (0, \hat{h})$, it follows that $(x_a, h_a) \in P_a$ implies $px_a + th_a \geq px_a + t\hat{h}_a$ ($\alpha$-a.e. in $A$).

This proves that the decentralization $px_a \leq m_a(p) := px_a$ derived in Section 6 can now be replaced with $px_a + th_a \leq m_a(p, t) := px_a + t\hat{h}_a$ for a suitable vector $t$ of net Pigou taxes. Thus, there is a decentralization by means of a Pigou–Walrasian equilibrium (or compensated equilibrium) in the sense of Hammond (1995).
8. Remedial Policy

Let \((\hat{x}, \hat{h}, \hat{y})\) be any intertemporal \(f\)-welfare optimum, as considered in Section 3.1. As explained in Sections 3.2–3.4, it can be decentralized \(\alpha\text{-a.e.} in A\) by budget constraints of the form

\[px_a + qy_a \leq w_a(p, q) := p\hat{x}_a + q\hat{y}_a\]

— either as compensated equilibria, or if the assumptions of Section 3.3 are met, as a Walrasian \(f\)-equilibrium. Let \(\hat{\nu} \in \Delta(A \times H)\) denote the history distribution that results when each agent \(a \in A\) is assigned personal history \(\hat{h}_a\).

As argued in Section 4.1, for each personal history distribution \(\nu \in \Delta(A \times H)\) there will be an \(f\)-optimal allocation specified by a function \(\eta_a(\nu, h)\) of \(a, \nu, \) and the personal history \(h \in H\). Of course, \(\hat{y}_a = \eta_a(\hat{\nu}, \hat{h}_a)\) for each \(a \in A\). For each \(h \in H\), let \(\hat{y}_a(h)\) denote \(\eta_a(\nu, h)\).

So far, there is no inducement for agents to choose \(h_a = \hat{h}_a\ \alpha\text{-a.e.} in A\), as required by the history constraint. Remedying this serious defect requires modifying the kind of first-period budget constraint \(px_a \leq m_a(p) := p\hat{x}_a\) found in Section 6 so that \(m_a(p)\) also depends on the choice of personal history. In fact, the new budget constraint to be considered takes the form \(px_a \leq m_a(p, q, h)\) where

\[m_a(p, q, h) := px_a + q\hat{y}_a(q) = w_a(p, q) - q\hat{y}_a(h_a)\]

which depends also on \(q\), the second-period price vector. The effect of the change is to make each agent \(a \in A\) pay a net tax \(q\hat{y}_a(h_a) - q\hat{y}_a\) for choosing personal history \(h_a \in H\). This net tax is the present discounted value of the increased net subsidy that agent \(a\) will receive in period 2 as a result of choosing \(h_a\) instead of \(\hat{h}_a\) in period 1.

Formally, this modified first-period decentralization works because \(\hat{y}_a(h_a) = \hat{y}_a\) implies that \(m_a(p, q, h_a) = p\hat{x}_a\), while for any pair \((x_a, h_a) \in X_a\) with

\[u_a(x_a, h_a) + v_a(\hat{y}_a(h_a); h_a) \geq u_a(\hat{x}_a, \hat{h}_a) + v_a(\hat{y}_a(\hat{h}_a); \hat{h}_a)\]

it must be true that \(px_a + q\hat{y}_a(h_a) \geq w_a(p, q) = p\hat{x}_a + q\hat{y}_a\) and so \(px_a \geq m_a(p, q, h_a)\). Thus, if \((\hat{x}, \hat{h}, \hat{y}, p, q)\) is an intertemporal compensated equilibrium relative to the transfer system specified by \(w_a(p, q) (a \in A)\), then \((\hat{x}, \hat{h}, p)\) is a first-period compensated equilibrium relative
to the transfer system specified by $m_a(p, q, h)$ ($a \in A, h \in H$). Similarly, if $(\hat{x}, \hat{h}, \hat{y}, p, q)$ is an intertemporal Walrasian equilibrium relative to $w_a(p, q)$ ($a \in A$), then $(\hat{x}, \hat{h}, p)$ is a first-period Walrasian equilibrium relative to $m_a(p, q, h)$ ($a \in A, h \in H$).

Informally, the decentralization works because it forces each agent to internalize entirely the effect that the choice of personal history has on the widespread externality caused by the transfer system. This is only possible, however, if each agent’s choice of personal history $h \in H$ can be observed in time to enforce the appropriate budget constraint $p x_a \leq m_a(p, q, h)$; otherwise, the economy may be restricted to some second-best allocation of the kind that arises when there are binding moral hazard constraints. Indeed, if no observations are possible in time to enforce this first period budget constraint, then $m_a$ must remain independent of personal history. In this case there can be no escape from the inferior allocation discussed at the end of Section 4.2, or even from the kind of disastrous allocation that arose in Section 5.2 of Hammond (1993b).

Note that even in the special framework of Section 7, there seems little use for linear Pigovian prices in decentralizing the optimum. Instead, the price system presented in this section uses what are in effect non-linear prices for personal history.

9. Concluding Remarks

As explained in the introduction, markets would be very unlikely to produce optimal allocations in the real world economy, even if every agent, including those in the governments of the world, were planning everything perfectly in advance. This paper has largely been concerned with additional reasons for doubting the efficacy of markets when it is recognized that prices and fiscal systems adjust period by period. Yet it is not intended to deny that markets may well be a powerful tool for allocating resources. What is emphatically claimed is that this has yet to be demonstrated by proper economic analysis. Also, the subgame imperfections which can arise in intertemporal economies make the usual claimed demonstration, based on the second fundamental efficiency theorem of welfare economics, even more suspect than in static economies.

Meanwhile an urgent task of economic analysis is to devise an alternative and much less market-oriented framework for evaluating economic policies and even for comparing entire economic systems. Too much existing work measures economic performance with reference
to the perfect market ideal, making just a few grudging allowances for those inevitable market failures which have to be corrected by some form of public intervention. In fact the ideal economic system is very much more elusive than most economists seem to realize. This seems to be especially true of intertemporal resource allocation, as considered in this paper.

REFERENCES


