Competitive Market Mechanisms as Social Choice Procedures

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1 Introduction and outline

1.1 Markets and social choice

Social choice theory concerns itself with the proper choice of a social state from a given feasible set of social states. The main question the theory addresses is how that social choice should depend on the profile of individual preferences. Early work by Arrow and various predecessors, including Condorcet, Borda, Dodgson and Black, was about the construction of a social preference ordering — i.e., a binary relation of weak preference that is complete and transitive. Later, Sen (1971, 1982, 1986) in particular initiated an investigation of more general social choice rules (SCRs) which may not maximize any binary relation, even one that may not be complete or transitive.

A "competitive" or *Walrasian* market mechanism is a prominent example of such a non-binary social choice rule, although it has rarely been regarded as such either by social choice theorists or by economists studying general equilibrium. Generally, this rule will be called the *Walrasian* SCR. It is typically defined for a special class of economic environments in which each social

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state is an economic allocation of private goods, and individuals' preferences concern only their own personal consumption. Moreover, the economic environments in the domain are also typically required to satisfy continuity and convexity assumptions guaranteeing the existence of a Walrasian equilibrium in the market economy.

Another major concern of social choice theory has been with various characterizations of a particular social choice rule. Accordingly, this chapter explores what sets of axioms are uniquely satisfied by the Walrasian mechanism, with or without various forms of lump-sum redistribution.

1.2 Finite economies

The most obvious social choice property satisfied by a Walrasian mechanism is Pareto efficiency. Indeed, as discussed in Section 4.3, for economies with just one "representative" agent, or alternatively with several identical agents who all receive the same consumption vector, Pareto efficiency offers a complete characterization of the Walrasian mechanism under standard continuity, convexity, and aggregate interiority assumptions. Beyond this special case, however, standard textbook examples with two consumers and two goods demonstrate that Pareto efficiency alone is insufficient to characterize the Walrasian mechanism. Nevertheless, once one allows lump-sum redistribution, then most Pareto efficient allocations can be characterized as Walrasian equilibria. The main exceptions are extreme or "oligarchic" allocations in which some agents are so well off that they cannot benefit even from free gifts of goods which would otherwise be consumed by agents outside the oligarchy. Also, even such oligarchic allocations are compensated Walrasian equilibria with lump-sum transfers. These and some related results are presented in Section 4.

Characterizations of Walrasian equilibria without lump-sum transfers remain much more elusive, however, especially for economic environments with a fixed finite number of agents. First Section 5 presents conditions sufficient to ensure the existence of Walrasian equilibrium. Then Section 6 considers characterizations that apply to one economic environment with a fixed set of agents having a fixed type profile. Next, Section 7 considers characterizations with a fixed set of agents but a variable type profile. To conclude the results for "finite economies" with a finite set of individual agents, Section 8 allows the number of agents to vary as well as their type profile. Included are asymptotic characterizations such as the Debreu–Scarf theorem which hold when the number of agents tends to infinity.

1.3 Continuum economies

In finite economies, each individual agent nearly always has influence over market prices. This calls into question the standard Walrasian hypothesis that agents take equilibrium prices as given. For this reason, attempts to provide social choice characterizations of Walrasian equilibria without lump-sum transfers become somewhat less problematic when this influence disappears because the economy has a continuum of agents.

Before considering how to characterize the Walrasian mechanism in a continuum economy, however, Section 9 suggests reasons for generalizing Aumann's (1964) standard concept to "statistical" continuum economies described by a joint distribution over "potential" agents' labels and their types. Efficiency and existence theorems for such economies are presented in Sections 10 and 11; these results are largely adaptations of the counterparts for finite economies in Sections 4 and 5. In continuum economies, however, most of the theorems hold even when agents have non-convex preferences. To allow indivisible goods and other non-convexities in agents' feasible sets, there are also extensions to the case when these sets are "piecewise convex" rather than convex. These extensions typically require an additional "dispersion" assumption.

Following these preliminary results, the next four sections offer several different characaterizations of Walrasian equilibrium allocations which are specific to continuum economies. First, Section 12 presents equivalence theorems for the core and some related solution concepts in a continuum economy. Next, given a continuum of agents whose possible types lie in a suitable smooth domain, Section 13 characterizes allocation mechanisms satisfying a strong Walrasian equilibrium condition as those which satisfy strong forms of both Pareto efficiency and absence of envy. Next, Section 14 offers characterizations in which strategyproofness replaces the absence of envy condition used in Section 13. Then Section 15 shows how, when anonymity is assumed, a "multilateral" version of strategyproofness on its own characterizes a Walrasian mechanism, without the need to assume any form of Pareto efficiency, or a smooth type domain.

Results based on the characterizations by Aumann and Shapley (1974) and by Aumann (1975) of Walrasian equilibria as value allocations will not be discussed here. Hart (2002) in particular offers an authoritative survey.

1.4 Public goods and externalities

The last Section 16 briefly discusses some possible extensions to accommodate public goods and externalities.

2 Agent types

This Section contains essential preliminaries, including various assumptions concerning economic agents and their types. Thereafter, Section 3 provides key definitions for economics with a finite set of agents. Corresponding definitions for continuum economies are provided in Sections 9 and 11.

2.1 Commodity space

For simplicity, and to help focus on the main issues in the existing literature, this chapter assumes throughout that there is a fixed finite set G of *goods* or *commodities*. The associated commodity space is the finite-dimensional Euclidean space $\mathbb{R}^G$.[2] The typical member of $\mathbb{R}^G$ is the vector $x = (x_g)_{g \in G}$. Let $\#G$ denote the number of goods $g \in G$, which is also the dimension of the commodity space $\mathbb{R}^G$.

2.2 Notation

The *Euclidean norm* of each $x \in \mathbb{R}^G$ will be denoted by $\|x\| := \sqrt{\sum_{g \in G} x_g^2}$.

Define the three inequalities $\geqq$, $>$ and $\gg$ on $\mathbb{R}^G$ so that, for each $a = (a_g)_{g \in G}$ and $b = (b_g)_{g \in G}$ in $\mathbb{R}^G$, one has:

(1) $a \geqq b$ iff $a_g \geq b_g$ for all $g \in G$;
(2) $a > b$ iff $a \geqq b$ and $a \neq b$;
(3) $a \gg b$ iff $a_g > b_g$ for all $g \in G$.

Given any set $S \subset \mathbb{R}^G$, let $\operatorname{cl} S$ denotes its closure, and $\operatorname{int} S$ its interior. Also, let:

(1) $p\,S := \{\, p\,x \mid x \in S \,\}$ for any $p \in \mathbb{R}^G$;
(2) $\lambda\,S := \{\, \lambda\,x \mid x \in S \,\}$ for any $\lambda \in \mathbb{R}$.

[2] This assumption excludes the overlapping generations models pioneered by Allais (1947) and Samuelson (1958), with both an infinite time horizon and an infinite set of agents. In particular, this is the main interesting class of economies in which, even though markets are ostensibly complete, nevertheless a Walrasian equilibrium allocation may well be Pareto inefficient. For further discussion, see the survey by Geanakoplos and Polemarchakis (1991) in particular.

Finally, given any two sets $A, B \subset \mathbb{R}^G$, define the *vector sum*

$$A + B := \{ c \in \mathbb{R}^G \mid \exists a \in A, b \in B : c = a + b \}.$$

and the *vector difference* $A - B := A + (-1)B$.

2.3 Consumption and production sets

As usual in classical "Arrow–Debreu" general equilibrium theory, assume any individual agent has a *consumption set* $X \subset \mathbb{R}^G$. Following Rader (1964, 1972, 1976, 1978), however, assume each agent is also endowed with a private *production set* $Y \subset \mathbb{R}^G$. One can also interpret Y as the agent's "opportunity set". The special case usually treated in general equilibrium theory is of a *pure exchange economy* in which an agent's set Y takes the form $\{e\}$ for a fixed initial *endowment vector* e. Often it is assumed that e is an interior point of X. Another common assumption is that X is the non-negative orthant $\mathbb{R}^G_+$.

Occasionally it will be assumed that either X or Y satisfies *free disposal*. In the case of the consumption set X, this means that, whenever $x \in X$ and $\tilde{x} \geqq x$, then $\tilde{x} \in X$. In the case of the production set Y, this means that, whenever $y \in Y$ and $\tilde{y} \leqq y$, then $\tilde{y} \in Y$.

Given the (net) consumption vector x and net production vector y, the agent must make up the difference between x and y by arranging to obtain the *net trade vector* $t := x - y$ from market purchases and sales, or from some more general kind of interaction with other agents.

Given the two sets X and Y, the agent has a *feasible set* of net trades given by the vector difference $T := X - Y$ of the consumption and production sets.

2.4 Preferences for consumption

Assume that each agent $i \in N$ has a (complete and transitive) *preference ordering* R on X. Let P and I denote the associated strict preference and indifference relations, respectively. Given any fixed $\bar{x} \in X$, let

$$R(\bar{x}) := \{ x \in X \mid x \; R \; \bar{x} \} \quad \text{and} \quad P(\bar{x}) := \{ x \in X \mid x \; P \; \bar{x} \}$$

denote the *preference set* and *strict preference set* respectively.

Preferences are said to be:

(i) (globally) *non-satiated* if $P(\bar{x})$ is non-empty for every $\bar{x} \in X$;

(ii) *locally non-satiated* (LNS) if, given any $\bar{x} \in X$ and any topogical neighbourhood V of $\bar{x}$ in $\mathbb{R}^G$, there exists $x \in P(\bar{x}) \cap V$ (implying that the preference ordering has no local maximum);

(iii) *convex* if X and Y are both convex sets, and if for all $\bar{x} \in X$, the preference set $R(\bar{x})$ is convex;

(iv) *strictly convex* if they are convex and moreover, for all $x, \bar{x} \in X$ with $x \in R(\bar{x})$ and $x \neq \bar{x}$, every *strictly convex* combination $\tilde{x} := \alpha x + (1-\alpha)\bar{x}$ with $0 < \alpha < 1$ satisfies $\tilde{x}\ P\ \bar{x}$;

(v) *continuous* if X and Y are both closed sets, as are $R(\bar{x})$ and also the "dispreference" set $R^{i-}(\bar{x}) := \{x \in X \mid \bar{x}\ R\ x\}$ for each $\bar{x} \in X$;

(vi) *weakly monotone* if $x \in R(\bar{x})$ whenever $\bar{x} \in X$ and $x \geqq \bar{x}$;

(vii) *monotone* if preferences are weakly monotone and $x \in P(\bar{x})$ whenever $x \gg \bar{x}$;

(viii) *strictly monotone* if $x \in P(\bar{x})$ whenever $\bar{x} \in X$ and $x > \bar{x}$.

The first two of these properties each have important implications.

Lemma 1 *Suppose preferences are LNS. Then $\bar{x} \in \operatorname{cl} P(\bar{x})$ for all $\bar{x} \in X$.*

PROOF. Given any $\bar{x} \in X$ and any $n = 1, 2, \ldots$, the LNS property implies that there exists $x_n \in P(\bar{x})$ with $\|x_n - \bar{x}\| < 2^{-n}$. Then the sequence x_n $(n = 1, 2, \ldots)$ converges to $\bar{x}$. So $\bar{x} \in \operatorname{cl} P(\bar{x})$. □

The following implication of convex preferences is used in many later proofs.

Lemma 2 *Suppose an agent's preferences are convex. Then $P(\bar{x})$ is convex for all $\bar{x} \in X$.*

PROOF. Suppose that $x, \tilde{x} \in P(\bar{x})$. Because preferences are complete, it loses no generality to suppose that $x, \tilde{x}$ have been chosen so that $x\ R\ \tilde{x}$. Let $\hat{x} := \alpha\, x + (1-\alpha)\,\tilde{x}$ where $0 \leq \alpha \leq 1$. Because preferences are convex, $\hat{x} \in R(\tilde{x})$. But $\tilde{x}\ P\ \bar{x}$ and so, because preferences are transitive, $\hat{x}\ P\ \bar{x}$. □

Later, Section 4.6 uses a somewhat stronger convexity condition that appeared in Arrow and Debreu (1954), as well as Debreu (1959) and McKenzie (1959). Following Arrow and Hahn (1971, p. 78), say that preferences are *semi-strictly convex* if the feasible set X is convex and, in addition, whenever $x, \bar{x} \in X$ with $x\ P\ \bar{x}$ and $0 < \alpha \leq 1$, then $\alpha\, x + (1-\alpha)\,\bar{x}\ P\ \bar{x}$.

A sufficient condition for preferences to be semi-strictly convex is that there exists a *concave* (not merely quasi-concave) utility function $u : X \to \mathbb{R}$ which *represents* R in the sense that $u(x) \geq u(\bar{x})$ if and only if $x \; R \; \bar{x}$.[3]

It is easy to construct examples of preferences that are convex and non-satiated, but are locally satiated. However, the same will not be true of semi-strictly convex preferences, as the following simple result shows.

Lemma 3 *Suppose that preferences are non-satiated and semi-strictly convex. Then preferences are LNS.*

PROOF. Suppose $\bar{x} \in X$ and let V denote any neighbourhood of $\bar{x}$. Because preferences are non-satiated, there exists $x \in P(\bar{x})$. Because preferences are semi-strictly convex, the convex combination $\tilde{x} := \alpha \, x + (1 - \alpha) \, \bar{x}$ (with $0 < \alpha \leq 1$) belongs to $P(\bar{x})$. But $\tilde{x} \in V$ for all sufficiently small $\alpha > 0$. □

Lemma 4 *Suppose that preferences are non-satiated, continuous, and semi-strictly convex. Then preferences are convex.*

PROOF. Suppose that $\hat{x}, \tilde{x} \in R(\bar{x})$ with $\hat{x} \; R \; \tilde{x}$. By Lemma 3, preferences are LNS. So by Lemma 1 there exists a sequence $x_n \in P(\hat{x})$ which converges to $\hat{x}$ as $n \to \infty$. Because preferences are transitive, each $x_n \in P(\tilde{x})$. But preferences are also semi-strictly convex, so for each $\alpha \in (0, 1)$ one has $\alpha \, x_n + (1 - \alpha) \, \tilde{x} \in P(\tilde{x}) \subset R(\bar{x})$ for $n = 1, 2, \ldots$. Because preferences are continuous, so $\alpha \, x + (1 - \alpha) \, \tilde{x} \in R(\bar{x})$ in the limit as $n \to \infty$, for each $\alpha \in (0, 1)$. □

2.5 Regular smooth preferences

Some results presented later depend on particular smoothness assumptions that are frequently used in general equilibrium theory. Specifically, say that the agent has *regular smooth preferences* when:

(a) $X = \mathbb{R}^G_+$;
(b) $Y = \{e\}$ where $e \gg 0$;
(c) R is continuous, convex, and strictly monotone on X;
(d) R satisfies the *boundary condition* that, for any $\bar{x} \gg 0$, one has $x \in P(\bar{x})$ only if $x \gg 0$;
(e) R can be represented on X by a continuous utility function $u : \mathbb{R}^G_+ \to \mathbb{R}$ that is $\mathcal{C}^1$ on $\mathbb{R}^G_{++}$.

[3] See Kannai (1977) for a comprehensive discussion of conditions guaranteeing that a convex preference relation can be represented by a concave utility function.

Condition (d) receives its name because it allows an indifference curve to intersect the boundary of $\mathbb{R}^G_+$ only if it is a subset of that boundary.

2.6 Preferences for net trades

The first part of this chapter, involving economies with finitely many agents, largely considers both agents' consumption and production vectors explicitly. Later, especially in the work involving a continuum of agents, notation will be reduced somewhat by considering only net trade vectors. Then it is convenient to ignore the distinction between an agent's consumption set X and production set Y. Instead, we focus on the *feasible net trade set* $T := X - Y$.

This reduction is facilitated by Rader's (1978) discussion of sufficient conditions for various important properties of an agent's preference relation R on X to carry over to the *derived preference relation for net trades*. This relation, denoted by $\succsim$, is defined on the feasible set $T = X - Y$ so that, for all $t, t' \in T$, one has $t \succsim t'$ if and only if, whenever $x' \in X$ and $y' \in Y$ with $t' = x' - y'$, there exist $x \in R(x')$ and $y \in Y$ such that $t = x - y$. Thus, $t \succsim t'$ if and only if, given any consumption vector $x' = y' + t' \in X$ that the agent can attain by combining a feasible production vector y' with t', there exists a weakly preferred consumption vector $x = y + t \in X$ that the agent can attain by combining an alternative feasible production vector y with t.

Suppose preferences are continuous and the set $\{ (x, y) \in X \times Y \mid x - y = t \}$ is bounded for each t — so compact because it must also be closed. Then it is not hard to show that $\succsim$ is a (complete and transitive) preference ordering over T. Let $\succ$ and $\sim$ denote the associated strict preference and indifference relations, respectively.

Thus, when concentrating on net trades, little is lost by regarding the agent as having a fixed consumption set T and a fixed endowment vector 0, though then the requirement that $T \subset \mathbb{R}^G_+$ has to be relaxed.

In a continuum economy, virtually none of the standard results require preferences to be convex. On the other hand, the assumption that T is a convex set will eventually play an important role. This is because the key "cheaper point" lemma of Section 3.7 may not hold when non-convex feasible sets are allowed, so results concerning "compensated" equilibria may not extend to Walrasian equilibria.

Section 11 uses several assumptions on preferences for net trades to prove that equilibrium exists in a continuum economy. Of these, the simplest is that T *allows autarky* in the sense that $0 \in T$ — i.e., feasibility does not require any net trade. Next, say that T is *bounded below* if there exists $\underline{t} \in \mathbb{R}^G$ such that

$t \geqq \underline{t}$ for all $t \in T$. Finally, say that preferences for net trades are *weakly monotone* if $t \in T$ and $t \succsim_\theta t'$ whenever $t \geqq t'$ and $t' \in T$.

2.7 A metric space of agents' types

When the preference ordering $\succsim$ on T is complete, note that $t \in T$ iff $t \succsim t$. Accordingly, each pair $(T, \succsim)$ for which $\succsim$ is complete corresponds uniquely to the *graph* of $\succsim$, defined as the set

$$G_\succsim := \{ (t, t') \in \mathbb{R}^G \times \mathbb{R}^G \mid t \succsim t' \}.$$

This graph can therefore be used to characterize the agent. We often assume that each possible $\succsim$ is continuous, which is true if and only if $G_\succsim$ is a closed set. Let Θ denote the domain of all possible continuous agent types — i.e., all possible closed subsets of $\mathbb{R}^G \times \mathbb{R}^G$ that correspond to complete and continuous preference orderings.

As discussed in Debreu (1969), Hildenbrand (1974, pp. 15–19) Mas-Colell (1985, Section A.5), and Aliprantis and Border (1999), the family of closed subsets of $\mathbb{R}^G \times \mathbb{R}^G$ can be given a topology of *closed convergence*, and thus made into a compact metric space. As each $\theta \in \Theta$ corresponds to a closed subset of $\mathbb{R}^G \times \mathbb{R}^G$, the space Θ of possible agent types is then also a compact metric space.

