Lecture Outline

Special Matrices
- Square, Symmetric, and Diagonal Matrices
- The Identity Matrix
- The Inverse Matrix
- Partitioned Matrices

Permutations and Their Signs
- Permutations
- Transpositions
- Signs of Permutations
- Basic Lemma on Signs of Permutations
- Some Implications

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- Determinants of Order 2
- Determinants of Order 3
- The Determinant Function
- Permutation and Transposition Matrices
- Triangular Matrices
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Square Matrices

A square matrix has an equal number of rows and columns, this number being called its dimension.

The (principal, or main) diagonal of a square matrix $A = (a_{ij})_{n \times n}$ of dimension $n$ is the list $(a_{ii})_{i=1}^{n} = (a_{11}, a_{22}, \ldots, a_{nn})$ of its $n$ diagonal elements.

The other elements $a_{ij}$ with $i \neq j$ are the off-diagonal elements.

A square matrix is often expressed in the form

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix}$$

with some extra dots along the diagonal.
Symmetric Matrices

Definition
A square matrix $A$ is symmetric just in case it is equal to its transpose — i.e., if $A^\top = A$.

Example
The product of two symmetric matrices need not be symmetric.

Using again our example of non-commuting $2 \times 2$ matrices, here are two examples where the product of two symmetric matrices is asymmetric:

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$;

$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$
Two Exercises with Symmetric Matrices

Exercise
Let \( \mathbf{x} \) be a column \( n \)-vector.

1. Find the dimensions of \( \mathbf{x}^\top \mathbf{x} \) and of \( \mathbf{x} \mathbf{x}^\top \).
2. Show that one is a non-negative number which is positive unless \( \mathbf{x} = \mathbf{0} \), and that the other is an \( n \times n \) symmetric matrix.

Exercise
Let \( \mathbf{A} \) be an \( m \times n \)-matrix.

1. Find the dimensions of \( \mathbf{A}^\top \mathbf{A} \) and of \( \mathbf{A} \mathbf{A}^\top \).
2. Show that both \( \mathbf{A}^\top \mathbf{A} \) and \( \mathbf{A} \mathbf{A}^\top \) are symmetric matrices.
3. Show that \( m = n \) is a necessary condition for \( \mathbf{A}^\top \mathbf{A} = \mathbf{A} \mathbf{A}^\top \).
4. Show that \( m = n \) with \( \mathbf{A} \) symmetric is a sufficient condition for \( \mathbf{A}^\top \mathbf{A} = \mathbf{A} \mathbf{A}^\top \).
Diagonal Matrices

A square matrix $A = (a_{ij})^{n \times n}$ is diagonal just in case all of its off diagonal elements are 0 — i.e., $i \neq j \implies a_{ij} = 0$.

A diagonal matrix of dimension $n$ can be written in the form

$$D = \begin{pmatrix} 
    d_1 & 0 & 0 & \cdots & 0 \\
    0 & d_2 & 0 & \cdots & 0 \\
    0 & 0 & d_3 & \cdots & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & 0 & \cdots & d_n 
\end{pmatrix} = \text{diag}(d_1, d_2, d_3, \ldots, d_n) = \text{diag} \mathbf{d}$$

where the $n$-vector $\mathbf{d} = (d_1, d_2, d_3, \ldots, d_n) = (d_i)_{i=1}^n$ consists of the diagonal elements of $D$.

Note that $\text{diag} \mathbf{d} = (d_{ij})_{n \times n}$ where each $d_{ij} = \delta_{ij}d_{ii} = \delta_{ij}d_{jj}$.

Obviously, any diagonal matrix is symmetric.
Multiplying by Diagonal Matrices

Example

Let $D$ be a diagonal matrix of dimension $n$.

Suppose that $A$ and $B$ are $m \times n$ and $n \times m$ matrices, respectively.

Then $E := AD$ and $F := DB$ are well defined matrices of dimensions $m \times n$ and $n \times m$, respectively.

By the law of matrix multiplication, their elements are

$$e_{ij} = \sum_{k=1}^{n} a_{ik} \delta_{kj} d_{jj} = a_{ij} d_{jj} \text{ and } f_{ij} = \sum_{k=1}^{n} \delta_{ik} d_{ii} b_{kj} = d_{ii} b_{ij}$$

Thus, post-multiplying $A$ by $D$ is the column operation of simultaneously multiplying every column $a_j$ of $A$ by its matching diagonal element $d_{jj}$.

Similarly, pre-multiplying $B$ by $D$ is the row operation of simultaneously multiplying every row $b_i^\top$ of $B$ by its matching diagonal element $d_{ii}$.
Two Exercises with Diagonal Matrices

Exercise

Let $D$ be a diagonal matrix of dimension $n$. Give conditions that are both necessary and sufficient for each of the following:

1. $AD = A$ for every $m \times n$ matrix $A$;
2. $DB = B$ for every $n \times m$ matrix $B$.

Exercise

Let $D$ be a diagonal matrix of dimension $n$, and $C$ any $n \times n$ matrix.

An earlier example shows that one can have $CD \neq DC$ even if $n = 2$.

1. Show that $C$ being diagonal is a sufficient condition for $CD = DC$.
2. Is this condition necessary?
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The identity matrix of dimension $n$ is the diagonal matrix

$$I_n = \text{diag}(1, 1, \ldots, 1)$$

whose $n$ diagonal elements are all equal to 1.

Equivalently, it is the $n \times n$-matrix $A = (a_{ij})^{n \times n}$ whose elements are all given by $a_{ij} = \delta_{ij}$ for the Kronecker delta function $(i, j) \mapsto \delta_{ij}$ defined on $\{1, 2, \ldots, n\}^2$.

Exercise

*Given any $m \times n$ matrix $A$, verify that $I_mA = A = AI_n = A$.**
Uniqueness of the Identity Matrix

Exercise

Suppose that the two $n \times n$ matrices $X$ and $Y$ respectively satisfy:

1. $AX = A$ for every $m \times n$ matrix $A$;
2. $YB = B$ for every $n \times m$ matrix $B$.

Prove that $X = Y = I_n$.

(Hint: Consider each of the $mn$ different cases where $A$ (resp. $B$) has exactly one non-zero element that is equal to 1.)

The results of the last two exercises together serve to prove:

Theorem

The identity matrix $I_n$ is the unique $n \times n$-matrix such that:

- $I_nB = B$ for each $n \times m$ matrix $B$;
- $AI_n = A$ for each $m \times n$ matrix $A$. 
Remark

The identity matrix $I_n$ earns its name because it represents a multiplicative identity on the “algebra” of all $n \times n$ matrices.

That is, $I_n$ is the unique $n \times n$-matrix with the property that $I_nA = AI_n = A$ for every $n \times n$-matrix $A$.

Typical notation suppresses the subscript $n$ in $I_n$ that indicates the dimension of the identity matrix.
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Left and Right Inverse Matrices

Definition
Let $A$ denote any $n \times n$ matrix.

1. The $n \times n$ matrix $X$ is a left inverse of $A$ just in case $XA = I_n$.

2. The $n \times n$ matrix $Y$ is a right inverse of $A$ just in case $AY = I_n$.

3. The $n \times n$ matrix $Z$ is an inverse of $A$ just in case it is both a left and a right inverse — i.e., $ZA = AZ = I_n$. 

The Unique Inverse Matrix

Theorem
Suppose that the $n \times n$ matrix $A$ has both a left and a right inverse. Then both left and right inverses are unique, and both are equal to a unique inverse matrix denoted by $A^{-1}$.

Proof.
If $XA = AY = I$, then $XAY = XI = X$ and $XAY = IY = Y$, implying that $X = XAY = Y$.

Similarly, if $X'$ is any alternative left inverse, then $X'A = I$ and so $X' = X'AY = Y = X$.

Also, if $Y'$ is any alternative right inverse, then $AY' = I$ and so $Y' = XAY' = X = Y$.

It follows that $X' = X = Y = Y'$, so one can define $A^{-1}$ as the unique common value of all these matrices.

Big question: when does the inverse exist?
Answer: if and only if the determinant is non-zero.
Rule for Inverting Products

Theorem
Suppose that \( A \) and \( B \) are two invertible \( n \times n \) matrices.

