Lecture Notes 7: Dynamic Equations Part B: Second and Higher-Order Linear Difference Equations in One Variable

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Lecture Outline

Solving Second-Order Equations

Inhomogeneous Equations

Particular Solutions in Two Special Cases

Method of Undetermined Coefficients

Higher-Order Linear Equations with Constant Coefficients

Stationary States and Stability for Second-Order Equations

Second-Order Equations

A general second-order difference equation
specifies the state x_t at each time t
as a function $x_t = F_t(x_{t-1}, x_{t-2})$ of the state at two previous times.

Suppose we define a new variable defined by $y_t := x_{t-1}$.
Then the equation $x_t = F_t(x_{t-1}, x_{t-2})$ can be converted
into the coupled pair

$$\begin{aligned} x_t &= F_t(x_{t-1}, y_{t-1}) \\ y_t &= x_{t-1} \end{aligned}$$

of first-order equations that express the vector $(x_t, y_t)^\top \in \mathbb{R}^2$
as a function of the vector $(x_{t-1}, y_{t-1})^\top \in \mathbb{R}^2$.

The Linear Case

We focus on linear equations in one variable
with constant coefficients, which take the form

$$x_{t+1} + a x_t + b x_{t-1} = f_t$$

Here a, b are scalars, and f_t is the forcing term.

We assume that $b \neq 0$ because otherwise
we have the first-order equation $x_{t+1} + a x_t = f_t$.

If we define $y_t = x_{t-1}$, the equation becomes the coupled pair

$$x_{t+1} = -a x_t - b y_t + f_t; \quad y_{t+1} = x_t$$

In matrix form, these can be written as

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} = \begin{pmatrix} -a & -b \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_t \\ y_t \end{pmatrix} + \begin{pmatrix} f_t \\ 0 \end{pmatrix}$$

Such vector difference equations are the subject of part C.

The Homogeneous Case

Nevertheless, consider the homogeneous case
when the vector equation is

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} - \begin{pmatrix} -a & -b \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_t \\ y_t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The solution in matrix form is evidently

$$\begin{pmatrix} x_t \\ y_t \end{pmatrix} = \begin{pmatrix} -a & -b \\ 1 & 0 \end{pmatrix}^t \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

for an arbitrary initial state $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$.

Inspired by our earlier discussion of matrix powers,
consider the case when $(\lambda, (x_0, y_0)^\top)$ is an eigenpair, that is

$$\begin{pmatrix} -a & -b \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \lambda \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \quad \text{where} \quad \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Solving the Homogeneous Case

In case $\begin{pmatrix} -a & -b \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \lambda \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$, the solution takes the form

$$\begin{pmatrix} x_t \\ y_t \end{pmatrix} = \begin{pmatrix} -a & -b \\ 1 & 0 \end{pmatrix}^t \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \lambda^t \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

For this to work, the initial vector $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ must solve the matrix equation $\begin{pmatrix} -a - \lambda & -b \\ 1 & -\lambda \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

For a non-trivial solution to exist,
the matrix $\begin{pmatrix} -a - \lambda & -b \\ 1 & -\lambda \end{pmatrix}$ must be singular, implying that

$$\begin{vmatrix} -a - \lambda & -b \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + a\lambda + b = 0$$

The Auxiliary Equation

Instead of treating the second-order equation as a coupled pair, consider directly the homogeneous second-order equation

$$x_{t+1} + a x_t + b x_{t-1} = 0$$

Inspired by our previous analysis
using eigenvalues of a suitable matrix,
we look for a solution of the form $x_t = \lambda^t x_0$,
for suitable constants λ and x_0.

It is a solution provided that $\lambda^{t+1} x_0 + a \lambda^t x_0 + b \lambda^{t-1} x_0 = 0$.

Ignoring the trivial solutions when $x_0 = 0$ or $\lambda = 0$,
cancel $\lambda^{t-1} x_0$ to obtain the auxiliary or characteristic equation

$$\lambda^2 + a\lambda + b = 0$$

This, of course, is the condition for λ to be an eigenvalue.

The Auxiliary Equation and Its Roots

The auxiliary equation $\lambda^2 + a\lambda + b = 0$ is quadratic.

It therefore has two roots λ_1, λ_2
satisfying $\lambda^2 + a\lambda + b = (\lambda - \lambda_1)(\lambda - \lambda_2)$.

In particular $\lambda_1 + \lambda_2 = -a$ and $\lambda_1 \lambda_2 = b$.

The assumption that $b \neq 0$ implies
that the two roots λ_1, λ_2 are both non-zero.

This leaves three cases:

1. two distinct real roots $\lambda_1, \lambda_2 \in \mathbb{R}$,
   which is true iff $a^2 > 4b$;
2. two complex conjugate roots $\lambda_1, \lambda_2 = re^{\pm i\theta} \in \mathbb{C}$,
   which is true iff $a^2 < 4b$;
3. two coincident real roots $\lambda = \lambda_1 = \lambda_2 \in \mathbb{R}$,
   which is true iff $a^2 = 4b$.

Case 1: Two Distinct Real Roots

In this case $\lambda^2 + a\lambda + b = (\lambda - \lambda_1)(\lambda - \lambda_2)$,
where $\lambda_1, \lambda_2 = -\frac{1}{2}a \pm \frac{1}{2}\sqrt{a^2 - 4b}$.

Note that $a = \lambda_1 + \lambda_2$ and $b = \lambda_1 \lambda_2$
with $a^2 - 4b = (\lambda_1 + \lambda_2)^2 - 4\lambda_1\lambda_2 = (\lambda_1 - \lambda_2)^2 > 0$.

There are two degrees of freedom in the difference equation,
so we look for two linearly independent solutions $x_t^{H(1)}$ and $x_t^{H(2)}$
of the homogeneous difference equation $x_{t+1} + ax_t + bx_{t-1} = 0$.
— that is two solutions for which $Ax_t^{H(1)} + Bx_t^{H(2)} \equiv 0$
implies that the two scalars A and B satisfy $A = B = 0$.

Two Linearly Independent Solutions

Note that $A\lambda_1^t + B\lambda_2^t = 0$ for both $t = 0$ and $t = 1$ if and only if

$$\begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This has a non-trivial solution in the two constants A and B iff $0 = \begin{vmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{vmatrix}$, or if and only if $0 = \lambda_2 - \lambda_1$.

So when $\lambda_1 \neq \lambda_2$, the only solution is trivial, with $A = B = 0$.