Given any agent type $\theta \in \Theta$, let T_θ and $\succsim_\theta$ denote the corresponding feasible set of net trades and preference ordering, respectively. The topology of closed convergence on Θ is useful precisely because it gives the two correspondences $\theta \mapsto\!\!\!\rightarrow T_\theta$ and $\theta \mapsto\!\!\!\rightarrow G_{\succsim_\theta}$ closed graphs, and also makes them both lower hemicontinuous. Specifically:

Lemma 5 *(1) Suppose that (θ, t, t') is the limit as $n \to \infty$ of a convergent sequence $(\theta_n, t_n, t'_n)_{n=1}^\infty$ in $\Theta \times \mathbb{R}^G \times \mathbb{R}^G$ satisfying $t_n, t'_n \in T_{\theta_n}$ as well as $t_n \succsim_{\theta_n} t'_n$ for all $n = 1, 2, \ldots$. Then $t, t' \in T_\theta$ and $t \succsim_\theta t'$.*

(2) Suppose that θ is the limit as $n \to \infty$ of a convergent sequence $(\theta_n)_{n=1}^\infty$ of agent types. Then any $t \in T_\theta$ is the limit of a convergent sequence $(t_n)_{n=1}^\infty$ of net trade vectors that satisfies $t_n \in T_{\theta_n}$ for each $n = 1, 2, \ldots$.

(3) Given any triple $(\theta, t, t') \in \Theta \times \mathbb{R}^G \times \mathbb{R}^G$ satisfying $t, t' \in T_\theta$ and $t \succ_\theta t'$, there exist neighbourhoods U of θ, V of t and V' of t' such that $\tilde{t} \succ_{\tilde{\theta}} \tilde{t}'$ for all $\tilde{\theta} \in U$, all $\tilde{t} \in V \cap T_{\tilde{\theta}}$, and all $\tilde{t}' \in V' \cap T_{\tilde{\theta}}$.

PROOF. See, for example, Hildenbrand (1974, p. 98, Corollaries 1 and 3). □

Say that *autarky is possible* for a type θ agent if $0 \in T_\theta$. Say that a type θ agent's feasible set T_θ has a *lower bound* $\underline{t}_\theta$ if $t \in T_\theta$ implies $t \geqq \underline{t}_\theta$.

From now on, given any $\theta \in \Theta$ and any $\bar{t} \in T_\theta$, let

$$R_\theta(\bar{t}) := \{\, t \in T_\theta \mid t \succsim_\theta \bar{t} \,\} \quad \text{and} \quad P_\theta(\bar{t}) := \{\, t \in T_\theta \mid t \succ_\theta \bar{t} \,\}$$

denote the associated weak and strict preference sets respectively.

Given any type $\theta \in \Theta$ and any price vector $p \in \mathbb{R}^G \setminus \{0\}$, the net trade *Walrasian budget set* of a θ agent is $B_\theta(p) := \{\, t \in T_\theta \mid p\, t \leq 0 \,\}$.

Lemma 6 *Suppose each T_θ has a lower bound $\underline{t}_\theta$, and $0 \in T_\theta$. Then for each $\theta \in \Theta$ and $p \gg 0$:*

(1) $0 \in B_\theta(p)$, which is a compact set;
(2) the mapping $\theta \mapsto \underline{w}_\theta(p) := \inf p\, T_\theta$ is continuous.

PROOF. (1) Obviously $0 \in B_\theta(p)$ for each p and θ. Because $\underline{t}_\theta$ is a lower bound and $p \gg 0$, each component t_g of any vector $t \in B_\theta(p)$ satisfies

$$p_g\, \underline{t}_g \leq p_g\, t_g \leq - \sum_{h \in G \setminus \{g\}} p_h\, t_h \leq - \sum_{h \in G \setminus \{g\}} p_h\, \underline{t}_{\theta h},$$

so $B_\theta(p)$ is bounded. Because $B_\theta(p)$ is evidently closed, it must be compact.

(2) By Lemma 5, the correspondence

$$\theta \mapsto\!\!\!\rightarrow T_\theta = \{\, t \in \mathbb{R}^G \mid (t, t) \in G_{\succsim_\theta} \,\}$$

has a closed graph and is lower hemi-continuous. Because each T_θ has a lower bound, continuity of the mapping $\theta \mapsto \underline{w}_\theta(p)$ follows from applying the maximum theorem to the problem of maximizing $-p\, t$ over the non-empty compact set $B_\theta(p)$. See, for example, Hildenbrand (1974, p. 30, corollary). □

2.8 Smooth type domains

The domain Θ of types is said to be *smooth* provided that it is a piecewise $\mathcal{C}^1$-arc connected[4] subset of a normed linear space satisfying:

[4] This means that, given any pair of types $\theta', \theta'' \in \Theta$, one can connect θ' to θ'' by an arc $s \mapsto \theta(s)$ mapping $[0, 1]$ to Θ such that $\theta(0) = \theta'$, $\theta(1) = \theta''$, where the function $\theta(\cdot)$ is continuous and piecewise continuously differentiable w.r.t. s. For a more general discussion of smooth preferences, see especially Mas-Colell (1985).

(a) for each $\theta \in \Theta$, the feasible set T_θ is closed, convex, satisfies free disposal, has 0 as an interior point, and has $\underline{t}_\theta$ as a lower bound;
(b) for each fixed $\theta \in \Theta$, the preference ordering $\succsim_\theta$ on T_θ can be represented by a $\mathcal{C}^1$ *utility function* $u_\theta(t)$ of (t, θ) which is strictly increasing and strictly quasi-concave as a function of t;
(c) on the domain D of triples $(p, w, \theta) \in \mathbb{R}^G_{++} \times \mathbb{R} \times \Theta$ such that $w > \underline{w}_\theta(p)$, maximizing $u_\theta(t)$ w.r.t. t over the *Walrasian budget set* $\{ t \in T_\theta \mid p\, t \leq w \}$ gives rise to a unique *Walrasian demand* vector $h_\theta(p, w)$ in the interior of T_θ that is a $\mathcal{C}^1$ function of (p, w, θ);
(d) the *indirect utility function* defined on the domain D by $v_\theta(p, w) := u_\theta(h_\theta(p, w))$ is continuous, and has a positive partial derivative $(v_\theta)'_w = \partial v_\theta / \partial w$ w.r.t. w that is a continuous function of (w, θ).
(e) for each fixed $p \gg 0$, if the sequences θ_n and w_n in Θ and $\mathbb{R}$ respectively tend to θ and $+\infty$, then for any $t \in T_\theta$ one has $u_\theta(h_{\theta_n}(p, w_n)) > u_\theta(t)$ for all sufficiently large n.

Condition (e) is not standard, nor is it implied by the other conditions. It is used in Sections 13 and 14 to ensure that self-selection or incentive constraints prevent lump-sum transfers from diverging to $+\infty$.[5] When conditions (a)–(d) all hold, a sufficient condition for (e) to hold as well is that, given any fixed $t \in \mathbb{R}^G$, $p \gg 0$, and $\theta \in \Theta$, one has $h_\theta(p, w) \gg t$ for all large w. This condition clearly holds, for example, in the special *homothetic* case when $T_\theta = \{ t \in \mathbb{R}^G \mid t + e_\theta \geqq 0 \}$ for some fixed endowment vector $e_\theta \gg 0$, and the preference ordering $\succsim_\theta$ on T_θ gives rise to a Walrasian demand function satisfying $h_\theta(p, w) \equiv (w + p\, e_\theta) b_\theta(p) - e_\theta$ where $b_\theta(p) \in \mathbb{R}^G_{++}$ and $p\, b_\theta(p) = 1$ for all $p \gg 0$. More generally, condition (e) holds whenever the $\mathcal{C}^1$ demand functions $h_{\theta g}(p, w)$ for each commodity $g \in G$ have income responses $\partial h_{\theta g} / \partial w$ that are all positive and bounded away from 0 — i.e., there exist functions $a_{\theta g}(p) > 0$ such that $\partial h_{\theta g}(p, w) / \partial w \geq a_{\theta g}(p)$ for all $w > \underline{w}_\theta(p)$.

3 Walrasian equilibrium and Pareto efficiency

3.1 Finite set of agents

As usual in social choice theory, let N denote the finite set of individuals. In the tradition of general equilibrium theory, these individuals may also be described

[5] Luis Corchón suggests there may be a relationship between preferences violating condition (e) and the phenomenon of "immiserizing growth" which Bhagwati (1958, 1987) in particular has explored thoroughly. This deserves later investigation.

as *agents* or *consumers*. Let $\#N$ denote the number of agents $i \in N$.[6]

Superscripts will indicate particular agents $i \in N$. Thus, X^i, Y^i, T^i, R^i, and $\succsim^i$ respectively denote agent i's consumption set, production set, set of feasible net trades, preference ordering over X^i, and preference ordering over T^i. Similarly, x^i, y^i, and $t^i = x^i - y^i$ respectively denote agent i's consumption, production, and net trade vectors.

3.2 Feasible and Pareto efficient allocations

Let X^N and Y^N denote the Cartesian products $\prod_{i \in N} X^i$ and $\prod_{i \in N} Y^i$ respectively. Then $X^N \times Y^N$ is the set of *individually feasible* collections (x^N, y^N) satisfying $x^i \in X^i$ and $y^i \in Y^i$ for all $i \in N$. An *allocation* is a collection $(x^N, y^N) \in X^N \times Y^N$ that also satisfies the *resource balance constraint* $\sum_{i \in N} (x^i - y^i) = 0$. Note that an allocation is automatically feasible, by definition, although even so we shall often write of a "feasible allocation". Also note that free disposal is not assumed, although it will be satisfied if at least one individual agent's production set has this property.

A particular feasible allocation $(\hat{x}^N, \hat{y}^N)$ is said to be:

- *weakly Pareto efficient* (WPE) if no alternative feasible allocation (x^N, y^N) satisfies $x^i \; P^i \; \hat{x}^i$ for all $i \in N$.
- *Pareto efficient* (PE) if no alternative feasible allocation (x^N, y^N) satisfies $x^i \; R^i \; \hat{x}^i$ for all $i \in N$, with $x^h \; P^h \; \hat{x}^h$ for some $h \in N$;

In an example with two agents labelled 1 and 2 and only one good, an allocation can be weakly Pareto efficient but not Pareto efficient provided that one agent has locally satiated preferences at that allocation. Indeed, suppose that agent 1 is made no worse off by giving up a small enough amount of the good, whereas agent 2 always prefers more of the good. Then transferring that small enough amount of the good from agent 1 to agent 2 makes the latter better off, but leaves agent 1 indifferent.

Another example shows that weak Pareto efficiency does not imply Pareto efficiency even when all agents have LNS preferences. Indeed, suppose there are three goods and three agents. Suppose agent 1 has preferences represented by the utility function $u^1(x^1) = x^1_1$, whereas agents 2 and 3 both have utility functions $u^i(x^i)$ that are strictly increasing in (x^i_2, x^i_3) $(i = 2, 3)$, but independent of x^i_1. Suppose each agent $i \in N$ has a fixed endowment vector $e^i \in \mathbb{R}^3$

[6] Standard "Arrow–Debreu" theory also admits a finite set J of private producers that are owned jointly by one or more consumers and have production sets $Y^j \subset \mathbb{R}^G$ (all $j \in J$). This survey will not consider such jointly owned producers.

of the three different goods. Then any allocation that allocates the total endowment of good 1 to agent 1 must be weakly Pareto efficient. Yet Pareto efficiency requires in addition that the total endowments of goods 2 and 3 must be efficiently distributed between agents 2 and 3.

3.3 Wealth distribution and Walrasian equilibrium

Let $p \in \mathbb{R}^G \setminus \{0\}$ denote the typical price vector. Note that the signs of the prices p_g $(g \in G)$ are not specified, as is appropriate without any assumption of either free disposal or monotone preferences.

Given any $p \in \mathbb{R}^G \setminus \{0\}$, the *Walrasian budget constraint* of any agent $i \in N$ is $p\,x^i \leq p\,y^i$. This requires the value of consumption at prices p not to exceed the value of production at those same prices.

A *Walrasian equilibrium* (or WE) is an allocation $(\hat{x}^N, \hat{y}^N)$ and a price vector $\hat{p} \in \mathbb{R}^G \setminus \{0\}$ such that, for each $i \in N$, both $\hat{p}\,\hat{x}^i \leq \hat{p}\,\hat{y}^i$ and also

$$x^i \in P^i(\hat{x}^i),\; y^i \in Y^i \Longrightarrow \hat{p}\,x^i > \hat{p}\,y^i.$$

Equivalently, each agent i's choices $(\hat{x}^i, \hat{y}^i)$ together maximize R^i w.r.t. (x^i, y^i), subject to the feasibility constraints $x^i \in X^i$, $y^i \in Y^i$, as well as the budget constraint $\hat{p}\,x^i \leq \hat{p}\,y^i$. Feasibility of the allocation guarantees that, at the equilibrium price vector $\hat{p}$, demand matches supply for each commodity.

As remarked in the introduction, even though a WE allocation is always at least weakly Pareto efficient in the framework considered here, the converse is not true. At best, most weakly Pareto efficient allocations will be Walrasian equilibria only if agents' budget constraints are modified by specifying a more general *wealth distribution rule* (or by imposing a *lump-sum transfer system*). This is defined as a collection $w^N(p)$ of functions $w^i(p)$ $(i \in N)$ which are homogeneous of degree one in p and satisfy $\sum_{i \in N} w^i(p) \equiv 0$. Here $w^i(p)$ represents agent i's net unearned wealth, which supplements the net profit $p\,y^i$ that i earns by producing and selling the net output vector y^i when the price vector is p. Often it is enough to consider a wealth distribution rule having the special form $w^i(p) \equiv p\,(\bar{x}^i - \bar{y}^i)$ for some fixed feasible allocation $(\bar{x}^N, \bar{y}^N)$.

Relative to the specified wealth distribution rule $w^N(p)$, a *Walrasian equilibrium with lump-sum transfers* (or WELT) consists of a (feasible) allocation $(\hat{x}^N, \hat{y}^N)$ together with a price vector $\hat{p} \in \mathbb{R}^G \setminus \{0\}$, such that, for each $i \in N$, both $\hat{p}\,(\hat{x}^i - \hat{y}^i) \leq w^i(\hat{p})$ and also

$$x^i \in P^i(\hat{x}^i),\; y^i \in Y^i \Longrightarrow \hat{p}\,(x^i - y^i) > w^i(\hat{p}). \tag{1}$$

Equivalently, for each $i \in N$, the pair $(\hat{x}^i, \hat{y}^i)$ must maximize R^i subject to $(x^i, y^i) \in X^i \times Y^i$ and the budget constraint $\hat{p}\,(x^i - y^i) \leq w^i(\hat{p})$.

Obviously, this definition implies that a WE is a WELT relative to the trivial wealth distribution rule given by $w^i(p) = 0$ for all $i \in N$ and all $p \in \mathbb{R}^G \setminus \{0\}$.

3.4 Compensated Walrasian equilibrium

In order to demonstrate that most weakly Pareto efficient allocations can be achieved as WELTs, for a suitable wealth distribution rule and equilibrium price vector, it is useful to introduce the following routine extension of a concept due to McKenzie (1957).[7]

3.4.1 Definition

Given the wealth distribution rule $w^N(p)$, the particular allocation $(\hat{x}^N, \hat{y}^N)$ and price vector $\hat{p} \in \mathbb{R}^G \setminus \{0\}$ form a *compensated equilibrium with lump-sum transfers* (or CELT) if, for each $i \in N$, both $\hat{p}\,(\hat{x}^i - \hat{y}^i) \leq w^i(\hat{p})$ and also

$$x^i \in R^i(\hat{x}^i),\ y^i \in Y^i \Longrightarrow \hat{p}\,(x^i - y^i) \geq w^i(\hat{p}). \tag{2}$$

The only differences between a CELT satisfying (2) and a WELT satisfying (1) are that $P^i(\hat{x}^i)$ has been replaced by $R^i(\hat{x}^i)$, and the strict inequality has become weak.

A *compensated equilibrium* (or CE), without lump-sum transfers, occurs when $w^i(\hat{p}) = 0$ for all $i \in N$.

3.4.2 Arrow's exceptional case

The difference between compensated and (uncompensated) Walrasian equilibrium is illustrated by the following example, often referred to as *Arrow's exceptional case.*[8] The agent's feasible set is taken to be the non-negative quadrant $X = \mathbb{R}^2_+ = \{\, (x_1, x_2) \mid x_1, x_2 \geq 0 \,\}$ in $\mathbb{R}^2$. Preferences are assumed to be represented by the utility function $u(x_1, x_2) = x_1 + \sqrt{x_2}$ restricted to

[7] See also Koopmans (1957, pp. 32–3). Debreu (1951, p. 281) mentions a similar concept, but in that paper fails to distinguish it from a WELT.

[8] The original example in Arrow (1951a) involved an Edgeworth box economy in which one of the two agents has non-monotone preferences. Koopmans (1957, p. 34) appears to have been the first to present a version of the example in which preferences are monotone.

the domain $X = \mathbb{R}^2_+$. So all the indifference curves are given by the equation $x_2 = (u - x_1)^2$ for $0 \leq x_1 \leq u$, and must therefore be parts of parabolae which touch the x_1-axis, as indicated by the dotted curves in Fig. 1.

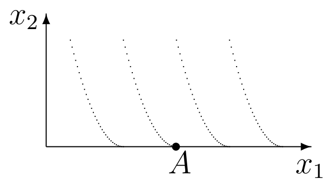

Fig. 1. Arrow's Exceptional Case

This agent has continuous, convex and strictly monotone preferences, as is easily checked. In fact, preferences are "quasi-linear", with the special feature that the marginal willingness to pay for the second good becomes infinite as $x_2 \to 0$.

In this example, trouble arises at any consumption vector of the form $(x_1, 0)$ with x_1 positive, such as the point A in Fig. 1. Any such consumption vector is clearly a compensated equilibrium for the agent at any price vector of the form $(0, p_2)$ where $p_2 > 0$. To make point A a Walrasian equilibrium at any price vector is impossible, however. The reason is that any Walrasian equilibrium price vector would have to be a compensated equilibrium price vector, and so take the form $(0, p_2)$. Hence, the budget constraint would have to be $p_2\, x_2 \leq 0$, which is equivalent to $x_2 \leq 0$. But then the agent could always move to preferred points by increasing x_1 indefinitely while keeping $x_2 = 0$.

3.5 Properties of compensated and Walrasian equilibria

Lemma 7 *In any CELT one has* $\hat{p}\,(\hat{x}^i - \hat{y}^i) = w^i(\hat{p})$ *for all* $i \in N$.

PROOF. By definition, $\hat{p}\,(\hat{x}^i - \hat{y}^i) \leq w^i(\hat{p})$ for all $i \in N$. On the other hand, because $\hat{x}^i \in R^i(\hat{x}^i)$ and $\hat{y}^i \in Y^i$, the definition also implies that $\hat{p}\,(\hat{x}^i - \hat{y}^i) \geq w^i(\hat{p})$ for all $i \in N$. □

From now on, let $\pi^i(p)$ denote $\sup p\, Y^i$, agent i's supremum profit at the price vector $p \neq 0$.

Lemma 8 *For any agent* $i \in N$ *who has LNS preferences, in any CELT (or CE) one must have* $\hat{p}\,\hat{y}^i = \pi^i(\hat{p}) = \max p\, Y^i$.

PROOF. Define $w := w^i(\hat{p}) + \pi^i(\hat{p})$. Suppose that $\bar{x}^i \in X^i$ satisfies $\hat{p}\,\bar{x}^i < w$. Then there exists a neighbourhood V of $\bar{x}^i$ such that $\hat{p}\,x^i < w$ for all $x^i \in V$. Because preferences are LNS, there exists $\tilde{x}^i \in P^i(\bar{x}^i) \cap V$. By the definitions of w and $\pi^i(\hat{p})$, then there also exists $y^i \in Y^i$ such that $\hat{p}\,\tilde{x}^i < w^i(\hat{p}) + \hat{p}\,y^i$. So $\bar{x}^i$ cannot be part of a CELT at the price vector $\hat{p}$.

