Lecture Notes 10: Dynamic Programming

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Outline

Stochastic Linear Difference Equations in One Variable

Explicit Solution

Gaussian Disturbances

Optimal Saving

Preferences and Constraints

The Two Period Problem

The $T$ Period Problem

A General Savings Problem

General Problems

Finite Horizon Case

Infinite Time Horizon

Stationarity and the Bellman Equation

Finding a Fixed Function

Successive Approximation and Policy Improvement

Unboundedness
A simple stochastic linear difference equation of the first order in one variable takes the form

\[ x_t = ax_{t-1} + \epsilon_t \quad (t \in \mathbb{N}) \]

Here \( a \) is a real parameter, and each \( \epsilon_t \) is a real random disturbance.

Assume that:

1. there is a given or pre-determined initial state \( x_0 \);
2. the random variables \( \epsilon_t \) are independent and identically distributed (IID) with mean \( \mathbb{E}\epsilon_t = 0 \) and variance \( \mathbb{E}\epsilon_t^2 = \sigma^2 \).

A special case is when the disturbances are all normally distributed — i.e., \( \epsilon_t \sim \mathcal{N}(0, \sigma^2) \).
Explicit Solution and Conditional Mean

For each fixed outcome $\epsilon^N = (\epsilon_t)_{t \in \mathbb{N}}$ of the random sequence, there is a unique solution which can be written as

$$x_t = a^t x_0 + \sum_{s=1}^{t} a^{t-s} \epsilon_s$$

The main stable case occurs when $|a| < 1$.

Then each term of the sum $a^t x_0 + \sum_{s=1}^{t} a^{t-s} \epsilon_s$ converges to 0 as $t \to \infty$.

This is what econometricians or statisticians call a first-order autoregressive (or AR(1)) process.

In fact, given $x_0$ at time 0, our assumption that $\mathbb{E}\epsilon_s = 0$ for all $s = 1, 2, \ldots, t$ implies that the conditional mean of $x_t$ is

$$m_t := \mathbb{E}[x_t | x_0] = \mathbb{E} \left[ a^t x_0 + \sum_{s=1}^{t} a^{t-s} \epsilon_s | x_0 \right] = a^t x_0$$
Conditional Variance

The conditional variance, however, is given by

\[ v_t := \mathbb{E} \left[ (x_t - m_t)^2 | x_0 \right] = \mathbb{E}[(x_t - a^t x_0)^2 | x_0] = \mathbb{E} \left[ \sum_{s=1}^{t} a^{t-s} \epsilon_s \right]^2 \]

In the case we are considering

with independently distributed disturbances \( \epsilon_s \),

the variance of a sum is the sum of the variances.

Hence

\[ v_t = \sum_{s=1}^{t} \mathbb{E} \left[ a^{t-s} \epsilon_s \right]^2 = \sum_{s=1}^{t} a^{2(t-s)} \mathbb{E} \epsilon_s^2 = \sigma^2 \sum_{s=1}^{t} a^{2(t-s)} \]

Using the rule for summing the geometric series \( \sum_{s=1}^{t} a^{2(t-s)} \),

we finally obtain

\[ v_t = \frac{1 - a^{2t}}{1 - a^2} \sigma^2 \]
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Unboundedness
Recall that if $X \sim N(\mu, \sigma^2)$, then the characteristic function defined by $\phi_X(t) = \mathbb{E}[e^{iXt}]$ takes the form

$$\phi_X(t) = \mathbb{E}[e^{iXt}] = \int_{-\infty}^{+\infty} e^{ixt} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2} \left(\frac{x-\mu}{\sigma}\right)^2\right) dx$$

This reduces to $\phi_X(t) = \exp \left( it\mu - \frac{1}{2} \sigma^2 t^2 \right)$.

Hence, if $Z = X + Y$ where $X \sim N(\mu_X, \sigma_X^2)$ and $Y \sim N(\mu_Y, \sigma_Y^2)$ are independent random variables, then

$$\phi_Z(t) = \mathbb{E}[e^{iZt}] = \mathbb{E}[e^{i(X+Y)t}] = \mathbb{E}[e^{iXt}e^{iYt}] = \mathbb{E}[e^{iXt}]\mathbb{E}[e^{iYt}]$$
Sums of Normally Distributed Random Variables, II

So

\[ \phi_Z(t) = \exp \left( it \mu_X - \frac{1}{2} \sigma_X^2 t^2 \right) \exp \left( it \mu_Y - \frac{1}{2} \sigma_Y^2 t^2 \right) \]
\[ = \exp \left( it (\mu_X + \mu_Y) - \frac{1}{2} (\sigma_X^2 + \sigma_Y^2) t^2 \right) \]
\[ = \exp \left( it \mu_Z - \frac{1}{2} \sigma_Z^2 t^2 \right) \]

where \( \mu_Z = \mu_X + \mu_Y = \mathbb{E}(X + Y) \) is the mean of \( X + Y \),
and \( \sigma_Z^2 = \sigma_X^2 + \sigma_Y^2 \) is the variance of \( X + Y \).

It follows that \( t \mapsto \phi_Z(t) \)
is the characteristic function of a random variable \( Z \sim N(\mu_Z, \sigma_Z^2) \)where \( \mu_Z = \mu_X + \mu_Y = \mathbb{E}(X + Y) \) and \( \sigma_Z^2 = \sigma_X^2 + \sigma_Y^2 \).

That is, the sum \( Z = X + Y \)
of two independent normally distributed random variables \( X \) and \( Y \)is also normally distributed, with:

1. mean equal to the sum of the means;
2. variance equal to the sum of the variances.
In the particular case when each $\epsilon_t$ is normally distributed as well as IID, then $x_t$ is also normally distributed with mean $m_t$ and variance $\nu_t$.

As $t \to \infty$, the conditional mean $m_t = a^t x_0 \to 0$ and the conditional variance

$$\nu_t = \frac{1 - a^{2t}}{1 - a^2} \sigma^2 \to \nu := \frac{\sigma^2}{1 - a^2}$$

In the case when each $\epsilon_t$ is normally distributed, this implies that the asymptotic distribution of $x_t$ is also normal, with mean 0 and variance $\nu = \sigma^2/(1 - a^2)$. 
Stationarity

Now suppose that \( x_0 \) itself has this asymptotic normal distribution — suppose that \( x_0 \sim N(0, \sigma^2/(1 - a^2)) \).

This is what the distribution of \( x_0 \) would be if the process had started at \( t = -\infty \) instead of at \( t = 0 \).

Then the unconditional mean of each \( x_t \) is \( \mathbb{E}x_t = a^t \mathbb{E}x_0 = 0 \).

On the other hand, because \( x_{t+k} = a^k x_t + \sum_{s=1}^{k} a^{k-s} \epsilon_{t+s} \), the unconditional covariance of \( x_t \) and \( x_{t+k} \) is

\[
\mathbb{E}(x_{t+k}x_t) = \mathbb{E}[a^k x_t^2] = a^k \nu = \frac{a^k}{1 - a^2} \sigma^2 \quad (k = 0, 1, 2 \ldots)
\]

In fact, given any \( t \), the joint distribution of the \( r \) random variables \( x_t, x_{t+1}, \ldots, x_{t+r-1} \) is multivariate normal with variance–covariance matrix having elements \( \mathbb{E}(x_{t+k}x_t) = a^k \sigma^2/(1 - a^2) \), independent of \( t \).

Because of this independence, the process is said to be stationary.
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Unboundedness
Intertemporal Utility

Consider a household which at time $s$ is planning its intertemporal consumption stream $\mathbf{c}_s^T := (c_s, c_{s+1}, \ldots, c_T)$ over periods $t$ in the set $\{s, s+1, \ldots, T\}$.

Its intertemporal utility function $\mathbb{R}^{T-s+1} \ni \mathbf{c}_s^T \mapsto U_s^T(\mathbf{c}_s^T) \in \mathbb{R}$ is assumed to take the additively separable form

$$U_s^T(\mathbf{c}_s^T) := \sum_{t=s}^{T} u_t(c_t)$$

where the one period felicity functions $c \mapsto u_t(c)$ are differentiably increasing and strictly concave (DISC) — i.e., $u_t'(c) > 0$, and $u_t''(c) < 0$ for all $t$ and all $c > 0$.

As before, the household faces:

1. fixed initial wealth $w_s$;
2. a terminal wealth constraint $w_{T+1} \geq 0$. 
Also as before, we assume a wealth accumulation equation \( w_{t+1} = \tilde{r}_t (w_t - c_t) \), where \( \tilde{r}_t \) is the household’s gross rate of return on its wealth in period \( t \).