Hence, the two functions $x_t^{(1)} = x_0\lambda_1^t$ and $x_t^{(2)} = x_0\lambda_2^t$ with $x_0 \neq 0$ are linearly independent solutions of $x_{t+1} + ax_t + bx_{t-1} = 0$.

There are two degrees of freedom in the difference equation.

Therefore, its general solution with these two degrees of freedom is $x_t = A\lambda_1^t + B\lambda_2^t$ for arbitrary real constants A and B.

Example: The Fibonacci Sequence

The Fibonacci sequence is

$$(x_t)_{t=0}^{\infty} = (0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \ldots)$$

It is the unique solution with $x_0 = 0$ and $x_1 = 1$
of the Fibonacci difference equation $x_{t+1} - x_t - x_{t-1} = 0$.

The characteristic equation is $\lambda^2 - \lambda - 1 = 0$,
with characteristic roots $\lambda_{1,2} = -\frac{1}{2}(-1 \pm \sqrt{5})$.

Its two roots are:
(i) the golden ratio $\varphi := \lambda_1 = \frac{1}{2}(1 + \sqrt{5}) \approx 1.61803398875$;
and (ii) $\lambda_2 = 1 - \lambda_1 = \frac{1}{2}(1 - \sqrt{5}) \approx -0.61803398875$.

The general solution of the Fibonacci difference equation
is $x_t = A\lambda_1^t + B\lambda_2^t$ for arbitrary constants A and B.

To obtain the Fibonacci sequence with $x_0 = 0$ and $x_1 = 1$
requires $B = -A$ and $1 = A(\lambda_1 - \lambda_2) = A\sqrt{5}$,
so $B = -A = -\frac{1}{5}\sqrt{5}$.

Hence $x_t = \frac{1}{5}\sqrt{5} \cdot 2^{-t} \left[(1 + \sqrt{5})^t - (1 - \sqrt{5})^t\right]$, so $x_t \in \mathbb{N}$.

Case 2: Two Complex Conjugate Roots

Consider next the case where the equation $\lambda^2 + a\lambda + b = 0$ has two complex conjugate roots that we write as

$$\lambda = re^{\pm i\theta} = r(\cos\theta \pm i\sin\theta) \quad \text{where } \sin\theta \neq 0$$

In this case $\lambda^2 + a\lambda + b = (\lambda - re^{i\theta})(\lambda - re^{-i\theta})$ where

$$a = re^{i\theta} + re^{-i\theta} = r(\cos\theta + i\sin\theta) + r(\cos\theta - i\sin\theta) = 2r\cos\theta$$

and $b = (re^{i\theta})(re^{-i\theta}) = r^2$ with $\sin\theta \neq 0$.

It follows that $a^2 - 4b = 4r^2\cos^2\theta - 4r^2 = -4r^2\sin^2\theta < 0$.
Note that $r = \sqrt{|b|}$ and $\theta = \arccos\left(\dfrac{a}{2r}\right) = \arccos\left(\frac{1}{2}a|b|^{-\frac{1}{2}}\right)$.

Case 2: Oscillating Solutions

In the complex plane $\mathbb{C}$, two possible solutions of the difference equation $x_{t+1} + ax_t + bx_{t-1} = 0$ with $x_0 \neq 0$ are

$$\begin{array}{rrcl} & x_t^{(1)} & = & x_0(re^{i\theta})^t = x_0 r^t e^{i\theta t} = x_0 r^t(\cos\theta t + i \sin\theta t) \\ \text{and} & x_t^{(2)} & = & x_0(re^{-i\theta})^t = x_0 r^t e^{-i\theta t} = x_0 r^t(\cos\theta t - i \sin\theta t) \end{array}$$

In the real line $\mathbb{R}$, two possible solutions are

$$x_t^{(1)} = r^t \cos\theta t \quad \text{and} \quad x_t^{(2)} = r^t \sin\theta t$$

These are linearly independent because

$$\begin{vmatrix} x_0^{(1)} & x_0^{(2)} \\ x_1^{(1)} & x_1^{(2)} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ r\cos\theta & r\sin\theta \end{vmatrix} = r\sin\theta \neq 0$$

The general solution is therefore $x_t = r^t(A\cos\theta t + B\sin\theta t)$ for arbitrary real constants A and B, where $A = x_0$.

Case 3: Two Coincident Roots

In this case $\lambda^2 + a\lambda + b = (\lambda - \bar{\lambda})^2$,
where $a = -2\bar{\lambda}$ and $b = \bar{\lambda}^2$.

Consider the perturbed equation $x_{t+1} + ax_t + \tilde{b}x_{t-1} = 0$ where $a = -2\bar{\lambda}$ still and $\tilde{b} = \bar{\lambda}^2 - \epsilon^2$ with ϵ a small positive number.

We consider the behaviour of its general solution as $\epsilon \to 0$.

The auxiliary equation $\lambda^2 + a\lambda + \tilde{b} = 0$
can be written as $\lambda^2 - 2\bar{\lambda}\lambda + \bar{\lambda}^2 - \epsilon^2 = 0$.

Note that $\lambda^2 - 2\bar{\lambda}\lambda + \bar{\lambda}^2 - \epsilon^2 = (\lambda - \bar{\lambda} + \epsilon)(\lambda - \bar{\lambda} - \epsilon)$.

So the perturbed auxiliary equation
has the two real roots $\lambda = \bar{\lambda} \pm \epsilon$.

The Solution with Fixed Initial Conditions

Fix $\bar{x}_0$ and $\bar{x}_1$.

The general solution satisfying $x_0 = \bar{x}_0$ and $x_1 = \bar{x}_1$ is $x_t = A(\bar{\lambda} + \epsilon)^t + B(\bar{\lambda} - \epsilon)^t$ where $\bar{x}_0 = A + B$ and $\bar{x}_1 = A(\bar{\lambda} + \epsilon) + B(\bar{\lambda} - \epsilon) = (A + B)\bar{\lambda} + (A - B)\epsilon$.

Hence $A + B = \bar{x}_0$ and $A - B = (1/\epsilon)(\bar{x}_1 - \bar{x}_0\bar{\lambda})$, implying that $A = \frac{1}{2}\left[\bar{x}_0 + (1/\epsilon)(\bar{x}_1 - \bar{x}_0\bar{\lambda})\right]$ and $B = \frac{1}{2}\left[\bar{x}_0 - (1/\epsilon)(\bar{x}_1 - \bar{x}_0\bar{\lambda})\right]$.