Then the inverse of the matrix product \( AB \) exists, and is the reverse product \( B^{-1}A^{-1} \) of the inverses.

Proof.
Using the associative law for matrix multiplication repeatedly gives:

\[
(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}(I)B = B^{-1}(IB) = B^{-1}B = I
\]
and

\[
(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = A(I)A^{-1} = (AI)A^{-1} = AA^{-1} = I.
\]

These equations confirm that \( X := B^{-1}A^{-1} \) is the unique matrix satisfying the double equality \((AB)X = X(AB) = I\).
Rule for Inverting Chain Products

Exercise

Prove that, if \( A, B \) and \( C \) are three invertible \( n \times n \) matrices, then \((ABC)^{-1} = C^{-1}B^{-1}A^{-1}\).

Then use mathematical induction to extend this result in order to find the inverse of the product \( A_1A_2 \cdots A_k \) of any finite chain of invertible \( n \times n \) matrices.
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Partitioned Matrices: Definition

A partitioned matrix is a rectangular array of different matrices.

Example
Consider the \((m + \ell) \times (n + k)\) matrix

\[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
\]

where the four submatrices \(A, B, C, D\) are of dimension \(m \times n, m \times k, \ell \times n\) and \(\ell \times k\) respectively.

Note: Matrix \(D\) may no longer be diagonal, or even square.

For any scalar \(\alpha \in \mathbb{R}\),
the scalar multiple of a partitioned matrix is

\[
\alpha \begin{pmatrix}
A & B \\
C & D
\end{pmatrix} = \begin{pmatrix}
\alpha A & \alpha B \\
\alpha C & \alpha D
\end{pmatrix}
\]
Partitioned Matrices: Addition

Suppose the two partitioned matrices

\[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
E & F \\
G & H
\end{pmatrix}
\]

have the property that the following four pairs of corresponding matrices have equal dimensions:
(i) \(A\) and \(E\); (ii) \(B\) and \(F\); (iii) \(C\) and \(G\); (iv) \(D\) and \(H\).

Then the sum of the two matrices is

\[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix} + \begin{pmatrix}
E & F \\
G & H
\end{pmatrix} = \begin{pmatrix}
A + E & B + F \\
C + G & D + H
\end{pmatrix}
\]
Partitioned Matrices: Multiplication

Suppose that the two partitioned matrices

\[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
\text{ and }
\begin{pmatrix}
E & F \\
G & H
\end{pmatrix}
\]

along with all the relevant pairs of their sub-matrices, are compatible for multiplication.

Then their product is defined as

\[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
\begin{pmatrix}
E & F \\
G & H
\end{pmatrix} =
\begin{pmatrix}
AE + BG & AF + BH \\
CE + DG & CF + DH
\end{pmatrix}
\]

This extends the usual multiplication rule for matrices: multiply the rows of sub-matrices in the first partitioned matrix by the columns of sub-matrices in the second partitioned matrix.
Transposes and Some Special Matrices

The rule for transposing a partitioned matrix is

\[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}^\top =
\begin{pmatrix}
A^\top & C^\top \\
B^\top & D^\top
\end{pmatrix}
\]

So the original matrix is symmetric iff \( A = A^\top, D = D^\top, \) and \( B = C^\top \iff C = B^\top. \)

It is diagonal iff \( A, D \) are both diagonal, while also \( B = 0 \) and \( C = 0. \)

The identity matrix is diagonal with \( A = I, D = I, \) possibly identity matrices of different dimensions.
Partitioned Matrices: Inverses, I

For an \((m + n) \times (m + n)\) partitioned matrix to have an inverse, the equation

\[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
\begin{pmatrix}
E & F \\
G & H
\end{pmatrix}
= 
\begin{pmatrix}
AE + BG & AF + BH \\
CE + DG & CF + DH
\end{pmatrix}
= 
\begin{pmatrix}
I_m & 0_{m \times n} \\
0_{n \times m} & I_n
\end{pmatrix}
\]

should have a solution for the matrices \(E, F, G, H\), given \(A, B, C, D\).

Assuming that the \(m \times m\) matrix \(A\) has an inverse, we can:

1. construct new first \(m\) equations
   by premultiplying the old ones by \(A^{-1}\);
2. construct new second \(n\) equations by:
   - premultiplying the new first \(m\) equations
     by the \(n \times m\) matrix \(C\);
   - then subtracting this product from the old second \(n\) equations.

The result is

\[
\begin{pmatrix}
I_m & A^{-1}B \\
0_{n \times m} & D - CA^{-1}B
\end{pmatrix}
\begin{pmatrix}
E & F \\
G & H
\end{pmatrix}
= 
\begin{pmatrix}
A^{-1} & 0_{m \times n} \\
-CA^{-1} & I_n
\end{pmatrix}
\]
Partitioned Matrices: Inverses, II

For the next step, assume the \( n \times n \) matrix \( X := D - CA^{-1}B \) also has an inverse \( X^{-1} = (D - CA^{-1}B)^{-1} \).

Given
\[
\begin{pmatrix}
I_m & A^{-1}B \\
0_{n \times m} & D - CA^{-1}B
\end{pmatrix}
\begin{pmatrix}
E & F \\
G & H
\end{pmatrix}
= \begin{pmatrix}
A^{-1} & 0_{m \times n} \\
-CA^{-1} & I_n
\end{pmatrix},
\]
we first premultiply the last \( n \) equations by \( X^{-1} \) to get
\[
\begin{pmatrix}
I_m & A^{-1}B \\
0_{n \times m} & I_n
\end{pmatrix}
\begin{pmatrix}
E & F \\
G & H
\end{pmatrix}
= \begin{pmatrix}
A^{-1} & 0_{m \times n} \\
-X^{-1}CA^{-1} & X^{-1}
\end{pmatrix}
\]

Next, we subtract \( A^{-1}B \) times the last \( n \) equations from the first \( m \) equations to obtain
\[
\begin{pmatrix}
E & F \\
G & H
\end{pmatrix}
= \begin{pmatrix}
I_m & 0_{m \times n} \\
0_{n \times m} & I_n
\end{pmatrix}
\begin{pmatrix}
E & F \\
G & H
\end{pmatrix}
= \begin{pmatrix}
A^{-1} + A^{-1}BX^{-1}CA^{-1} & -A^{-1}BX^{-1} \\
-X^{-1}CA^{-1} & X^{-1}
\end{pmatrix}
\]
Final Exercises

Exercise

1. Assume that $A^{-1}$ and $X^{-1} = (D - CA^{-1}B)^{-1}$ exist.

Given $Z := \begin{pmatrix} A^{-1} + A^{-1}BX^{-1}CA^{-1} & -A^{-1}BX^{-1} \\ -X^{-1}CA^{-1} & X^{-1} \end{pmatrix}$, use direct multiplication twice in order to verify that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} Z = Z \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I_m & 0_{m \times n} \\ 0_{n \times m} & I_n \end{pmatrix}$$

2. Let $A$ be any invertible $m \times m$ matrix.

Show that the bordered $(m + 1) \times (m + 1)$ matrix $\begin{pmatrix} A & b \\ c^T & d \end{pmatrix}$ is invertible provided that $d \neq c^T A^{-1} b$, and find its inverse in this case.
Partitioned Matrices: Extension

Exercise

Suppose that the two partitioned matrices

\[
A = (A_{ij})^{k \times \ell} \quad \text{and} \quad B = (B_{ij})^{k \times \ell}
\]

are both \( k \times \ell \) arrays of respective \( m_i \times n_j \) matrices \( A_{ij}, B_{ij} \).

1. Under what conditions can the product \( AB \) be defined as a \( k \times \ell \) array of matrices?

2. Under what conditions can the product \( BA \) be defined as a \( k \times \ell \) array of matrices?

3. When either \( AB \) or \( BA \) can be so defined, give a formula for its product, using summation notation.

4. Express \( A^\top \) as a partitioned matrix.

5. Under what conditions is the matrix \( A \) symmetric?
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Permutations

Definition

Given $\mathbb{N}_n = \{1, \ldots, n\}$ for any $n \in \mathbb{N}$ with $n \geq 2$, a permutation of $\mathbb{N}_n$ is a bijective mapping $\mathbb{N}_n \ni i \mapsto \pi(i) \in \mathbb{N}_n$.