Conversely, if $(\hat{x}^i, \hat{y}^i)$ is part of a CELT at the price vector $\hat{p}$ and given the wealth level $w^i(\hat{p})$, then $\hat{p}\,\hat{x}^i \geq w$. By Lemma 7 and the definition of w, it follows that

$$\hat{p}\,\hat{y}^i = \hat{p}\,\hat{x}^i - w^i(\hat{p}) \geq w - w^i(\hat{p}) = \pi^i(\hat{p}).$$

The result follows from the definition of $\pi^i(\hat{p})$. □

Thus, in any CELT, each agent $i \in N$ with LNS preferences must have a net output vector $\hat{y}^i$ which maximizes net profit at the equilibrium price vector $\hat{p}$ over i's production set Y^i.

Lemma 9 *If all agents have LNS preferences, then any WELT is a CELT.*

PROOF. Suppose that $(\hat{x}^N, \hat{y}^N, \hat{p})$ is a WELT relative to the wealth distribution rule $w^N(p)$. Consider any $i \in N$ and any $(x^i, y^i) \in R^i(\hat{x}^i) \times Y^i$. Because preferences are LNS, Lemma 1 implies that there exists a sequence $x^i_n \in P^i(\hat{x}^i)$ $(n = 1, 2, \ldots)$ which converges to x^i. The fact that $(\hat{x}^N, \hat{y}^N, \hat{p})$ is a WELT implies that $\hat{p}\,(x^i_n - y^i) > w^i(\hat{p})$ for all n. Taking the limit as $n \to \infty$ yields $\hat{p}\,(x^i - y^i) \geq w^i(\hat{p})$. This is enough to show that $(\hat{x}^N, \hat{y}^N, \hat{p})$ is a CELT as well as a WELT relative to the wealth distribution rule $w^N(p)$. □

Obviously Lemma 9 implies that if preferences are LNS, then Lemmas 7 and 8 are also true for any WELT. The following summarizes the results of Lemmas 7–9 in this case:

Theorem 10 *Suppose all agents have LNS preferences. Then the feasible allocation $(\hat{x}^N, \hat{y}^N)$ is a WELT at the price vector $\hat{p} \neq 0$ relative to the wealth distribution rule $w^N(p)$ if and only if, for all $i \in N$, one has:*

(1) $\hat{p}\,(\hat{x}^i - \hat{y}^i) = w^i(\hat{p})$;
(2) $\hat{p}\,\hat{y}^i = \pi^i(\hat{p})$;
(3) $x^i \in P^i(\hat{x}^i)$ *implies that* $\hat{p}\,x^i > \hat{p}\,\hat{x}^i$.

The same allocation is a CELT at the price vector $\hat{p}$ if and only if, for all $i \in N$, properties (1) and (2) are satisfied, but (3) is replaced by:

(3′) $x^i \in R^i(\hat{x}^i)$ *implies that* $\hat{p}\,x^i \geq \hat{p}\,\hat{x}^i$.

PROOF. Properties (1), (2) and (3) together imply that

$$\hat{p}\,(x^i - y^i) > \hat{p}\,\hat{x}^i - \pi^i(\hat{p}) = \hat{p}\,(\hat{x}^i - \hat{y}^i) = w^i(\hat{p})$$

whenever $x^i \in P^i(\hat{x}^i)$ and $y^i \in Y^i$, as required for a WELT. When (3) is replaced by (3′), there is a CELT instead of a WELT. So the listed properties are sufficient.

Conversely, Lemmas 7–9 show the necessity of properties (1) and (2). In addition, if $(\hat{x}^N, \hat{y}^N, \hat{p})$ is indeed a WELT, then for any $i \in N$, putting $y^i = \hat{y}^i$ in the definition and using Lemma 8 implies that, whenever $x^i \in P^i(\hat{x}^i)$, then

$$\hat{p}\,(x^i - \hat{y}^i) > w^i(\hat{p}) = \hat{p}\,(\hat{x}^i - \hat{y}^i).$$

In particular, $\hat{p}\,x^i > \hat{p}\,\hat{x}^i$, thus verifying property (3). The proof that a CELT must satisfy property (3′) is similar. □

3.6 Walrasian equilibrium with equal budgets

A *Walrasian equilibrium with equal budgets* (or WEEB) is a WELT $(\hat{x}^N, \hat{y}^N, \hat{p})$ relative to some wealth distribution rule satisfying the restriction that, in equilibrium, one has

$$\hat{p}\,\hat{x}^i = \frac{1}{\#N} \sum_{h \in N} \hat{p}\,\hat{y}^h \tag{3}$$

for all $i \in N$. In other words, all agents $i \in N$ have the same amount to spend on their respective (net) consumption vectors x^i. Such a WELT will be an equilibrium relative to the specific "egalitarian" wealth distribution rule defined by $w^i(p) := -\pi^i(p) + (1/\#N) \sum_{h \in N} \pi^h(p)$ for all price vectors $p \neq 0$. This rule collects each agent's supremum profit, and then distributes the total profit equally among all agents.

In the case of a standard exchange economy where $Y^i = \{e^i\}$ with $e^i \in \mathbb{R}^G_+$ for each $i \in N$, a WEEB is equivalent to a WE in the "equal split" exchange economy where each agent is endowed with an equal share $(1/\#N) \sum_{h \in N} e^h$ of the aggregate endowment, as discussed by Thomson (1983) and many others. More generally, let $\tilde{Y}$ denote the *mean production set* $(1/\#N) \sum_{h \in N} Y^h$. The *equivalent equal opportunity economy* is defined as one where each agent $i \in N$ is endowed with the production set $\tilde{Y}$ (instead of with Y^i). Say that the feasible allocation (x^N, y^N) in this equivalent economy has *symmetric production* if $y^h = y^i$ for all $h, i \in N$.

Theorem 11 *Corresponding to each WEEB $(\hat{x}^N, \hat{y}^N, \hat{p})$ in the original economy is a WE $(\hat{x}^N, \tilde{y}^N, \hat{p})$ with symmetric production in the equivalent equal opportunity economy.*

PROOF. Suppose $(\hat{x}^N, \hat{y}^N, \hat{p})$ is a WEEB. Let $\tilde{y} := (1/\#N) \sum_{h \in N} \hat{y}^h \in \tilde{Y}$ and let $\tilde{y}^i := \tilde{y}$ (all $i \in N$), so that $\tilde{y}^N$ has symmetric production. Then $\sum_{i \in N} \hat{x}^i = \sum_{i \in N} \hat{y}^i = \sum_{i \in N} \tilde{y}^i$, and (3) implies that

$$\hat{p}\, \hat{x}^i = \frac{1}{\#N} \sum_{h \in N} \hat{p}\, \hat{y}^h = \hat{p}\, \tilde{y} = \hat{p}\, \tilde{y}^i \quad (\text{all } i \in N) \ .$$

Also, given any $(x^i, y) \in P^i(\hat{x}^i) \times \tilde{Y}$, Theorem 10 implies that $\hat{p}\, x^i > \hat{p}\, \hat{x}^i$ and

$$\begin{aligned} \hat{p}\, y &\leq \sup \hat{p}\, \tilde{Y} = \sup \hat{p}\, \frac{1}{\#N} \sum_{i \in N} Y^i = \frac{1}{\#N} \sum_{i \in N} \sup \hat{p}\, Y^i \\ &= \frac{1}{\#N} \sum_{i \in N} \pi^i(\hat{p}) = \frac{1}{\#N} \sum_{i \in N} \hat{p}\, \hat{y}^i = \hat{p}\, \tilde{y}. \end{aligned}$$

Applying Theorem 10 once again shows that $(\hat{x}^N, \tilde{y}^N, \hat{p})$ is a WE in the equivalent equal opportunity economy.

Conversely, suppose that $(\hat{x}^N, \tilde{y}^N, \hat{p})$ is a WE with symmetric production in the equivalent equal opportunity economy. Let $\tilde{y}$ denote the common value of $\tilde{y}^i$ (all $i \in N$). Then $\hat{p}\, \hat{x}^i = \hat{p}\, \tilde{y}$ for all $i \in N$.

Furthermore, Theorem 10 implies that $\tilde{y} \in \arg\max \hat{p}\, \tilde{Y}$. By definition of $\tilde{Y}$, it follows that $\tilde{y} = (1/\#N) \sum_{i \in N} \hat{y}^i$ for some collection $\hat{y}^N \in Y^N$ satisfying $\hat{y}^i \in \arg\max \hat{p}\, Y^i$ for each $i \in N$. Then $\sum_{i \in N} \hat{x}^i = \#N \tilde{y} = \sum_{i \in N} \hat{y}^i$. Finally, because $(\hat{x}^N, \tilde{y}^N, \hat{p})$ is a WE, Theorem 10 also implies that $\hat{p}\, x^i > \hat{p}\, \hat{x}^i$ whenever $x^i \in P^i(\hat{x}^i)$. This completes the proof that $(\hat{x}^N, \hat{y}^N, \hat{p})$ is a WEEB. □

When $\tilde{Y}$ is convex, it loses no generality to assume that any WE in the equivalent equal opportunity economy has symmetric production. This need not be true when $\tilde{Y}$ is non-convex, however.

3.7 The cheaper point lemma

Given the wealth distribution rule $w^i(p)$ ($i \in N$), say that agent i has *a cheaper point* at the price vector $\hat{p}$ if there exists $(\bar{x}^i, \bar{y}^i) \in X^i \times Y^i$ such that $\hat{p}\,(\bar{x}^i - \bar{y}^i) < w^i(\hat{p})$.

The usual way to prove that a CELT is a WELT involves the following fundamental result:[9]

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[9] Koopmans (1957, p. 34) ascribes this to the Remark in Section 5 of Debreu (1954), for whose "essence" Debreu gives credit to Arrow (1951a, Lemma 5).

Lemma 12 (The Cheaper Point Lemma) *Given any $i \in N$, suppose the feasible sets X^i and Y^i are convex, while i's preferences are continuous. Suppose that agent i has a cheaper point $(\bar{x}^i, \bar{y}^i)$ at the price vector $\hat{p}$. Then, if $(\hat{x}^i, \hat{y}^i)$ is any compensated equilibrium for i satisfying (2), it follows that $(\hat{x}^i, \hat{y}^i)$ is a Walrasian equilibrium for i satisfying (1).*

PROOF. Suppose that $(\hat{x}^i, \hat{y}^i)$ is a compensated equilibrium for i. Consider any $(x^i, y^i) \in P^i(\hat{x}^i) \times Y^i$. Because X^i and Y^i are convex, while R^i is continuous, there exists λ with $0 < \lambda < 1$ small enough to ensure that $(1-\lambda)x^i + \lambda \bar{x}^i \in R^i(\hat{x}^i)$. Then $(1-\lambda)y^i + \lambda \bar{y}^i \in Y^i$. The hypothesis that $(\hat{x}^i, \hat{y}^i)$ is a compensated equilibrium for i implies that

$$\hat{p}\,[(1-\lambda)x^i + \lambda \bar{x}^i - (1-\lambda)y^i - \lambda \bar{y}^i] \geq w^i(\hat{p})$$

or equivalently,

$$(1-\lambda)\hat{p}\,(x^i - y^i) \geq w^i(\hat{p}) - \lambda\, \hat{p}\,(\bar{x}^i - \bar{y}^i). \tag{4}$$

But $(\bar{x}^i, \bar{y}^i)$ is a cheaper point for agent i, so $\hat{p}\,(\bar{x}^i - \bar{y}^i) < w^i(\hat{p})$. Because $\lambda > 0$, it follows that $w^i(\hat{p}) - \lambda\, \hat{p}\,(\bar{x}^i - \bar{y}^i) > (1-\lambda) w^i(\hat{p})$. Thus, (4) implies that

$$(1-\lambda)\hat{p}\,(x^i - y^i) > (1-\lambda) w^i(\hat{p}). \tag{5}$$

Dividing each side of (5) by $1 - \lambda$, which is positive, yields $\hat{p}\,(x^i - y^i) > w^i(\hat{p})$, as required. □

This result motivates the following obvious adaptation of a definition suggested by Debreu (1962). Given the wealth distribution rule $w^N(p)$, a feasible allocation $(\hat{x}^N, \hat{y}^N)$ and price vector $\hat{p} \neq 0$ are said to be a *quasi-equilibrium with lump-sum transfers* (or QELT) provided that:

(1) for all $i \in N$, one has $\hat{p}\,(\hat{x}^i - \hat{y}^i) \leq w^i(\hat{p})$;
(2) for every agent $i \in N$ who has a cheaper point at prices $\hat{p}$, whenever $x^i \in P^i(\hat{x}^i)$ and $y^i \in Y^i$, then $\hat{p}\,(x^i - y^i) > w^i(\hat{p})$.

When preferences are LNS, it is easy to see that any QELT is a CELT. When the feasible set is convex and preferences are continuous, Lemma 12 clearly implies that any CELT allocation must be a QELT.

3.8 The unique cheapest point case

Arrow's exceptional case in Section 3.4.2 has the key feature that there is a whole line segment of cheapest points $(x_1, 0)$ at any relevant price vector of the form $(0, p_2)$ (with $p_2 > 0$). The next property explicitly rules this out.

Say that agent $i \in N$ has a *unique cheapest point* at the price vector $p \neq 0$ provided there is a unique $\underline{x}^i \in X^i$ such that $p\,x^i > p\,\underline{x}^i$ for all $x^i \in X^i \setminus \{\underline{x}^i\}$.

Provided that $p \gg 0$, it is easy to check that a unique cheapest point exists in each of the following cases:

(1) X^i is closed, bounded below, and strictly convex;[10]
(2) $X^i = \mathbb{R}^G_+$;
(3) there is only one good, and X^i is a closed half-line that is bounded below.

Lemma 13 *Suppose agent $i \in N$ has convex feasible sets X^i and Y^i, as well as preferences that are LNS and continuous. Suppose that $(\hat{x}^i, \hat{y}^i) \in X^i \times Y^i$ is a CELT for agent i at the price vector $\hat{p} \neq 0$, and that agent i has a unique cheapest point at $\hat{p}$. Then $(\hat{x}^i, \hat{y}^i)$ is a WELT for agent i at the price vector $\hat{p}$.*

PROOF. Because of Lemma 12, it is enough to consider the case when agent i has no cheaper point — that is, when $\hat{p}\,(x^i - y^i) \geq w^i(\hat{p})$ for all $(x^i, y^i) \in X^i \times Y^i$. In particular, putting $y^i = \hat{y}^i$ and using Lemma 7 yields

$$\hat{p}\,x^i \geq \hat{p}\,\hat{y}^i + w^i(\hat{p}) = \hat{p}\,\hat{x}^i$$

for all $x^i \in X^i$, so that $\hat{x}^i$ is a cheapest point of X^i.

Now consider any $x^i \in P^i(\hat{x}^i)$. Then $x^i \neq \hat{x}^i$ and so $\hat{p}\,x^i > \hat{p}\,\hat{x}^i$ because of the hypothesis that agent i has a unique cheapest point at $\hat{p}$. Theorem 10 implies that the CELT $(\hat{x}^i, \hat{y}^i)$ for agent i at the price vector $\hat{p}$ must be a WELT. □

3.9 Aggregate interiority

The rest of this chapter makes frequent use of the *aggregate interiority* assumption. This requires 0 to be an interior point of the set $\sum_{i \in N}(X^i - Y^i)$ of *feasible aggregate net trade vectors* $\sum_{i \in N}(x^i - y^i)$ satisfying $x^i \in X^i$ and $y^i \in Y^i$ for all $i \in N$. This interiority assumption by itself ensures that there always exists at least one agent with a cheaper point:

Lemma 14 *Suppose aggregate interiority is satisfied. Then, given any wealth distribution rule $w^N(p)$ and any price vector $\hat{p} \in \mathbb{R}^G \setminus \{0\}$, at least one agent $h \in N$ has a feasible pair $(\bar{x}^h, \bar{y}^h) \in X^h \times Y^h$ satisfying $\hat{p}\,(\bar{x}^h - \bar{y}^h) < w^h(\hat{p})$.*

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[10] This case is mentioned by Koopmans (1957, p. 32).

PROOF. Because $0 \in \operatorname{int} \sum_{i \in N} (X^i - Y^i)$, there exist $(\bar{x}^i, \bar{y}^i) \in X^i \times Y^i$ for all $i \in N$ such that $\hat{p} \sum_{i \in N} (\bar{x}^i - \bar{y}^i) < 0 = \sum_{i \in N} w^i(\hat{p})$. The result follows immediately. □

4 Characterizing Pareto efficient allocations

4.1 First efficiency theorem

The first result, based on Arrow (1951a), is very simple.

Theorem 15 *Any WELT is weakly Pareto efficient, and is Pareto efficient if all agents have LNS preferences.*

PROOF. Let $(\hat{x}^N, \hat{y}^N, \hat{p})$ be any WELT. We show that $\sum_{i \in N} (x^i - y^i) \neq 0$ for any $(x^N, y^N) \in \prod_{i \in N} [P^i(\hat{x}^i) \times Y^i]$, so no such (x^N, y^N) is feasible.

Indeed, if $(x^i, y^i) \in P^i(\hat{x}^i) \times Y^i$ for all $i \in N$, the definition of WELT implies that $\hat{p}\,(x^i - y^i) > w^i(\hat{p})$. Summing over i gives

$$\hat{p} \sum_{i \in N} (x^i - y^i) > \sum_{i \in N} w^i(\hat{p}) = 0, \tag{6}$$

and so $\sum_{i \in N} (x^i - y^i) \neq 0$.