The solution for fixed ϵ is therefore

$$x_t^\epsilon = \frac{1}{2}\left[\bar{x}_0 + (1/\epsilon)(\bar{x}_1 - \bar{x}_0\bar{\lambda})\right](\bar{\lambda} + \epsilon)^t + \frac{1}{2}\left[\bar{x}_0 - (1/\epsilon)(\bar{x}_1 - \bar{x}_0\bar{\lambda})\right](\bar{\lambda} - \epsilon)^t$$

which can be rewritten as

$$x_t^\epsilon = \frac{1}{2}\bar{x}_0\left[(\bar{\lambda} + \epsilon)^t + (\bar{\lambda} - \epsilon)^t\right] + \frac{1}{2}(\bar{x}_1 - \bar{x}_0\bar{\lambda})(1/\epsilon)\left[(\bar{\lambda} + \epsilon)^t - (\bar{\lambda} - \epsilon)^t\right]$$

The Limiting Solution as $\epsilon \to 0$

The limit of x_t^ϵ as $\epsilon \to 0$ takes the form

$$\bar{x}_0 \bar{\lambda}^t + \tfrac{1}{2}(\bar{x}_1 - \bar{x}_0 \bar{\lambda}) \lim_{\epsilon \to 0} (1/\epsilon) \left[(\bar{\lambda} + \epsilon)^t - (\bar{\lambda} - \epsilon)^t \right]$$

To evaluate the last limit, apply l'Hôpital's rule to obtain

$$\begin{aligned} & \lim\nolimits_{\epsilon \to 0} \left[(\bar{\lambda} + \epsilon)^t - (\bar{\lambda} - \epsilon)^t \right] / \epsilon \\ = \ & \lim\nolimits_{\epsilon \to 0} \left[t(\bar{\lambda} + \epsilon)^{t-1} + t(\bar{\lambda} - \epsilon)^{t-1} \right] / 1 \\ = \ & 2t\bar{\lambda}^{t-1} \ = \ (2t/\bar{\lambda})\, \bar{\lambda}^t \end{aligned}$$

Two linearly independent possible solutions
of the difference equation $x_{t+1} + a x_t + b x_{t-1} = 0$
with $x_0 \neq 0$ are $x_t^{(1)} = x_0 \lambda^t$ and $x_t^{(2)} = x_0 t \lambda^t$.

There are two degrees of freedom in the difference equation.

Its general solution is $x_t = (C + Dt)\lambda^t$
for arbitrary real constants C and D.

A Simpler Approach, I

We are trying to solve the homogeneous second-order difference equation with a repeated root λ, taking the form

$$x_{t+1} - 2\lambda x_t + \lambda^2 x_{t-1} = 0$$

We know that one solution is $x_t = x_0 \lambda^t$ for arbitrary x_0.

To find a second linearly independent solution
that we know must exist, try putting $x_t = \lambda^t y_t$.

Substituting into the original equation gives

$$\lambda^{t+1} y_{t+1} - 2\lambda^{t+1} y_t + \lambda^{t+1} y_{t-1} = 0$$

Disregarding the trivial case when $\lambda = 0$,
one has $y_{t+1} - 2y_t + y_{t-1} = 0$.

A Simpler Approach, II

To solve $y_{t+1} - 2y_t + y_{t-1} = 0$,
try introducing yet another new variable $z_t = y_{t+1} - y_t$.

This leads to the new difference equation $z_t - z_{t-1} = 0$
whose solution is obviously $z_t = z_0$ for all $t = 1, 2, \ldots$.

Then $y_{t+1} - y_t = z_0$ for all t, implying that $y_t = y_0 + z_0 t$.

It follows that $x_t = \lambda^t y_t = (y_0 + z_0 t)\lambda^t$.

To conclude, two solutions are $x_t^{(1)} = \lambda^t$ and $x_t^{(2)} = t\lambda^t$.

These are linearly independent because

$$\begin{vmatrix} x_0^{(1)} & x_0^{(2)} \\ x_1^{(1)} & x_1^{(2)} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ \lambda & \lambda \end{vmatrix} = \lambda \neq 0$$

The general solution is therefore $x_t = (A + Bt)\lambda^t$
for arbitrary real constants A and B, where $A = x_0$.

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From Particular to General Solutions

The homogeneous equation with constant coefficients takes the form

$$x_{t+1} + a x_t + b x_{t-1} = 0$$

The associated inhomogeneous equation takes the form

$$x_{t+1} + a x_t + b x_{t-1} = f_t$$

for a general forcing term f_t on the RHS.

Let x_t^P denote a particular solution,
and x_t^G any alternative general solution,
of the inhomogeneous equation.

Characterizing the General Solution

Our assumptions imply that, for each $t = 1, 2, \ldots$, one has

$$\begin{aligned} x^P_{t+1} + a x^P_t + b x^P_{t-1} &= f_t \\ x^G_{t+1} + a x^G_t + b x^G_{t-1} &= f_t \end{aligned}$$

Subtracting the first equation from the second implies that

$$x^G_{t+1} - x^P_{t+1} + a(x^G_t - x^P_t) + b(x^G_{t-1} - x^P_{t-1}) = 0$$

This shows that $x^H_t := x^G_t - x^P_t$
solves the homogeneous equation $x_{t+1} + a x_t + b x_{t-1} = 0$.

So the general solution x^G_t
of the inhomogeneous equation $x_{t+1} + a x_t + b x_{t-1} = f_t$
with forcing term f_t is the sum $x^P_t + x^H_t$ of

- any particular solution x^P_t of the inhomogeneous equation;
- the general solution x^H_t of the homogeneous equation.

Linearity in the Forcing Term

Theorem

Suppose that x_t^P and y_t^P are particular solutions of the two respective difference equations

$$x_{t+1} + ax_t + bx_{t-1} = d_t \quad \textit{and} \quad y_{t+1} + ay_t + by_{t-1} = e_t$$

Then, for any scalars α and β,
the linear combination $z_t^P := \alpha x_t^P + \beta y_t^P$ is a particular solution of the equation $z_{t+1} + az_t + bz_{t-1} = \alpha d_t + \beta e_t$.

Proof.

Routine algebra. □

Consider any equation of the form $x_{t+1} + ax_t + bx_{t-1} = f_t$ where f_t is a linear combination $\sum_{k=1}^n \alpha_k f_t^k$ of n forcing terms.