That is, the mapping $\mathbb{N}_n \ni i \mapsto \pi(i) \in \mathbb{N}_n$ is both:

1. a surjection, or mapping of $\mathbb{N}_n$ onto $\mathbb{N}_n$, in the sense that the range set satisfies $\pi(\mathbb{N}_n) := \{j \in \mathbb{N}_n \mid \exists i \in \mathbb{N}_n : j = \pi(i)\} = \mathbb{N}_n$;

2. an injection, or a one to one mapping, in the sense that $\pi(i) = \pi(j) \implies i = j$ or, equivalently, $i \neq j \implies \pi(i) \neq \pi(j)$.

Exercise

Prove that the mapping $\mathbb{N}_n \ni i \mapsto f(i) \in \mathbb{N}_n$ is a bijection, and so a permutation, if and only if its range set $f(\mathbb{N}_n) := \{j \in \mathbb{N}_n \mid \exists i \in \mathbb{N}_n : j = f(i)\}$ has cardinality $\#f(\mathbb{N}_n) = \#\mathbb{N}_n = n$. 
Products of Permutations

Definition
The product $\pi \circ \rho$ of two permutations $\pi, \rho \in \Pi_n$ is the composition mapping $\mathbb{N}_n \ni i \mapsto (\pi \circ \rho)(i) := \pi[\rho(i)] \in \mathbb{N}_n$.

Exercise
Prove that the product $\pi \circ \rho$ of any two permutations $\pi, \rho \in \Pi_n$ is a permutation.

Hint: Show that $\#(\pi \circ \rho)(\mathbb{N}_n) = \#\rho(\mathbb{N}_n) = \#\mathbb{N}_n = n$.

Example
1. If you shuffle a pack of 52 playing cards once, without dropping any on the floor, the result will be a permutation $\pi$ of the cards.
2. If you shuffle the same pack a second time, the result will be a new permutation $\rho$ of the shuffled cards.
3. Overall, the result of shuffling the cards twice will be the single permutation $\rho \circ \pi$. 
Finite Permutation Groups

Definition
Given any $n \in \mathbb{N}$, the family $\Pi_n$ of all permutations of $\mathbb{N}_n$ includes:

- the identity permutation $\iota$ defined by $\iota(h) = h$ for all $h \in \mathbb{N}_n$;
- because the mapping $\mathbb{N}_n \ni i \mapsto f(i) \in \mathbb{N}_n$ is bijective, for each $\pi \in \Pi_n$, a unique inverse permutation $\pi^{-1} \in \Pi_n$ satisfying $\pi^{-1} \circ \pi = \pi \circ \pi^{-1} = \iota$.

Definition
The associative law for functions says that, given any three functions $h : X \to Y$, $g : Y \to Z$ and $f : Z \to W$, the composite function $f \circ g \circ h : X \to W$ satisfies

$$(f \circ g \circ h)(x) \equiv f(g(h(x))) \equiv [(f \circ g) \circ h](x) \equiv [f \circ (g \circ h)](x)$$

Exercise
Given any $n \in \mathbb{N}$, show that $(\Pi_n, \pi, \iota)$ is an algebraic group — i.e., the group operation $(\pi, \rho) \mapsto \pi \circ \rho$ is well-defined, associative, with $\iota$ as the unit, and an inverse $\pi^{-1}$ for every $\pi \in \Pi_n$. 
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Definition
For each disjoint pair $k, \ell \in \{1, 2, \ldots, n\}$, the transposition mapping $i \mapsto \tau_{k\ell}(i)$ on $\{1, 2, \ldots, n\}$ is the permutation defined by

$$
\tau_{k\ell}(i) := \begin{cases} 
\ell & \text{if } i = k; \\
k & \text{if } i = \ell; \\
i & \text{otherwise}; 
\end{cases}
$$

That is, $\tau_{k\ell}$ transposes the order of $k$ and $\ell$, leaving all $i \notin \{k, \ell\}$ unchanged.

Evidently $\tau_{k\ell} = \tau_{\ell k}$ and $\tau_{k\ell} \circ \tau_{\ell k} = \iota$, the identity permutation, and so $\tau \circ \tau = \iota$ for every transposition $\tau$. 


Transposition is Not Commutative

Any \((j_1, j_2, \ldots, j_n) \in \mathbb{N}_n^n\) whose components are all different corresponds to a unique permutation, denoted by \(\pi^{j_1j_2\ldots j_n} \in \Pi_n\), that satisfies \(\pi(i) = j_i\) for all \(i \in \mathbb{N}_n^n\).

Example

Two transpositions defined on a set containing more than two elements may not commute because, for example,

\[
\tau_{12} \circ \tau_{23} = \pi^{231} \neq \tau_{23} \circ \tau_{12} = \pi^{312}
\]
Adjacency Transpositions

Definition
For each $k \in \{1, 2, \ldots, n - 1\}$, the transposition $\tau_{k,k+1}$ of row $k$ with its successor is an adjacency transposition.
Reduction to Adjacency Transpositions

Lemma

The transposition $\tau_{k\ell}$ (with $k < \ell$)
is the composition of $2(\ell - k) - 1$ adjacency transpositions.

Proof.

First, we apply in order the $\ell - k$ successive adjacency transpositions $\tau_{k,k+1}, \tau_{k+1,k+2}, \ldots, \tau_{\ell-1,\ell}$.
The result is the permutation $\pi^{1,2,\ldots,k-1,k+1,k+2,\ldots,\ell,k,k+1,\ldots,n}$
that moves $k$ to the $\ell$th position,
while moving each element between $k + 1$ and $\ell$ down one.

Second, we apply in order the $\ell - k - 1$ successive adjacency transpositions $\tau_{\ell-1,\ell}, \tau_{\ell-2,\ell-1} \ldots, \tau_{k,k+1}$
that moves $\ell$ to the $k$th position,
while moving each element $k + 1, k + 2, \ldots, \ell - 1$
back up one position to where it was originally.

The final result is the transposition $\tau_{k\ell}$.
Theorem

Any permutation $\pi \in \Pi_n$ on $\mathbb{N}_n := \{1, 2, \ldots, n\}$ is the product of at most $n - 1$ transpositions.

We will prove the result by induction on $n$.

As the induction hypothesis, suppose the result holds for permutations on $\mathbb{N}_{n-1}$.

Any permutation $\pi$ on $\mathbb{N}_2 := \{1, 2\}$ is either the identity, or the transposition $\tau_{12}$, so the result holds for $n = 2$. 
Proof of Induction Step

For general $n$, let $j := \pi^{-1}(n)$ denote the element that $\pi$ moves to the end.

By construction, the permutation $\pi \circ \tau_{jn}$ must satisfy $\pi \circ \tau_{jn}(n) = \pi(\tau_{jn}(n)) = \pi(j) = n$.

So the restriction $\tilde{\pi}$ of $\pi \circ \tau_{jn}$ to $\mathbb{N}_{n-1}$ is a permutation on $\mathbb{N}_{n-1}$.

By the induction hypothesis, for all $k \in \mathbb{N}_{n-1}$, there exist transpositions $\tau^1, \tau^2, \ldots, \tau^q$ such that $\tilde{\pi}(k) = (\pi \circ \tau_{jn})(k) = (\tau^1 \circ \tau^2 \circ \ldots \circ \tau^q)(k)$ where $q \leq n - 2$ is the number of transpositions in the product.

For $p = 1, \ldots, q$, because $\tau^p$ interchanges only elements of $\mathbb{N}_{n-1}$, one can extend its domain to include $n$ by letting $\tau^p(n) = n$.

Then $(\pi \circ \tau_{jn})(k) = (\tau^1 \circ \tau^2 \circ \ldots \circ \tau^q)(k)$ for $k = n$ as well, so $\pi = (\pi \circ \tau_{jn}) \circ \tau_{jn}^{-1} = \tau^1 \circ \tau^2 \circ \ldots \circ \tau^q \circ \tau_{jn}^{-1}$.