When all agents have LNS preferences, Lemma 7 implies that the WELT is a CELT. So if $(x^N, y^N) \in \prod_{i \in N} [R^i(\hat{x}^i) \times Y^i]$, then $\hat{p}\,(x^i - y^i) \geq w^i(\hat{p})$ for each i. If in addition $x^h \; P^h \; \hat{x}^h$ for any $h \in N$, then $\hat{p}\,(x^h - y^h) > w^h(\hat{p})$. Once again, summing over i gives (6). □

4.2 Second efficiency theorem

The second result, also based on Arrow (1951a), is much more involved than the first. Indeed, Arrow's exceptional case described in Section 3.4.2 is just one example showing the need for extra assumptions. We therefore begin with a simpler result:

Theorem 16 *Suppose agents' preferences are LNS and convex. Then any weakly Pareto efficient allocation* $(\hat{x}^N, \hat{y}^N)$ *is a CELT.*

PROOF. For each $i \in N$, define the set $Z^i := P^i(\hat{x}^i) - Y^i$ of net trade vectors $x^i - y^i$ allowing i to achieve a consumption vector $x^i \in P^i(\hat{x}^i)$. By Lemma 2, the set $P^i(\hat{x}^i)$ is convex, and so therefore is Z^i, as the vector difference of two convex sets. Then weak Pareto efficiency implies that $0 \notin Z := \sum_{i \in N} Z^i$. Moreover, Z must be convex as the sum of convex sets. So there exists a price vector $\hat{p} \neq 0$ which defines a hyperplane $\hat{p}\, z = 0$ through the origin with the property that $\hat{p}\, z \geq 0$ for all $z \in Z$. Because $Z = \sum_{i \in N} Z^i$, it follows that

$$0 \leq \inf \hat{p} \sum_{i \in N} Z^i = \sum_{i \in N} \inf \hat{p}\, Z^i. \tag{7}$$

By Lemma 1, LNS preferences imply that $\hat{x}^i \in \operatorname{cl} P^i(\hat{x}^i)$, and so $\hat{x}^i - \hat{y}^i \in \operatorname{cl} Z^i$. It follows that

$$\inf \hat{p}\, Z^i \leq \hat{w}^i := \hat{p}\,(\hat{x}^i - \hat{y}^i) \tag{8}$$

for all $i \in N$. Now (7) and (8) together imply that

$$0 \leq \inf \hat{p} \sum_{i \in N} Z^i = \sum_{i \in N} \inf \hat{p}\, Z^i \leq \sum_{i \in N} \hat{w}^i = 0 \tag{9}$$

because $\sum_{i \in N} (\hat{x}^i - \hat{y}^i) = 0$ and so $\sum_{i \in N} \hat{w}^i = \hat{p} \sum_{i \in N} (\hat{x}^i - \hat{y}^i) = 0$. Hence, both inequalities in (9) must be equalities, which is consistent with (8) only if

$$\inf\{\, \hat{p}\, z^i \mid z^i \in Z^i \,\} = \hat{w}^i \quad \text{for all } i \in N. \tag{10}$$

Now suppose that $(x^i, y^i) \in R^i(\hat{x}^i) \times Y^i$. Because preferences are transitive, $P^i(x^i) \subset P^i(\hat{x}^i)$. Because preferences are LNS, Lemma 1 implies that there is a sequence of points $(x^i_n)_{n=1}^\infty$ in $P^i(x^i)$ which converges to x^i. But then $x^i_n - y^i \in P^i(\hat{x}^i) - Y^i = Z^i$ for all n. By (10), it follows that $\hat{p}\,(x^i_n - y^i) \geq \hat{w}^i$. So $\hat{p}\,(x^i - y^i) \geq \hat{w}^i = \hat{p}\,(\hat{x}^i - \hat{y}^i)$ in the limit as $n \to \infty$.

These properties imply that $(\hat{x}^N, \hat{y}^N, \hat{p})$ is a CELT. □

4.3 Identical agents

Obviously, we say that *all agents are identical* in case the feasible sets X^i, Y^i and preference orderings R^i are all independent of i. Suppose too that the identical agents have LNS, continuous, and convex preferences. Following Lucas (1978) and Stokey, Lucas and Prescott (1989, ch. 1), it has been common in macroeconomics to consider symmetric allocations in an economy with many identical agents. There is a particularly simple relationship between Walrasian equilibrium and Pareto efficiency in such a framework.

Consider any WE $(\hat{x}^N, \hat{y}^N, \hat{p})$. By Theorem 10, this must satisfy the budget equation $\hat{p}\, \hat{x}^i = \hat{p}\, \hat{y}^i$ for all $i \in N$ because preferences are LNS. Consider also the symmetric allocation $(\bar{x}^N, \bar{y}^N)$ defined for all $i \in N$ by

$$\bar{x}^i := \bar{x} := \frac{1}{\#N} \sum_{i \in N} \hat{x}^i \quad \text{and} \quad \bar{y}^i := \bar{y} := \frac{1}{\#N} \sum_{i \in N} \hat{y}^i$$

Because preferences are convex, it is easy to see that this is also a WE at the same price vector $\hat{p}$. It must therefore satisfy $\hat{x}^i \; I^i \; \bar{x}$ for all $i \in N$. Hence the original WE is equivalent to a symmetric WE that is also Pareto efficient.

More interesting is the converse. Suppose $(\hat{x}^N, \hat{y}^N)$ is a symmetric Pareto efficient allocation, with $\hat{x}^i = \hat{x}$ and $\hat{y}^i := \hat{y}$ for all $i \in N$. Of course, feasibility requires that $\hat{x} = \hat{y}$. Theorem 16 states that this allocation must be a CELT, for some price vector $\hat{p} \neq 0$, and given a wealth distribution rule $w^N(p)$ satisfying $w^i(\hat{p}) = \hat{p}\,(\hat{x}^i - \hat{y}^i)$ for all $i \in N$. But then $w^i(\hat{p}) = \hat{p}\,(\hat{x} - \hat{y}) = 0$ for all $i \in N$. So this CELT is actually a CE, without lump-sum transfers.

Finally, as mentioned in Section 3.9, we impose the standard aggregate interiority assumption requiring that $0 \in \operatorname{int} \sum_{i \in N}(X^i - Y^i)$. By Lemma 14, at least one agent $h \in N$ has a cheaper point $(\bar{x}^h, \bar{y}^h) \in X^h \times Y^h$ satisfying $\hat{p}\,(\bar{x}^h - \bar{y}^h) < 0$. Since all agents are identical, all must have cheaper points. By Lemma 12, it follows that $(\hat{x}, \hat{y}, \hat{p})$ is a Walrasian equilibrium. This proves that a symmetric Pareto efficient allocation is a WE under the assumptions stated above — namely, that agents have identical LNS, continuous and convex preferences, while $0 \in \operatorname{int} \sum_{i \in N}(X^i - Y^i)$.

4.4 Non-oligarchic allocations

The following general conditions for a CELT to be a WELT, and so for a Pareto efficient allocation to be a WELT, originated in Hammond (1993), following ideas pioneered by McKenzie (1959, 1961).

For the weakly Pareto efficient allocation $(\hat{x}^N, \hat{y}^N)$, the proper subset $K \subset N$ (with both K and $N \setminus K$ non-empty) is said to be an *oligarchy* if there is no alternative feasible allocation satisfying $x^i \; P^i \; \hat{x}^i$ for all $i \in K$. In the case when $K = \{d\}$ for some $d \in N$, we may speak of d being a *dictator* who is unable to find any preferred alternative. On the other hand, a *non-oligarchic* weakly Pareto efficient (or NOWPE) allocation occurs if there is no oligarchy.

One way to interpret this definition is that the members of any oligarchy K are already so well off at the relevant allocation $(\hat{x}^N, \hat{y}^N)$ that they cannot benefit from any gift that an outside individual might make. Indeed, the members of K would not even wish to steal whatever little consumption is left to those agents who are excluded from the set K.

It may be instructive to consider these definitions in the context of an economy with 3 agents and just one good. As usual, agents are assumed always to prefer more of the good for their own consumption. Each agent is assumed to have a fixed positive endowment of the only good. Then the set of feasible non-negative consumption allocations is a triangle in 3-dimensional space. The corners correspond to the dictatorial allocations which give all the total endowment to one of the three agents. On the other hand, those oligarchic allocations which are not also dictatorial occur on the relative interior of each edge of the feasible triangle.

Theorem 17 *Suppose agents' preferences are LNS, convex and continuous. Suppose too that* $0 \in \operatorname{int} \sum_{i \in N} (X^i - Y^i)$*. Then any non-oligarchic weakly Pareto efficient (NOWPE) allocation* $(\hat{x}^N, \hat{y}^N)$ *is a WELT.*

PROOF. By Theorem 16, the hypotheses here guarantee that $(\hat{x}^N, \hat{y}^N)$ is a CELT at some price vector $\hat{p} \neq 0$ w.r.t. some wealth distribution rule $w^N(p)$ satisfying $w^i(\hat{p}) = \hat{p}\,(\hat{x}^i - \hat{y}^i)$ for all $i \in N$, and so $\sum_{i \in N} w^i(\hat{p}) = 0$. Let K be the set of all agents having cheaper points. By Lemma 14, K is non-empty.

Consider any collection $(x^N, y^N) \in X^N \times Y^N$ with $x^i \; P^i \; \hat{x}^i$ for all $i \in K$. Because each agent $i \in K$ has a cheaper point, Lemma 12 implies that

$$\hat{p}\,(x^i - y^i) > w^i(\hat{p}) \quad \text{for all } i \in K. \tag{11}$$

But no agent outside K has a cheaper point, so

$$\hat{p}\,(x^i - y^i) \geq w^i(\hat{p}) \quad \text{for all } i \in N \setminus K. \tag{12}$$

Because K must be non-empty, adding the sum of (11) over all $i \in K$ to the sum of (12) over all $i \in N \setminus K$ yields $\hat{p} \sum_{i \in N} (x^i - y^i) > \sum_{i \in N} w^i(\hat{p}) = 0$. Hence, (x^N, y^N) cannot be a feasible allocation. Unless $K = N$, it follows that K is an oligarchy.

Conversely, if $(\hat{x}^N, \hat{y}^N)$ is non-oligarchic, then $K = N$, so all agents have cheaper points. By Lemma 12, the CELT is actually a WELT. □

Corollary 18 *Under the hypotheses of Theorem 17, one has*

$$\mathit{NOWPE} \subset \mathit{WELT} \subset \mathit{PE} \subset \mathit{WPE} \subset \mathit{CELT} = \mathit{QELT}.$$

PROOF. It has just been proved that NOWPE ⊂ WELT. Theorem 15 implies that WELT ⊂ PE. Obviously PE ⊂ WPE from the definitions in Section 3.2. Theorem 16 implies that WPE ⊂ CELT. Finally, CELT = QELT as discussed at the end of Section 3.7. □

The next result is prompted by an idea due to Spivak (1978).

Theorem 19 *Suppose agents' preferences are LNS, convex and continuous. Let $(\hat{x}^N, \hat{y}^N)$ be a weakly Pareto efficient allocation in which K is an oligarchy. Then there is a price vector $\hat{p} \neq 0$ such that $(\hat{x}^N, \hat{y}^N, \hat{p})$ is a CELT in which each agent $i \in N \setminus K$ is at a cheapest point.*

PROOF. The proof is similar to that of Theorem 16. For each $i \in K$, define $Z^i := P^i(\hat{x}^i) - Y^i$; for each $i \in N \setminus K$, however, define $Z^i := X^i - Y^i$. Then let $Z := \sum_{i \in K} Z^i$, which is obviously non-empty and convex. The definition of oligarchy implies that $0 \notin Z$. Hence there exists $\hat{p} \neq 0$ such that $\hat{p}\, z \geq 0$ for all $z \in Z$.

As in the proof of Theorem 16, for all $i \in N$ the net trade vector $\hat{x}^i - \hat{y}^i \in \operatorname{cl} Z^i$. Repeating the derivation of (10), one has

$$\inf \hat{p}\, Z^i = \hat{w}^i := \hat{p}\,(\hat{x}^i - \hat{y}^i) \quad \text{for all } i \in N. \tag{13}$$

For all $i \in K$, (13) implies that $\hat{p}\,(x^i - y^i) \geq \hat{w}^i$ whenever $(x^i, y^i) \in P^i(\hat{x}^i) \times Y^i$, and so, because preferences are LNS, whenever $(x^i, y^i) \in R^i(\hat{x}^i) \times Y^i$. For all $i \in N \setminus K$, (13) implies that $\hat{p}\,(x^i - y^i) \geq \hat{w}^i$ whenever $(x^i, y^i) \in X^i \times Y^i$, so $(\hat{x}^i, \hat{y}^i)$ is a cheapest point of $X^i \times Y^i$. Hence, relative to the specific wealth distribution rule $w^N(p)$ defined by $w^i(p) := p\,(\hat{x}^i - \hat{y}^i)$ (all $i \in N$, $p \in \mathbb{R}^G$), the triple $(\hat{x}^N, \hat{y}^N, \hat{p})$ is a CELT with each $i \in N \setminus K$ at a cheapest point. □

4.5 Exact characterization of non-oligarchic allocations

Let CELT* indicate the subset of CELT allocations with the special property that *every* equilibrium price vector allows *every* agent a cheaper point. Let WELT* and QELT* be the corresponding subsets of WELT and QELT.

Theorem 20 *Suppose that agents' preferences are LNS, convex and continuous, and that $0 \in \operatorname{int} \sum_{i \in N} (X^i - Y^i)$. Then one has the exact characterization NOWPE = WELT* = CELT* = QELT*, and also every NOWPE allocation is (fully) Pareto efficient.*

PROOF. The cheaper point Lemma 12 makes the equality QELT* = CELT* = WELT* obvious. Also, the proof of Theorem 17 actually demonstrates the stronger result that NOWPE ⊂ CELT* = WELT* ⊂ WELT. And, of course, any WELT* allocation must be Pareto efficient, by Theorem 15. Finally, the contrapositive of Theorem 19 obviously implies that CELT* ⊂ NOWPE. □

Theorem 20 characterizes only the non-oligarchic part of the Pareto frontier, avoiding the oligarchic extremes. A recent result due to Florig (2001a) goes far toward characterizing the oligarchic extremes that are not WELT allocations. As in Shafer and Sonnenschein's (1976) very general framework, Florig considers incomplete preferences over personal consumption which depend on other agents' consumption and production activities. These preferences may even be intransitive. By contrast, this section remains within the standard framework of this chapter, in which each agent has complete and transitive preferences that are independent of all other agents' consumption and production activities. The only difference will be the use of the stronger semi-strict convexity condition that was briefly discussed in Section 2.4.

Florig's characterization relies on extended "hierarchical" price systems. It also uses (p. 532) a notion of weak Pareto efficiency which departs from the standard definition. To avoid confusion, here we use a different term and say that the feasible allocation $(\hat{x}^N, \hat{y}^N)$ is *incompletely Pareto efficient* if any Pareto superior allocation (x^N, y^N) satisfies both $x^h \; I^h \; \hat{x}^h$ and $x^h - y^h \neq \hat{x}^h - \hat{y}^h$ for at least one $h \in N$. Equivalently, no feasible allocation (x^N, y^N) in $\prod_{i \in N} [R^i(\hat{x}^i) \times Y^i]$ satisfies $\emptyset \neq N^{\neq} \subset N^{\succ}$, where

$$N^{\neq} := \{\, i \in N \mid x^i - y^i \neq \hat{x}^i - \hat{y}^i \,\} \text{ and } N^{\succ} := \{\, i \in N \mid x^i \in P^i(\hat{x}^i) \,\}. \quad (14)$$

Obviously, any Pareto efficient allocation is incompletely Pareto efficient, and any incompletely Pareto efficient is weakly Pareto efficient. But in neither case does the converse hold, in general.

A *kth order price system* is a $k \times \#G$ matrix $\mathbf{P}^k$ whose rows $p^r \in \mathbb{R}^G$ $(r = 1, 2, \ldots, k)$ are price vectors satisfying the *orthogonality condition* $p^r \cdot p^s := \sum_{g \in G} p^r_g \, p^s_g = 0$ whenever $r \neq s$. Obviously, this implies that $\mathbf{P}^k$ has rank k, where $k \leq \#G$. Then any commodity vector $z \in \mathbb{R}^G$ has a k-dimensional value given by the matrix product $\mathbf{P}^k z \in \mathbb{R}^k$.

Define the *lexicographic strict ordering* $>^k_L$ on $\mathbb{R}^k$ so that $a >^k_L b$ iff there exists $r \in \{\, 1, 2, \ldots, k \,\}$ such that $a_s = b_s$ for $s = 1, \ldots, r-1$, but $a_r > b_r$. Let $\geq^k_L$ be the corresponding weak ordering. Of course, $>^k_L$ is a *total ordering* in the sense that $a \neq b$ implies either $a >^k_L b$ or $b >^k_L a$.

The feasible allocation $(\hat{x}^N, \hat{y}^N)$ is said to be a *hierarchical* WELT at the kth order price system $\mathbf{P}^k$ if, for all $i \in N$, whenever $(x^i, y^i) \in P^i(\hat{x}^i) \times Y^i$, then $\mathbf{P}^k(x^i - y^i) >^k_L \mathbf{P}^k(\hat{x}^i - \hat{y}^i)$. Equivalently, because $\leq^k_L$ is a total ordering, for each $i \in N$ the pair $(\hat{x}^i, \hat{y}^i)$ must maximize R^i subject to individual feasibility and the *hierarchical Walrasian budget constraint* $\mathbf{P}^k(x^i - y^i) \leq^k_L \mathbf{P}^k(\hat{x}^i - \hat{y}^i)$.

Theorem 21 *Any hierarchical WELT allocation is incompletely Pareto efficient.*

PROOF. Suppose the allocation $(\hat{x}^N, \hat{y}^N)$ is a hierarchical WELT at the kth order price system $\mathbf{P}^k$. Suppose too the collection $(x^N, y^N) \in \prod_{i \in N}[R^i(\hat{x}^i) \times Y^i]$ satisfies $\emptyset \neq N^{\neq} \subset N^{\succ}$, where $N^{\neq}$ and $N^{\succ}$ are defined by (14). By definition of a hierarchical WELT, one has

$$\mathbf{P}^k(x^i - y^i) >_L^k \mathbf{P}^k(\hat{x}^i - \hat{y}^i) \quad \text{for all } i \in N^{\succ}. \tag{15}$$

Of course

$$\mathbf{P}^k(x^i - y^i) = \mathbf{P}^k(\hat{x}^i - \hat{y}^i) \quad \text{for all } i \in N \setminus N^{\neq}. \tag{16}$$

For (15) and (16) to be consistent, one must have $N^{\succ} \subset N^{\neq}$ and so, by hypothesis, $N^{\succ} = N^{\neq} \neq \emptyset$. Now one can sum (15) and (16) over all $i \in N$ in order to obtain $\mathbf{P}^k \sum_{i \in N}(x^i - y^i) >_L^k \mathbf{P}^k \sum_{i \in N}(\hat{x}^i - \hat{y}^i) = 0$, as is easy to check. Hence, (x^N, y^N) cannot be a feasible allocation when $\emptyset \neq N^{\neq} \subset N^{\succ}$. So $(\hat{x}^N, \hat{y}^N)$ is incompletely Pareto efficient. $\square$

Theorem 22 *Suppose all agents' preferences are LNS, semi-strictly convex and continuous. Then any incompletely Pareto efficient allocation* $(\hat{x}^N, \hat{y}^N)$ *is a hierarchical WELT.*

PROOF. The proof is by recursive construction of successive orthogonal price vectors $p^r \in \mathbb{R}^G \setminus \{0\}$ $(r = 1, 2, \ldots)$. Let $\mathbf{P}^k := (p^1, \ldots, p^k)$ denote the resulting kth order price system, and N^k the set of all agents $i \in N$ for whom $(\hat{x}^i, \hat{y}^i)$ is a hierarchical WELT at $\mathbf{P}^k$. Obviously $N^k \subset N^{k+1}$ $(k = 1, 2, \ldots)$. By Theorem 16, there exists a non-zero price vector $p^1 \in \mathbb{R}^G$ at which the allocation $(\hat{x}^N, \hat{y}^N)$ is a CELT. This is the first step.

It will be proved by induction on k that, for all $i \in N \setminus N^k$, whenever $(x^i, y^i) \in R^i(\hat{x}^i) \times Y^i$ and $p^r(x^i - y^i) = p^r(\hat{x}^i - \hat{y}^i)$ for $r = 1, \ldots, k-1$, then $p^k(x^i - y^i) \geq p^k(\hat{x}^i - \hat{y}^i)$. This is true when $k = 1$ because $(\hat{x}^N, \hat{y}^N, p^1)$ is a CELT.