The theorem implies that a particular solution is the corresponding linear combination $\sum_{k=1}^n \alpha_k x_t^{Pk}$ of particular solutions to the equations $x_{t+1} + ax_t + bx_{t-1} = f_t^k$.

Deriving an Explicit Particular Solution, I

In part A we were able to derive an explicit solution to the general first-order linear equation $x_t - a_t x_{t-1} = f_t$.

Here, for the special case of constant coefficients, we derive an explicit particular solution satisfying $x_0 = x_1 = 0$ to the general second-order linear equation $x_{t+1} + a x_t + b x_{t-1} = f_t$.

Indeed, suppose that $\lambda^2 + a\lambda + b = (\lambda - \lambda_1)(\lambda - \lambda_2)$ because λ_1 and λ_2 are the roots (possibly coincident, or possibly complex conjugates) of the auxiliary equation $\lambda^2 + a\lambda + b = 0$.

Introduce the new variable $y_t = x_t - \lambda_1 x_{t-1}$, implying that

$$\begin{aligned} y_{t+1} - \lambda_2 y_t &= x_{t+1} - \lambda_1 x_t - \lambda_2 x_t + \lambda_1 \lambda_2 x_{t-1} \\ &= x_{t+1} - (\lambda_1 + \lambda_2) x_t + \lambda_1 \lambda_2 x_{t-1} \\ &= x_{t+1} + a x_t + b x_{t-1} \ = \ f_t \end{aligned}$$

Deriving an Explicit Particular Solution, II

Instead of the second-order equation $x_{t+1} + ax_t + bx_{t-1} = f_t$, we have the recursive pair of first-order equations

$$x_t - \lambda_1 x_{t-1} = y_t \quad \text{and} \quad y_{t+1} - \lambda_2 y_t = f_t \quad (\text{for } t = 1, 2, \ldots)$$

where λ_1 and λ_2 are the roots of $\lambda^2 + a\lambda + b = 0$.

Given the initial conditions $x_0 = x_1 = 0$ and so $y_1 = 0$, the explicit solutions like those derived in Part A are the sums

$$y_t = \sum\nolimits_{k=1}^{t-1} \lambda_2^{t-k-1} f_k \quad \text{and} \quad x_t = \sum\nolimits_{s=2}^{t} \lambda_1^{t-s} y_s \quad \text{for } t = 1, 2, \ldots$$

Substituting the first equation in the second yields the double sum

$$x_t = \sum\nolimits_{s=2}^{t} \lambda_1^{t-s} \sum\nolimits_{k=1}^{s-1} \lambda_2^{s-k-1} f_k$$

which we would like to reduce to $x_t = \sum_{k=1}^{t-1} \xi_{t-k-1} f_k$
— i.e., a linear combination of the forcing terms $(f_1, f_2, \ldots, f_{t-1})$.

Deriving an Explicit Particular Solution, III

We begin by introducing
the mapping $\mathbb{N} \times \mathbb{N} \ni (k, s) \mapsto 1_{ks}\{k < s\} \in \{0, 1\}$ defined by

$$1_{ks}\{k < s\} := \begin{cases} 1 & \text{if } k < s \\ 0 & \text{if } k \geq s \end{cases}$$

Then we can rewrite $x_t = \sum_{s=2}^{t} \lambda_1^{t-s} \sum_{k=1}^{s-1} \lambda_2^{s-k-1} f_k$
as the double sum $x_t = \sum_{s=2}^{t} \sum_{k=1}^{t-1} 1_{ks}\{k < s\} \lambda_1^{t-s} \lambda_2^{s-k-1} f_k$.

Interchanging the order of summation gives

$$\begin{aligned} x_t &= \sum_{k=1}^{t-1} \sum_{s=2}^{t} 1_{ks}\{k < s\} \lambda_1^{t-s} \lambda_2^{s-k-1} f_k \\ &= \sum_{k=1}^{t-1} \left(\sum_{s=k+1}^{t} \lambda_1^{t-s} \lambda_2^{s-k-1} \right) f_k \\ &= \sum_{k=1}^{t-1} \left(\lambda_1^{t-k-1} + \lambda_1^{t-k-2} \lambda_2 + \ldots + \lambda_1 \lambda_2^{t-k-2} + \lambda_2^{t-k-1} \right) f_k \end{aligned}$$

This reduces to $x_t = \sum_{k=1}^{t-1} \xi_{t-k-1} f_k$ where $\xi_m := \sum_{j=0}^{m} \lambda_1^{m-j} \lambda_2^{j}$.

Deriving an Explicit Particular Solution: IV

The value of the sum $\xi_m = \sum_{j=0}^{m} \lambda_1^{m-j} \lambda_2^j$ depends on whether:

- we are in the general case when $\lambda_1 \neq \lambda_2$;
- we are in the degenerate case when $\lambda_1 = \lambda_2 = \lambda$.

In the general case one has

$$(\lambda_1 - \lambda_2)\xi_m = \sum_{j=0}^{m} \left(\lambda_1^{m+1-j} \lambda_2^j - \lambda_1^{m-j} \lambda_2^{j+1} \right) = \lambda_1^{m+1} - \lambda_2^{m+1}$$

implying the particular solution

$$x_t^P = \frac{1}{\lambda_1 - \lambda_2} \sum_{k=1}^{t-1} \left(\lambda_1^{t-k} - \lambda_2^{t-k} \right) f_k$$

In the degenerate case one has $\xi_m = (m+1)\lambda^m$,
implying the particular solution

$$x_t^P = \sum_{k=1}^{t-1} (t-k) \lambda^{t-k} f_k$$

Lecture Outline

First Special Case with Distinct Real Roots, I

Consider the equation $x_{t+1} + ax_t + bx_{t-1} = f_t$
in the first special case when $f_t = \mu^t$ with $\mu \neq 0$.

In the general case when the two roots λ_1 and λ_2
of the auxiliary equation $\lambda^2 + a\lambda + b = 0$ are distinct,
the particular solution with $x_0^P = x_1^P = 0$ is

$$x_t^P = \frac{1}{\lambda_1 - \lambda_2} \sum_{k=1}^{t-1} \left(\lambda_1^{t-k} - \lambda_2^{t-k} \right) \mu^k$$

But $(\lambda - \mu) \sum_{k=1}^{t-1} \lambda^{t-k} \mu^k = \sum_{k=1}^{t-1} \left(\lambda^{t-k+1} \mu^k - \lambda^{t-k} \mu^{k+1} \right)$, so

$$x_t^P = \frac{1}{\lambda_1 - \lambda_2} \left(\frac{\lambda_1^t \mu - \lambda_1 \mu^t}{\lambda_1 - \mu} - \frac{\lambda_2^t \mu - \lambda_2 \mu^t}{\lambda_2 - \mu} \right)$$

in case $\mu \notin \{\lambda_1, \lambda_2\}$.

