Outline

Introduction
Vickrey–Mirrlees Model
Typical Problem
Economic Application
Vickrey–Mirrlees Model

Problem: how much to pay workers of different skills.

Goal: achieve fairness while preserving incentives.

References: William S. Vickrey (1945)
“Measuring Marginal Utility by Reactions to Risk”

James A. Mirrlees (1971)
“An Exploration in the Theory of Optimal Income Taxation”

Let \( n \in \mathbb{R}_+ \) denote a person’s skill level, defined to mean that there is a constant rate of marginal substitution of \( n_1/n_2 \) between hours of work supplied by workers of the two skill levels \( n_1 \) and \( n_2 \).

Thus, a worker’s productivity is proportional to \( n \), personal skill.

Assume that the distribution of workers’ skills can be described by a continuous density function \( \mathbb{R}_+ \ni n \mapsto f(n) \in \mathbb{R}_+ \) which, like a probability density function, satisfies \( \int_0^{\infty} f(n)\,dn = 1 \).
Objective and Constraints

“Macro” model with a “representative consumer/worker” whose preferences for consumption/labour supply pairs \((c, \ell) \in \mathbb{R}^2_+\) are represented by the utility function \(u(c) - v(\ell)\), where \(u' > 0, v' > 0, u'' < 0, v'' > 0\).

The **social objective** is to maximize the utility integral \(\int_0^\infty [u(c(n)) - v(\ell(n))]f(n)dn\) w.r.t. the functions \(\mathbb{R}_+ \ni n \mapsto (c(n), \ell(n)) \in \mathbb{R}^2_+\).

The **resource balance constraint** takes the form \(C \leq F(L)\) where

- \(C := \int_0^\infty c(n)f(n)dn\) is mean consumption;
- \(L := \int_0^\infty n\ell(n)f(n)dn\) is mean effective labour supply.

The aggregate production function \(\mathbb{R}_+ \ni L \mapsto F(L) \in \mathbb{R}_+\) is assumed to satisfy \(F'(L) > 0\) and \(F''(L) \leq 0\) for all \(L \geq 0\).
Pseudo First-Order Conditions

Consider the Lagrangian

\[ L(c(\cdot), \ell(\cdot)) := \int_0^\infty \left[ u(c(n)) - v(\ell(n)) \right] f(n) dn \]

\[ - \lambda \left[ \int_0^\infty c(n) f(n) dn - F \left( \int_0^\infty n \ell(n) f(n) dn \right) \right] \]

as a functional (rather than a mere function) of the functions \( \mathbb{R}_+ \ni n \mapsto (c(n), \ell(n)) \in \mathbb{R}^2_+ \).

We derive “pseudo” first-order conditions by pretending that the derivatives \( \frac{\partial L}{\partial c(n)} \) and \( \frac{\partial L}{\partial \ell(n)} \) both exist, for all \( n \geq 0 \).

This gives the pseudo first-order conditions

\[ 0 = \frac{\partial L}{\partial c(n)} = \left[ u'(c(n)) - \lambda \right] f(n) \]

\[ 0 = \frac{\partial L}{\partial \ell(n)} = -v'(\ell(n)) f(n) + \lambda F'(L) nf(n) \]
For any skill level $n$ such that $f(n) > 0$, these two equations

$$0 = [u'(c(n)) - \lambda]f(n) = -v'(\ell(n))f(n) + \lambda F'(L)nf(n)$$

imply that:

- $u'(c(n)) = \lambda$ and so $c(n) = c^*$, where the constant $c^*$ uniquely solves $u'(c^*) = \lambda$ ("to each according to their need");

- $v'(\ell(n)) = \lambda F'(L)n$, implying that $v''(\ell(n)) \cdot \frac{d\ell}{dn} = \lambda F' > 0$, so $\frac{d\ell}{dn} > 0$ ("from each according to their ability")