Hence $\pi$ is the product of at most $q + 1 \leq n - 1$ transpositions.

This completes the proof by induction on $n$. \qed
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**Signs of Permutations**
- Basic Lemma on Signs of Permutations
- Some Implications

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Inversions and Signs of Permutations

Definition

1. Given any permutation \( \pi \in \Pi_n \), an inversion of \( \pi \) is a pair \((i, j) \in \mathbb{N}_n\) that \( \pi \) “reorders” in the sense that \( i < j \) and \( \pi(i) > \pi(j) \).
2. For each permutation \( \pi \in \Pi_n \), let
   \[
   \mathcal{N}(\pi) := \{(i, j) \in \mathbb{N}_n \times \mathbb{N}_n \mid (i - j)[\pi(i) - \pi(j)] < 0\}
   \]
   denote the set of inversions of \( \pi \).
3. Define \( n(\pi) := \#\mathcal{N}(\pi) \in \mathbb{N} \cup \{0\} \) as the number of inversions.
4. A permutation \( \pi : \mathbb{N}_n \rightarrow \mathbb{N}_n \) is either even or odd according as \( n(\pi) \) is an even or odd number.
5. The sign or signature of a permutation \( \pi \), is defined as \( \text{sgn}(\pi) := (-1)^{n(\pi)} \), which is:
   (i) \(+1\) if \( \pi \) is even; (ii) \(-1\) if \( \pi \) is odd.
Three Exercises

Exercise

Show that:

1. \( n(\pi) = 0 \iff \pi = \iota, \) the identity permutation;

2. if \( \pi \) is an adjacency transposition \( \tau_{k,k+1}, \) then \( n(\pi) = 1 \) and so \( \text{sgn}(\pi) = -1; \)

3. \( n(\pi) \leq \frac{1}{2} n(n - 1), \) with equality if and only if \( \pi(i) = n - i + 1 \) for all \( i \in \mathbb{N}_n. \)

**Hint:** Consider the number of pairs \( (i, j) \in \mathbb{N}_n \times \mathbb{N}_n \) that satisfy \( i < j. \)
The Sign of the Inverse Permutation

**Theorem**

*For the inverse $\pi^{-1}$ of any permutation $\pi$, one has: (i) $n(\pi^{-1}) = n(\pi)$; (ii) $\text{sgn}(\pi^{-1}) = \text{sgn}(\pi)$.***

**Proof.**

For Part (i), the number of inversions of $\pi$ can be calculated as

$$n(\pi) = \#\{(i, j, k, \ell) \in \mathbb{N}_n^4 | i < j \land k = \pi(i) > \pi(j) = \ell\}$$

$$= \#\{(i, j, k, \ell) \in \mathbb{N}_n^4 | \pi^{-1}(k) = i < j = \pi^{-1}(\ell) \land k > \ell\}$$

$$= \#\{(\ell, k, j, i) \in \mathbb{N}_n^4 | \ell < k \land j = \pi^{-1}(\ell) > \pi^{-1}(k) = i\}$$

Interchanging $i$ with $\ell$ and $j$ with $k$ in the last expression gives

$$n(\pi) = \#\{(i, j, k, \ell) \in \mathbb{N}_n^4 | i < j \land k = \pi^{-1}(i) > \pi^{-1}(j) = \ell\}$$

which equals $n(\pi^{-1})$, by definition.

For Part (ii), notice how part (i) implies that $\text{sgn}(\pi^{-1}) = (-1)^{n(\pi^{-1})} = (-1)^{n(\pi)} = \text{sgn}(\pi)$. ☐
Exercise

*Use the argument of the last proof to establish a bijection*

\[
\mathbb{N}_n^2 \supset \mathcal{N}(\pi) \ni (i, j) \mapsto (\pi(i), \pi(j)) \in \mathcal{N}(\pi^{-1}) \subset \mathbb{N}_n^2
\]

*between the two sets \( \mathcal{N}(\pi) \) and \( \mathcal{N}(\pi^{-1}) \) of inversions, whose inverse is the bijection*

\[
\mathbb{N}_n^2 \supset \mathcal{N}(\pi^{-1}) \ni (k, \ell) \mapsto (\pi^{-1}(k), \pi^{-1}(\ell)) \in \mathcal{N}(\pi) \subset \mathbb{N}_n^2
\]
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Lemma

Given any $k \in \mathbb{N}_{n-1}$, let $\tau$ be any adjacency transposition $\tau_{k,k+1}$, and $\pi \in \Pi_n$ any permutation on $\mathbb{N}_n$.

Then $\text{sgn}(\tau \circ \pi) = -\text{sgn}(\pi)$.

The proof follows on the next 7 slides.
Commencing the Proof of the Basic Lemma

Throughout the following proof, suppose that the labels $i, j \in \mathbb{N}_n$ of any pair $\{i, j\} \subset \mathbb{N}_n$ are chosen to satisfy $i < j$.

Our proof will treat three different main cases, depending on whether the set $\{i, j\} \cap \{\pi^{-1}(k), \pi^{-1}(k + 1)\}$ has 0, 2, or 1 member.

**Case 1:** Suppose that $\#\{i, j\} \cap \{\pi^{-1}(k), \pi^{-1}(k + 1)\} = 0$, or equivalently, that $\{\pi(i), \pi(j)\} \cap \{k, k + 1\} = \emptyset$.

In this case $(\tau \circ \pi)(i) = \pi(i)$ and $(\tau \circ \pi)(j) = \pi(j)$, so $\pi(i) > \pi(j) \iff (\tau \circ \pi)(i) > (\tau \circ \pi)(j)$.

Hence $\pi$ and $\tau \circ \pi$ have identical sets of inversions $(i, j)$ that satisfy $\{i, j\} \cap \{\pi^{-1}(k), \pi^{-1}(k + 1)\} = \emptyset$. 
Proof: Case 2

Case 2: Suppose that \(#\{i, j\} \cap \{\pi^{-1}(k), \pi^{-1}(k + 1)\} = 2.\) In this case there are two different subcases.

Subcase (a): This subcase occurs when \(\pi^{-1}(k) < \pi^{-1}(k + 1).\) Here, because \(i < j,\) one has \(i = \pi^{-1}(k)\) and \(j = \pi^{-1}(k + 1).\) Then \(\pi(i) = k < k + 1 = \pi(j),\) so \((i, j)\) is not an inversion of \(\pi.\)
Yet \((\tau \circ \pi)(i) = k + 1 > k = (\tau \circ \pi)(j),\) so \((i, j)\) is an inversion of \(\tau \circ \pi.\)

Subcase (b): This subcase occurs when \(\pi^{-1}(k) > \pi^{-1}(k + 1).\) Here, because \(i < j,\) one has \(i = \pi^{-1}(k + 1)\) and \(j = \pi^{-1}(k).\) Then \(\pi(i) = k + 1 > k = \pi(j),\) so \((i, j)\) is an inversion of \(\pi.\)
Yet \((\tau \circ \pi)(i) = k < k + 1 = (\tau \circ \pi)(j),\) so \((i, j)\) is not an inversion of \(\tau \circ \pi.\)
Proof: Case 3(a)

Case 3: Suppose that \( \#\{i, j\} \cap \{\pi^{-1}(k), \pi^{-1}(k + 1)\} = 1 \).

We will show that \( \pi \) and \( \tau \circ \pi \) have the same set of inversions.

To show this, we consider four different subcases.

Subcase (a): \( \pi(i) = k \) and \( \pi(j) \notin \{k, k + 1\} \).

Here \( (\tau \circ \pi)(i) = k + 1 \) and \( (\tau \circ \pi)(j) = \pi(j) \).

Now, if \((i, j)\) is an inversion of \( \pi \), then \( k > \pi(j) \),
implying that \( (\tau \circ \pi)(i) = k + 1 > k > \pi(j) = (\tau \circ \pi)(j) \),
so \((i, j)\) is an inversion of \( \tau \circ \pi \).

Conversely, if \((i, j)\) is an inversion of \( \tau \circ \pi \),
then \( k + 1 = (\tau \circ \pi)(i) > (\tau \circ \pi)(j) = \pi(j) \).