Disregarding the terms in λ_1^t and λ_2^t
that solve the corresponding homogeneous equation,
the solution reduces to $x_t^P = \alpha \mu^t$ for a suitable constant α.

First Special Case with Distinct Real Roots, II

The degenerate case when $\mu \in \{\lambda_1, \lambda_2\}$ is more complicated.

In case $\lambda_1 \neq \lambda_2 = \mu$, the particular solution with $x_0^P = x_1^P = 0$ is still

$$x_t^P = \frac{1}{\lambda_1 - \lambda_2} \sum_{k=1}^{t-1} \left(\lambda_1^{t-k} - \lambda_2^{t-k} \right) \mu^k$$

Because $\lambda_2 = \mu$, this reduces to

$$\begin{aligned} x_t^P &= \frac{1}{\lambda_1 - \mu} \sum_{k=1}^{t-1} \left(\lambda_1^{t-k} \mu^k - \mu^t \right) \\ &= \frac{1}{\lambda_1 - \mu} \left[\frac{\lambda_1^t \mu - \lambda_1 \mu^t}{\lambda_1 - \mu} - (t-1)\mu^t \right] \end{aligned}$$

Disregarding the terms in λ_1^t and in $\lambda_2^t = \mu^t$
that solve the corresponding homogeneous equation,
the solution reduces to $x_t^P = \alpha t \mu^t$ for a suitable constant α.

First Special Case with Coincident Real Roots

Consider now the degenerate case
with coincident real roots $\lambda_1 = \lambda_2 = \lambda$.

So the inhomogeneous equation is $x_{t+1} - 2\lambda x_t + \lambda^2 x_{t-1} = \mu^t$.

As before, put $y_t = x_t - \lambda x_{t-1}$ so that

$$y_{t+1} - \lambda y_t = x_{t+1} - \lambda x_t - \lambda x_t + \lambda^2 x_{t-1} = \mu^t$$

We consider again the particular solution
with $x_0 = x_1 = 0$ and so $y_1 = 0$.

First Special Case with Coincident Real Roots: $\lambda \neq \mu$

Provided that $\lambda \neq \mu$, for $t = 2, 3, \ldots$ one has

$$
\begin{aligned}
y_t^P &= \textstyle\sum_{k=2}^t \lambda^{t-k} \mu^{k-1} = \dfrac{\mu(\lambda^{t-1} - \mu^{t-1})}{\lambda - \mu} \\
\text{and then} \quad x_t^P &= \textstyle\sum_{k=2}^t \lambda^{t-k} y_k^P = \sum_{k=2}^t \lambda^{t-k} \mu \dfrac{\lambda^{k-1} - \mu^{k-1}}{\lambda - \mu} \\
&= \textstyle\sum_{k=2}^t \dfrac{\mu \lambda^{t-1} - \lambda^{t-k} \mu^k}{\lambda - \mu} \\
&= \dfrac{\mu(t-1)\lambda^{t-1}}{\lambda - \mu} - \dfrac{\lambda^{t-1}\mu^2 - \mu^{t+1}}{(\lambda - \mu)^2}
\end{aligned}
$$

Hence $x_t^P = (\alpha + \beta t)\lambda^t + \gamma \mu^t$ for suitable constants α, β and γ that depend on λ and μ, but not on t.

Because $(\alpha + \beta t)\lambda^t$ is a complementary solution
of the homogeneous equation,
the particular solution can be reduced to $x_t^P = \gamma \mu^t$.

First Special Case with Coincident Real Roots: $\lambda = \mu$

In case $\lambda = \mu$, however, for $t = 2, 3, \ldots$ one has

$$\begin{array}{rrcl} & y_t^P & = & \sum_{k=2}^t \lambda^{t-k} \mu^{k-1} \; = \; (t-1)\lambda^{t-1} \\ \text{and then} & x_t^P & = & \sum_{k=2}^t \lambda^{t-k} y_k^P \; = \; \sum_{k=2}^t \lambda^{t-k} (k-1) \lambda^{k-1} \\ & & = & \sum_{k=2}^t (k-1) \lambda^{t-1} \; = \; \frac{1}{2} t(t-1) \lambda^{t-1} \end{array}$$

Hence $x_t^P = (\alpha t + \beta t^2) \lambda^t$ for suitable constants α and β that depend on $\lambda = \mu$, but not on t.

Because $\alpha t \lambda^t$ is a complementary solution
of the homogeneous equation,
the particular solution can be reduced to $x_t^P = \beta t^2 \mu^t$.

Second Special Case: General Theorem

Consider next the equation $x_{t+1} + ax_t + bx_{t-1} = f_t$
in the second special case when $f_t = t^r \mu^t$ with $\mu \neq 0$ and $r \in \mathbb{N}$.

As before, let λ_1 and λ_2 denote the roots
of the auxiliary equation $\lambda^2 + a\lambda + b = 0$.

Theorem

The difference equation $x_{t+1} + ax_t + bx_{t-1} = t^r \mu^t$
has a particular solution of the form $x_t^P = \xi^P(t)\mu^t$
where $\xi^P(t) = \sum_{j=0}^{d} \xi_{rj} t^j$ is a polynomial in t which has degree:

- *$d = r$ in case $\mu \notin \{\lambda_1, \lambda_2\}$;*
- *$d = r + 2$ in case $\mu = \lambda_1 = \lambda_2$;*
- *$d = r + 1$ otherwise.*

We begin the proof by introducing, as before,
the new variable $y_t := x_t - \lambda_1 x_{t-1}$, implying that

$$\begin{aligned} y_{t+1} - \lambda_2 y_t &= x_{t+1} - \lambda_1 x_t - \lambda_2 x_t + \lambda_1 \lambda_2 x_{t-1} \\ &= x_{t+1} + ax_t + bx_{t-1} \ = \ t^r \mu^t \end{aligned}$$

Continuing the Proof of the General Theorem

By the result in part A, the first-order equation $y_{t+1} - \lambda_2 y_t = t^r \mu^t$
has a particular solution of the form $y_t = Q(t)\mu^t$,
where $Q(t) = \sum_{j=0}^{d} q_{rj} t^j$ is a polynomial in t which has degree:

(i) $d = r$ in case $\mu \neq \lambda_2$; (ii) $d = r + 1$ in case $\mu = \lambda_2$.