It is assumed that:

1. the return \( \tilde{r}_t \) in each period \( t \) is a random variable with positive values;
2. the return distributions for different times \( t \) are stochastically independent;
3. starting with predetermined wealth \( w_s \) at time \( s \), the household seeks to maximize the expectation \( \mathbb{E}_s[U_s^T(c_s^T)] \) of its intertemporal utility.
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Unboundedness
Two Period Case

We work backwards from the last period, when \( s = T \).

In this last period the household will obviously choose \( c_T = w_T \), yielding a maximized utility equal to \( V_T(w_T) = u_T(w_T) \).

Next, consider the penultimate period, when \( s = T - 1 \). The consumer will want to choose \( c_{T-1} \) in order to maximize

\[
\begin{align*}
&\underbrace{u_{T-1}(c_{T-1})}_{\text{period } T-1} + \underbrace{E_{T-1}V_T(w_T)}_{\text{result of an optimal policy in period } T} \\
\end{align*}
\]

subject to the wealth constraint

\[
W_T = \underbrace{\tilde{r}_{T-1}}_{\text{random gross return}} \underbrace{(W_{T-1} - c_{T-1})}_{\text{saving}}
\]
First-Order Condition

Substituting both the function $V_T(w_T) = u_T(w_T)$ and the wealth constraint into the objective reduces the problem to

$$\max_{c_{T-1}} \left\{ u_{T-1}(c_{T-1}) + \mathbb{E}_{T-1}[u_T(\tilde{r}_{T-1}(w_{T-1} - c_{T-1}))] \right\}$$

subject to $0 \leq c_{T-1} \leq w_{T-1}$ and $\tilde{c}_T := \tilde{r}_{T-1}(w_{T-1} - c_{T-1})$.

Assume we can differentiate under the integral sign, and that there is an interior solution with $0 < c_{T-1} < w_{T-1}$.

Then the first-order condition (FOC) is

$$0 = u'_{T-1}(c_{T-1}) + \mathbb{E}_{T-1}[(−\tilde{r}_{T-1})u'_T(\tilde{r}_{T-1}(w_{T-1} - c_{T-1}))]$$
The Stochastic Euler Equation

Rearranging the first-order condition while recognizing that $\tilde{c}_T := \tilde{r}_{T-1}(w_{T-1} - c_{T-1})$, one obtains

$$u'_{T-1}(c_{T-1}) = \mathbb{E}_{T-1}[\tilde{r}_{T-1}u'_{T}(\tilde{r}_{T-1}(w_{T-1} - c_{T-1}))]$$

Dividing by $u'_{T-1}(c_{T-1})$ gives the stochastic Euler equation

$$1 = \mathbb{E}_{T-1} \left[ \tilde{r}_{T-1} \frac{u'_T(\tilde{c}_T)}{u'_T(c_{T-1})} \right] = \mathbb{E}_{T-1} \left[ \tilde{r}_{T-1} \text{MRS}_{T-1}^{T}(c_{T-1}; \tilde{c}_T) \right]$$

involving the marginal rate of substitution function

$$\text{MRS}_{T-1}^{T}(c_{T-1}; \tilde{c}_T) := \frac{u'_T(\tilde{c}_T)}{u'_T(c_{T-1})}$$
For the marginal utility function $c \mapsto u'(c)$, its **elasticity of substitution** is defined for all $c > 0$ by $\eta(c) := d \ln u'(c) / d \ln c$.

Then $\eta(c)$ is both the degree of relative risk aversion, and the degree of relative fluctuation aversion.

A **constant elasticity of substitution** (or CES) utility function satisfies $d \ln u'(c) / d \ln c = -\epsilon < 0$ for all $c > 0$.

The marginal rate of substitution satisfies $u'(c)/u'(\bar{c}) = (c/\bar{c})^{-\epsilon}$ for all $c, \bar{c} > 0$. 
Normalized Utility

Normalize by putting $u'(1) = 1$, implying that $u'(c) \equiv c^{-\epsilon}$.

Then integrating gives

$$u(c; \epsilon) = u(1) + \int_1^c x^{-\epsilon} dx$$

$$= \begin{cases} 
  u(1) + \frac{c^{1-\epsilon} - 1}{1-\epsilon} & \text{if } \epsilon \neq 1 \\
  u(1) + \ln c & \text{if } \epsilon = 1 
\end{cases}$$

Introduce the final normalization

$$u(1) = \begin{cases} 
  \frac{1}{1-\epsilon} & \text{if } \epsilon \neq 1 \\
  0 & \text{if } \epsilon = 1 
\end{cases}$$

The utility function is reduced to

$$u(c; \epsilon) = \begin{cases} 
  \frac{c^{1-\epsilon}}{1-\epsilon} & \text{if } \epsilon \neq 1 \\
  \ln c & \text{if } \epsilon = 1 
\end{cases}$$
The Stochastic Euler Equation in the CES Case

Consider the CES case when $u_t'(c) \equiv \delta_t c^{-\epsilon}$, where each $\delta_t$ is the discount factor for period $t$.

Definition
The one-period discount factor in period $t$ is defined as $\beta_t := \delta_{t+1}/\delta_t$.

Then the stochastic Euler equation takes the form

$$1 = \mathbb{E}_{T-1} \left[ \tilde{r}_{T-1} \beta_{T-1} \left( \frac{\tilde{c}_T}{c_{T-1}} \right)^{-\epsilon} \right]$$

Because $c_{T-1}$ is being chosen at time $T-1$, this implies that

$$(c_{T-1})^{-\epsilon} = \mathbb{E}_{T-1} \left[ \tilde{r}_{T-1} \beta_{T-1} (\tilde{c}_T)^{-\epsilon} \right]$$
The Two Period Problem in the CES Case

In the two-period case, we know that

\[ \tilde{c}_T = \tilde{w}_T = \tilde{r}_T - 1 (w_T - 1 - c_T - 1) \]

in the last period, so the Euler equation becomes

\[ (c_{T-1})^{-\epsilon} = \mathbb{E}_{T-1} [\tilde{r}_{T-1} \beta_{T-1} (\tilde{c}_T)^{-\epsilon}] = \beta_{T-1} (w_{T-1} - c_{T-1})^{-\epsilon} \mathbb{E}_{T-1} [(\tilde{r}_{T-1})^{1-\epsilon}] \]

Take the \((-1/\epsilon)\) th power of each side and define

\[ \rho_{T-1} := \left( \beta_{T-1} \mathbb{E}_{T-1} [(\tilde{r}_{T-1})^{1-\epsilon}] \right)^{-1/\epsilon} \]

to reduce the Euler equation to \( c_{T-1} = \rho_{T-1} (w_{T-1} - c_{T-1}) \)

whose solution is evidently \( c_{T-1} = \gamma_{T-1} w_{T-1} \) where

\[ \gamma_{T-1} := \rho_{T-1} / (1 + \rho_{T-1}) \quad \text{and} \quad 1 - \gamma_{T-1} = 1 / (1 + \rho_{T-1}) \]

are respectively the optimal consumption and savings ratios. It follows that \( \rho_{T-1} = \gamma_{T-1} / (1 - \gamma_{T-1}) \) is the consumption/savings ratio.
Optimal Discounted Expected Utility

The optimal policy in periods $T$ and $T - 1$ is $c_t = \gamma_t w_t$ where $\gamma_T = 1$ and $\gamma_{T-1}$ has just been defined.

In this CES case, the discounted utility of consumption in period $T$ is $V_T(w_T) := \delta_T u(w_T; \epsilon)$.

The discounted expected utility at time $T - 1$ of consumption in periods $T$ and $T - 1$ together is

$$V_{T-1}(w_{T-1}) = \delta_{T-1} u(\gamma_{T-1} w_{T-1}; \epsilon) + \delta_T \mathbb{E}_{T-1}[u(\tilde{w}_T; \epsilon)]$$

where $\tilde{w}_T = \tilde{r}_{T-1} (1 - \gamma_{T-1}) w_{T-1}$. 
Discounted Expected Utility in the Logarithmic Case

In the logarithmic case when $\epsilon = 1$, one has

$$V_{T-1}(w_{T-1}) = \delta_{T-1} \ln(\gamma_{T-1} w_{T-1})$$

$$+ \delta_T \mathbb{E}_{T-1}[\ln(\tilde{r}_{T-1}(1 - \gamma_{T-1}) w_{T-1})]$$

It follows that

$$V_{T-1}(w_{T-1}) = \alpha_{T-1} + (\delta_{T-1} + \delta_T) u(w_{T-1}; \epsilon)$$

where

$$\alpha_{T-1} := \delta_{T-1} \ln \gamma_{T-1} + \delta_T \{\ln(1 - \gamma_{T-1}) + \mathbb{E}_{T-1}[\ln \tilde{r}_{T-1}]\}$$
In the CES case when $\epsilon \neq 1$, one has

$$(1 - \epsilon)V_{T-1}(w_{T-1}) = \delta_{T-1}(\gamma_{T-1}w_{T-1})^{1-\epsilon}$$

$$+ \delta_T[(1 - \gamma_{T-1})w_{T-1}]^{1-\epsilon} \mathbb{E}_{T-1}[(\tilde{r}_{T-1})^{1-\epsilon}]$$

so $V_{T-1}(w_{T-1}) = v_{T-1}u(w_{T-1}; \epsilon)$ where

$$v_{T-1} := \delta_{T-1}(\gamma_{T-1})^{1-\epsilon} + \delta_T(1 - \gamma_{T-1})^{1-\epsilon} \mathbb{E}_{T-1}[(\tilde{r}_{T-1})^{1-\epsilon}]$$