**Exercise**

*Use concavity arguments to prove that this is the (essentially unique) solution.*

*What makes this solution practically infeasible?*
Sufficiency Theorem: Statement

**Theorem**

*Suppose that there exists \( \lambda > 0 \) such that \( c^* \) and the function \( \mathbb{R}_+ \ni n \mapsto \ell^*(n) \) jointly satisfy the first-order conditions:*

\[
  u'(c^*) = \lambda \quad \text{and} \quad v'((\ell^*)(n)) = \lambda F'(L^*)n \quad \text{for all } n \in \mathbb{R}_+
\]

*where \( c^* = F(L^*) \) and \( L^* = \int_0^\infty n\ell^*(n)f(n)\,dn \).*

*Let \( \mathbb{R}_+ \ni n \mapsto (c(n), \ell(n)) \in \mathbb{R}_+^2 \) be any other policy satisfying \( C = F(L) \) where \( C = \int_0^\infty c(n)f(n)\,dn \) and \( L = \int_0^\infty n\ell(n)f(n)\,dn \).*

*Then*

\[
\int_0^\infty [u(c(n)) - v(\ell(n))]f(n)dn \leq u(c^*) - \int_0^\infty v(\ell^*(n))f(n)dn
\]

*with strict inequality unless \( c(n) = c^* \) wherever \( f(n) > 0 \).*
Sufficiency Theorem: Proof, 1

Because $u'' < 0$ and so $u$ is strictly concave, the supergradient property of concave functions implies that

$$u(c(n)) - u(c^*) \leq u'(c^*)[c(n) - c^*] = \lambda[c(n) - c^*]$$

for all $n$, with strict inequality unless $c(n) = c^*$.

Integrating gives

$$\int_0^\infty [u(c(n)) - u(c^*)] f(n) \, dn \leq \lambda(C - c^*),$$

with strict inequality unless $c(n) = c^*$ wherever $f(n) > 0$.

Similarly, because $v'' \geq 0$ and so $v$ is convex, the subgradient property of convex functions implies that

$$v(\ell(n)) - v(\ell^*(n)) \geq v'(\ell^*(n))[\ell(n) - \ell^*(n)] = \lambda F'(L^*)[\ell(n) - \ell^*(n)]$$

Integrating gives

$$\int_0^\infty [v(\ell(n)) - v(\ell^*(n))] f(n) \, dn \geq \lambda F'(L^*)(L - L^*)$$
Sufficiency Theorem: Proof, II

Subtracting the second inequality from the first, then rearranging, one has

\[ \int_{0}^{\infty} \left\{ [u(c(n)) - v(\ell(n))] - [u(c^*) - v(\ell^*(n))] \right\} f(n) \, dn \leq \lambda [(C - c^*) - F'(L^*)(L - L^*)] \]

Next, because \( F'' \leq 0 \) and so \( F \) is concave, one has

\[ C - c^* = F(L) - F(L^*) \leq F'(L^*)(L - L^*) \]

Finally, because \( \lambda > 0 \), it follows that

\[ \int_{0}^{\infty} [u(c(n)) - v(\ell(n))] f(n) \, dn \leq \int_{0}^{\infty} [u(c^*) - v(\ell^*(n))] f(n) \, dn \]

as required for \( \mathbb{R}_+ \ni n \mapsto (c^*, \ell^*(n)) \in \mathbb{R}^2_+ \) to be optimal.
Outline

Introduction

Vickrey–Mirrlees Model

Typical Problem

Economic Application
Problem Formulation

The calculus of variations is used to optimize a functional that maps functions into real numbers.

A typical problem is to choose a function \([t_0, t_1] \ni t \mapsto x(t) \in \mathbb{R}\), often denoted simply by \(x\), in order to maximize the integral objective functional

\[
J(x) = \int_{t_0}^{t_1} F(t, x(t), \dot{x}(t)) \, dt
\]

subject to the fixed end point conditions \(x(t_0) = x_0, x(t_1) = x_1\).