By the linearity property of particular solutions, the equation

$$x_t - \lambda_1 x_{t-1} = y_t = Q(t)\mu^t = \sum_{j=0}^{d} q_{rj} t^j \mu^t$$

has a particular solution $x_t^P = \xi^P(t)\mu^t$ where

$$x_t^P = \xi^P(t)\mu^t = \sum_{j=0}^{d} q_{rj} P_j(t)\mu^t$$

is the appropriate linear combination
of the particular solutions $x_t = P_j(t)\mu^t$ ($j = 0, 1, 2, \ldots, d$)
of the $d + 1$ first-order equations $x_t - \lambda_1 x_{t-1} = t^j \mu^t$.

Ending the Proof of the General Theorem

Again, using the result in part A,
for each $j = 0, 1, 2, \ldots, r$, the solution $x_t = P_j(t)\mu^t$
of the first-order difference equation $x_t - \lambda_1 x_{t-1} = t^j \mu^t$
involves a polynomial $P_j(t)$ in t which has degree:

(i) j in case $\mu \neq \lambda_1$; (ii) $j + 1$ in case $\mu = \lambda_1$.

So the degree of the highest order polynomial $P_d(t)$ is
(i) d in case $\mu \neq \lambda_1$; (ii) $d + 1$ in case $\mu = \lambda_1$.

Combined with our previous result on whether $d = r$ or $d = r + 1$,
the degree d of $\xi^P(t)$ is now easily seen to be

- $d = r$ in case $\mu \notin \{\lambda_1, \lambda_2\}$;
- $d = r + 2$ in case $\mu = \lambda_1 = \lambda_2$;
- $d = r + 1$ otherwise. □

Using the notation $\#S$ for the number of elements in a set S,
these three cases can be summarized as $d = r + 3 - \#\{\lambda_1, \lambda_2, \mu\}$.

Lecture Outline

First Special Case: A Simpler Approach

We have proved that the second-order difference equation

$$x_{t+1} + ax_t + bx_{t-1} = \mu^t$$

has a particular solution of the form $x_t^P = \alpha\mu^t$.

But there is a much easier way to find x_t^P,
treating the parameter α as an undetermined coefficient.

Indeed, for $x_t = \alpha\mu^t$ to be a solution,
one needs $\alpha\mu^{t+1} + a\alpha\mu^t + b\alpha\mu^{t-1} = \mu^t$.

Dividing each side by μ^{t-1}
yields the equation $\alpha(\mu^2 + a\mu + b) = \mu$.

In the non-degenerate case when $\mu^2 + a\mu + b \neq 0$
because μ is not a root
of the characteristic equation $\lambda^2 + a\lambda + b = 0$,
one has $\alpha = \mu(\mu^2 + a\mu + b)^{-1}$.

Hence, a particular solution is $x_t^P = (\mu^2 + a\mu + b)^{-1}\mu^{t+1}$.

Degenerate Case When μ is a Characteristic Root

The simple degenerate case occurs when $\mu^2 + a\mu + b = 0$ because μ equals one of the two distinct roots λ_1 and λ_2 of the characteristic equation $\lambda^2 + a\lambda + b = 0$.

Then we have proved that the second-order difference equation

$$x_{t+1} + ax_t + bx_{t-1} = \mu^t$$

has a particular solution of the form $x_t^P = \alpha t \mu^t$.

To determine the undetermined coefficient α, we must solve

$$\alpha(t+1)\mu^{t+1} + a\alpha t\mu^t + b\alpha(t-1)\mu^{t-1} = \mu^t$$

Dividing each side by μ^{t-1} and gathering terms yields the equation $\alpha t(\mu^2 + a\mu + b) + \alpha(\mu^2 - b) = \mu$.

Provided that $\mu^2 \neq b$, this reduces to $\alpha = (\mu^2 - b)^{-1}\mu$.

Doubly Degenerate Case

When $\mu^2 = b$, however, the degenerate case is more complicated.

Indeed, the equation $\mu^2 + a\mu + b = 0$ implies that $a\mu + 2b = 0$.

Hence $\mu = -2b/a$, so $\mu^2 = b = 4b^2/a^2$ implying that $a^2 = 4b$.

Then the characteristic equation $\lambda^2 + a\lambda + b = 0$
reduces to $(\lambda - \mu)^2 = 0$, with μ as its repeated root.

Inspired by the earlier theorem,
we look for a particular solution of the form $x_t^P = \alpha t^2 \mu^t$.

To determine the undetermined coefficient α, we must solve

$$\alpha(t+1)^2\mu^{t+1} + a\alpha t^2 \mu^t + b\alpha(t-1)^2\mu^{t-1} = \mu^t$$

Dividing each side by μ^{t-1} and gathering terms yields

$$\alpha t^2(\mu^2 + a\mu + b) + \alpha(2t+1)\mu^2 + \alpha b(-2t+1) = \mu$$

Because $\mu^2 + a\mu + b = 0$ and $0 \neq b = \mu^2$,
this equation reduces to $2\alpha\mu^2 = \mu$, implying that $\alpha = 1/2\mu$.

Second Special Case

Again, inspired by earlier theorems, we can apply the method of undetermined coefficients to the equation

$$x_{t+1} + a x_t + b x_{t-1} = \sum_{k=1}^{m} \sum_{j=1}^{r_k} \alpha_{kj} t^j \mu_k^t$$

where we naturally assume that the constants μ_k $(k = 1, 2, \ldots, m)$ are all different.

A particular solution takes the form

$$x_t^P = \sum_{k=1}^{m} \sum_{j=1}^{d_k} \beta_{kj} t^j \mu_k^t$$

where the degree d_k of each polynomial $\sum_{j=1}^{d_k} \beta_{kj} t^j$ with undetermined coefficients $\langle \langle \beta_{kj} \rangle_{j=1}^{d_k} \rangle_{k=1}^{m}$ is

- r_k in case $\mu_k \notin \{\lambda_1, \lambda_2\}$;
- $r_k + 2$ in case $\mu_k = \lambda_1 = \lambda_2$;
- $r_k + 1$ otherwise.