In both cases, one can write $V_{T-1}(w_{T-1}) = \alpha_{T-1} + v_{T-1}u(w_{T-1}; \epsilon)$ for a suitable additive constant $\alpha_{T-1}$ (which is 0 in the CES case) and a suitable multiplicative constant $v_{T-1}$. 
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Unboundedness
The Time Line

In each period $t$, suppose:

- the consumer starts with known wealth $w_t$;
- then the consumer chooses consumption $c_t$, along with savings or residual wealth $w_t - c_t$;
- there is a cumulative distribution function $F_t(r)$ on $\mathbb{R}$ that determines the gross return $\tilde{r}_t$ as a positive-valued random variable.

After these three steps have been completed, the problem starts again in period $t + 1$, with the consumer’s wealth known to be $w_{t+1} = \tilde{r}_t(w_t - c_t)$. 
Expected Conditionally Expected Utility

Starting at any $t$, suppose the consumer’s choices, together with the random returns, jointly determine a cdf $F_t^T$ over the space of intertemporal consumption streams $c_t^T$.

The associated expected utility is $\mathbb{E}_t[U_t^T(c_t^T)]$, using the shorthand $\mathbb{E}_t$ to denote integration w.r.t. the cdf $F_t^T$.

Then, given that the consumer has chosen $c_t$ at time $t$, let $\mathbb{E}_{t+1}[\cdot|c_t]$ denote the conditional expected utility.

This is found by integrating w.r.t. the conditional cdf $F_{t+1}^T(c_{t+1}^T|c_t)$.

The law of iterated expectations allows us to write the unconditional expectation $\mathbb{E}_t[U_t^T(c_t^T)]$ as the expectation $\mathbb{E}_t[\mathbb{E}_{t+1}[U_t^T(c_t^T)|c_t]]$ of the conditional expectation.
The Expectation of Additively Separable Utility

Our hypothesis is that the intertemporal von Neumann–Morgenstern utility function takes the additively separable form

\[ U_t^T(c_t^T) = \sum_{\tau=t}^{T} u_\tau(c_\tau) \]

The conditional expectation given \( c_t \) must then be

\[ \mathbb{E}_{t+1}[U_t^T(c_t^T)|c_t] = u_t(c_t) + \mathbb{E}_{t+1} \left[ \sum_{\tau=t+1}^{T} u_\tau(c_\tau)|c_t \right] \]

whose expectation is

\[ \mathbb{E}_t \left[ \sum_{\tau=t}^{T} u_\tau(c_\tau) \right] = u_t(c_t) + \mathbb{E}_t \left[ \mathbb{E}_{t+1} \left[ \sum_{\tau=t+1}^{T} u_\tau(c_\tau) \right] |c_t \right] \]
The Continuation Value

Let $V_{t+1}(w_{t+1})$ be the state valuation function expressing the maximum of the continuation value

$$
E_{t+1} \left[ U_{t+1}^T(c_{t+1}^T) | w_{t+1} \right] = E_{t+1} \left[ \sum_{\tau=t+1}^{T} u_\tau(c_\tau) | w_{t+1} \right]
$$

as a function of the wealth level or state $w_{t+1}$.

Assume this maximum value is achieved by following an optimal policy from period $t + 1$ on.

Then total expected utility at time $t$ will then reduce to

$$
E_t \left[ U_t^T(\tilde{c}_t^T) | c_t \right] = u_t(c_t) + E_t \left[ E_{t+1} \left[ \sum_{\tau=t+1}^{T} u_\tau(c_\tau) | w_{t+1} \right] | c_t \right]
$$

$$
= u_t(c_t) + E_t[V_{t+1}(\tilde{w}_{t+1}) | c_t]
$$

$$
= u_t(c_t) + E_t[V_{t+1}(\tilde{r}_t(w_t - c_t))]
$$
The Principle of Optimality

Maximizing \( \mathbb{E}_s [U_s^T(c_s^T)] \) w.r.t. \( c_s \), taking as fixed
the optimal consumption plans \( c_t(w_t) \) at times \( t = s + 1, \ldots, T \),
therefore requires choosing \( c_s \) to maximize

\[
    u_s(c_s) + \mathbb{E}_s[V_{s+1}(\tilde{r}_s(w_s - c_s))]
\]

Let \( c_s^*(w_s) \) denote a solution to this maximization problem.

Then the value of an optimal plan \( (c_t^*(w_t))^T \)
that starts with wealth \( w_s \) at time \( s \) is

\[
    V_s(w_s) := u_s(c_s^*(w_s)) + \mathbb{E}_s[V_{s+1}(\tilde{r}_s(w_s - c_s^*(w_s)))]
\]

Together, these two properties can be expressed as

\[
    V_s(w_s) = \arg \max_{0 \leq c_s \leq w_s} \{ u_s(c_s) + \mathbb{E}_s[V_{s+1}(\tilde{r}_s(w_s - c_s))] \}
\]

which can be described as the principle of optimality.
An Induction Hypothesis

Consider once again the case when \( u_t(c) \equiv \delta_t u(c; \epsilon) \) for the CES (or logarithmic) utility function that satisfies \( u'(c; \epsilon) \equiv c^{-\epsilon} \) and, specifically

\[
u(c; \epsilon) = \begin{cases} 
    c^{1-\epsilon}/(1 - \epsilon) & \text{if } \epsilon \neq 1; \\
    \ln c & \text{if } \epsilon = 1.
\end{cases}
\]

Inspired by the solution we have already found for the final period \( T \) and penultimate period \( T - 1 \), we adopt the induction hypothesis that there are constants \( \alpha_t, \gamma_t, \nu_t \) \( (t = T, T - 1, \ldots, s + 1, s) \) for which

\[
c_t^*(w_t) = \gamma_t w_t \quad \text{and} \quad V_t(w_t) = \alpha_t + \nu_t u(w_t; \epsilon)
\]

In particular, the consumption ratio \( \gamma_t \) and savings ratio \( 1 - \gamma_t \) are both independent of the wealth level \( w_t \).
Applying Backward Induction

Under the induction hypotheses that

\[ c_t^*(w_t) = \gamma_t w_t \quad \text{and} \quad V_t(w_t) = \alpha_t + v_t u(w_t; \epsilon) \]

the maximand

\[ u_s(c_s) + \mathbb{E}_s[V_{s+1}(\tilde{r}_s(w_s - c_s))] \]

takes the form

\[ \delta_s u(c_s; \epsilon) + \mathbb{E}_s[\alpha_{s+1} + v_{s+1} u(\tilde{r}_s(w_s - c_s); \epsilon)] \]

The first-order condition for this to be maximized w.r.t. \( c_s \) is

\[ 0 = \delta_s u'(c_s; \epsilon) - v_{s+1} \mathbb{E}_s[\tilde{r}_s u'(\tilde{r}_s(w_s - c_s); \epsilon)] \]

or, equivalently, that

\[ \tilde{\delta}_s(c_s)^{-\epsilon} = v_{s+1} \mathbb{E}_s[\tilde{r}_s(\tilde{r}_s(w_s - c_s))^{-\epsilon}] = v_{s+1}(w_s - c_s)^{-\epsilon} \mathbb{E}_s[(\tilde{r}_s)^{1-\epsilon}] \]
Solving the Logarithmic Case

When \( \epsilon = 1 \) and so \( u(c; \epsilon) = \ln c \),
the first-order condition reduces to \( \delta_s(c_s)^{-1} = v_{s+1}(w_s - c_s)^{-1} \).
Its solution is indeed \( c_s = \gamma_s w_s \) where \( \delta_s(\gamma_s)^{-1} = v_{s+1}(1 - \gamma_s)^{-1} \),
implying that \( \gamma_s = \delta_s / (\delta_s + v_{s+1}) \).
The state valuation function then becomes

\[
V_s(w_s) = \delta_s u(\gamma_s w_s; \epsilon) + \alpha_{s+1} + v_{s+1}E_s[u(\tilde{r}_s(1 - \gamma_s)w_s; \epsilon)] \\
= \delta_s \ln(\gamma_s w_s) + \alpha_{s+1} + v_{s+1}E_s[\ln(\tilde{r}_s(1 - \gamma_s)w_s)] \\
= \delta_s \ln(\gamma_s w_s) + \alpha_{s+1} + v_{s+1}\{\ln(1 - \gamma_s)w_s + \ln R_s\}
\]

where we define the geometric mean certainty equivalent return \( R_s \)
so that \( \ln R_s := E_s[\ln(\tilde{r}_s)] \).
The State Valuation Function