Because \( \pi(j) \neq k \), it follows that \( k > \pi(j) \).

So \( \pi(i) = k > \pi(j) \), implying that \((i, j)\) is an inversion of \( \pi \).
Subcase (b): $\pi(i) = k + 1$ and $\pi(j) \not\in \{k, k + 1\}$.

Here $(\tau \circ \pi)(i) = k$ and $(\tau \circ \pi)(j) = \pi(j)$.

Now, if $(i, j)$ is an inversion of $\pi$, then $k + 1 > \pi(j)$.

Because $\pi(j) \neq k$, it follows that $k > \pi(j)$.

But then $(\tau \circ \pi)(i) = k > \pi(j) = (\tau \circ \pi)(j)$,

so $(i, j)$ is an inversion of $\tau \circ \pi$.

Conversely, if $(i, j)$ is an inversion of $\tau \circ \pi$,
then $\pi(i) = k + 1 > k = (\tau \circ \pi)(i) > (\tau \circ \pi)(j) = \pi(j)$.

It follows that $(i, j)$ is an inversion of $\pi$. 


Proof: Case 3(c)

Subcase (c): $\pi(j) = k$ and $\pi(i) \notin \{k, k + 1\}$.

Here $(\tau \circ \pi)(j) = k + 1$ and $(\tau \circ \pi)(i) = \pi(i)$.

Now, if $(i, j)$ is an inversion of $\pi$, then $\pi(i) > k$.

Because $\pi(i) \neq k + 1$, it follows that $\pi(i) > k + 1$.

Hence $(\tau \circ \pi)(i) = \pi(i) > k + 1 = (\tau \circ \pi)(j)$,

implying that $(i, j)$ is an inversion of $\tau \circ \pi$.

Conversely, if $(i, j)$ is an inversion of $\tau \circ \pi$,

then $\pi(i) = (\tau \circ \pi)(i) > (\tau \circ \pi)(j) = k + 1 > k = \pi(j)$,

so $(i, j)$ is an inversion of $\pi$. 
Proof: Case 3(d)

Subcase (d): \( \pi(j) = k + 1 \) and \( \pi(i) \notin \{k, k + 1\} \).

Here \( (\tau \circ \pi)(i) = \pi(i) \) and \( (\tau \circ \pi)(j) = k \).

Now, if \((i, j)\) is an inversion of \(\pi\), then \(\pi(i) > \pi(j) = k + 1\).

This implies that \( (\tau \circ \pi)(i) = \pi(i) > k + 1 > k = (\tau \circ \pi)(j) \),
so \((i, j)\) is an inversion of \(\tau \circ \pi\).

Conversely, if \((i, j)\) is an inversion of \(\tau \circ \pi\),
then \(\pi(i) = (\tau \circ \pi)(i) > (\tau \circ \pi)(j) = k \).

Because \(\pi(i) \neq k + 1\), it follows that \(\pi(i) > k + 1 = \pi(j)\).
Hence \((i, j)\) is an inversion of \(\pi\).
Completing the Proof

To summarize what has been proved so far:

1. The two permutations $\pi$ and $\tau \circ \pi$
   have identical sets of inversions $(i, j)$
   satisfying $\{i, j\} \neq \{\pi^{-1}(k), \pi^{-1}(k + 1)\}$.

2. As for the unique pair $(i, j)$ satisfying
   both $i < j$ and $\{i, j\} = \{\pi^{-1}(k), \pi^{-1}(k + 1)\}$:
   
   2.1 in case $\pi^{-1}(k) < \pi^{-1}(k + 1)$
      and so $i = \pi^{-1}(k)$, $j = \pi^{-1}(k + 1)$,
      the pair $(i, j)$ is an inversion of $\tau \circ \pi$, but not of $\pi$;
   2.2 in case $\pi^{-1}(k) > \pi^{-1}(k + 1)$
      and so $i = \pi^{-1}(k + 1)$, $j = \pi^{-1}(k)$,
      the pair $(i, j)$ is an inversion of $\pi$, but not of $\tau \circ \pi$.

Hence $n(\tau \circ \pi) = n(\pi) \pm 1$ according as $\pi^{-1}(k) \leq \pi^{-1}(k + 1)$.

In particular, this implies that $\text{sgn}(\tau \circ \pi) = -\text{sgn}(\pi)$. \qed
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Lemma

If \( \rho, \pi \in \Pi_n \) where \( \rho \) is the product of \( m \) adjacency transpositions, then
\[
\text{sgn}(\rho \circ \pi) = (-1)^m \text{sgn}(\pi) = \text{sgn}(\rho) \text{sgn}(\pi).
\]

Proof.

1. In case \( m = 1 \), apply the basic lemma once, while noting that \( \text{sgn}(\rho) = -1 \).
2. In case \( m > 1 \), apply the basic lemma iteratively \( m \) times to obtain \( \text{sgn}(\rho \circ \pi) = (-1)^m \text{sgn}(\pi) \).

Then put \( \pi = \iota \) to obtain \( \text{sgn}(\rho) = \text{sgn}(\rho \circ \iota) = (-1)^m \).
Hence \( \text{sgn}(\rho \circ \pi) = (-1)^m \text{sgn}(\pi) = \text{sgn}(\rho) \text{sgn}(\pi) \).
Lemma
If $\rho, \pi \in \Pi_n$ where $\rho$ is the product of $m$ general transpositions, then $\text{sgn}(\rho \circ \pi) = (-1)^m \text{sgn}(\pi) = \text{sgn}(\rho) \text{sgn}(\pi)$.

Proof.

1. In case $m = 1$ and $\rho$ is the general transposition $\tau_{k,\ell}$, recall that $\tau_{k,\ell}$ is the product of $2(\ell - k) - 1$ adjacency transpositions. Because $2(\ell - k) - 1$ is odd, it follows that $\text{sgn}(\rho \circ \pi) = (-1)^{2(\ell-k)-1} \text{sgn}(\pi) = - \text{sgn}(\pi)$. Then put $\pi = \iota$ to obtain $\text{sgn}(\rho) = \text{sgn}(\rho \circ \iota) = -1$. Hence $\text{sgn}(\rho \circ \pi) = - \text{sgn}(\pi) = \text{sgn}(\rho) \text{sgn}(\pi)$.

2. In case $\rho$ is the product of $m$ general transpositions, apply the result of Part 1 iteratively $m$ times, while noting that $\text{sgn}(\rho) = (-1)^m \text{sgn}(\iota) = (-1)^m$. 


Signing Product and Inverse Permutations

**Theorem**

For all \( \rho, \pi \in \Pi_n \) one has \( \text{sgn}(\rho \circ \pi) = \text{sgn}(\rho) \text{sgn}(\pi) \), and also \( \text{sgn}(\pi^{-1}) = \text{sgn}(\pi) \).

**Proof.**

Recall that each permutation is the product of finitely many transpositions.

Suppose that \( \rho \) and \( \pi \) are the products of \( r \) and \( p \) transpositions respectively.

Then \( \rho \circ \pi \) is the product of \( r + p \) transpositions.

Hence \( \text{sgn}(\rho \circ \pi) = (-1)^{r+p} = (-1)^r(-1)^p = \text{sgn}(\rho) \text{sgn}(\pi) \).

Then putting \( \rho = \pi^{-1} \) implies that

\[
\text{sgn}(\pi^{-1}) \text{sgn}(\pi) = \text{sgn}(\pi^{-1} \circ \pi) = \text{sgn}(\iota) = 1
\]

It follows that \( \text{sgn}(\pi^{-1}) \neq -\text{sgn}(\pi) \), so \( \text{sgn}(\pi^{-1}) = \text{sgn}(\pi) \). \( \square \)
For each $k, \ell \in \mathbb{N}_n$ with $k \leq \ell$, define $\pi^{k \to \ell} \in \Pi_n$ so that for each $i \in \mathbb{N}_n$ one has

$$
\pi^{k \to \ell}(i) := \begin{cases} 
  i & \text{if } i < k \text{ or } i > \ell; \\
  i - 1 & \text{if } k < i \leq \ell; \\
  \ell & \text{if } i = k.
\end{cases}
$$

That is, $\pi^{k \to \ell}$ moves $k$ “up to” $\ell$, and then to compensate, shifts each $i$ between $k + 1$ and $\ell$ “down one”.