Determining the Coefficients

The coefficients $\langle\langle\beta_{kj}\rangle_{j=1}^{d_k}\rangle_{k=1}^{m}$ of the particular solution

$$x_t^P = \sum\nolimits_{k=1}^{m} \sum\nolimits_{j=1}^{d_k} \beta_{kj} t^j \mu_k^t$$

can be found (in principle!) by solving, for $k = 1, 2, \ldots, m$,
the m independent systems of linear equations
that result from equating coefficients of powers of t
in the expansions

$$\sum\nolimits_{j=1}^{d_k} \beta_{kj}[(t+1)^j \mu_k^2 + a t^j \mu_k^t + b(t-1)^j] = \sum\nolimits_{j=1}^{r_k} \alpha_{kj} t^j \mu_k$$

Lecture Outline

Higher-Order Linear Equations with Constant Coefficients

An *n*th order linear equation with constant coefficients takes the form

$$x_t + \sum_{r=1}^{n} a_r x_{t-r} = f_t$$

in the inhomogeneous case, and

$$x_t + \sum_{r=1}^{n} a_r x_{t-r} = 0$$

in the homogeneous case.

The corresponding auxiliary equation is $\lambda^n + \sum_{r=1}^{n} a_r \lambda^{n-r} = 0$.

Roots of the Auxiliary Equation

The auxiliary equation can be written as $p_n(\lambda) = 0$
whose LHS is the polynomial $\lambda^n + \sum_{r=1}^n a_r \lambda^{t-r}$ of degree n.

By the fundamental theorem of algebra,
this equation has at least one root λ_1, which may be complex.

Then $p_n(\lambda)$ can be factored as $p_n(\lambda) \equiv (\lambda - \lambda_1) p_{n-1}(\lambda)$.

But now the equation $p_{n-1}(\lambda) = 0$
also has at least one root λ_2, which may also be complex.

Repeating this argument n times,
the auxiliary equation $p_n(\lambda) = 0$ has n roots $\lambda_1, \lambda_2, \ldots, \lambda_n$,
some of which may be repeated.

In particular, $p_n(\lambda) \equiv \prod_{i=1}^n (\lambda - \lambda_i)$.

Solving the Homogeneous Equation

Theorem

Consider the homogeneous equation $x_t + \sum_{r=1}^{n} a_r x_{t-r} = 0$, *and suppose that the auxiliary equation can be written as*

$$0 = \lambda^n + \sum\nolimits_{r=1}^{n} a_r \lambda^{t-r} = \prod\nolimits_{j=1}^{k} (\lambda - \rho_j)^{m_j}$$

with k *distinct roots* ρ_j $(j = 1, 2, \ldots, k)$
whose respective multiplicities m_j *satisfy* $\sum_{j=1}^{k} m_j = n$.

Then the general solution of the homogeneous equation takes the form

$$x_t = \sum\nolimits_{j=1}^{k} \sum\nolimits_{h=1}^{m_j} \alpha_{jh} t^{h-1} \rho_j^t$$

for n *arbitrary constants* α_{jh} $(h = 1, 2, \ldots, m_j$ *and* $j = 1, 2, \ldots, k)$.

That is, the general solution is an arbitrary linear combination of the functions $t^{h-1}\rho_j^t$ $(h = 1, 2, \ldots, m_j$ and $j = 1, 2, \ldots, k)$.

Solving the Inhomogeneous Equation

Theorem

The general solution of the inhomogeneous equation

$$x_t + \sum_{r=1}^{n} a_r x_{t-r} = \sum_{h=1}^{i} \sum_{j=1}^{q_h} \alpha_{hj} t^j \mu_h^t$$

is the sum of: (i) the general complementary solution to the corresponding homogeneous equation $x_t + \sum_{r=1}^{n} a_r x_{t-r} = 0$*; and (ii) any particular solution.*

One particular solution takes the form $x_t^P = \sum_{h=1}^{i} \sum_{j=1}^{d_h} \beta_{hj} t^j \mu_h^t$ *where the degree* d_h *of each polynomial* $\sum_{j=1}^{d_h} \beta_{hj} t^j$ *with undetermined coefficients* $\langle \langle \beta_{hj} \rangle_{j=1}^{d_h} \rangle_{h=1}^{i}$ *is*

- q_h *in case* $\mu_h \notin \{\rho_1, \rho_2, \ldots, \rho_k\}$*;*
- $q_h + m_j$ *in case* $\mu_h = \rho_j$*, a root of multiplicity* m_j*.*

Lecture Outline

Stationary States of a Linear Equation

Consider the second-order equation $x_{t+1} + ax_t + bx_{t-1} = f$ for a constant forcing term $f \in \mathbb{R}$.

Here a stationary state $x^* \in \mathbb{R}$ has the defining property that $x_{t-1} = x_t = x^* \Longrightarrow x_{t+1} = x^*$.

This is satisfied if and only if $x^* + ax^* + bx^* = f$, or equivalently, if and only if $(1 + a + b)x^* = f$.

In case $a + b = -1$, there is:

- no stationary state unless $f = 0$;
- a whole real line $\mathbb{R}$ of stationary states if $f = 0$.

Otherwise, if $a + b \neq -1$, the only stationary state is $x^* = (1 + a + b)^{-1} f$.

Stability of a Linear Equation

If $a + b \neq -1$, let $y_t := x_t - x^*$ denote the deviation of state x_t from the stationary state $x^* = (1 + a + b)^{-1} f$. Then

$$\begin{aligned} y_{t+1} = x_{t+1} - x^* &= -a x_t - b x_{t-1} f - x^* \\ &= -a(y_t + x^*) - b(y_{t-1} + x^*) + f - x^* = -a y_t - b y_{t-1} \end{aligned}$$

Thus y_t solves the homogenous equation $x_{t+1} + a x_t + b x_{t-1} = 0$.

As already seen, the solution to this homogeneous equation depends on the two roots $\lambda_{1,2} = -\frac{1}{2} a \pm \frac{1}{2}\sqrt{a^2 - 4b}$ of the quadratic characteristic equation

$$f(\lambda) \equiv \lambda^2 + a\lambda + b \equiv (\lambda - \lambda_1)(\lambda - \lambda_2) = 0$$

There are three cases to consider:

1. two distinct real roots because $a^2 - 4b > 0$;
2. two complex conjugate roots because $a^2 - 4b < 0$;
3. two coincident real roots because $a^2 - 4b = 0$.

Stability Condition

With two distinct roots λ_1 and λ_2, real or complex, the general solution of the homogeneous equation is $x_t = A\lambda_1^t + B\lambda_2^t$.