The formula

\[ V_s(w_s) = \delta_s \ln(\gamma_s w_s) + \alpha_{s+1} + \nu_{s+1}\{\ln(1 - \gamma_s)w_s + \ln R_s\} \]

reduces to the desired form \( V_s(w_s) = \alpha_s + \nu_s \ln w_s \) provided we take \( \nu_s := \delta_s + \nu_{s+1} \), which implies that \( \gamma_s = \delta_s/\nu_s \), and also

\[
\alpha_s := \delta_s \ln \gamma_s + \alpha_{s+1} + \nu_{s+1}\{\ln(1 - \gamma_s) + \ln R_s\} \\
= \delta_s \ln(\delta_s/\nu_s) + \alpha_{s+1} + \nu_{s+1}\{\ln(\nu_{s+1}/\nu_s) + \ln R_s\} \\
= \delta_s \ln \delta_s + \alpha_{s+1} - \nu_s \ln \nu_s + \nu_{s+1}\{\ln \nu_{s+1} + \ln R_s\}
\]

This confirms the induction hypothesis for the logarithmic case.

The relevant constants \( \nu_s \) are found by summing backwards, starting with \( \nu_T = \delta_T \), implying that \( \nu_s = \sum_{T=s}^T \delta_s \).
The Stationary Logarithmic Case

In the stationary logarithmic case:

- the felicity function in each period $t$ is $\beta^t \ln c_t$, so the one period discount factor is the constant $\beta$;
- the certainty equivalent return $R_t$ is also a constant $R$.

Then $v_s = \sum_{T=s}^T \delta_s = \sum_{T=s}^T \beta^\tau = (\beta^s - \beta^{T+1})/(1 - \beta)$, implying that $\gamma_s = \beta^s/v_s = \beta^s(1 - \beta)/(\beta^s - \beta^{T+1})$.

It follows that

$$c_s = \gamma_s w_s = \frac{(1 - \beta)w_s}{1 - \beta^{T-s+1}} = \frac{(1 - \beta)w_s}{1 - \beta^{H+1}}$$

when there are $H := T - s$ periods left before the horizon $T$.

As $H \to \infty$, this solution converges to $c_s = (1 - \beta)w_s$, so the savings ratio equals the constant discount factor $\beta$.

Remarkably, this is also independent of the gross return to saving.
First-Order Condition in the CES Case

Recall that the first-order condition in the CES Case is

\[ \delta_s(c_s)^{-\epsilon} = v_{s+1}(w_s - c_s)^{-\epsilon} \mathbb{E}_s[(\tilde{r}_s)^{1-\epsilon}] = v_{s+1}(w_s - c_s)^{-\epsilon} R_s^{1-\epsilon} \]

where we have defined the certainty equivalent return \( R_s \) as the solution to \( R_s^{1-\epsilon} := \mathbb{E}_s[(\tilde{r}_s)^{1-\epsilon}] \).

The first-order condition indeed implies that \( c_s^*(w_s) = \gamma_s w_s \), where \( \delta_s(\gamma_s)^{-\epsilon} = v_{s+1}(1 - \gamma_s)^{-\epsilon} R_s^{1-\epsilon} \).

This implies that

\[ \frac{\gamma_s}{1 - \gamma_s} = \left( v_{s+1} R_s^{1-\epsilon} / \delta_s \right)^{-1/\epsilon} \]

or

\[ \gamma_s = \frac{\left( v_{s+1} R_s^{1-\epsilon} / \delta_s \right)^{-1/\epsilon}}{1 + \left( v_{s+1} R_s^{1-\epsilon} / \delta_s \right)^{-1/\epsilon}} = \frac{\left( v_{s+1} R_s^{1-\epsilon} \right)^{-1/\epsilon}}{(\delta_s)^{-1/\epsilon} + \left( v_{s+1} R_s^{1-\epsilon} \right)^{-1/\epsilon}} \]
Completing the Solution in the CES Case

Under the induction hypothesis that $V_{s+1}(w) = v_{s+1} w^{1-\epsilon}/(1 - \epsilon)$, one also has

$$(1 - \epsilon) V_s(w_s) = \delta_s(\gamma_s w_s)^{1-\epsilon} + v_{s+1} \bar{E}_s[(\tilde{r}_s(1 - \gamma_s)w_s)^{1-\epsilon}]$$

This reduces to the desired form $(1 - \epsilon) V_s(w_s) = v_s(w_s)^{1-\epsilon}$, where

$$v_s := \delta_s(\gamma_s)^{1-\epsilon} + v_{s+1} \bar{E}_s[(\tilde{r}_s)^{1-\epsilon}](1 - \gamma_s)^{1-\epsilon}$$

$$= \frac{\delta_s(v_{s+1} R_s^{1-\epsilon})^{1-1/\epsilon} + v_{s+1} R_s^{1-\epsilon}(\delta_s)^{1-1/\epsilon}}{[(\delta_s)^{-1/\epsilon} + (v_{s+1} R_s^{1-\epsilon})^{-1/\epsilon}]^{1-\epsilon}}$$

$$= \delta_s v_{s+1} R_s^{1-\epsilon} \frac{(v_{s+1} R_s^{1-\epsilon})^{-1/\epsilon} + (\delta_s)^{-1/\epsilon}}{[(\delta_s)^{-1/\epsilon} + (v_{s+1} R_s^{1-\epsilon})^{-1/\epsilon}]^{1-\epsilon}}$$

$$= \delta_s v_{s+1} R_s^{1-\epsilon} [(\delta_s)^{-1/\epsilon} + (v_{s+1} R_s^{1-\epsilon})^{-1/\epsilon}]^{1-\epsilon}$$

This confirms the induction hypothesis for the CES case.

Again, the relevant constants are found by working backwards.
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Unboundedness
Histories and Strategies

For each time $t = s, s + 1, \ldots, T$ between the start $s$ and the horizon $T$, let $h^t$ denote a known history $(w^t, c^t, \tilde{r}^t)^{t}_{t=s}$ of the triples $(w^\tau, c^\tau, \tilde{r}^\tau)$ at successive times $\tau = s, s + 1, \ldots, t$ up to time $t$.

A general policy the consumer can choose involves a measurable function $h^t \mapsto \psi_t(h^t)$ mapping each known history up to time $t$, which determines the consumer’s information set, into a consumption level at that time.

The collection of successive functions $\psi_s^T = \langle \psi_t \rangle^{T}_{t=s}$ is what a game theorist would call the consumer’s strategy in the extensive form game “against nature”.

Markov Strategies

We found an optimal solution for the two-period problem when $t = T - 1$.

It took the form of a Markov strategy $\psi_t(h^t) := c^*_t(w_t)$, which depends only on $w_t$ as the particular state variable.

The following analysis will demonstrate in particular that at each time $t = s, s + 1, \ldots, T$, under the induction hypothesis that the consumer will follow a Markov strategy in periods $\tau = t + 1, t + 2, \ldots, T$, there exists a Markov strategy that is optimal in period $t$.

It will follow by backward induction that there exists an optimal strategy $h^t \mapsto \psi_t(h^t)$ for every period $t = s, s + 1, \ldots, T$ that takes the Markov form $h^t \mapsto w_t \mapsto c^*_t(w_t)$.

This treats history as irrelevant, except insofar as it determines current wealth $w_t$ at the time when $c_t$ has to be chosen.
A Stochastic Difference Equation

Accordingly, suppose that the consumer pursues a Markov strategy taking the form \( w_t \mapsto c_t^*(w_t) \).

Then the Markov state variable \( w_t \) will evolve over time according to the stochastic difference equation

\[
w_{t+1} = \phi_t(w_t, \tilde{r}_t) := \tilde{r}_t(w_t - c_t^*(w_t)).
\]

Starting at any time \( t \), conditional on initial wealth \( w_t \), this equation will have a random solution \( \tilde{w}_{t+1}^T = (\tilde{w}_\tau)^T_{\tau=t+1} \) described by a unique joint conditional cdf \( F_{t+1}^T(\mathbf{w}_{t+1}^T | w_t) \) on \( \mathbb{R}^{T-s} \).