If $N^k = N$ the proof is already complete. Otherwise, define the $(\#G - k)$-dimensional linear subspace $L^k := \{ z \in \mathbb{R}^G \mid \mathbf{P}^k z = 0 \}$ of vectors that are orthogonal to all the k mutually orthogonal price vectors $p^1, p^2, \ldots, p^k$. Consider the set $Z^k := \sum_{i \in N \setminus N^k}[P^i(\hat{x}^i) - Y^i]$ and the vector $\hat{z}^k := \sum_{i \in N \setminus N^k}(\hat{x}^i - \hat{y}^i) = -\sum_{i \in N^k}(\hat{x}^i - \hat{y}^i)$. Incomplete Pareto efficiency implies that $\hat{z}^k \notin Z^k$, so 0 is not a member of the convex set $(Z^k - \{\hat{z}^k\}) \cap L^k$. Also, for each $i \in N \setminus N^k$, nonsatiation and semi-strictly convex preferences together imply that $\hat{x}^i - \hat{y}^i$ is a boundary point of $[P^i(\hat{x}^i) - Y^i] \cap L^k$, so 0 is a boundary point of $(Z^k - \{\hat{z}^k\}) \cap L^k$. It follows that there exists a hyperplane $p^{k+1}z = 0$ in L^k that separates $(Z^k - \{\hat{z}^k\}) \cap L^k$ from 0, with $p^{k+1}z \geq 0$ for all $z \in (Z^k - \{\hat{z}^k\}) \cap L^k$. Arguing as in the proof of Theorem 16, for each $i \in N \setminus N^k$, this implies that $p^{k+1}(x^i - y^i) \geq p^{k+1}(\hat{x}^i - \hat{y}^i)$ whenever $(x^i, y^i) \in R^i(\hat{x}^i) \times Y^i$ satisfies $(x^i - y^i) - (\hat{x}^i - \hat{y}^i) \in L^k$. This completes the induction step.

Continue the construction, if necessary, until $k = \#G$. Arguing as in the proof of Lemmas 12 and 13, for each $i \in N \setminus N^{\#G-1}$, because $L^{\#G-1}$ has dimension one, one has $p^{\#G}(x^i - y^i) > p^{\#G}(\hat{x}^i - \hat{y}^i)$ whenever $(x^i, y^i) \in P^i(\hat{x}^i) \times Y^i$ satisfies $(x^i - y^i) - (\hat{x}^i - \hat{y}^i) \in L^{\#G-1}$. This implies that the feasible allocation $(\hat{x}^N, \hat{y}^N)$ is a hierarchical WELT given the price system $P = (p^1, \ldots, p^{\#G})$. □

5 Walrasian equilibrium in finite economies

This section provides sufficient conditions for the Walrasian SCR to give a non-empty choice set. It will also be shown that, when WE allocations do exist, they must satisfy the "individual rationality" or weak gains from trade constraints requiring each agent to be no worse off than under autarky. Indeed, unless autarky happens to be at least weakly Pareto efficient, any WE allocation must make at least one agent strictly better off than under autarky.

5.1 Autarky

Autarky means that agents rely only on their own resources, including their own production possibilities, and do not trade with other agents at all. Accordingly, for each $i \in N$, say that $X^i \cap Y^i$ is agent i's *autarky consumption set*. We assume throughout this section that each $X^i \cap Y^i$ is non-empty and bounded, and also that agents' preferences are continuous. These assumptions imply that the sets X^i and Y^i are both closed, so $X^i \cap Y^i$ is obviously compact. It follows that there is a non-empty set

$$A^i := \{ a^i \in X^i \cap Y^i \mid x^i \in X^i \cap Y^i \Longrightarrow a^i \; R^i \; x^i \}$$

of *optimal autarky consumption vectors*, all of which must be indifferent to each other. It follows that the set $R^i(a^i)$ is the same for all $a^i \in A^i$. From now on, we denote this set by $\hat{R}^i$.

5.2 Gains from trade

Following both cooperative game theory and also social choice theory, the *individual rationality* or *participation* constraint of each agent $i \in N$ requires the consumption vector x^i to satisfy $x^i \in \hat{R}^i$. Thus, agent i is no worse off than under autarky. Following the literaure on general equilibrium theory, one can also say that agent i experiences *weak gains from trade*.

Let $\hat{R}^N := \prod_{i \in N} \hat{R}^i$, and then define the *collective gains from trade set*

$$W := \{(x^N, y^N) \in \hat{R}^N \times Y^N \mid \sum_{i \in N} (x^i - y^i) = 0\}$$

of feasible allocations allowing each agent weak gains from trade. We assume that, like each agent's autarky set $X^i \cap Y^i$, the set W is also bounded. When preferences are continuous, it follows that W is also closed, so compact.

The following result ensures that any WE allocation $(\hat{x}^N, \hat{y}^N)$ confers collective gains from trade. In fact, a slightly stronger result is proved: except in trivial cases, at least one agent must experience strict gains from trade.

Theorem 23 (Gains from Trade Lemma) *Any WE $(\hat{x}^N, \hat{y}^N, \hat{p})$ satisfies $(\hat{x}^N, \hat{y}^N) \in W$. Furthermore, unless there happens to be an autarky allocation (x^N, y^N) with $x^i = y^i = a^i \in A^i$ (all $i \in N$) which is weakly Pareto efficient (or Pareto efficient if preferences are LNS), there must exist $h \in N$ such that $\hat{x}^h\ P^h\ a^h$ for all $a^h \in A^h$.*

PROOF. For each $i \in N$, let a^i be any member of A^i. Because $(x^i, y^i) = (a^i, a^i)$ satisfies the Walrasian budget constraint $\hat{p}\,(x^i - y^i) \leq 0$, revealed preference implies that $\hat{x}^i\ R^i\ a^i$ in the Walrasian equilibrium. So $(\hat{x}^N, \hat{y}^N) \in W$.

Next, suppose that $\hat{x}^i\ I^i\ a^i$ for all $i \in N$. By the first efficiency theorem (Theorem 15), the WE allocation $(\hat{x}^N, \hat{y}^N)$ is weakly Pareto efficient (and Pareto efficient if preferences are LNS). So therefore is the autarky allocation $(\bar{x}^N, \bar{y}^N)$ with $\bar{x}^i = \bar{y}^i = a^i$ for all $i \in N$, because preferences are transitive. □

5.3 Existence of compensated Walrasian equilibrium

Theorem 24 (Compensated Equilibrium Existence) *Suppose that each agent $i \in N$ has LNS, convex and continuous preferences, as well as a non-empty and bounded autarky consumption set $X^i \cap Y^i$. Suppose too that the collective gains from trade set W is bounded. Then there exists a compensated equilibrium $(\hat{x}^N, \hat{y}^N, \hat{p})$ with $(\hat{x}^N, \hat{y}^N) \in W$.*

PROOF. For each $i \in N$, let $\tilde{X}^i$ and $\tilde{Y}^i$ be compact convex subsets of $\mathbb{R}^G$ so large that the Cartesian product $\prod_{i \in N} (\tilde{X}^i \times \tilde{Y}^i)$ contains the bounded set

W within its interior.[11] Then define the two constrained sets

$$\bar{X}^i := \tilde{X}^i \cap \hat{R}^i \text{ and } \bar{Y}^i := \tilde{Y}^i \cap Y^i \tag{17}$$

which are both convex, as intersections of convex sets. Because $\tilde{X}^i$ and $\tilde{Y}^i$ are compact while $\hat{R}^i$ and Y^i are closed, the sets $\bar{X}^i$ and $\bar{Y}^i$ must be compact.

Let D denote the domain of all price vectors $p \in \mathbb{R}^G$ (including 0) such that $\|p\| \leq 1$. Then, for each $i \in N$ and each $p \in D$, define the modified budget set

$$\bar{B}^i(p) := \{ (x^i, y^i) \in \bar{X}^i \times \bar{Y}^i \mid p\,(x^i - y^i) \leq w(p) \}$$

where $w(p) := (\#N)^{-1}(1 - \|p\|)$.[12] This set is evidently compact and convex. It is also non-empty because it includes (a^i, a^i) for each $a^i \in A^i$.

Next, define the modified compensated demand set

$$\begin{aligned} \bar{\alpha}^i_C(p) := \{ & (x^i, y^i) \in \bar{B}^i(p) \mid \\ & (\tilde{x}^i, \tilde{y}^i) \in \bar{X}^i \times \bar{Y}^i,\ \tilde{x}^i\ P^i\ x^i \Longrightarrow p\,(\tilde{x}^i - \tilde{y}^i) \geq w(p) \}. \end{aligned}$$

This set is evidently non-empty because it includes the non-empty set

$$\{ (x^i, y^i) \in \bar{B}^i(p) \mid (\tilde{x}^i, \tilde{y}^i) \in \bar{B}^i(p) \Longrightarrow x^i\ R^i\ \tilde{x}^i \}$$

of pairs (x^i, y^i) which maximize agent i's continuous preference relation R^i over the compact set $\bar{B}^i(p)$. Also, to see that $\bar{\alpha}^i_C(p)$ is convex, suppose that $(x^i, y^i), (\hat{x}^i, \hat{y}^i) \in \bar{\alpha}^i_C(p)$ with $x^i\ R^i\ \hat{x}^i$, and let $(\bar{x}^i, \bar{y}^i) = \alpha(x^i, y^i) + (1-\alpha)(\hat{x}^i, \hat{y}^i)$ with $0 \leq \alpha \leq 1$ be any convex combination. Then $(\bar{x}^i, \bar{y}^i) \in \bar{B}^i(p)$, obviously. The definition of $\bar{X}^i$ in (17) implies that $\bar{x}^i\ R^i\ \hat{x}^i$, because preferences are convex. Hence, whenever $(\tilde{x}^i, \tilde{y}^i) \in \bar{X}^i \times \bar{Y}^i$ with $\tilde{x}^i\ P^i\ \bar{x}^i$, then $\tilde{x}^i\ P^i\ \hat{x}^i$. By definition of $\bar{\alpha}^i_C(p)$, it follows that $p\,(\tilde{x}^i - \tilde{y}^i) \geq w(p)$. This confirms that $(\bar{x}^i, \bar{y}^i) \in \bar{\alpha}^i_C(p)$, which must therefore be convex.

Also, for each $i \in N$ and each price vector $p \in D$, define the modified compensated net trade set

$$\bar{\zeta}^i_C(p) := \{ z^i \in \mathbb{R}^G \mid \exists (x^i, y^i) \in \bar{\alpha}^i_C(p) : z^i = x^i - y^i \}.$$

This is also non-empty and convex for each p. So is the aggregate modified compensated net trade set defined by $\zeta(p) := \sum_{i \in N} \bar{\zeta}^i_C(p)$ for each $p \in D$.

[11] Since Arrow and Debreu (1954), this has become a standard approach in general equilibrium existence proofs.

[12] This modified wealth distribution rule adapts an approach that Bergstrom (1976) also uses in order to prove existence without assuming free disposal, while allowing prices to be negative as well as positive.

The next step is to show that the correspondence $p \mapsto\!\!\!\!\to \zeta(p)$ has a closed graph in $D \times \mathbb{R}^G$. To do so, suppose that the sequence of pairs $(p_n, z_n) \in D \times \mathbb{R}^G$ satisfies $z_n \in \zeta(p_n)$ for $n = 1, 2, \ldots$ and converges to a limit (p, z) as $n \to \infty$. Then for each $i \in N$ there exists a corresponding sequence $(x^i_n, y^i_n) \in \bar{\alpha}^i_C(p_n)$ such that $z_n = \sum_{i \in N}(x^i_n - y^i_n)$ for $n = 1, 2, \ldots$. Now, each profile (x^N_n, y^N_n) of consumption and production vectors is restricted to the compact set $\bar{X}^N \times \bar{Y}^N := \prod_{i \in N}(\bar{X}^i \times \bar{Y}^i)$, so there is a convergent subsequence with limit point $(x^N, y^N) \in \bar{X}^N \times \bar{Y}^N$ satisfying $z = \sum_{i \in N}(x^i - y^i)$. It loses no generality to retain only the terms of this convergent subsequence, implying that $(x^N_n, y^N_n) \to (x^N, y^N)$ as $n \to \infty$. Then, because $(x^i_n, y^i_n) \in \bar{\alpha}^i_C(p_n) \subset \bar{B}^i(p_n)$ for each n, the budget constraint $p_n\,(x^i_n - y^i_n) \leq w(p_n)$ is satisfied. But $p_n \to p$ implies that $\|p_n\| \to \|p\|$ and so $w(p_n) \to w(p)$. Then, taking the limit as $n \to \infty$ shows that $p\,(x^i - y^i) \leq w(p)$. Hence, $(x^i, y^i) \in \bar{B}^i(p)$ for each $i \in N$. Furthermore, whenever $(\tilde{x}^i, \tilde{y}^i) \in \bar{X}^i \times \bar{Y}^i$ satisfies $\tilde{x}^i \; P^i \; x^i$, then continuity of preferences implies that $\tilde{x}^i \; P^i \; x^i_n$ for each large n. Because $(x^i_n, y^i_n) \in \bar{\alpha}^i_C(p_n)$, it follows that $p_n\,(\tilde{x}^i - \tilde{y}^i) \geq w(p_n)$, so taking the limit as $n \to \infty$ shows that $p\,(\tilde{x}^i - \tilde{y}^i) \geq w(p)$. This confirms that $(x^i, y^i) \in \bar{\alpha}^i_C(p)$ for each $i \in N$. But then $x^i - y^i \in \bar{\zeta}^i_C(p)$ for each $i \in N$, and so $z = \sum_{i \in N}(x^i - y^i) \in \zeta(p) = \sum_{i \in N} \bar{\zeta}^i_C(p)$. This confirms that the correspondence ζ has a closed graph.

Next, let $\bar{Z} := \sum_{i \in N}(\bar{X}^i - \bar{Y}^i)$. Because all the sets $\bar{X}^i$ and $\bar{Y}^i$ are compact and convex, so are $\bar{Z}$ and $D \times \bar{Z}$.

The successive definitions of $\bar{B}^i(p)$, $\bar{\alpha}^i_C(p)$, $\bar{\zeta}^i_C(p)$, and $\zeta(p)$ together imply that $\zeta(p) \subset \bar{Z}$ for all $p \in D$. Define the correspondence $\phi : \bar{Z} \twoheadrightarrow D$ by

$$\phi(z) := \arg\max_p \; \{\, p\, z \mid p \in D \,\}.$$

Note that ϕ also has a closed graph in $\bar{Z} \times D$. To see this, suppose that (z_n, p_n) is a sequence in $\bar{Z} \times D$ that satisfies $p_n \in \phi(z_n)$ for $n = 1, 2, \ldots$ and converges to (p, z). Then $(p, z) \in \bar{Z} \times D$. Also, for all $p' \in D$ one has $p_n\, z_n \geq p'\, z_n$, so taking limits as $n \to \infty$ gives $p\, z \geq p'\, z$. This proves that $p \in \phi(z)$, thus confirming that ϕ has a closed graph. In addition, it is obvious that $\phi(z)$ is non-empty and convex for all $z \in \bar{Z}$.

Consider finally the product correspondence $\psi : D \times \bar{Z} \twoheadrightarrow D \times \bar{Z}$ defined by $\psi(p, z) := \phi(z) \times \zeta(p)$. Because ϕ and ζ both have non-empty convex values throughout their respective domains $\bar{Z}$ and D, so does ψ throughout its domain $D \times \bar{Z}$. Also, the graph of the correspondence ψ is the Cartesian product of the graphs of ϕ and ζ, both of which are closed. So ψ has a closed graph. This graph is a subset of the compact set $(D \times \bar{Z}) \times (D \times \bar{Z})$, so ψ actually has a compact graph.

To summarize, we have shown that $\psi : D \times \bar{Z} \twoheadrightarrow D \times \bar{Z}$ has a convex domain, non-empty convex values, and a compact graph. These properties allow Kakutani's fixed-point theorem to be applied. So there must exist $(\hat{p}, \hat{z}) \in D \times \bar{Z}$ with $(\hat{p}, \hat{z}) \in \psi(\hat{p}, \hat{z})$. This implies that $\hat{p} \in \phi(\hat{z})$ and $\hat{z} \in \zeta(\hat{p})$.

Given any $z \in \bar{Z}$, the definition of ϕ implies that $\|p\| < 1$ for some $p \in \phi(z)$ only if $z = 0$. On the other hand, if $p \in D$ is any price vector satisfying $\|p\| = 1$, then $z \in \zeta(p)$ only if $z = \sum_{i \in N}(x^i - y^i)$ for some collection $(x^N, y^N) \in \prod_{i \in N} \bar{\alpha}^i_C(p)$. But then $p\,(x^i - y^i) \leq w(p)$ for all $i \in N$, implying that $p\,z \leq \#N w(p) = 1 - \|p\| = 0$. Yet $p\,z \leq 0$ for some $p \in \phi(z)$ only if $z = 0$. These arguments show that both $\hat{p} \in \phi(\hat{z})$ and $\hat{z} \in \zeta(\hat{p})$ are possible only if $\hat{z} = 0$. It follows that $0 = \sum_{i \in N}(\hat{x}^i - \hat{y}^i)$ for some collection $(\hat{x}^N, \hat{y}^N) \in \prod_{i \in N} \bar{\alpha}^i_C(\hat{p})$. In particular, $\hat{x}^i \in \bar{X}^i \subset \hat{R}^i$ for all $i \in N$. It follows that $(\hat{x}^N, \hat{y}^N)$ is a feasible allocation in the set W, a subset of the interior of $\prod_{i \in N}(\tilde{X}^i \times \tilde{Y}^i)$.

Because each agent $i \in N$ has LNS preferences, by Lemma 1 there exists a sequence $x^i_n \in P^i(\hat{x}^i)$ which converges to $\hat{x}^i$. Because preferences are also transitive, one has $x^i_n \in \hat{R}^i$ for all $i \in N$ and for $n = 1, 2, \ldots$. Furthermore, $\hat{x}^i$ belongs to the interior of $\tilde{X}^i$, so $x^i_n \in \tilde{X}^i$ for large n, implying that $(x^i_n, \hat{y}^i) \in \bar{X}^i \times Y^i$. Because $(\hat{x}^i, \hat{y}^i) \in \bar{\alpha}^i_C(\hat{p})$, it follows that $\hat{p}\,(x^i_n - \hat{y}^i) \geq w(\hat{p})$ for large n. Taking the limit as $n \to \infty$ yields $\hat{p}\,(\hat{x}^i - \hat{y}^i) \geq w(\hat{p})$. But $(\hat{x}^i, \hat{y}^i) \in \bar{B}^i(\hat{p})$, so $\hat{p}\,(\hat{x}^i - \hat{y}^i) \leq w(\hat{p})$. This shows that $\hat{p}\,(\hat{x}^i - \hat{y}^i) = w(\hat{p})$ for all $i \in N$. Because $0 = \sum_{i \in N}(\hat{x}^i - \hat{y}^i)$, summing over all $i \in N$ implies $w(\hat{p}) = 0$. By definition of $w(\hat{p})$, it follows that $\|\hat{p}\| = 1$.

Given any agent $i \in N$, suppose that $(x^i, y^i) \in R^i(\hat{x}^i) \times Y^i$. For small enough $\lambda > 0$, the convex combination $(1 - \lambda)\,(\hat{x}^i, \hat{y}^i) + \lambda(x^i, y^i)$ belongs to both $\tilde{X}^i \times \tilde{Y}^i$ and $X^i \times Y^i$. In fact, because preferences are convex and transitive, this convex combination also belongs to $R^i(\hat{x}^i) \times Y^i \subset \hat{R}^i \times Y^i$, and so to $\bar{X}^i \times \bar{Y}^i$. But $(\hat{x}^i, \hat{y}^i) \in \bar{\alpha}^i_C(\hat{p})$ and $\|\hat{p}\| = 1$, so

$$\hat{p}\,[(1 - \lambda)\,(\hat{x}^i - \hat{y}^i) + \lambda\,(x^i - y^i)] \geq w(\hat{p}) = 0.$$

Because $\hat{p}\,(\hat{x}^i - \hat{y}^i) = 0$ and $\lambda > 0$, this implies that $\hat{p}\,(x^i - y^i) \geq 0$.