Stability is satisfied if and only if for all $A, B \in \mathbb{R}$ one has $x_t \to 0$ as $t \to \infty$.

This is true if and only if the absolute values of both roots satisfy $|\lambda_1| < 1$ and $|\lambda_2| < 1$.

With two coincident roots $\lambda_1 = \lambda_2 = -\frac{1}{2}a = \sqrt{b}$, the general solution of the homogeneous equation is $x_t = (A + Bt)\lambda^t$.

Again, stability is satisfied if and only if for all $A, B \in \mathbb{R}$ one has $x_t \to 0$ as $t \to \infty$.

This is true if and only if the absolute value of the double root satisfies $|\lambda| < 1$.

Two Distinct Real Roots

Here $\lambda^2 + a\lambda + b = (\lambda - \lambda_1)(\lambda - \lambda_2)$
where λ_1 and λ_2 are both real.

Note that the quadratic function $f(\lambda) \equiv \lambda^2 + a\lambda + b$ is convex and satisfies $f(\lambda) \to +\infty$ as $\lambda \to \pm\infty$.

So the real roots of $f(\lambda) = 0$ satisfy $|\lambda_1| < 1$ and $|\lambda_2| < 1$ iff

$$f(-1) > 0 \text{ and } f(1) > 0 \text{ with } f'(-1) < 0 \text{ and } f'(1) > 0$$

These conditions are equivalent to

$$1 - a + b > 0 \text{ and } 1 + a + b > 0 \text{ with } -2 + a < 0 \text{ and } 2 + a > 0$$

or to $|a| < 2$ and $|a| < 1 + b$.

Together with the condition $a^2 > 4b$
for the equation $f(\lambda) = 0$ to have two distinct real roots,
these inequalities are equivalent to $|a| - 1 < b < 1$.

Two Complex Conjugate Roots

The characteristic equation $\lambda^2 + a\lambda + b = 0$
has two complex conjugate roots when $a^2 - 4b < 0$.

In this case, these characteristic roots are

$$\lambda_{1,2} = -\tfrac{1}{2}a \pm \tfrac{1}{2}i\sqrt{4b - a^2} = r\, e^{\pm i\theta} = r(\cos\theta \pm i\sin\theta)$$

where $r = \sqrt{b}$ and $\theta = \arccos(a/2\sqrt{b})$

Then the general solution of the homogeneous equation
can be written as $x_t = r^t(A\cos\theta t + B\sin\theta t)$.

Stability is satisfied if and only if
for all $A, B \in \mathbb{R}$ one has $x_t \to 0$ as $t \to \infty$.

This is true if and only if $b < 1$,
as well as $a^2 - 4b < 0$ which implies that $b > 0$.

A Repeated Real Root

The characteristic equation $\lambda^2 + a\lambda + b = 0$
has two coincident real roots roots when $a^2 = 4b$.

In this case, $\lambda^2 + a\lambda + b = (\lambda + \frac{1}{2}a)^2$.
The coincident real roots both equal $-\frac{1}{2}a$.
Then the general solution of the homogeneous equation
is $x_t = (A + Bt)(-\frac{1}{2}a)^t$.

Stability is satisfied if and only if
for all $A, B \in \mathbb{R}$ one has $x_t \to 0$ as $t \to \infty$.

This is true if and only if the modulus
of the repeated root $\lambda = -\frac{1}{2}a$ satisfies $|\lambda| < 1$,
and so if and only if $|a| < 2$.

A Simpler Stability Condition

Theorem

The linear autonomous equation $x_{t+1} + ax_t + bx_{t-1} = f$ *is stable, both locally and globally, if and only if* $|a| < 1 + b < 2$.

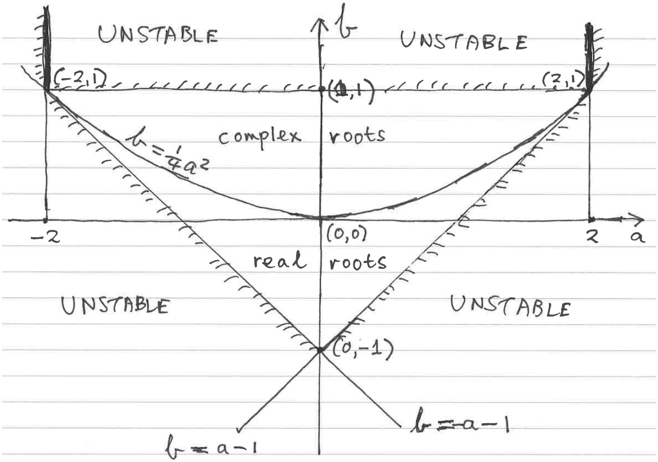
Proof.

Stability requires one of the following three to hold:

1. distinct real roots because $a^2 > 4b$, with $|a| - 1 < b < 1$;
2. complex conjugate roots because $a^2 < 4b$, with $b < 1$;
3. a repeated real root because $a^2 = 4b$, with $|a| < 2$.

A diagram in the (a, b)-plane shows that one of these three holds if and only if $|a| < 1 + b < 2$. □

Diagram of Stable Region



The stable region occurs where $|a| - 1 < b < 1$,
in the interior of an isosceles right-angled triangle
with corners at $(a, b) = (0, -1)$ and $(a, b) = (\pm 2, 1)$.

Stability with a Variable Forcing Term

Consider now the second-order equation $x_{t+1} + ax_t + bx_{t-1} = f_t$ for a variable forcing term f_t.

The general solution takes the form $x_t^G = x_t^H + x_t^P$ where:

- x_t^P is one particular solution of $x_{t+1} + ax_t + bx_{t-1} = f_t$;
- x_t^H is any one of a two-dimension continuum of solutions to the homogeneous equation $x_{t+1} + ax_t + bx_{t-1} = 0$.

The stability condition $|a| < 1 + b < 2$ is necessary and sufficient for any solution of the homogeneous equation to satisfy $x_t^H \to 0$ as $t \to \infty$.

It is therefore also necessary and sufficient for the difference between any two solutions $x_t^{(1)}$ and $x_t^{(2)}$ of the inhomogeneous equation $x_{t+1} + ax_t + bx_{t-1} = f_t$ to satisfy $x_t^{(1)} - x_t^{(2)} \to 0$ as $t \to \infty$.

In the long run, this means that there is an asymptotically unique solution to $x_{t+1} + ax_t + bx_{t-1} = f_t$.