Combined with the Markov strategy \( w_t \mapsto c_t^*(w_t) \), this generates a random consumption stream \( \tilde{c}_{t+1}^T = (\tilde{c}_\tau)^T_{\tau=t+1} \) described by a unique joint conditional cdf \( G_{t+1}^T(\mathbf{c}_{t+1}^T | w_t) \) on \( \mathbb{R}^{T-s} \).
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Unboundedness
General Finite Horizon Problem

Consider the objective of choosing
the sequence \((y_s, y_{s+1}, \ldots, y_{T-2}, y_{T-1})\) of controls
in order to maximize

\[
\mathbb{E}_s \left[ \sum_{t=s}^{T-1} u_s(x_s, y_s) + \phi_T(x_T) \right]
\]

subject to the law of motion \(x_{t+1} = \xi_t(x_t, y_t, \epsilon_t)\),
where the random shocks \(\epsilon_t\)
at different times \(t = s, s + 1, s + 2, \ldots, T - 1\)
are conditionally independent given \(x_t, y_t\).
Here \(x_T \mapsto \phi_T(x_T)\) is the terminal state valuation function.

The stochastic law of motion can also be expressed
through successive conditional probabilities \(P_{t+1}(x_{t+1}|x_t, y_t)\).
The choices of \(y_t\) at successive times determine
a controlled Markov process governing the stochastic transition
from each state \(x_t\) to its immediate successor \(x_{t+1}\).
Backward Recurrence Relation

To find the optimal solution, solve the backward recurrence relation

\[
\begin{align*}
V_s(x_s) &= \max \left\{ u_s(x_s, y_s) + \mathbb{E}_s [ V_{s+1}(x_{s+1}) | x_s, y_s ] \right\} \\
y^*_s(x_s) &= \arg \max_{y_s \in F_s(x_s)} \left\{ u_s(x_s, y_s) + \mathbb{E}_s [ V_{s+1}(x_{s+1}) | x_s, y_s ] \right\}
\end{align*}
\]

where, for each start time \( s \),

1. \( x_s \) denotes the “inherited state” at time \( s \);
2. \( V_s(x_s) \) is the current value in state \( x_s \)
   of the state value function \( X \ni x \mapsto V_s(x) \in \mathbb{R} \);
3. \( X \ni x \mapsto F_s(x) \subset Y \) is the feasible set correspondence,
   with graph \( G_s := \{(x, y) | x \in X \& y \in F_s(x)\} \);
4. \( G_s \ni (x, y) \mapsto u_s(x, y) \) denotes the immediate return function;
5. \( X \ni x \mapsto y^*_s(x) \in F_s(x_s) \) is the optimal “strategy”
   or policy function;
6. The relevant terminal condition is that \( V_T(x_T) \)
   is given by the exogenously specified function \( \phi_T(x_T) \).
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Unboundedness
Game theorists speak of the "one-shot" deviation principle. This states that if any deviation from a particular policy or strategy improves a player’s payoff, then there exists a one-shot deviation that improves the payoff.

We consider the infinite horizon extension of the consumption/investment problem already considered. Given the initial time $s$ and initial wealth $w_s$, this takes the form of choosing a consumption policy $c_t(w_t)$ at times $t = s, s + 1, s + 2, \ldots$ in order to maximize the discounted sum of total utility, given by

$$\sum_{t=s}^{\infty} \beta^{t-s} u(c_t)$$

subject to the accumulation equation $w_{t+1} = \tilde{r}_t(w_t - c_t)$ as well as the inequality constraint $w_t \geq 0$ for $t = s + 1, s + 2, \ldots$. 
Some Assumptions

The parameter $\beta \in (0, 1)$ is the **constant discount factor**. Note that utility function $\mathbb{R} \ni c \mapsto u(c)$ is independent of $t$; its first two derivatives are assumed to satisfy the inequalities $u'(c) > 0$ and $u''(c) < 0$ for all $c \in \mathbb{R}_+$. The **investment returns** $\tilde{r}_t$ in successive periods are assumed to be i.i.d. random variables.

It is assumed that $w_t$ in each period $t$ is known at time $t$, but not before.
Terminal Constraint

There has to be an additional constraint that imposes a lower bound on wealth at some time $t$. Otherwise there would be no optimal policy — the consumer can always gain by increasing debt (negative wealth), no matter how large existing debt may be.

In the finite horizon, there was a constraint $w_T \geq 0$ on terminal wealth. But here $T$ is effectively infinite.

One might try an alternative like

$$\lim_{t \to \infty} \inf \beta^t w_t \geq 0$$

But this places no limit on wealth at any finite time. We use the alternative constraint requiring that $w_t \geq 0$ for all time.
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Unboundedness
The Stationary Problem

Our modified problem can be written in the following form that is independent of $s$:

$$\max_{c_0, c_1, \ldots, c_t, \ldots} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to the constraints $0 \leq c_t \leq w_t$ and $w_{t+1} = \tilde{r}_t(w_t - c_t)$ for all $t = 0, 1, 2, \ldots$, with $w_0 = w$, where $w$ is given.

Because the starting time $s$ is irrelevant, this is a stationary problem.

Define the state valuation function $w \mapsto V(w)$ as the maximum value of the objective, as a function of initial wealth $w$.

It is independent of $s$ because the problem is stationary.
Bellman’s Equation

For the finite horizon problem, the principle of optimality was

\[
\begin{align*}
V_s(w_s) &= \max_{0 \leq c_s \leq w_s} \left\{ u_s(c_s) + \mathbb{E}_s[V_{s+1}(\tilde{r}_s(w_s - c_s))] \right\} \\
c_s^*(w_s) &= \arg\max_{0 \leq c_s \leq w_s} \left\{ u_s(c_s) + \mathbb{E}_s[V_{s+1}(\tilde{r}_s(w_s - c_s))] \right\}
\end{align*}
\]

For the stationary infinite horizon problem, however, the time starting time \(s\) is irrelevant.

So the principle of optimality can be expressed as

\[
\begin{align*}
V(w) &= \max_{0 \leq c \leq w} \left\{ u(c) + \beta \mathbb{E}[V(\tilde{r}(w - c))] \right\} \\
c^*(w) &= \arg\max_{0 \leq c \leq w} \left\{ u(c) + \beta \mathbb{E}[V(\tilde{r}(w - c))] \right\}
\end{align*}
\]

The state valuation function \(w \mapsto V(w)\) appears on both left and right hand sides of this equation.

Solving it therefore involves finding a fixed point, or function, in an appropriate function space.
Isoelastic Case

We consider yet again the isoelastic case with a CES (or logarithmic) utility function that satisfies $u'(c; \epsilon) \equiv c^{-\epsilon}$ and, specifically

$$u(c; \epsilon) = \begin{cases} 
    c^{1-\epsilon}/(1 - \epsilon) & \text{if } \epsilon \neq 1; \\
    \ln c & \text{if } \epsilon = 1.
\end{cases}$$

Recall the corresponding finite horizon case, where we found that the solution to the corresponding equations takes the form: (i) $V_s(w_s) = \alpha_s + \nu_s u(w; \epsilon)$ for suitable real constants $\alpha_s$ and $\nu_s > 0$, where $\alpha_s = 0$ if $\epsilon \neq 1$; (ii) $c_s^*(w_s) = \gamma_s w_s$ for a suitable constant $\gamma_s \in (0, 1)$. 
First-Order Condition

Accordingly, we look for a solution to the stationary problem

\[
\begin{align*}
V(w) &= c^*(w) = \arg \max_{0 \leq c \leq w} \left\{ u(c; \epsilon) + \beta \mathbb{E}[V(\tilde{r}(w - c))] \right\}
\end{align*}
\]

taking the isoelastic form \( V(w) = \alpha + \nu u(w; \epsilon) \)
for suitable real constants \( \alpha \) and \( \nu > 0 \), where \( \alpha = 0 \) if \( \epsilon \neq 1 \).

The first-order condition for solving this concave maximization problem is

\[
c^{-\epsilon} = \beta \mathbb{E}[\tilde{r}(\tilde{r}(w - c))^{-\epsilon}] = \zeta(1 - \gamma)^{-\epsilon}
\]

where \( \zeta \) := \( \beta R^{1-\epsilon} \) with \( R \) as the certainty equivalent return defined by \( R^{1-\epsilon} := \mathbb{E}[\tilde{r}^{1-\epsilon}] \).