A variation involves moving away from the first path \(x\) to the variant path \(x + \epsilon u\), where \(u\) denotes the differentiable function \([t_0, t_1] \ni t \mapsto u(t) \in \mathbb{R}\), and \(\epsilon \in \mathbb{R}\) is a scalar.

To ensure that the end point conditions \(x(t_0) + \epsilon u(t_0) = x_0\) and \(x(t_1) + \epsilon u(t_1) = x_1\) remain satisfied by \(x + \epsilon u\), one imposes the conditions \(u(t_0) = u(t_1) = 0\).
Toward a Necessary First-Order Condition

A maximum is a path $\mathbf{x}^*$ satisfying the end point conditions such that $J(\mathbf{x}^*) \geq J(\mathbf{x})$ for any alternative path $\mathbf{x}$ that also satisfies the end point conditions.

A necessary condition for $\mathbf{x}^*$ to maximize $J(\mathbf{x})$ w.r.t. $\mathbf{x}$ is that $J(\mathbf{x}^*) \geq J(\mathbf{x}^* + \epsilon \mathbf{u})$ for all small $\epsilon$.

Alternatively, the function

$$\mathbb{R} \ni \epsilon \mapsto f_{\mathbf{x}^*,\mathbf{u}}(\epsilon) := J(\mathbf{x}^* + \epsilon \mathbf{u})$$

must satisfy, for all small $\epsilon$, the inequality

$$f_{\mathbf{x}^*,\mathbf{u}}(0) = J(\mathbf{x}^*) \geq J(\mathbf{x}^* + \epsilon \mathbf{u}) = f_{\mathbf{x}^*,\mathbf{u}}(\epsilon)$$

In case the function $\epsilon \mapsto f_{\mathbf{x}^*,\mathbf{u}}(\epsilon)$ is differentiable at $\epsilon = 0$, a necessary first-order condition is therefore $f'_{\mathbf{x}^*,\mathbf{u}}(0) = 0$. 
Evaluating the Derivative

Our definitions of the functions $J$ and $f_{x^*,u}$ imply that

$$f_{x^*,u}(\epsilon) = J(x^* + \epsilon u) = \int_{t_0}^{t_1} F(t, x^*(t) + \epsilon u(t), \dot{x}^*(t) + \epsilon \dot{u}(t)) dt$$

Differentiating the integrand w.r.t. $\epsilon$ at $\epsilon = 0$ implies that

$$f'_{x^*,u}(0) = \int_{t_0}^{t_1} [F'_x(t)u(t) + F'_\dot{x}(t)\dot{u}(t)] dt$$

where for each $t \in [t_0, t_1]$, the partial derivatives $F'_x(t)$ and $F'_\dot{x}(t)$ of $F(t, x, \dot{x})$ are evaluated at the triple $(t, x^*(t), \dot{x}^*(t))$. 
Integrating by Parts

The product rule for differentiation implies that
\[
\frac{d}{dt}[F'_x(t)u(t)] = \left[\frac{d}{dt} F'_x(t)\right] u(t) + F'_x(t) \dot{u}(t)
\]

and so, integrating by parts, one has
\[
\int_{t_0}^{t_1} F'_x(t) \dot{u}(t) dt = \left|_t^{t_1} F'_x(t)u(t) \right| - \int_{t_0}^{t_1} \left[\frac{d}{dt} F'_x(t)\right] u(t) dt
\]

But the end point conditions imply that \(u(t_0) = u(t_1) = 0\), so the first term on the right-hand side vanishes.
The Euler Equation

Substituting $-\int_{t_0}^{t_1} \left[ \frac{d}{dt} F'_x(t) \right] u(t) dt$ for the term $\int_{t_0}^{t_1} F'_x(t) \dot{u}(t) dt$ in the equation $f'_{x^*, u}(0) = \int_{t_0}^{t_1} [F'_x(t) u(t) + F'_x(t) \dot{u}(t)] dt$, then recognizing the common factor $u(t)$, we finally obtain

$$f'_{x^*, u}(0) = \int_{t_0}^{t_1} \left[ F'_x(t) - \frac{d}{dt} F'_x(t) \right] u(t) dt$$

The first-order condition is $f'_{x^*, u}(0) = 0$ for every differentiable function $t \mapsto u(t)$ satisfying the two end point conditions $u(t_0) = u(t_1) = 0$.