Evidently when $k = \ell$, one has $\pi^{k \to k} = \iota$.

The lemma on the next slide shows that, provided $\ell < n$, because $\pi^{k \to \ell + 1}$ moves $k$ up one more step to $\ell + 1$ rather than $\ell$, one could achieve it by applying $\pi^{k \to \ell}$ first, followed by $\tau_{\ell, \ell + 1}$.

That is $\pi^{k \to \ell + 1} = \tau_{\ell, \ell + 1} \circ \pi^{k \to \ell}$.
Lemma

For all \( k, \ell \in \mathbb{N}_n \) with \( k \leq \ell < n \), one has \( \pi^{k \uparrow \ell+1} = \tau_{\ell, \ell+1} \circ \pi^{k \uparrow \ell} \).

Proof.

The definitions of \( \pi^{k \uparrow \ell} \) and \( \pi^{k \uparrow \ell+1} \) evidently imply that:

1. \( \pi^{k \uparrow \ell+1}(k) = \ell + 1 = \tau_{\ell, \ell+1}(\ell) = (\tau_{\ell, \ell+1} \circ \pi^{k \uparrow \ell})(k) \)
2. \( \pi^{k \uparrow \ell+1}(\ell + 1) = \ell = \tau_{\ell, \ell+1}(\ell + 1) = (\tau_{\ell, \ell+1} \circ \pi^{k \uparrow \ell})(\ell + 1) \)
3. if \( i < k \) or \( i > \ell + 1 \), then

\[
\pi^{k \uparrow \ell+1}(i) = i = \pi^{k \uparrow \ell}(i) = (\tau_{\ell, \ell+1} \circ \pi^{k \uparrow \ell})(i)
\]
4. if \( k < i \leq \ell \), then \( k \leq i - 1 = \pi^{k \uparrow \ell}(i) < \ell \) and so

\[
\pi^{k \uparrow \ell+1}(i) = i - 1 = \pi^{k \uparrow \ell}(i) = (\tau_{\ell, \ell+1} \circ \pi^{k \uparrow \ell})(i)
\]

This proves that \( \pi^{k \uparrow \ell+1}(i) = (\tau_{\ell, \ell+1} \circ \pi^{k \uparrow \ell})(i) \) for all \( i \in \mathbb{N}_n \).
Proposition

For all \( k, \ell \in \mathbb{N}_n \) with \( k < \ell < n \), one has:

1. \( \pi^{k \to \ell} \) is the product \( \tau_{\ell-1, \ell} \circ \tau_{\ell-2, \ell-1} \circ \cdots \circ \tau_{k+1, k+2} \circ \tau_{k, k+1} \) of \( \ell - k \) successive adjacency transpositions;

2. \( \text{sgn } \pi^{k \to \ell} = (-1)^{\ell-k} \).

Proof.

The proof of Part 1 will be by induction on \( \ell \).

When \( \ell = k \), one has \( \pi^{k \to k} = \iota \) by definition.

Suppose that \( \tau_{\ell-1, \ell} \circ \tau_{\ell-2, \ell-1} \circ \cdots \circ \tau_{k+1, k+2} \circ \tau_{k, k+1} \).

Combining the result of the Lemma with this hypothesis implies

\[
\pi^{k \to \ell+1} = \tau_{\ell, \ell+1} \circ \pi^{k \to \ell} = \tau_{\ell, \ell+1} \circ \tau_{\ell-1, \ell} \circ \cdots \circ \tau_{k+1, k+2} \circ \tau_{k, k+1}
\]

This proves Part 1 by induction; Part 2 follows immediately.
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Determinants of Order 2: Definition

Consider again the pair of linear equations

\[ \begin{align*}
    a_{11}x_1 + a_{12}x_2 &= b_1 \\
    a_{21}x_1 + a_{12}x_2 &= b_2
\end{align*} \]

with its associated coefficient matrix

\[ A = \begin{pmatrix}
    a_{11} & a_{12} \\
    a_{21} & a_{22}
\end{pmatrix} \]

Let us define the number \( D := a_{11}a_{22} - a_{21}a_{12} \).

Provided that \( D \neq 0 \), the equations have a unique solution given by

\[ \begin{align*}
    x_1 &= \frac{1}{D}(b_1a_{22} - b_2a_{12}), \\
    x_2 &= \frac{1}{D}(b_2a_{11} - b_1a_{21})
\end{align*} \]

The number \( D \) is called the **determinant** of the matrix \( A \).

It is denoted by either \( \text{det}(A) \), or more concisely, by \( |A| \).
Determinants of Order 2: Simple Rule

Thus, for any $2 \times 2$ matrix $A$, its determinant $D$ is

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

For this special case of order 2 determinants, a simple rule is:

1. multiply the diagonal elements together;
2. multiply the off-diagonal elements together;
3. subtract the product of the off-diagonal elements from the product of the diagonal elements.

Exercise

Show that the determinant satisfies

$$|A| = a_{11}a_{22} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + a_{21}a_{12} \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$$
Cramer’s Rule in the 2 × 2 Case

Using determinant notation, the solution to the equations

\[ a_{11}x_1 + a_{12}x_2 = b_1 \]
\[ a_{21}x_1 + a_{12}x_2 = b_2 \]

can be written in the alternative form

\[ x_1 = \frac{1}{D} \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}, \quad x_2 = \frac{1}{D} \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix} \]

This accords with Cramer’s rule,
which says that the solution to \(Ax = b\) is the vector \(x = (x_i)^n_{i=1}\) each of whose components \(x_i\) is the fraction with:

1. denominator equal to the determinant \(D\) of the coefficient matrix \(A\) (provided, of course, that \(D \neq 0\));
2. numerator equal to the determinant of the matrix \([A_i/b]\)
formed from \(A\) by replacing its \(i\)th column
with the \(b\) vector of right-hand side elements,
while keeping all the columns in their original order.
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Determinants of Order 3: Definition

Determinants of order 3 can be calculated from those of order 2 according to the formula

\[
\begin{vmatrix} A \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\
= \sum_{j=1}^{3} (-1)^{1+j} a_{1j} |C_{1j}|
\]

where, for \( j = 1, 2, 3 \), the \( 2 \times 2 \) matrix \( C_{1j} \) is the \((1, j)\)-cofactor obtained by removing both row 1 and column \( j \) from the matrix \( A \).

The result is the following sum

\[
\begin{vmatrix} A \end{vmatrix} = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} \\
- a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}
\]

of \( 3! = 6 \) terms, each the product of 3 elements chosen so that each row and each column is represented just once.
Determinants of Order 3: Cofactor Expansion

The determinant expansion

\[ |A| = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} - a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} \]

is very symmetric, suggesting (correctly) that the cofactor expansion along the first row \((a_{11}, a_{12}, a_{13})\)

\[ |A| = \sum_{j=1}^{3} (-1)^{1+j} a_{1j} |C_{1j}| \]

gives the same answer as the other cofactor expansions

\[ |A| = \sum_{j=1}^{3} (-1)^{r+j} a_{rj} |C_{rj}| = \sum_{i=1}^{3} (-1)^{i+s} a_{is} |C_{is}| \]

along, respectively:

- the rth row \((a_{r1}, a_{r2}, a_{r3})\)
- the sth column \((a_{1s}, a_{2s}, a_{3s})\)
Determinants of Order 3: Alternative Expressions

One way of condensing the notation

\[ |A| = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} \]

\[ - a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} \]

is to reduce it to

\[ |A| = \sum_{\pi \in \Pi_3} \text{sgn}(\pi) \prod_{i=1}^{3} a_{i\pi(i)} \]

for the sign function \( \Pi_3 \ni \pi \mapsto \text{sgn}(\pi) \in \{-1, +1\} \).