Hence \( c = \gamma w \) where \( \gamma^{-\epsilon} = \zeta(1 - \gamma)^{-\epsilon} \),
implying that \( \zeta = (1 - \gamma) / \gamma \), the savings–consumption ratio.
Then \( \gamma = 1/(1 + \zeta) \), so \( 1 - \gamma = \zeta/(1 + \zeta) \).
Solution in the Logarithmic Case

When $\epsilon = 1$ and so $u(c; \epsilon) = \ln c$, one has

$$V(w) = u(\gamma w; \epsilon) + \beta \{\alpha + v \mathbb{E}[u(\tilde{r}(1 - \gamma)w; \epsilon)]\}$$

$$= \ln(\gamma w) + \beta \{\alpha + v \mathbb{E}[\ln(\tilde{r}(1 - \gamma)w)]\}$$

$$= \ln \gamma + (1 + \beta v) \ln w + \beta \{\alpha + v \ln(1 - \gamma) + \mathbb{E}[\ln \tilde{r}]\}$$

This is consistent with $V(w) = \alpha + v \ln w$ in case:

1. $v = 1 + \beta v$, implying that $v = (1 - \beta)^{-1}$;

2. and also $\alpha = \ln \gamma + \beta \{\alpha + v \ln(1 - \gamma) + \mathbb{E}[\ln \tilde{r}]\}$, which implies that

$$\alpha = (1 - \beta)^{-1} \left[\ln \gamma + \beta \{(1 - \beta)^{-1} \ln(1 - \gamma) + \mathbb{E}[\ln \tilde{r}]\}\right]$$

This confirms the solution for the logarithmic case.
Solution in the CES Case

When $\epsilon \neq 1$ and so $u(c; \epsilon) = c^{1-\epsilon}/(1 - \epsilon)$, the equation

$$V(w) = u(\gamma w; \epsilon) + \beta v \mathbb{E}[u(\tilde{r}(1 - \gamma)w; \epsilon)]$$

implies that

$$(1 - \epsilon)V(w) = (\gamma w)^{1-\epsilon} + \beta v \mathbb{E}[(\tilde{r}(1 - \gamma)w)^{1-\epsilon}] = vw^{1-\epsilon}$$

where $v = \gamma^{1-\epsilon} + \beta v (1 - \gamma)^{1-\epsilon} R^{1-\epsilon}$ and so

$$v = \frac{\gamma^{1-\epsilon}}{1 - \beta (1 - \gamma)^{1-\epsilon} R^{1-\epsilon}} = \frac{\gamma^{1-\epsilon}}{1 - (1 - \gamma)^{1-\epsilon} \zeta^{\epsilon}}$$

But optimality requires $\gamma = 1/(1 + \zeta)$, implying finally that

$$v = \frac{(1 + \zeta)^{\epsilon-1}}{1 - \zeta (1 + \zeta)^{\epsilon-1}} = \frac{1}{(1 + \zeta)^{1-\epsilon} - \zeta}$$

This confirms the solution for the CES case.
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Unboundedness
Uniformly Bounded Returns

Suppose that the stochastic transition from each state $x$ to the immediately succeeding state $\tilde{x}$ is specified by a conditional probability measure $B \mapsto \mathbb{P}(\tilde{x} \in B|x, u)$ on a $\sigma$-algebra of the state space.

Consider the stationary problem of choosing a policy $x \mapsto u^*(x)$ in order to maximize the infinite discounted sum of utility

$$\mathbb{E} \sum_{t=1}^{\infty} \beta^{t-1} f(x_t, u_t)$$

where $0 < \beta < 1$, with $x_1$ given and subject to $u_t \in U(x_t)$ for $t = 1, 2, \ldots$.

The return function $(x, u) \mapsto f(x, u) \in \mathbb{R}$ is uniformly bounded provided there exist a uniform lower bound $M_*$ and a uniform upper bound $M^*$ such that

$$M_* \leq f(x, u) \leq M^* \quad \text{for all} \ (x, u)$$
The Function Space

The boundedness assumption $M_* \leq f(x, u) \leq M^*$ for all $(x, u)$ ensures that, because $0 < \beta < 1$ and so $\sum_{t=1}^{\infty} \beta^{t-1} = \frac{1}{1 - \beta}$, the infinite discounted sum of utility

$$W := \mathbb{E} \sum_{t=1}^{\infty} \beta^{t-1} f(x_t, u_t)$$

satisfies $(1 - \beta) W \in [M_*, M^*]$.

This makes it natural to consider the linear space $V$ of all bounded functions $X \ni x \mapsto V(x) \in \mathbb{R}$ equipped with its sup norm defined by $\|V\| := \sup_{x \in X} |V(x)|$.

We will pay special attention to the subset

$$V_M := \{ V \in V \mid x \in X \mapsto (1 - \beta) V(x) \in [M_*, M^*] \}$$

of state valuation functions whose values $V(x)$ all lie within the range of the possible values of $W$. 
Existence and Uniqueness

**Theorem**

Consider the Bellman equation system

\[
\begin{align*}
V(x) &= u^*(x) \\
u^*(x) &\in \arg\max_{u \in F(x)} \left\{ f(x, u) + \beta \mathbb{E} [V(\tilde{x})|x, u] \right\}
\end{align*}
\]

Under the assumption of uniformly bounded returns satisfying \( M_* \leq f(x, u) \leq M^* \) for all \((x, u)\):

1. among the set \( \mathcal{V}_M \) of state valuation functions that satisfy the inequalities \( M_* \leq (1 - \beta)V(x) \leq M^* \) for all \( x \), there is a unique state valuation function \( x \mapsto V(x) \) that satisfies the Bellman equation system.

2. any associated policy solution \( x \mapsto u^*(x) \) determines an optimal policy that is stationary — i.e., independent of time.
Two Mappings

Given any measurable policy function $X \ni x \mapsto u(x)$ denoted by $u$, define the mapping $T^u : \mathcal{V}_M \rightarrow \mathcal{V}$ by

$$[T^u V](x) := f(x, u(x)) + \beta \mathbb{E} [V(\tilde{x})|x, u(x)]$$

When the state is $x$, this gives the value $[T^u V](x)$ of choosing the policy $u(x)$ for one period, and then experiencing a future discounted return $V(\tilde{x})$ after reaching each possible subsequent state $\tilde{x} \in X$.

Define also the mapping $T^* : \mathcal{V}_M \rightarrow \mathcal{V}$ by

$$[T^* V](x) := \max_{u \in F(x)} \{ f(x, u) + \beta \mathbb{E} [V(\tilde{x})|x, u] \}$$

These definitions allow the Bellman equation system to be rewritten as

$$V(x) = [T^* V](x)$$

$$u^*(x) \in \arg \max_{u \in F(x)} [T^u V](x)$$
Two Mappings of $\mathcal{V}_M$ into Itself

For all $V \in \mathcal{V}_M$, policies $u$, and $x \in X$, we have defined

$$[T^u V](x) := f(x, u(x)) + \beta \mathbb{E} [V(\tilde{x}) | x, u(x)]$$

and

$$[T^* V](x) := \max_{u \in F(x)} \{ f(x, u) + \beta \mathbb{E} [V(\tilde{x}) | x, u] \}$$

Recall the uniform boundedness condition $M_* \leq f(x, u) \leq M^*$, together with the assumption that $V$ belongs to the domain $\mathcal{V}_M$ of functions satisfying $M_* \leq (1 - \beta) V(\tilde{x}) \leq M^*$ for all $\tilde{x}$.

So these two definitions jointly imply that

$$[T^u V](x) \geq M_* + \beta (1 - \beta)^{-1} M_* = (1 - \beta)^{-1} M_*$$

and

$$[T^u V](x) \leq M^* + \beta (1 - \beta)^{-1} M^* = (1 - \beta)^{-1} M^*$$

Similarly, given any $V \in \mathcal{V}_M$, one has $M_* \leq (1 - \beta) [T^* V](x) \leq M^*$ for all $x \in X$.

Therefore both $V \mapsto T^u V$ and $V \mapsto T^* V$ map $\mathcal{V}_M$ into itself.
A First Contraction Mapping

The definition \( [T^u V](x) := f(x, u(x)) + \beta \mathbb{E} [V(\tilde{x})|x, u(x)] \) implies that for any two functions \( V_1, V_2 \in \mathcal{V}_M \), one has

\[
[T^u V_1](x) - [T^u V_2](x) = \beta \mathbb{E} [V_1(\tilde{x}) - V_2(\tilde{x})|x, u(x)]
\]

The definition of the sup norm therefore implies that

\[
\| T^u V_1 - T^u V_2 \| = \sup_{x \in X} \| [T^u V_1](x) - [T^u V_2](x) \|
\]

\[
= \sup_{x \in X} \| \beta \mathbb{E} [V_1(\tilde{x}) - V_2(\tilde{x})|x, u(x)] \|
\]

\[
= \beta \sup_{x \in X} \| \mathbb{E} [V_1(\tilde{x}) - V_2(\tilde{x})|x, u(x)] \|
\]

\[
\leq \beta \sup_{x \in X} | V_1(\tilde{x}) - V_2(\tilde{x}) |
\]

\[
= \beta \| V_1 - V_2 \|
\]

Hence \( \mathcal{V}_M \ni V \mapsto T^u V \in \mathcal{V}_M \) is a contraction mapping with factor \( \beta < 1 \).
For each fixed policy \( u \), the contraction mapping

\[
\mathcal{V}_M \ni V \mapsto T^u V \in \mathcal{V}_M
\]

has a unique fixed point in the form of a function \( V^u \in \mathcal{V}_M \).