This completes the proof that $(\hat{x}^N, \hat{y}^N, \hat{p})$ is a compensated equilibrium satisfying $\hat{x}^i \in \hat{R}^i$ for all $i \in N$. □

5.4 Directional irreducibility and existence of Walrasian equilibrium

To show that the CE of Theorem 24 is a WE, the obvious procedure is to apply the cheaper point Lemma 12. A sufficient condition for this to be valid is that each agent's autarky set $X^i \cap Y^i$ should have an interior point. Yet this seems unduly restrictive because, for example, it requires each agent to have the capacity to supply a positive net quantity of all goods simultaneously.

Instead, we start with the standard aggregate interiority assumption of Section 3.9 requiring that $0 \in \operatorname{int} \sum_{i \in N} (X^i - Y^i)$. This ensures that at least one agent has a cheaper point. Some additional condition is still required, however, in order to ensure that *every* individual agent has a cheaper point. Any such additional condition will differ somewhat from the non-oligarchy assumption used in the latter part of Section 4. For one thing, we cannot simply assume that all feasible allocations are non-oligarchic because our continuity and boundedness assumptions actually guarantee the existence of oligarchic allocations. Assume instead that, given any feasible allocation and any proper subset of agents $K \subset N$, the agents in $N \setminus K$ start with resources allowing them to offer an aggregate net supply vector which is desired by the agents in K. In case the agents in K all have cheaper points in some compensated equilibrium, this condition ensures that this aggregate net supply vector of the other agents has positive value at the equilibrium price vector.

Arrow and Debreu (1954) were the first to introduce a condition of this kind. Adapted to the framework used here, their condition requires the existence of two non-empy sets $G_D, G_P \subset G$ — of *desirable commodities* and *productive inputs* respectively — with the properties:

(1) given any $i \in N$, any $x^i \in X^i$ and any $g \in G_D$, there exists $\tilde{x}^i \in P^i(x^i)$ such that $\tilde{x}^i_h = x^i_h$ for all $h \in G \setminus \{g\}$, while $\tilde{x}^i_g > x^i_g$;
(2) given any $g \in G_P$ and any $y \in \sum_{i \in N} Y^i$, there exists $\tilde{y} \in \sum_{i \in N} Y^i$ such that $\tilde{y}_h \geq y_h$ for all $h \in G \setminus \{g\}$, with $\tilde{y}_h > y_h$ for at least one $h \in G_D$;
(3) given any $i \in N$, there exist $g \in G_P$ and $(\bar{x}^i, \bar{y}^i) \in X^i \times Y^i$ such that $\bar{x}^i \leqq \bar{y}^i$ with $\bar{x}^i_g < \bar{y}^i_g$.

McKenzie (1959, 1961) introduced a more general sufficient condition. He defined an economy as *irreducible* provided that, for any proper subset $K \subset N$ and any feasible allocation $(\hat{x}^N, \hat{y}^N)$, there exist $(x^N, y^N) \in X^N \times Y^N$ such that $x^i \; P^i \; \hat{x}^i$ for all $i \in K$ and $\sum_{i \in K} [(x^i - y^i) - (\hat{x}^i - \hat{y}^i)] = - \sum_{i \in N \setminus K} (x^i - y^i)$ — or, equivalently, $\sum_{i \in N} (x^i - y^i) + \sum_{i \in N \setminus K} (\hat{x}^i - \hat{y}^i) = 0$.[13] This condition can be interpreted as requiring the existence of appropriate consumption and production vectors $(x^i, y^i) \in X^i \times Y^i$ for all $i \in N \setminus K$ such that, if the feasible aggregate net supply vector $- \sum_{i \in N \setminus K} (x^i - y^i)$ became available from outside the economy, these additional exogenous resources could be distributed as incremental net trade vectors $(x^i - y^i) - (\hat{x}^i - \hat{y}^i)$ to the agents $i \in K$ in a way which benefits them all simultaneously, without affecting the other agents $i \in N \setminus K$ at all. It is easy to check that an economy satisfying the Arrow–Debreu condition described above must be irreducible.

In order to ensure that a CE is a WE, a number of variations of McKenzie's original definitions have been propounded, including Arrow and Hahn's (1971)

[13] This follows an idea due to Gale (1957) — see also Gale (1976), Eaves (1976) and Hammond (1993).

concept of "resource relatedness" — direct or indirect. Some systematic exploration of these concepts was attempted in Spivak (1978) and Hammond (1993). The discussion has since been advanced by Florig (2001b) and by McKenzie (2002) himself.[14] Rather than pursue this further, the discussion here concentrates on a version of irreducibility that seems weaker than all other versions, yet remains sufficient for any CE to be a WE. Following the recent suggestion of Florig (2001b, p. 189), this weakened definition has the property that "only directions matter and not magnitudes".

First, given the particular feasible allocation $(\hat{x}^N, \hat{y}^N)$, define $U^i(\hat{x}^i, \hat{y}^i)$ as the convex cone of vectors $\lambda[(x^i - y^i) - (\hat{x}^i - \hat{y}^i)]$ with $\lambda > 0$, $x^i \in P^i(\hat{x}^i)$, and $y^i \in Y^i$. Thus, $U^i(\hat{x}^i, \hat{y}^i)$ consists of directions which allow agent i's net trade vector to be improved by moving an appropriate distance away from $\hat{x}^i - \hat{y}^i$.

Second, define $V^i(\hat{x}^i, \hat{y}^i)$ as the closed convex cone of vectors $\lambda[(x^i - y^i) - (\hat{x}^i - \hat{y}^i)]$ with $\lambda \geq 0$ and $(x^i, y^i) \in X^i \times Y^i$. Thus, $V^i(\hat{x}^i, \hat{y}^i)$ consists of directions in which it is feasible to change agent i's net trade vector by moving an appropriate distance away from $\hat{x}^i - \hat{y}^i$.

Third, define W^i as the convex cone of vectors $-\mu(x^i - y^i)$ with $\mu \geq 0$, $x^i \in X^i$, and $y^i \in Y^i$. Thus, W^i consists of directions in which resources can be removed from agent i without violating individual feasibility.

Finally, define Z^i as the convex cone of vectors $-\nu(\hat{x}^i - \hat{y}^i)$ with $\nu \geq 0$. Thus, Z^i is the half-line of non-positive multiples of $\hat{x}^i - \hat{y}^i$ when $\hat{x}^i \neq \hat{y}^i$; but $Z^i = \{0\}$ when $\hat{x}^i = \hat{y}^i$.

With these definitions, the economy is said to be *directionally irreducible* provided that, for any proper subset $K \subset N$ (with both K and $N \setminus K$ non-empty) and any feasible allocation $(\hat{x}^N, \hat{y}^N)$, the two sets

$$\sum_{i \in K} U^i(\hat{x}^i, \hat{y}^i) + \sum_{i \in N \setminus K} V^i(\hat{x}^i, \hat{y}^i) \quad \text{and} \quad \sum_{i \in N \setminus K} W^i + \sum_{i \in N} Z^i$$

intersect. This is obviously a weaker condition than McKenzie's original version of the irreducibility assumption that was stated above.

[14] Translated to the present context, McKenzie (2002, p. 172) defines an economy as irreducible when, for any proper subset $K \subset N$ and any feasible allocation $(\hat{x}^N, \hat{y}^N)$, there exist $(x^i, y^i) \in X^i \times Y^i$ for all $i \in K$ and a scalar $\lambda > 0$ such that $x^i \; P^i \; \hat{x}^i$ for all $i \in K$ and

$$\sum_{i \in K} [(x^i - y^i) - (\hat{x}^i - \hat{y}^i)] + \lambda \sum_{i \in N \setminus K} (x^i - y^i) = 0.$$

Theorem 25 *Suppose agents have convex consumption and production sets, as well as LNS and continuous preferences. Suppose too* $0 \in \operatorname{int} \sum_{i \in N} (X^i - Y^i)$, *and the economy is directionally irreducible. Then any CE is a WE.*

PROOF. Let $(\hat{x}^N, \hat{y}^N, \hat{p})$ be a CE. Because $0 \in \operatorname{int} \sum_{i \in N} (X^i - Y^i)$, Lemma 14 implies that there exist $h \in N$ and $(\underline{x}^h, \underline{y}^h) \in X^h \times Y^h$ such that $\hat{p}\,(\underline{x}^h - \underline{y}^h) < 0$.

Let K be any proper subset of N whose members all have such cheaper points. By directional irreducibility, there exist:

(1) $(x^N, y^N) \in X^N \times Y^N$ and $\lambda^i \geq 0$ (all $i \in N$) with $x^i \; P^i \; \hat{x}^i$ and $\lambda^i > 0$ for all $i \in K$;
(2) $(\bar{x}^i, \bar{y}^i) \in X^i \times Y^i$ and $\mu^i \geq 0$ for all $i \in N \setminus K$;
(3) $\nu^i \geq 0$ for all $i \in N$;

such that

$$\sum_{i \in N} \lambda^i [(x^i - y^i) - (\hat{x}^i - \hat{y}^i)] + \sum_{i \in N \setminus K} \mu^i (\bar{x}^i - \bar{y}^i) + \sum_{i \in N} \nu^i (\hat{x}^i - \hat{y}^i) = 0. \quad (18)$$

Because $(\hat{x}^N, \hat{y}^N, \hat{p})$ is a CE, Lemma 12 implies that $\hat{p}\,(x^i - y^i) > 0$ for all $i \in K$. Also, because preferences are LNS, Theorem 10 implies that $\hat{p}\,(\hat{x}^i - \hat{y}^i) = 0$ for all $i \in N$. From (18) it follows that

$$\hat{p} \sum_{i \in N \setminus K} [\lambda^i (x^i - y^i) + \mu^i (\bar{x}^i - \bar{y}^i)] = -\hat{p} \sum_{i \in K} \lambda^i (x^i - y^i) < 0.$$

So at least one $i \in N \setminus K$ has either $\lambda^i \, \hat{p}\,(x^i - y^i) < 0$ or $\mu^i \, \hat{p}\,(\bar{x}^i - \bar{y}^i) < 0$. Because $\lambda^i \geq 0$ and $\mu^i \geq 0$ for all $i \in N \setminus K$, at least one $i \in N \setminus K$ has either (x^i, y^i) or $(\bar{x}^i, \bar{y}^i)$ as a cheaper point. Hence, K cannot include all agents with cheaper points. The only remaining possibility is that all agents have cheaper points. By Lemma 12, this implies that the CE is actually a WE. $\square$

5.5 Extended irreducibility

Developing a suggestion of Spivak (1978) along the lines of Hammond (1993), it will be shown that a slight weakening of irreducibility is a necessary condition for the line of reasoning used to prove Theorem 25 to apply.

The economy is said to be *extended irreducible* provided that, for any proper subset $K \subset N$ and any feasible allocation $(\hat{x}^N, \hat{y}^N)$, there exist $(x^N, y^N) \in X^N \times Y^N$ and $\nu^i \in [0, 1]$ (all $i \in N$) such that $x^i \; P^i \; \hat{x}^i$ for all $i \in K$ and $\sum_{i \in N} (x^i - y^i) + \sum_{i \in N \setminus K} (\hat{x}^i - \hat{y}^i) = \sum_{i \in N} \nu^i \, (\hat{x}^i - \hat{y}^i)$.

Theorem 26 *Suppose autarky is feasible for each agent, while preferences are LNS, convex and continuous. Unless the economy is extended irreducible, there exists a CE with a non-empty set of agents at their cheapest points.*

PROOF. Unless the economy is extended irreducible, there exists a proper subset $K \subset N$ and a feasible allocation $(\hat{x}^N, \hat{y}^N)$ such that

$$0 \notin Z := \sum_{i \in K} [P^i(\hat{x}^i) - Y^i] + \sum_{i \in N \setminus K} \Big(\{\hat{x}^i - \hat{y}^i\} + X^i - Y^i \Big) \\ - \sum_{i \in N} \{\, \nu^i (\hat{x}^i - \hat{y}^i) \mid \nu^i \in [0,1] \,\}. \qquad (19)$$

Because Z is the sum of convex sets, it is convex. So it can be separated from the origin. Hence, there exists a price vector $p \neq 0$ such that

$$0 \leq \inf p\, Z. \qquad (20)$$

For all $i \in K$, let

$$\alpha^i := \inf p\, [P^i(\hat{x}^i) - Y^i] \leq w^i := p\, (\hat{x}^i - \hat{y}^i), \qquad (21)$$

where the inequality holds because preferences are LNS and so $(\hat{x}^i, \hat{y}^i) \in \mathrm{cl}\, P^i(\hat{x}^i) \times Y^i$. Next, for all $i \in N \setminus K$, let

$$\beta^i := \inf p\, (X^i - Y^i) \leq 0, \qquad (22)$$

where the inequality holds because autarky is feasible. From (19)–(22),

$$0 \leq \sum_{i \in K} \alpha^i + \sum_{i \in N \setminus K} (w^i + \beta^i) + \sum_{i \in N} \inf\{\, -\nu^i\, w^i \mid \nu^i \in [0,1] \,\} \\ \leq \sum_{i \in N} (w^i + \min\{\, 0, -w^i \,\}) = \sum_{i \in N} \min\{\, 0, w^i \,\} \leq 0. \qquad (23)$$

Hence $0 = \sum_{i \in N} \min\{\, 0, w^i \,\}$, which is only possible when $w^i \geq 0$ for all $i \in N$. But (21) and feasibility of the allocation $(\hat{x}^N, \hat{y}^N)$ together imply that $\sum_{i \in N} w^i = \sum_{i \in N} p\, (\hat{x}^i - \hat{y}^i) = 0$, so $w^i = 0$ for all $i \in N$. Then (23) reduces to

$$0 = \sum_{i \in K} \alpha^i + \sum_{i \in N \setminus K} \beta^i. \qquad (24)$$

But (21) implies that $\alpha^i \leq w^i = 0$ for all $i \in K$, whereas (22) implies that $\beta^i \leq 0$ for all $i \in N \setminus K$. From (24), these inequalities imply that $\alpha^i = 0$ for all $i \in K$ and $\beta^i = 0$ for all $i \in N \setminus K$. Substituting these into the definitions (21) and (22), one obtains

$$\begin{aligned} \inf p\,[P^i(\hat{x}^i) - Y^i] &= 0 \quad \text{for all } i \in K \\ \text{and} \quad \inf p\,(X^i - Y^i) &= 0 \quad \text{for all } i \in N \setminus K. \end{aligned}$$

Because preferences are LNS, these properties together imply that $(\hat{x}^N, \hat{y}^N, p)$ is a CE in which each agent $i \in N \setminus K$ is at a cheapest point. □

6 Characterizations of WE with a fixed profile of agents' types

6.1 Walrasian acceptability

Thomson (1983) offers a characterization of Walrasian equilibrium with equal budgets (actually, from equal division) based on a notion of equity related to a criterion he calls "acceptability". The following adaptation and simplification treats Walrasian equilibrium more generally, based on net trade vectors.

Let Π^N denote the set of permutations $\sigma : N \to N$. Say that the feasible allocation $(\hat{x}^N, \hat{y}^N)$ is *Walrasian acceptable* if it is a WELT and, for each $\sigma \in \Pi^N$, relative to the *permuted wealth distribution rule* $\tilde{w}^N(p)$ defined by

$$\tilde{w}^i(p) := p\,(\hat{x}^{\sigma(i)} - \hat{y}^{\sigma(i)}) \tag{25}$$

for all $p \in \mathbb{R}^G \setminus \{0\}$, there exists a WELT $(\tilde{x}^N, \tilde{y}^N, \tilde{p})$ such that $\hat{x}^i \; R^i \; \tilde{x}^i$ for all $i \in N$. Thus, no matter how the equilibrium net trade vectors $\hat{x}^i - \hat{y}^i$ are permuted in order to determine an alternative wealth distribution rule, there always exists a new WELT relative to this alternative rule which no agent prefers to the original WELT.

Theorem 27 *Provided that preferences are LNS, any WE allocation is Walrasian acceptable.*

PROOF. If $(\hat{x}^N, \hat{y}^N, \hat{p})$ is a WE and preferences are LNS, Theorem 10 implies that $\hat{p}\,(\hat{x}^i - \hat{y}^i) = 0$ for all $i \in N$. It follows that the same WE is a WELT relative to each permuted wealth distribution rule defined by (25). □

Theorem 28 *If $(\hat{x}^N, \hat{y}^N)$ is Walrasian acceptable, then there exists a price vector $\tilde{p} \neq 0$ such that $(\hat{x}^N, \hat{y}^N, \tilde{p})$ is a WE.*

PROOF. Label the set N of individuals as i_k $(k = 1, 2, \ldots, \#N)$. Then let $\sigma \in \Pi^N$ be the particular permutation defined by $\sigma(i_k) := i_{k+1}$ for all $k = 1, 2, \ldots, \#N - 1$ and $\sigma(i_{\#N}) := i_1$. By hypothesis, there must exist a

WELT $(\tilde{x}^N, \tilde{y}^N, \tilde{p})$ relative to the permuted rule (25) such that $\hat{x}^i \; R^i \; \tilde{x}^i$ for all $i \in N$. Because preferences are LNS, Lemma 9 and Theorem 10 imply that

$$\tilde{p}\,(\hat{x}^i - \hat{y}^i) \geq w^i(\tilde{p}) = \tilde{p}\,(\hat{x}^{\sigma(i)} - \hat{y}^{\sigma(i)})$$

for all $i \in N$. By definition of σ, it follows that $\tilde{p}\,(\hat{x}^i - \hat{y}^i)$ is independent of i. But $\sum_{i \in N} (\hat{x}^i - \hat{y}^i) = 0$, so $\tilde{p}\,(\hat{x}^i - \hat{y}^i) = 0$ for all $i \in N$.

Finally, suppose $(x^i, y^i) \in P^i(\hat{x}^i) \times Y^i$. Then $x^i \in P^i(\tilde{x}^i)$ because $\hat{x}^i \; R^i \; \tilde{x}^i$ and preferences are transitive. But $(\tilde{x}^N, \tilde{y}^N, \tilde{p})$ is a WELT, so

$$\tilde{p}\,(x^i - y^i) > w^i(\tilde{p}) = \tilde{p}\,(\hat{x}^{\sigma(i)} - \hat{y}^{\sigma(i)}) = 0.$$

This proves that $(\hat{x}^N, \hat{y}^N, \tilde{p})$ is a WE. □

6.2 Equal rights to multiple proportional trade

The following discussion of "equal rights to trade" develops some of the ideas in Schmeidler and Vind (1972). It offers a different fairness characterization of Walrasian equilibrium without lump-sum transfers.

6.2.1 Definitions

In this section we assume that each agent $i \in N$ has a feasible set of net trades T^i and an associated preference ordering $\succsim^i$ on T^i, as described in Section 2.6. Let $F := \{\, t^N \in T^N \mid \sum_{i \in N} t^i = 0 \,\}$ denote the feasible set of *balanced allocations* of net trade vectors to the different agents in the economy.