This condition holds

iff the integrand is zero for (almost) all $t \in [t_0, t_1]$, which is true iff the Euler equation $\frac{d}{dt} F'_x(t) = F'_x(t)$ holds for (almost) all $t \in [t_0, t_1]$. 
Outline

Introduction
  Vickrey–Mirrlees Model
  Typical Problem
  Economic Application
Macroeconomic variation of the Solow–Swan growth model. Given a capital stock $K$, output $Y$ is given by the production function $Y = f(K)$, where $f' > 0$, and $f'' \leq 0$. Net investment = gross investment, without depreciation. So given capital $K$ and consumption $C$, investment $I$ is given by

$$I = \dot{K} = f(K) - C$$
The Ramsey Problem and Beyond

The economy’s **intertemporal objective** is taken to be

\[
\int_0^T e^{-rt} U(C(t)) dt = \int_0^T e^{-rt} U(f(K) - \dot{K}) dt
\]

Frank Ramsey (EJ, 1928) assumed \( T = \infty \) (infinite horizon) and \( r = 0 \) (no discounting).

Nicholas Stern (of the *Stern Report on Climate Change*) and others advocate:

- \( T = \infty \);
- \( r \) as the hazard rate in a Poisson process that determines when extinction occurs; this implies that \( e^{-rt} \) is the probability that the human race has not become extinct by time \( t \).

Chichilnisky, Hammond, and Stern (2018) TWERPS 1174
Applying the Calculus of Variations

We apply the calculus of variations to the objective
\[ \int_0^T e^{-rt} U(f(K) - \dot{K}) \, dt \]
with the end conditions \( K(0) = \bar{K} \), which is exogenous, and \( K(T) = 0 \) at the finite time horizon \( T \).

Euler’s equation takes the form \( \frac{d}{dt} F'_{\dot{K}}(t) = F'_{K}(t) \)
where \( F(t, K, \dot{K}) = e^{-rt} U(f(K) - \dot{K}) = e^{-rt} U(C) \).
So Euler’s equation becomes \( \frac{d}{dt} [-e^{-rt} U'(C)] = e^{-rt} U'(C)f'(K) \). Equivalently, after evaluating the time derivative,
\[ -U''(C) \dot{C} e^{-rt} + rU'(C)e^{-rt} = e^{-rt} U'(C)f'(K) \]
Cancelling the common factor \( e^{-rt} \) and dividing by \( U'(C) > 0 \), then rearranging, one obtains
\[ -\frac{U''(C)}{U'(C)} \dot{C} = f'(K) - r \]
Define the (negative) elasticity of marginal utility as

$$
\eta(C) := -\frac{d \ln U'(C)}{d \ln C} = -\frac{U''(C)C}{U'(C)}
$$

This is related to the curvature of the utility function, and to how quickly marginal utility $U'(C)$ decreases as $C$ increases.

Rearranging the equation $-U''(C) \dot{C} / U'(C) = f'(K) - r$ yet again, one obtains the equation

$$
\eta(C) \frac{\dot{C}}{C} = f'(K) - r
$$

whose left hand side is the proportional rate of consumption growth multiplied by: (i) the elasticity of marginal utility; or (ii) the elasticity of an intertemporal MRS; or (iii) the degree of relative fluctuation aversion.
Final Recommendation

Morton I. Kamien and Nancy L. Schwartz (2012)