The six values of \( \text{sgn}(\pi) \) can be read off as

\[
\begin{align*}
\text{sgn}(\pi^{123}) &= +1; & \text{sgn}(\pi^{132}) &= -1; & \text{sgn}(\pi^{231}) &= +1; \\
\text{sgn}(\pi^{213}) &= -1; & \text{sgn}(\pi^{312}) &= +1; & \text{sgn}(\pi^{321}) &= -1.
\end{align*}
\]

Exercise

Verify these values for each of the six \( \pi \in \Pi_3 \) by:

1. calculating the number of inversions directly;
2. expressing each \( \pi \) as the product of transpositions, and then counting these.
Outline

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  The Inverse Matrix
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Permutations and Their Signs
  Permutations
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  Signs of Permutations
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The Determinant Function

Let \( D_n \) denote the domain \( \mathbb{R}^{n \times n} \) of \( n \times n \) matrices.

For all \( n \in \mathbb{N} \), the determinant mapping

\[
D_n \ni A \mapsto |A| := \sum_{\pi \in \Pi_n} \text{sgn} (\pi) \prod_{i=1}^{n} a_{i \pi(i)} \in \mathbb{R}
\]

specifies the determinant \( |A| \) of each \( n \times n \) matrix \( A \) as a function of its \( n \) row vectors \( (a_i^\top)_{i=1}^{n} \).

For a general natural number \( n \in \mathbb{N} \), consider any mapping

\[
D_n \ni A \mapsto D(A) = D \left( \langle a_i^\top \rangle_{i=1}^{n} \right) \in \mathbb{R}
\]

defined on the domain \( D_n \) of \( n \times n \) matrices \( A \) with row vectors \( \langle a_i^\top \rangle_{i=1}^{n} \).

**Notation:** For each fixed \( r \in \mathbb{N}_n \), let \( D(A/b_r^\top) \) denote the new value \( D(a_1^\top, \ldots, a_{r-1}^\top, b_r^\top, a_{r+1}^\top, \ldots, a_n^\top) \) of the function \( D \) after the \( r \)th row \( a_r^\top \) of the matrix \( A \) has been replaced by the new row vector \( b_r^\top \in \mathbb{R}^n \).
Row Multilinearity

Definition
The function $D_n \ni A \mapsto D(A)$ of $A$’s $n$ rows $\langle a_i^\top \rangle_{i=1}^n$ is (row) multilinear just in case, for each row number $i \in \{1, 2, \ldots, n\}$, each pair $b_i^\top, c_i^\top \in \mathbb{R}^n$ of new versions of row $i$, and each pair of scalars $\lambda, \mu \in \mathbb{R}$, one has

$$D(A/\lambda b_i^\top + \mu c_i^\top) = \lambda D(A/b_i^\top) + \mu D(A/c_i^\top)$$

Formally, the mapping $\mathbb{R}^n \ni a_i^\top \mapsto D(A/a_i^\top) \in \mathbb{R}$ is required to be linear, for fixed each row $i \in \mathbb{N}_n$.

That is, $D$ is a linear function of the $i$th row vector $a_i^\top$ on its own, when all the other rows $a_h^\top (h \neq i)$ are fixed.
Determinants are Row Multilinear

**Theorem**

*For all* \( n \in \mathbb{N} \), the determinant mapping

\[
\mathcal{D}_n \ni \mathbf{A} \mapsto |\mathbf{A}| := \sum_{\pi \in \Pi_n} \text{sgn}(\pi) \prod_{i=1}^{n} a_{i\pi(i)} \in \mathbb{R}
\]

*is a row multilinear function of its* \( n \) *row vectors* \((\mathbf{a}_i^\top)_{i=1}^n\).

**Proof.**

For each fixed row \( r \in \mathbb{N} \), we have

\[
\det(\mathbf{A}/\lambda \mathbf{b}_r^\top + \mu \mathbf{c}_r^\top)
= \sum_{\pi \in \Pi_n} \text{sgn}(\pi) \left( \lambda b_{r\pi(r)} + \mu c_{r\pi(r)} \right) \prod_{i \neq r} a_{i\pi(i)}
= \sum_{\pi \in \Pi_n} \text{sgn}(\pi) \left[ \lambda b_{r\pi(r)} \prod_{i \neq r} a_{i\pi(i)} + \mu c_{r\pi(r)} \prod_{i \neq r} a_{i\pi(i)} \right]
= \lambda \det(\mathbf{A}/\mathbf{b}_r^\top) + \mu \det(\mathbf{A}/\mathbf{c}_r^\top)
\]

as required for multilinearity.
Permutation Matrices: Definition

Definition
Given any permutation $\pi \in \Pi_n$ on \{1, 2, \ldots, n\}, define $P^\pi$ as the $n \times n$ permutation matrix whose elements satisfy $p^\pi_{\pi(i),j} = \delta_{i,j}$ or equivalently $p^\pi_{i,j} = \delta_{\pi^{-1}(i),j}$.

That is, the rows of the identity matrix $I_n$ are permuted so that for each $i = 1, 2, \ldots, n$, its $i$th row vector is moved to become row $\pi(i)$ of $P^\pi$.

Lemma
For each permutation matrix one has $(P^\pi)^\top = P^{\pi^{-1}}$.

Proof.
Because $\pi$ is a permutation, $i = \pi(j) \iff j = \pi^{-1}(i)$.

Then the definitions imply that for all $(i,j) \in \mathbb{N}_n^2$ one has

$$(P^\pi)^\top_{i,j} = p^\pi_{j,i} = \delta_{\pi(j),i} = \delta_{\pi^{-1}(i),j} = p^{\pi^{-1}}(i,j)$$
Permutation Matrices: Examples

Example

There are two $2 \times 2$ permutation matrices, which are given by:

$$P^{12} = I_2; \quad P^{21} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$ 

Their signs are respectively $+1$ and $-1$.

There are $3! = 6$ permutation matrices in 3 dimensions given by:

$$P^{123} = I_3; \quad P^{132} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}; \quad P^{213} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix};$$

$$P^{231} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}; \quad P^{312} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}; \quad P^{321} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$ 

Their signs are respectively $+1$, $-1$, $-1$, $+1$, $+1$ and $-1$. 
Multiplying by a Matrix Permutes Its Rows or Columns

Lemma
Given any $n \times n$ matrix $A$, for each permutation $\pi \in \Pi_n$ the corresponding permutation matrix $P^\pi$ satisfies

$$(P^\pi A)_{\pi(i),j} = a_{ij} = (AP^\pi)_{i,\pi(j)}$$

That is, premultiplying $A$ by $P^\pi$ permutes the rows of $A$, whereas postmultiplying $A$ by $P^\pi$ permutes the columns of $A$.

Proof.
For each pair $(i, j) \in \mathbb{N}_n^2$, one has

$$(P^\pi A)_{\pi(i),j} = \sum_{k=1}^{n} p^\pi_{\pi(i),k}a_{kj} = \sum_{k=1}^{n} \delta_{ik}a_{kj} = a_{ij}$$

and also

$$(AP^\pi)_{i,\pi(j)} = \sum_{k=1}^{n} a_{ik}p^\pi_{k,\pi(j)} = \sum_{k=1}^{n} a_{ik}\delta_{kj} = a_{ij} \quad \square$$
Multiplying Permutation Matrices

Theorem

Given the composition \( \pi \circ \rho \) of two permutations \( \pi, \rho \in \Pi_n \), the associated permutation matrices satisfy \( P^\pi P^\rho = P^{\pi \circ \rho} \).

Proof.

For each pair \((i, j) \in \mathbb{N}_n^2\), one has

\[
(P^\pi P^\rho)_{ij} = \sum_{k=1}^n p_{ik}^\pi p_{kj}^\rho = \sum_{k=1}^n \delta_{\pi^{-1}(i), k} \delta_{\rho^{-1}(k), j}
\]

\[
= \sum_{k=1}^n \delta_{(\rho^{-1} \circ \pi^{-1})(i), \rho^{-1}(k)} \delta_{\rho^{-1}(k), j}
\]

\[
= \sum_{\ell=1}^n \delta_{(\pi \circ \rho)^{-1}(i), \ell} \delta_{\ell, j} = \delta_{(\pi \circ \rho)^{-1}(i), j} = p_{ij}^{\pi \circ \rho} \quad \square
\]

Corollary

If \( \pi = \pi^1 \circ \pi^2 \circ \cdots \circ \pi^q \), then \( P^\pi = P^{\pi^1} P^{\pi^2} \cdots P^{\pi^q} \).