Furthermore, given any initial function \( V \in \mathcal{V}_M \), consider the infinite sequence of mappings \([T^u]^k V \ (k \in \mathbb{N})\) that result from applying the operator \( T^u \) iteratively \( k \) times.

The contraction mapping property of \( T^u \) implies that \( \| [T^u]^k V - V^u \| \to 0 \) as \( k \to \infty \).
Characterizing the Fixed Point, I

Starting from \( V_0 = 0 \) and given any initial state \( x \in X \), note that

\[
[T^u]^k V_0(x) = [T^u] ( [T^u]^{k-1} V_0 ) (x) \\
= f(x, u(x)) + \beta \mathbb{E} \left[ ( [T^u]^{k-1} V_0 ) (\tilde{x}) | x, u(x) \right]
\]

It follows by induction on \( k \) that \( [T^u]^k V_0(\tilde{x}) \) equals the expected discounted total payoff \( \mathbb{E} \sum_{t=1}^{k} \beta^{t-1} f(x_t, u_t) \) of starting from \( x_0 = \tilde{x} \) and then following the policy \( x \mapsto u(x) \) for \( k \) subsequent periods.

Taking the limit as \( k \to \infty \), it follows that for any state \( \tilde{x} \in X \), the value \( V^u(\tilde{x}) \) of the fixed point in \( \mathcal{V}_M \) is the expected discounted total payoff

\[
\mathbb{E} \sum_{t=1}^{\infty} \beta^{t-1} f(x_t, u_t)
\]

of starting from \( x_0 = \tilde{x} \) and then following the policy \( x \mapsto u(x) \) for ever thereafter.
A Second Contraction Mapping

Recall the definition

\[ [T^* V](x) := \max_{u \in F(x)} \{ f(x, u) + \beta \mathbb{E} [V(\tilde{x})|x, u] \} \]

Given any state \( x \in X \) and any two functions \( V_1, V_2 \in \mathcal{V}_M \), define \( u_1, u_2 \in F(x) \) so that for \( k = 1, 2 \) one has

\[ [T^* V_k](x) = f(x, u_k) + \beta \mathbb{E} [V_k(\tilde{x})|x, u_k] \]

Note that \( [T^* V_2](x) \geq f(x, u_1) + \beta \mathbb{E} [V_2(\tilde{x})|x, u_1] \) implying that

\[ [T^* V_1](x) - [T^* V_2](x) \leq \beta \mathbb{E} [V_1(\tilde{x}) - V_2(\tilde{x})|x, u_1] \]
\[ \leq \beta \| V_1 - V_2 \| \]

Similarly, interchanging 1 and 2 in the above argument gives \( [T^* V_2](x) - [T^* V_1](x) \leq \beta \| V_1 - V_2 \| \).

Hence \( \| T^* V_1 - T^* V_2 \| \leq \beta \| V_1 - V_2 \| \), so \( T^* \) is also a contraction.
Applying the Contraction Mapping Theorem, II

Similarly the contraction mapping $V \mapsto T^* V$
has a unique fixed point in the form of a function $V^* \in \mathcal{V}_M$
such that $V^*(\bar{x})$ is the maximized expected discounted total payoff
of starting in state $x_0 = \bar{x}$
and following an optimal policy for ever thereafter.

Moreover, $V^* = T^* V^* = T^{u^*} V$.

This implies that $V^*$ is also the value
of following the policy $x \mapsto u^*(x)$ throughout,
which must therefore be an optimal policy.
Characterizing the Fixed Point, II

Starting from $V_0 = 0$ and given any initial state $x \in X$, note that

$$\left[T^*\right]^k V_0(x) = \left[T^*\right]\left(\left[T^*\right]^{k-1} V_0\right)(x)$$
$$= \max_{u \in F(x)} \{ f(x, u) + \beta \mathbb{E} \left[ \left(\left[T^*\right]^{k-1} V_0\right)(\tilde{x}) | x, u \right] \}$$

It follows by induction on $k$ that $\left[T^*\right]^k V_0(\bar{x})$ equals the maximum possible expected discounted total payoff $\mathbb{E} \sum_{t=1}^{k} \beta^{t-1} f(x_t, u_t)$ of starting from $x_1 = \bar{x}$ and then following the “backward” sequence of optimal policies $(u_k^*, u_{k-1}^*, u_{k-2}^*, \ldots, u_2^*, u_1^*)$, where for each $k$ the policy $x \mapsto u_k^*(x)$ is optimal when $k$ periods remain.
Outline

Stochastic Linear Difference Equations in One Variable
   Explicit Solution
   Gaussian Disturbances

Optimal Saving
   Preferences and Constraints
   The Two Period Problem
   The $T$ Period Problem
   A General Savings Problem

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Unboundedness
Method of Successive Approximation

The method of successive approximation starts with an arbitrary function $V_0 \in \mathcal{V}_M$.

For $k = 1, 2, \ldots$, it then repeatedly solves the pair of equations $V_k = T^* V_{k-1} = T^{u_k^*} V_{k-1}$ to construct sequences of:

1. state valuation functions $X \ni x \mapsto V_k(x) \in \mathbb{R}$;
2. policies $X \ni x \mapsto u_k^*(x) \in F(x)$ that are optimal given that one applies the preceding state valuation function $X \ni \tilde{x} \mapsto V_{k-1}(\tilde{x}) \in \mathbb{R}$ to each immediately succeeding state $\tilde{x}$.

Because the operator $V \mapsto T^* V$ on $\mathcal{V}_M$ is a contraction mapping, the method produces a convergent sequence $(V_k)_{k=1}^{\infty}$ of state valuation functions whose limit satisfies $V^* = T^* V^* = T^{u^*} V^*$ for a suitable policy $X \ni x \mapsto u^*(x) \in F(x)$. 


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Monotonicity

For all functions $V \in V_M$, policies $u$, and states $x \in X$, we have defined

$$[T^u V](x) := f(x, u(x)) + \beta \mathbb{E}[V(\tilde{x})|x, u(x)]$$

and

$$[T^* V](x) := \max_{u \in F(x)} \{f(x, u) + \beta \mathbb{E}[V(\tilde{x})|x, u]\}$$

Notation

Given any pair $V_1, V_2 \in V_M$, we write $V_1 \succeq V_2$ to indicate that the inequality $V_1(x) \geq V_2(x)$ holds for all $x \in X$.

Definition

An operator $\mathcal{V}_M \ni V \mapsto TV \in \mathcal{V}_M$ is monotone just in case whenever $V_1, V_2 \in \mathcal{V}_M$ satisfy $V_1 \succeq V_2$, one has $TV_1 \succeq TV_2$.

Theorem

The following operators on $\mathcal{V}_M$ are monotone:

1. $V \mapsto T^u V$ for all policies $u$;
2. $V \mapsto T^* V$ for the optimal policy.
Proof that $T^u$ is Monotone

Given any state $x \in X$ and any two functions $V_1, V_2 \in \mathcal{V}_M$, the definition of $T^u$ implies that

$$[T^u V_1](x) := f(x, u(x)) + \beta \mathbb{E} [V_1(\tilde{x}) | x, u(x)]$$

and

$$[T^u V_2](x) := f(x, u(x)) + \beta \mathbb{E} [V_2(\tilde{x}) | x, u(x)]$$

Subtracting the second equation from the first implies that

$$[T^u V_1](x) - [T^u V_2](x) = \beta \mathbb{E} [V_1(\tilde{x}) - V_2(\tilde{x}) | x, u(x)]$$

If $V_1 \geq V_2$ and so the inequality $V_1(\tilde{x}) \geq V_2(\tilde{x})$ holds for all $\tilde{x} \in X$, it follows that $[T^u V_1](x) \geq [T^u V_2](x)$.

Since this holds for all $x \in X$, we have proved that $T^u V_1 \geq T^u V_2$. \qed
Proof that $T^*$ is Monotone

Given any state $x \in X$ and any two functions $V_1, V_2 \in \mathcal{V}_M$, define $u_1, u_2 \in F(x)$ so that for $k = 1, 2$ one has

$$[T^* V_k](x) = \max_{u \in F(x)} \{ f(x, u) + \beta \mathbb{E} [V_k(\tilde{x})|x, u] \}$$

$$= [T^{u_k} V_k](x) = f(x, u_k) + \beta \mathbb{E} [V_k(\tilde{x})|x, u_k]$$

It follows that

$$[T^* V_1](x) \geq f(x, u_2) + \beta \mathbb{E} [V_1(\tilde{x})|x, u_2]$$

and

$$[T^* V_2](x) = f(x, u_2) + \beta \mathbb{E} [V_2(\tilde{x})|x, u_2]$$

Subtracting the second equation from the first inequality gives

$$[T^* V_1](x) - [T^* V_2](x) \geq \beta \mathbb{E} [V_1(\tilde{x}) - V_2(\tilde{x})|x, u_2]$$

If $V_1 \geq V_2$ and so the inequality $V_1(\tilde{x}) \geq V_2(\tilde{x})$ holds for all $\tilde{x} \in X$, it follows that $[T^* V_1](x) \geq [T^* V_2](x)$.