Given the allocation $\bar{t}^N \in F$, agent i is said to *envy* agent h if $\bar{t}^h \in T^i$ with $\bar{t}^h \succ^i \bar{t}^i$. On the other hand, the allocation $\bar{t}^N \in F$ is *envy free* if $\bar{t}^i \succsim^i \bar{t}^h$ for all $h, i \in N$ such that $\bar{t}^h \in T^i$. Say that the allocation $\bar{t}^N \in F$ is *fair* if it is both envy free and Pareto efficient. Say that the allocation $\bar{t}^N \in F$ is *strongly envy free* if $\bar{t}^i \succsim^i \sum_{h \in N} n^h \, \bar{t}^h$ for all $i \in N$ and all collections of non-negative integers $n^h \in \mathbb{Z}_+$ $(h \in N)$ such that $\sum_{h \in N} n^h \, \bar{t}^h \in T^i$. [15]

Next, say that the feasible allocation $\bar{t}^N \in F$ offers *equal rights to trade* if there exists a common *trading set* $B \subset \mathbb{R}^G$ such that, for all $i \in N$, both $\bar{t}^i \in B$ and also $t^i \in B \cap T^i \Longrightarrow \bar{t}^i \succsim^i t^i$.

[15] This currently accepted terminology follows Varian (1974). It departs from Schmeidler and Vind (1972) who use "fair" to mean what is here called "envy free". I have also replaced their "strongly fair" by "strongly envy free". Note too that what they describe as a "Walras net trade" has only to satisfy the budget constraint $p\,t = 0$ at the specified price vector $p \neq 0$; if preference maximization also holds, even in a restricted set, they call the net trade vector "competitive".

Because $\bar{t}^h \in B$ for all $h \in N \setminus \{i\}$, offering the common trading set B gives all agents $i \in N$ the right to choose, in particular, any other agent's net trade vector $\bar{t}^h$ instead of their own, provided that $\bar{t}^h \in T^i$. Accordingly, equal rights to trade imply that the allocation $\bar{t}^N$ is envy free.

Say that there are *equal rights to multiple trade* if the set B is closed under addition — i.e., $B + B \subset B$. Thus, an agent $i \in N$ who has the right to trade either of the net trade vectors $t^i, \tilde{t}^i$ in B also has the right to the combined net trade vector $t^i + \tilde{t}^i$. As Schmeidler and Vind (1972, Theorem 1) demonstrate, if $\bar{t}^N \in F$ offers equal rights to multiple trade within a set B satisfying $0 \in B$, then $\bar{t}^N$ is strongly envy free.

Also, say that there are *equal rights to proportional trade* if the set B is closed under multiplication by any non-negative scalar — i.e., $\lambda B \subset B$ for all $\lambda \geq 0$, implying that B is a cone. Thus, each agent enjoys the right to any multiple of an allowable net trade vector, with both supplies and demands re-scaled in the same proportion. This extension is related to, but somewhat different from Schmeidler and Vind's (1972) divisibility condition. Finally, say that there are *equal rights to multiple proportional trade* if both the last two properties are satisfied, implying that $\lambda B + \mu B \subset B$ whenever $\lambda, \mu \geq 0$, so B must be a convex cone.

Obviously, if $(\bar{t}^N, p)$ is a WE, then there are equal rights to multiple proportional trade within the *Walrasian budget set* $B_p := \{ t \in \mathbb{R}^G \mid p\,t = 0 \}$, a linear subspace of $\mathbb{R}^G$.

6.2.2 Two preliminary lemmas

Given any allocation $\bar{t}^N \in F \subset (\mathbb{R}^G)^N$, let $L(\bar{t}^N)$ denote the linear subspace of $\mathbb{R}^G$ spanned by the associated set $\{ \bar{t}^i \mid i \in N \}$ of net trade vectors. The following simple result will be used later:

Lemma 29 *Suppose preferences are non-satiated, and the feasible allocation $\bar{t}^N \in F$ offers equal rights to multiple proportional trade within the convex cone B. Then $L(\bar{t}^N) \subset B$, and $L(\bar{t}^N)$ is of dimension $\#G - 1$ at most.*

PROOF. First, because $\bar{t}^N \in F$ and so $\sum_{i \in N} \bar{t}^i = 0$, note that $-\bar{t}^i = \sum_{h \in N \setminus \{i\}} \bar{t}^h$. Second, by definition of equal rights to trade within B, one has $\bar{t}^h \in B$ for all $h \in N$. Because B is a convex cone, this implies that $-\bar{t}^i \in B$ for all $i \in N$. It follows that B must contain every linear combination $\sum_{i \in N} \lambda^i \bar{t}^i$ of the set $\{ \bar{t}^i \mid i \in N \}$ of net trade vectors, no matter what the sign of each scalar $\lambda^i \in \mathbb{R}$ may be. This proves that $L(\bar{t}^N) \subset B$.

Next, because $\succsim^i$ is non-satiated, there exists $\hat{t}^i \in T^i$ with $\hat{t}^i \succ^i \bar{t}^i$. Because there are equal rights to trade within the set B, one has $t^i \in B \cap T^i \Longrightarrow \bar{t}^i \succsim^i t^i$, so B cannot include $\hat{t}^i$. Nor therefore can the subset $L(\bar{t}^N)$ of B. This proves that $L(\bar{t}^N)$ must be of dimension less than $\#G$. □

The next result is in the spirit of Schmeidler and Vind (1972, Theorem 4).

Lemma 30 *Suppose that the feasible allocation $\bar{t}^N \in F$ offers equal rights to multiple proportional trade within some common convex cone B that contains a linear subspace $L \subset \mathbb{R}^G$ of dimension $\#G - 1$. Then $\bar{t}^N$ is a WE at some price vector $p \neq 0$.*

PROOF. If L is a subspace of dimension $\#G - 1$, it is a hyperplane through the origin, so $L = B_p := \{ t \in \mathbb{R}^G \mid p\, t = 0 \}$ for some $p \neq 0$. But each agent $i \in N$ has the right to trade within B_p, so $\bar{t}^i \in B_p$ and also $t^i \in B_p \cap T^i \Longrightarrow \bar{t}^i \succsim^i t^i$. Because $\bar{t}^N \in F$, it follows that $(\bar{t}^N, p)$ is a WE. □

6.2.3 Pareto Efficiency

So far, equal rights to multiple proportional trade are consistent with the common budget set B being a linear subspace of low dimension — in fact, even $B = \{0\}$ is possible, with enforced autarky. Supplementing equal rights to trade with Pareto efficiency avoids this trivial case, unless autarky happens to be Pareto efficient anyway. This leads to our characterization result. [16]

Theorem 31 *Suppose agents' preferences for net trades are LNS and convex. Let $\bar{t}^N \in F$ be any weakly Pareto efficient allocation offering equal rights to multiple proportional trade. Assume that at least one agent $h \in N$ has preferences represented by a utility function $u^h(t^h)$ which is differentiable at $\bar{t}^h$. Then $\bar{t}^N$ is a CE at some price vector $p \neq 0$.*

PROOF. Let p denote the gradient vector of u^h at $\bar{t}^h$. Given any $v \in L(\bar{t}^N)$, one has $\bar{t}^h + \lambda\, v \in L(\bar{t}^N)$ for all $\lambda > 0$ because $\bar{t}^h \in L(\bar{t}^N)$ and $L(\bar{t}^N)$ is a linear subspace. By Lemma 29, equal rights to multiple proportional trade within the convex cone B imply that $L(\bar{t}^N) \subset B$. So $\bar{t}^h + \lambda\, v \in B$ for all $\lambda > 0$, from which it follows that $\bar{t}^h \succsim^h \bar{t}^h + \lambda\, v$. By definition of p, it follows that $p\, v \leq 0$. But this is true for all v in the linear subspace $L(\bar{t}^N)$, so $p\, v = 0$ for all $v \in L(\bar{t}^N)$. In particular, $p\, \bar{t}^i = 0$ for all $i \in N$.

[16] More general results of this kind appear in Hammond (2003).

By Theorem 16, the Pareto efficient allocation $\bar{t}^N$ must be a CELT at some common supporting price vector $\bar{p} \neq 0$. Because p is the gradient vector of u^h at $\bar{t}^h$, the common supporting price vector $\bar{p}$ must be a positive multiple of p, so can be replaced by p. But $p\,\bar{t}^i = 0$ for all $i \in N$, so $\bar{t}^N$ is a CE at the price vector $p \neq 0$. □

6.3 Interactive opportunity sets

One of the oldest ideas in economics is that agents who exploit arbitrage or other trading opportunities will be driven toward a Walrasian equilibrium. Some relatively recent attempts to formalize this idea have appeared in Fisher (1981, 1983) — see also Fisher and Saldanha (1982), as well as Stahl and Fisher (1988). That work, however, typically assumes that all agents trade at a common price vector, and asks when awareness of disequilibrium will lead that price vector to change.[17] Instead, this section summarizes some striking recent results due to Serrano and Volij (2000) that derive rather than presume the existence of (uniform) market price vectors.

Consider the general framework of Section 3.1 in which agents $i \in N$ have types described by consumption sets X^i, production sets Y^i and preference orderings R^i. Serrano and Volij suggest constructing recursively a sequence $Z^{i,m}(x^N)$ $(m = 0, 1, 2, \ldots)$ of *multilateral interactive opportunity sets* defined for each agent $i \in N$ and each profile of consumption vectors $x^N \in X^N$.[18] The construction starts when $m = 0$ with the obvious sets $Z^{i,0}(x^N) := Y^i$ of consumption vectors which each agent $i \in N$ can achieve without any trade at all. Given any fixed m and $x^N \in X^N$, as well as the previously constructed sets $Z^{i,m}(x^N)$ $(i \in N)$, the next step is to construct the set

$$Z^{i,m+1}(x^N) := Z^{i,m}(x^N) - \sum_{h \in N \setminus \{i\}} \Big([R^h(x^h) - Z^{h,m}(x^N)] \cup \{0\} \Big) \qquad (26)$$

for each agent $i \in N$. Thus $\hat{x}^i \in Z^{i,m+1}(x^N)$ iff $\hat{x}^i = \tilde{x}^i - \sum_{h \in K} t^h$ for some combination of a consumption vector $\tilde{x}^i \in Z^{i,m}(x^N)$, a set of trading partners $K \subset N \setminus \{i\}$, and a collection of incremental net trade vectors $t^h \in R^h(x^h) - Z^{h,m}(x^N)$ that leave all the agents $h \in K$ no worse off than they are at x^N provided they also make appropriate use of their mth order opportunity sets $Z^{h,m}(x^N)$. This construction evidently implies that $Z^{i,m}(x^N) \subset Z^{i,m+1}(x^N)$ for

[17] For a different approach to arbitrage opportunities, see Makowski and Ostroy (1995, 1998, 2001), as well as the discussion in Section 15.2.

[18] Serrano and Volij (2000) use the term "interactive choice set". In social choice theory, however, it is common to reserve the term "choice set" for the set of *chosen* options rather than the feasible set of available options.

all $i \in N$ and for $m = 0, 1, 2, \ldots$. So the limit sets $Z^i(x^N) := \cup_{m=0}^{\infty} Z^{i,m}(x^N)$ are well defined, for each $i \in N$ and $x^N \in X^N$.

An alternative construction leads to a sequence $\tilde{Z}^{i,m}(x^N)$ ($m = 0, 1, 2, \ldots$) of *bilateral interactive opportunity sets*, with $\tilde{Z}^{i,0}(x^N) := Y^i$ as before, but with (26) replaced by

$$\tilde{Z}^{i,m+1}(x^N) := \tilde{Z}^{i,m}(x^N) - \left(\cup_{h \in N \setminus \{i\}} [R^h(x^h) - Z^{h,m}(x^N)] \cup \{0\}\right). \tag{27}$$

This is equivalent to letting i trade with at most one other agent $h \in N \setminus \{i\}$ when adding an incremental net trade vector to the elements of $\tilde{Z}^{i,m}(x^N)$. As before, $\tilde{Z}^{i,m}(x^N) \subset \tilde{Z}^{i,m+1}(x^N)$ for $m = 0, 1, 2, \ldots$, so one can define each limit set $\tilde{Z}^i(x^N) := \cup_{m=0}^{\infty} \tilde{Z}^{i,m}(x^N)$ (for all $i \in N$ and $x^N \in X^N$). Given any $i \in N$ and $x^N \in X^N$, an obvious argument by induction on m shows that $\tilde{Z}^{i,m}(x^N) \subset Z^{i,m}(x^N)$ for $m = 0, 1, 2, \ldots$, so $\tilde{Z}^i(x^N) \subset Z^i(x^N)$.

Serrano and Volij derive their results under the assumption that there is a pure exchange economy with $X^i = \mathbb{R}^G_+$ and $Y^i = \{e^i\} \subset \mathbb{R}^G_{++}$ for all $i \in N$, and with continuous monotone preference orderings R^i on $\mathbb{R}^G_+$. Their Proposition 1 then states that the multilateral and bilateral interactive opportunity sets $Z^i(x^N)$ and $\tilde{Z}^i(x^N)$ are always identical. The following summarizes their Theorem 1′:

Theorem 32 *In a pure exchange economy with strictly positive endowments and continuous monotone preference orderings, the allocation $\hat{x}^N$ is a WE if and only if the consumption vector $\hat{x}^i$ of each agent $i \in N$ maximizes R^i over either of the (identical) opportunity sets $Z^i(\hat{x}^N)$ and $\tilde{Z}^i(\hat{x}^N)$.*

The key idea of the proof they provide is expressed in their Theorem 4. This states that if the consumption profile x^N admits the existence of any $\tilde{x}^i \in P^i(x^i) \cap Z^i(x^N)$, then in a large enough replica economy of the kind described in Section 8.1, agent i can gain by joining a coalition that blocks x^N. So the result follows from the well-known Debreu–Scarf limit theorem for the core of an infinitely replicated economy.

Note that if preferences are not convex, Theorem 32 still holds formally, though there may be no WE.

7 Characterizations of WE with a variable profile of agents' types

7.1 *Minimal message spaces*

Consider pure exchange economies in which the type of each agent $i \in N$ can be expressed as $\theta^i = (X^i, e^i, R^i)$, where $X^i \subset \mathbb{R}^G$ is a consumption set, the production set Y^i is $\{e^i\}$ for some fixed endowment vector in $e^i \in \mathbb{R}^G_{++}$, and the preference ordering R^i on X^i can be represented by a continuous, strictly increasing, and quasi-concave utility function $u^i : X^i \to \mathbb{R}$. Following Jordan (1982), assume also that the consumption set X^i is $\mathbb{R}^G_{++}$ if the closure of the weak preference set $R^i(\bar{x}^i)$ is contained in $\mathbb{R}^G_{++}$ for each $\bar{x}^i \in \mathbb{R}^G_{++}$; otherwise $X^i = \mathbb{R}^G_+$. Finally, assume that each utility function u^i is either concave or strictly quasi-concave. Let Θ^i denote the domain of such types.

Let $\theta^N := (\theta^i)_{i \in N}$ denote the typical type profile, and $\Theta^N := \prod_{i \in N} \Theta^i$ the associated domain of all such profiles. Because the relevant conditions of Section 5 are satisfied, a Walrasian equilibrium exists for each profile $\theta^N \in \Theta^N$.

A *message process* is a pair (μ, M), where M is an abstract topological *message space*, and $\mu : \Theta^N \twoheadrightarrow M$ is a non-empty valued correspondence on the domain Θ^N of type profiles. As in Section 6.2, let $F := \{\, t^N \in (\mathbb{R}^G)^N \mid \sum_{i \in N} t^i = 0 \,\}$ denote the feasible set of balanced allocations of net trade vectors to the different agents in the economy. Given the message space M, an *outcome function* is a mapping $g : M \to F$. Then say that the triple (μ, M, g) is an *allocation mechanism* if (μ, M) is a message process and g is an outcome function. Finally, say that two mechanisms (μ, M, g) and (μ', M', g') are *equivalent* if there exists a homeomorphism $h : M \to M'$ between the two message spaces such that $g'(h(m)) = g(m)$ for all $m \in M$, and

$$\mu'(\theta^N) = h(\mu(\theta^N)) = \{\, m' \in M' \mid \exists m \in \mu(\theta^N) : m' = h(m) \,\}$$

for all $\theta^N \in \Theta^N$. In particular, equivalence implies that

$$g'(\mu'(\theta^N)) = g'(h(\mu(\theta^N))) = g(\mu(\theta^N))$$

for all $\theta^N \in \Theta^N$, so the two equivalent mechanisms must have an identical range of possible outcomes.

Let $\Delta^0 := \{\, p \in \mathbb{R}^G_{++} \mid \sum_{g \in G} p_g = 1 \,\}$ denote the relative interior of the unit simplex in $\mathbb{R}^G$, whose members are normalized strictly positive price vectors. Given any price vector $p \in \Delta^0$, agent $i \in N$, and type $\theta^i = (X^i, e^i, R^i)$, define

$$\begin{aligned} & \beta(p; \theta^i) := \{\, t \in \mathbb{R}^G \mid p\, t \leq 0,\ t + e^i \in X^i \,\} \\ \text{and} \quad & \tau(p; \theta^i) := \{\, t \in \beta(p; \theta^i) \mid \tilde{t} \in \beta(p; \theta^i) \Longrightarrow t + e^i \ R^i \ \tilde{t} + e^i \,\} \end{aligned}$$

as the Walrasian budget and demand sets, respectively. The *Walrasian* allocation mechanism is the triple (μ_W, M_W, g_W), where:

$$
\begin{aligned}
M_W &:= \{\, (p, t^N) \in \Delta^0 \times F \mid p\, t^i = 0 \text{ (all } i \in N) \,\}; \\
\mu_W(\theta^N) &:= \cap_{i \in N} \mu_W^i(\theta^i) \\
&\qquad \text{with } \mu_W^i(\theta^i) := \{\, (p, t^N) \in M_W \mid t^i \in \tau(p; \theta^i) \,\} \text{ (all } i \in N); \\
g_W(p, t^N) &:= t^N.
\end{aligned}
$$

Note that the space M_W is a connected manifold in $\mathbb{R}^G \times (\mathbb{R}^G)^N$ of dimension $d := \#N(\#G - 1)$. Also, because $t^N \in F$ implies $\sum_{i \in N} t^i = 0$ and

$$
\mu_W(\theta^N) = \{\, (p, t^N) \in \Delta^0 \times F \mid t^i \in \tau(p; \theta^i) \text{ (all } i \in N) \,\},
$$

it follows that $\mu_W(\theta^N)$ is the set of all Walrasian equilibria.

Following Hurwicz's (1960, 1972) original ideas, Mount and Reiter (1974) and Hurwicz (1977) pioneered the formal study of such allocation mechanisms. They required that all possible outcomes $t^N \in g(\mu(\theta^N))$ of the mechanism should be Pareto efficient allocations satisfying weak gains from trade or "individual rationality" — i.e., $t^i + e^i \; R^i \; e^i$ for all $i \in N$. They also required the mechanism to be *informationally decentralized* in the sense that $\mu(\theta^N) = \cap_{i \in N} \mu^i(\theta^i)$ for a profile of correspondences $\mu^i : \Theta^i \twoheadrightarrow M$ defined on Θ^i, the domain of possible types θ^i for agent i. Obviously the Walrasian mechanism satisfies all these conditions. They showed that any other mechanism with all these properties requires a message space of dimension at least d, the dimension of the Walrasian message space M_W. Various corrections, elaborations and extensions appear in Reiter (1977), Walker (1977), Osana (1978), Sato (1981), and Chander (1983).