Proof.

By induction on \( q \), using the result of the Theorem. \( \square \)
Another Multiplication Property

Proposition

If $P^\pi$ is any $n \times n$ permutation matrix, then $P^\pi(P^\pi)^\top = (P^\pi)^\top P^\pi = I_n$.

Proof.

For each pair $(i, j) \in \mathbb{N}_n^2$, one has

$$
[P^\pi(P^\pi)^\top]_{ij} = \sum_{k=1}^n p_{ik}^\pi p_{jk}^\pi = \sum_{k=1}^n \delta_{\pi^{-1}(i), k} \delta_{\pi^{-1}(j), k}
$$

$$
= \delta_{\pi^{-1}(i), \pi^{-1}(j)} = \delta_{ij}
$$

and also

$$
[(P^\pi)^\top P^\pi]_{ij} = \sum_{k=1}^n p_{ki}^\pi p_{kj}^\pi = \sum_{k=1}^n \delta_{\pi^{-1}(k), i} \delta_{\pi^{-1}(k), j}
$$

$$
= \sum_{\ell=1}^n \delta_{\ell, i} \delta_{\ell, j} = \delta_{ij} \quad \square
$$
Transposition Matrices

A special case of a permutation matrix is a transposition $T_{rs}$ of rows $r$ and $s$.

As the matrix $I$ with rows $r$ and $s$ transposed, it satisfies

$$
(T_{rs})_{ij} = \begin{cases} 
\delta_{ij} & \text{if } i \not\in \{r, s\} \\
\delta_{sj} & \text{if } i = r \\
\delta_{rj} & \text{if } i = s
\end{cases}
$$

Exercise

Prove that:

1. any transposition matrix $T = T_{rs}$ is symmetric;
2. $T_{rs} = T_{sr}$;
3. $T_{rs} T_{sr} = T_{sr} T_{rs} = I$;
Determinants with Permutated Rows

Theorem

Given any $n \times n$ matrix $A$ and any permutation $\pi \in \mathbb{N}_n$, one has $|P^{\pi}A| = \text{sgn}(\pi) |A|$.

Proof.

The expansion formula for determinants gives

$$|P^{\pi}A| = \sum_{\rho \in \Pi_n} \text{sgn}(\rho) \prod_{i=1}^{n} (P^{\pi}A)_{i,\rho(i)}$$

But for each $i \in \mathbb{N}_n$, $\rho \in \Pi_n$, one has $(P^{\pi}A)_{i,\rho(i)} = a_{\pi^{-1}(i),\rho(i)}$, so

$$|P^{\pi}A| = \sum_{\rho \in \Pi_n} \text{sgn}(\rho) \prod_{i=1}^{n} a_{\pi^{-1}(i),\rho(i)}$$

$$= \left[1 / \text{sgn}(\pi) \right] \sum_{\pi \circ \rho \in \Pi_n} \text{sgn}(\pi \circ \rho) \prod_{i=1}^{n} a_{i,(\pi \circ \rho)(i)}$$

$$= \text{sgn}(\pi) \sum_{\sigma \in \Pi_n} \text{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)} = \text{sgn}(\pi) |A|$$

because $1 / \text{sgn}(\pi) = \text{sgn}(\pi)$, whereas there is an obvious bijection $\rho \leftrightarrow \pi \circ \rho = \sigma$ on the set of permutations $\Pi_n$. 

University of Warwick, EC9A0 Maths for Economists

Peter J. Hammond
The Alternation Rule for Determinants

Corollary
Given any $n \times n$ matrix $A$ and any transposition $\tau_{rs}$ with associated transposition matrix $T_{rs}$, one has $|T_{rs}A| = -|A|$.

Proof.
Apply the previous theorem in the special case when $\pi = \tau_{rs}$ and so $P^\pi = T_{rs}$.

Then, because $\text{sgn}(\pi) = \text{sgn}(\tau_{rs}) = -1$, the equality $|P^\pi A| = \text{sgn}(\pi)|A|$ implies that $|T_{rs}A| = -|A|$.

☐
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Triangular Matrices
Triangular Matrices: Definition

Definition
A square matrix is upper (resp. lower) triangular if all its non-zero off diagonal elements are above and to the right (resp. below and to the left) of the diagonal — i.e., in the upper (resp. lower) triangle bounded by the principal diagonal.

▶ The elements of an upper triangular matrix $U$ satisfy $(U)_{ij} = 0$ whenever $i > j$.
▶ The elements of a lower triangular matrix $L$ satisfy $(L)_{ij} = 0$ whenever $i < j$. 
Products of Upper Triangular Matrices

Theorem
The product $W = UV$ of any two upper triangular matrices $U, V$ is upper triangular, with diagonal elements $w_{ii} = u_{ii} v_{ii}$ $(i = 1, \ldots, n)$ equal to the product of the corresponding diagonal elements of $U, V$.

Proof.
Given any two upper triangular $n \times n$ matrices $U$ and $V$, one has $u_{ik} v_{kj} = 0$ unless both $i \leq k$ and $k \leq j$.
So the elements $(w_{ij})^{n \times n}$ of their product $W = UV$ satisfy

$$w_{ij} = \begin{cases} \sum_{k=i}^{j} u_{ik} v_{kj} & \text{if } i \leq j \\ 0 & \text{if } i > j \end{cases}$$

Hence $W = UV$ is upper triangular.

Finally, when $j = i$ the above sum collapses to just one term, and $w_{ii} = u_{ii} v_{ii}$ for $i = 1, \ldots, n$. □
Products of Lower Triangular Matrices

Theorem
The product of any two lower triangular matrices is lower triangular.

Proof.
Given any two lower triangular matrices \( L, M \), taking transposes shows that \((LM)^\top = M^\top L^\top = U\), where the product \( U \) is upper triangular, as the product of upper triangular matrices.

Hence \( LM = U^\top \) is lower triangular, as the transpose of an upper triangular matrix.
Triangular Matrices: Exercises

Exercise
Prove that the transpose:

1. $\mathbf{U}^\top$ of any upper triangular matrix $\mathbf{U}$ is lower triangular;
2. $\mathbf{L}^\top$ of any lower triangular matrix $\mathbf{L}$ is upper triangular.

Exercise
Consider the matrix $\mathbf{E}_{r+\alpha q}$ that represents the elementary row operation of adding a multiple of $\alpha$ times row $q$ to row $r$, with $r \neq q$.

Under what conditions is $\mathbf{E}_{r+\alpha q}$
(i) upper triangular? (ii) lower triangular?

Hint: Apply the row operation to the identity matrix $\mathbf{I}$.

Answer: (i) iff $q < r$; (ii) iff $q > r$. 
Determinants of Triangular Matrices

Theorem

The determinant of any $n \times n$ upper triangular matrix $U$ equals the product of all the elements on its principal diagonal.

Proof.

Recall the expansion formula $|U| = \sum_{\pi \in \Pi} \text{sgn}(\pi) \prod_{i=1}^{n} u_{i\pi(i)}$ where $\Pi$ denotes the set of permutations on $\{1, 2, \ldots, n\}$.

Because $U$ is upper triangular, one has $u_{i\pi(i)} = 0$ unless $i \leq \pi(i)$.

So $\prod_{i=1}^{n} u_{i\pi(i)} = 0$ unless $i \leq \pi(i)$ for all $i = 1, 2, \ldots, n$.

But the identity $\iota$ is the only permutation $\pi \in \Pi$ that satisfies $i \leq \pi(i)$ for all $i \in \mathbb{N}_n$.

Because $\text{sgn}(\iota) = +1$, the expansion reduces to the single term

$$|U| = \text{sgn}(\iota) \prod_{i=1}^{n} u_{i\iota(i)} = \prod_{i=1}^{n} u_{ii}$$

This is the product of the $n$ diagonal elements, as claimed.
Invertible Triangular Matrices

Similarly $|L| = \prod_{i=1}^{n} \ell_{ii}$ for any lower triangular matrix $L$.

Evidently:

**Corollary**

* A triangular matrix (upper or lower) is invertible *if and only if no element on its principal diagonal is 0.*