Since this holds for all $x \in X$, we have proved that $T^* V_1 \geq T^* V_2$. \qed
Starting Policy Improvement

The method of policy improvement starts with any fixed policy \( u_0 \) or \( X \ni x \mapsto u_0(x) \in F(x) \), along with the value \( V^{u_0} \in \mathcal{V}_M \) of following that policy for ever.

The value \( V^{u_0} \) is the unique fixed point satisfying \( V^{u_0} = T^{u_0} V^{u_0} \) which belongs to the domain \( \mathcal{V}_M \) of suitably bounded functions.

At each step \( k = 1, 2, \ldots \), given the previous policy \( u_{k-1} \) and associated value \( V^{u_{k-1}} \) satisfying \( V^{u_{k-1}} = T^{u_{k-1}} V^{u_{k-1}} \):

1. the policy \( u_k \) is chosen so that \( T^* V^{u_{k-1}} = T^{u_k} V^{u_{k-1}} \);
2. the state valuation function \( x \mapsto V_k(x) \) is chosen as the unique fixed point in \( \mathcal{V}_M \) of the operator \( T^{u_k} \).
Policy Improvement Theorem

Theorem

The infinite sequence \((u_k, V^{u_k})_{k \in \mathbb{N}}\)
consisting of pairs of policies \(u_k\)
with their associated valuation functions \(V^{u_k} \in \mathcal{V}_M\) satisfies

1. \(V^{u_k} \geq V^{u_{k-1}}\) for all \(k \in \mathbb{N}\) (policy improvement);

2. \(\|V^{u_k} - V^*\| \to 0\) as \(k \to \infty\),
   where \(V^*\) is the infinite-horizon optimal
   state valuation function in \(\mathcal{V}_M\) that satisfies \(T^*V^* = V^*\).
Proof of Policy Improvement

By definition of the optimality operator $T^*$, one has $T^* V \geq T^u V$ for all functions $V \in \mathcal{V}_M$ and all policies $u$.

So at each step $k$ of the policy improvement routine, one has

$$T^{u_k} V^{u_{k-1}} = T^* V^{u_{k-1}} \geq T^{u_{k-1}} V^{u_{k-1}} = V^{u_{k-1}}$$

In particular, $T^{u_k} V^{u_{k-1}} \geq V^{u_{k-1}}$.

Now, applying successive iterations of the monotonic operator $T^{u_k}$ implies that

$$V^{u_{k-1}} \leq T^{u_k} V^{u_{k-1}} \leq [T^{u_k}]^2 V^{u_{k-1}} \leq \ldots$$

$$\ldots \leq [T^{u_k}]^r V^{u_{k-1}} \leq [T^{u_k}]^{r+1} V^{u_{k-1}} \leq \ldots$$

But the definition of $V^{u_k}$ implies that for all $V \in \mathcal{V}_M$, including $V = V^{u_{k-1}}$, one has $\|[T^{u_k}]^r V - V^{u_k}\| \to 0$ as $r \to \infty$.

Hence $V^{u_k} = \sup_r [T^{u_k}]^r V^{u_{k-1}} \geq V^{u_{k-1}}$, thus confirming that the policy $u_k$ does improve $u_{k-1}$.  

\[\square\]
Proof of Convergence

Recall that at each step $k$ of the policy improvement routine, one has $T^u_k V^{u_{k-1}} = T^* V^{u_{k-1}}$ and also $T^u_k V^{u_k} = V^{u_k}$.

Now, for each state $x \in X$, define $\hat{V}(x) := \sup_{k \in \mathbb{N}} V^u_k(x)$.

Because $V^u_k \geq V^u_{k-1}$ and $T^u_k$ is monotonic, one has $V^u_k = T^u_k V^u_k \geq T^u_k V^u_{k-1} = T^* V^u_{k-1}$.

Next, because $T^*$ is monotonic, it follows that

$$\hat{V} = \sup_k V^u_k \geq \sup_k T^* V^u_{k-1} = T^*(\sup_k V^u_{k-1}) = T^* \hat{V}$$

Similarly, monotonicity of $T^*$ and its definition together imply that

$$\hat{V} = \sup_k V^u_k = \sup_k T^u_k V^u_k \leq \sup_k T^* V^u_k = T^*(\sup_k V^u_k) = T^* \hat{V}$$

Hence $\hat{V} = T^* \hat{V} = V^*$, because $T^*$ has a unique fixed point.

Therefore $V^* = \sup_k V^u_k$ and so, because the sequence $V^u_k(x)$ is non-decreasing, one has $V^u_k(x) \to V^*(x)$ for each $x \in X$. 

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Lecture Outline

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Unboundedness
Unbounded Utility

In economics the boundedness condition $M_* \leq f(x, u) \leq M^*$ is rarely satisfied!

Consider for example the isoelastic utility function

$$u(c; \epsilon) = \begin{cases} 
  \frac{c^{1-\epsilon}}{1-\epsilon} & \text{if } \epsilon > 0 \text{ and } \epsilon \neq 1 \\
  \ln c & \text{if } \epsilon = 1
\end{cases}$$

This function is obviously:

1. bounded below but unbounded above in case $0 < \epsilon < 1$;
2. unbounded both above and below in case $\epsilon = 1$;
3. bounded above but unbounded below in case $\epsilon > 1$.

Also commonly used is the negative exponential utility function defined by $u(c) = -e^{-\alpha c}$

where $\alpha$ is the constant absolute rate of risk aversion (CARA).

This function is bounded above and, provided that $c \geq 0$, also below.
Warning Example: Statement of Problem

The following example shows that there can be irrelevant **unbounded** solutions to the Bellman equation.

**Example**

Consider the problem of maximizing \( \sum_{t=0}^{\infty} \beta^t (1 - u_t) \)

where \( u_t \in [0, 1] \), \( 0 < \beta < 1 \), and \( x_{t+1} = \frac{1}{\beta} (x_t + u_t) \), with \( x_0 > 0 \).

Notice that \( x_{t+1} \geq \frac{1}{\beta} x_t \) implying that \( x_t \geq \beta^{-t} x_0 \to \infty \) as \( t \to \infty \).

Of course the return function \([0, 1] \ni u \mapsto f(x, u) = 1 - u \in [0, 1]\) is uniformly bounded.

Warning Example: Unbounded Spurious Solution

The Bellman equation is

\[
J(x) = u^*(x) = \arg \max_{u \in [0,1]} \left\{ 1 - u + \beta J \left( \frac{1}{\beta} (x + u) \right) \right\}
\]

Even though the return function is uniformly bounded, this Bellman equation has an unbounded spurious solution.

Indeed, we find a spurious solution with \( J(x) \equiv \gamma + x \) for a suitable constant \( \gamma \).

The condition for this to solve the Bellman equation is that

\[
\gamma + x = \max_{u \in [0,1]} \left\{ 1 - u + \beta \left[ \gamma + \frac{1}{\beta} (x + u) \right] \right\} = \max_{u \in [0,1]} \{ 1 + \beta \gamma + x \} = 1 + \beta \gamma + x
\]

which is true iff \( \gamma = 1 + \beta \gamma \) and so \( \gamma = (1 - \beta)^{-1} \).
Warning Example: True Solution

The problem is to maximize \( \sum_{t=0}^{\infty} \beta^t (1 - u_t) \)
where \( u_t \in [0, 1] \), \( 0 < \beta < 1 \), and \( x_{t+1} = \frac{1}{\beta} (x_t + u_t) \), with \( x_0 > 0 \).

The obvious optimal policy is to choose \( u_t = 0 \) for all \( t \), giving the maximized value \( J(x) = \sum_{t=0}^{\infty} \beta^t = (1 - \beta)^{-1} \).

Indeed the bounded function \( J(x) = (1 - \beta)^{-1} \), together with \( u^* = 0 \), both independent of \( x \), do indeed solve the Bellman equation

\[
J(x) = \max_{u \in [0,1]} \left\{ 1 - u + \beta J \left( \frac{1}{\beta} (x + u) \right) \right\} = \max_{u \in [0,1]} \left\{ 1 - u + \beta (1 - \beta)^{-1} \right\} = 1 + \frac{\beta}{1 - \beta} = \frac{1}{1 - \beta}
\]