These results do not characterize the Walrasian allocation mechanism uniquely because they fail to exclude other mechanisms which might use a message space of dimension d to generate as outcomes general WELT allocations satisfying weak gains from trade, rather than WE allocations specifically. Jordan (1982), however, provides conditions guaranteeing that *only* mechanisms equivalent to the Walrasian allocation mechanism are informationally decentralized and use a message space of dimension not exceeding d in order to generate Pareto efficient allocations satisfying weak gains from trade. In this sense, his results characterize the Walrasian mechanism uniquely.

To derive their results, Mount and Reiter in particular imposed a "local threadedness" condition on the correspondence $\mu : \Theta^N \twoheadrightarrow M$ requiring that, for each $\bar{\theta}^N \in \Theta^N$, there should exist a neighbourhood U of $\bar{\theta}^N$ and a continuous selection $f : U \to M$ satisfying $f(\theta^N) \in \mu(\theta^N)$ for all $\theta^N \in U$.[19] Jordan

[19] Mount and Reiter (1974, Definition 6, p. 173) originally described this property as being "locally sliced", and the continuous selection f as a "local slice". Reiter (1977, p. 230) introduces the new term. Later, Jordan (1982) demonstrates a version of Mount and Reiter's main result using the weaker condition that a continuous selection $f : U \to M$ exists for just one open set $U \subset \Theta^N$.

imposes a different "regularity" assumption requiring μ to be a continuous single-valued function on the restricted domain Θ^N_{CD} of "Cobb–Douglas" environments in which each $X^i = \mathbb{R}^G_+$ and each utility function takes the form $u^i(x^i) \equiv \prod_{g \in G}(x^i_g)^{\alpha^i_g}$ for some parameter vector $\alpha^i \in \mathbb{R}^G_{++}$.[20] This regularity assumption ensures that the characterization result holds for the domain Θ^N_{CD}; it is extended to the whole of Θ^N by requiring the range set $\mu(\Theta^N_{\text{CD}})$ to be a relatively closed subset of the message space M. Jordan provides examples showing that these two extra assumptions are indispensable.

Recent work by Hurwicz and Marschak (2003) demonstrates an analogous form of informational superiority for the Walrasian mechanism when it is approximated by a mechanism using a finite message space, and the size rather than the dimension of that space is used to measure the cost of the mechanism.

7.2 Monotonicity

Consider a pure exchange economy where each agent $i \in N$ has $X^i = \mathbb{R}^G_+$ and $Y^i = \{e^i\}$ for a fixed endowment vector $e^i \in \mathbb{R}^G_{++}$. Let $\mathcal{D}^i$ denote the domain of convex, continuous and monotone preferences on X^i. Let $\mathcal{D}^N := \prod_{i \in N} \mathcal{D}^i$.

Given the fixed *endowment profile* $e^N := (e^i)_{i \in N}$, define the set

$$F_{e^N} := \{\, x^N \in X^N = (\mathbb{R}^G)^N \mid \sum_{i \in N}(x^i - e^i) = 0 \,\}$$

of all feasible consumption profiles. Then let $\Phi_{e^N} : \mathcal{D}^N \twoheadrightarrow X^N$ denote a *social choice rule* (or SCR) which, for each *preference profile* $R^N \in \mathcal{D}^N$, specifies a non-empty choice set $\Phi_{e^N}(R^N) \subset F_{e^N}$. Say that Φ_{e^N} is *singleton valued* if there exists a mapping $\phi_{e^N} : \mathcal{D}^N \to X^N$ such that $\Phi_{e^N}(R^N) \equiv \{\phi_{e^N}(R^N)\}$.

Following Maskin (1999) and Dasgupta, Hammond and Maskin (1979), say that the SCR is *monotonic* provided that any $x^N \in \Phi_{e^N}(R^N)$ also belongs to $\Phi_{e^N}(\tilde{R}^N)$ whenever the two preference profiles R^N and $\tilde{R}^N$ satisfy $x^i \; R^i \; \bar{x}^i \Longrightarrow x^i \; \tilde{R}^i \; \bar{x}^i$ for all $i \in N$ and all $\bar{x}^i \in X^i$.[21] One reason for being interested in this property is that it is necessary and sufficient for Nash implementability

[20] It is easy to show that the gross substitutability condition $\partial x^i_g / \partial p_h \geq 0$ (all $h \neq g$) is satisfied in every environment $\theta^N \in \Theta^N_{\text{CD}}$, so $\mu_W(\theta^N)$ must be singleton-valued — see Arrow and Hahn (1971).

[21] An early version of Maskin's paper was widely circulated in 1977.

of Walrasian equilibrium when $\#N \geq 3$.[22] The following result is suggested by Hurwicz (1986, p. 1473).

Theorem 33 *Suppose preferences in a pure exchange economy satisfy the smoothness conditions (a)–(d) of Section 2.5. Suppose $\phi_{e^N}(R^N)$ is a singleton-valued social choice rule which is continuous, monotonic, and generates Pareto efficient allocations satisfying weak gains from trade. Then $\phi_{e^N}(R^N)$ must be a Walrasian equilibrium.*

Without the boundary condition (d) of Section 2.5, this result would be false. A counter-example announced by Hurwicz (1986) eventually appeared in Hurwicz, Maskin and Postlewaite (1995) — see also Thomson (1999). Without the boundary condition, the SCR $\phi_{e^N}(R^N)$ could select a *constrained Walrasian equilibrium* instead. This is defined as a pair $(\hat{x}^N, \hat{p})$ consisting of a feasible allocation and a price vector with the properties that, for all $i \in N$ one has $\hat{p}\,\hat{x}^i \leq \hat{p}\,\hat{e}^i$, and also $\hat{p}\,x^i > \hat{p}\,\hat{e}^i$ whenever both $x^i \in P^i(\hat{x}^i)$ and $x^i \leqq \sum_{h \in N} e^h_g$. The latter is an additional constraint on agents' demands. For details, see Hurwicz (1986, p. 1473) and also Nagahisa (1994). When preferences satisfy the boundary condition, this additional constraint is irrelevant, and the constrained and unconstrained Walrasian equilibria coincide.

Game forms typically admit multiple Nash equilibria for some preference profiles. Uniqueness of the equilibrium set is unlikely to be a generic property. Accordingly, it is natural to consider an SCR Φ_{e^N} which is not singleton valued. Then, if Φ_{e^N} is monotonic, results such as those of Gevers (1986) and Nagahisa (1991) provide sufficient conditions for $\Phi_{e^N}(R^N)$ to include every Walrasian allocation for each preference profile R^N. The SCR may also include some non-Walrasian allocations, however.

7.3 Local independence in smooth economies

Nagahisa and Suh (1995) refine an earlier characterization of the Walras rule proposed by Nagahisa (1991). They also characterize Walras equilibrium with equal budgets.

Let $\mathcal{D}^i$ now denote the domain of regular smooth preferences satisfying all the smoothness conditions of Section 2.5. Let $\mathcal{D}^N := \prod_{i \in N} \mathcal{D}^i$. Let $\Phi_{e^N} : \mathcal{D}^N \twoheadrightarrow X^N$ be the SCR.

[22] Maskin's demonstration of sufficiency relies on a condition he calls "no veto power". This is vacuously satisfied in an exchange economy with $\#N \geq 3$ when agents have strictly monotone preferences.

Say that the SCR Φ_{e^N} gives *envy-free* allocations if, for each R^N and each $x^N \in \Phi_{e^N}(R^N)$, one has $x^i \; R^i \; x^h$ for all $h, i \in N$. Say that the SCR gives *fair* allocations if it gives allocations that are both envy-free and Pareto efficient.

For each agent $i \in N$, for each smooth preference ordering $R^i \in \mathcal{D}^i$ represented by a utility function u^i which is $\mathcal{C}^1$ on $\mathbb{R}^G_{++}$, for each net consumption vector $x^i \in \mathbb{R}^G_{++}$, and for each pair of goods $f, g \in G$, let

$$s^i_{fg}(R^i, x^i) := \frac{\partial u^i}{\partial x^i_f}(x^i) \Big/ \frac{\partial u^i}{\partial x^i_g}(x^i)$$

denote agent i's *marginal rate of substitution* between f and g at x^i. Note that this depends only on agent i's preference ordering R^i, not on the utility function used to represent it.

Say that the SCR satisfies *local independence* provided that, for each pair of preference profiles $R^N, \tilde{R}^N \in \mathcal{D}^N$ and each allocation $x^N \in X^N$, satisfying $s^i_{fg}(R^i, x^i) = s^i_{fg}(\tilde{R}^i, x^i)$ for all $i \in N$ and all $f, g \in G$, then $x^N \in \Phi_{e^N}(R^N)$ if and only if $x^N \in \Phi_{e^N}(\tilde{R}^N)$. This is clearly a local version of the familiar "independence of irrevelant alternatives" axiom in social choice theory. Also, if individuals do have smooth preferences, and if $x^i \; R^i \; \bar{x}^i$ implies $x^i \; \tilde{R}^i \; \bar{x}^i$ for all $\bar{x}^i \in X^i$, then $s^i_{fg}(R^i, x^i) = s^i_{fg}(\tilde{R}^i, x^i)$ for all $f, g \in G$. When preferences are smooth, it follows that local independence implies the monotonicity condition described in Section 7.2.

With these definitions, Nagahisa and Suh's main characterization results can be stated as follows:

Theorem 34 *The SCR $R^N \mapsto \Phi_{e^N}(R^N)$ is Walrasian if and only if it is locally independent, Pareto efficient, and satisfies weak gains from trade.*

Theorem 35 *The SCR $R^N \mapsto \Phi_{e^N}(R^N)$ is WEEB if and only if it is fair and locally independent.*

In each case it is easy to check that a Walrasian (resp., WEEB) has the relevant properties. Proofs that these properties are complete characterizations can be found in Nagahisa and Suh (1995). They also provide examples that go most of the way toward showing how the smoothness and other conditions are required for these results to be valid.

7.4 Strategyproofness with exogenous prices

7.4.1 Strategyproof mechanisms in a finite economy

A *net trade allocation mechanism* $t^N(\theta^N)$ specifies the profile of agents' net trade vectors $t^N \in T^N$ satisfying $\sum_{i \in N} t^i = 0$ as a function of their type profile θ^N. The mechanism is said to be (individually) *strategyproof* if the *incentive constraint* $t^i(\theta^N) \succsim_{\theta^i} t^i(\tilde{\theta}^i, \theta^{-i})$ is satisfied whenever $t^i(\tilde{\theta}^i, \theta^{-i}) \in T_{\theta^i}$, where θ^{-i} denotes $\langle \theta^h \rangle_{h \in N \setminus \{i\}}$, with i's type omitted. This definition implies that no individual $i \in N$ whose true type is $\theta^i \in \Theta$ has the incentive to manipulate the mechanism by acting as a type $\tilde{\theta}^i$ agent would.

In finite economies, a mechanism that produces a Walrasian equilibrium allocation for every type profile will rarely be strategyproof. This is because agents can typically manipulate prices to their own advantage by acting in the market as if they were a different type of agent who is willing to undertake only a smaller volume of trade. Moreover, unless the mechanism allows allocations arbitrarily close to the extremes of the feasible set, strategyproofness is usually inconsistent with Pareto efficiency — see Serizawa and Weymark (2003).

Makowski, Ostroy and Segal (1999) present some more positive results concerning WE allocations on a restricted domain. These involve cases where at least one agent has a flat indifference surface in some neighbourhood of a Walrasian equilibrium allocation. If this neighbourhood is sufficiently large, then individual agents cannot manipulate prices except by distorting their desired net trades so much that they become worse off.

7.4.2 A linear technology

Similar positive results might be expected in economies with production where equilibrium prices happen to be determined entirely by exogenous supply conditions, independent of demand. The "non-substitution theorem" due to Samuelson (1951), Koopmans (1951), Arrow (1951b) and Georgescu-Roegen (1951) describes an important case when this is true — see also Mirrlees (1969) for a "dynamic" extension to steady growth paths. For recent work on sufficient conditions for a technology to be linear, at least in some neighbourhood of a given aggregate demand vector, see Bergstrom (1996) and Villar (2003). One line of work not cited there explores relevant links between factor price equalization and price invariance (or what Bhagwati and Wan (1979) call "stationarity") in the theory of international trade for small open economies — see, for example, Diewert (1983) and Hammond (1986).

Formally, suppose there is a linear technology represented by the half-space $Y := \{ y \in \mathbb{R}^G \mid \bar{p}\, y \leq \bar{p}\, e \}$ for some fixed price vector $\bar{p} \gg 0$ and some exogenous aggregate endowment vector $e \in \mathbb{R}^G_{++}$. Suppose too that each type $\theta \in \Theta$ of agent has the same consumption set $\mathbb{R}^G_+$, the same production set Y, and a variable preference ordering R_θ which is continuous and strictly monotone on $\mathbb{R}^G_+$. For this important special case, under extra assumptions discussed below, Maniquet and Sprumont (1999) do indeed provide two different characterizations of consumption mechanisms $x^N(\theta^N)$ that generate WE allocations:

(1) they are strategyproof, Pareto efficient, and give identical agents equally good consumption vectors (see their Theorem 2);
(2) they are coalitionally strategyproof, Pareto efficient, and give all agents equally good consumption allocations whenever all agents have identical preferences (see their Theorem 4).

Given the fixed price vector $\bar{p} \gg 0$, let

$$\gamma_\theta(w) := \{x \in \mathbb{R}^G_+ \mid \bar{p}\, x \leq w;\ x' \in P_\theta(x) \Longrightarrow \bar{p}\, x' > w\}$$

denote the Walrasian demand set of a type θ agent when confronted with the budget constraint $\bar{p}\, x \leq w$, where $w \in \mathbb{R}_+$. The first extra assumption requires the range of different correspondences $\gamma_\theta : \mathbb{R}_+ \twoheadrightarrow \mathbb{R}^G_+$ to have the property that, given any continuous and non-decreasing *wealth consumption curve* $c : \mathbb{R}_+ \to \mathbb{R}^G_+$ satisfying $\bar{p}\, c(w) = w$ for all $w \in \mathbb{R}_+$, there exists $\theta \in \Theta$ such that $\gamma_\theta(w) = \{c(w)\}$. Finally, result (2) relies on a second extra assumption requiring that for all $\theta, \theta' \in \Theta$ and all $w \in \mathbb{R}_+$, there exists $\theta'' \in \Theta$ such that $\gamma_\theta(w) \cup \gamma_{\theta'}(w) \subset \gamma_{\theta''}(w)$. Of course, this last assumption rules out single-valued demand functions except in the trivial case where every type of agent has exactly the same correspondence $w \mapsto\!\!\!\rightarrow \gamma_\theta(w)$.

8 Characterizations of WE with a varying number of agents

8.1 The core and Edgeworth equilibrium

Let $\mathcal{E}$ denote the economy described in Section 3.1, with a finite set of agents N whose consumption and production sets are X^i and Y^i respectively, and whose preference orderings are R^i, for all $i \in N$. Consider any feasible allocation $(\bar{x}^N, \bar{y}^N)$ in $\mathcal{E}$. Given a coalition $K \subset N$, say that K *blocks* $(\bar{x}^N, \bar{y}^N)$, and that K is a *blocking coalition*, if there is a *blocking allocation* $(x^K, y^K) \in \prod_{i \in K}[P^i(\bar{x}^i) \times Y^i]$ satisfying $\sum_{i \in K}(x^i - y^i) = 0$.[23] On the other hand, the feasible allocation $(\hat{x}^N, \hat{y}^N)$ is in the *core* if there is no blocking coalition.

[23] Alternatively, a *weak blocking allocation* satisfies $(x^K, y^K) \in \prod_{i \in K}[R^i(\bar{x}^i) \times Y^i]$, as well as $\sum_{i \in K}(x^i - y^i) = 0$ and $x^h\ P^h\ \bar{x}^h$ for some $h \in K$. This weakening would make little difference to the results presented here.

Obviously, any core allocation is weakly Pareto efficient because otherwise the "grand coalition" N would block it.

Theorem 36 *Suppose agents' preferences are LNS and* $(\hat{x}^N, \hat{y}^N, \hat{p})$ *is a WE. Then the equilibrium allocation* $(\hat{x}^N, \hat{y}^N)$ *is in the core.*

PROOF. Let $K \subset N$ be any coalition. Suppose $(x^K, y^K) \in \prod_{i \in K}[P^i(\bar{x}^i) \times Y^i]$. Because $(\hat{x}^N, \hat{y}^N, \hat{p})$ is a WE, it follows that $\hat{p}\,(x^i - y^i) > 0$ for all $i \in K$. This contradicts $\sum_{i \in K}(x^i - y^i) = 0$, so there can be no blocking coalition. □

Given the economy $\mathcal{E}$ and any natural number $r \in \mathbb{N}$, let $\mathcal{E}^r$ denote the rth *replica economy* in which each agent $i \in N$ is replicated r times. These replicated agents have labels ik $(k = 1, 2, \ldots, r)$. Their respective consumption and production sets and preference orderings satisfy $X^{ik} = X^i$, $Y^{ik} = Y^i$, and $R^{ik} = R^i$ for all $k = 1, 2, \ldots, r$.

Following Aliprantis, Brown and Burkinshaw (1987a, b), define an *Edgeworth equilibrium* in the economy $\mathcal{E}$ as a feasible allocation $(\hat{x}^N, \hat{y}^N)$ such that the core of each replica economy $\mathcal{E}^r$ $(r = 1, 2, \ldots)$ includes the *replica allocation* $(x^{N \times \{1,2,\ldots,r\}}, y^{N \times \{1,2,\ldots,r\}})$ satisfying $x^{ik} = \hat{x}^i$ and $y^{ik} = \hat{y}^i$ for all $i \in N$ and all $k = 1, 2, \ldots, r$.[24] We denote this rth replica allocation by $(\hat{x}^N, \hat{y}^N)^r$.

Theorem 37 *Suppose agents' preferences are LNS and* $(\hat{x}^N, \hat{y}^N, \hat{p})$ *is a WE. Then the equilibrium allocation* $(\hat{x}^N, \hat{y}^N)$ *is an Edgeworth equilibrium.*

PROOF. Because $(\hat{x}^N, \hat{y}^N, \hat{p})$ is a WE in the economy $\mathcal{E}$, so is $((\hat{x}^N, \hat{y}^N)^r, \hat{p})$ in the replica economy $\mathcal{E}^r$. Theorem 36 implies that $(\hat{x}^N, \hat{y}^N)^r$ belongs to the core of $\mathcal{E}^r$ for each $r = 1, 2, \ldots$. So $(\hat{x}^N, \hat{y}^N)$ is an Edgeworth equilibrium. □

The following converse result is based on Debreu and Scarf's (1963) limit theorem for the core. Agents are assumed to have convex consumption and production sets. Preferences need not be convex, however, though if they are not there may be neither a CE nor an Edgeworth equilibrium.

Theorem 38 *Suppose agents have convex consumption and production sets for which autarky is feasible, as well as preferences that are LNS and continuous. Then a feasible allocation* $(\hat{x}^N, \hat{y}^N)$ *is an Edgeworth equilibrium only if there exists a price vector* $\hat{p} \neq 0$ *such that* $(\hat{x}^N, \hat{y}^N, \hat{p})$ *is a CE.*

[24] Vind (1995) carefully discusses the (rather tenuous) relationship between perfect competition, the core, and Edgeworth's (1881) concept of "final equilibrium".

PROOF. Given the feasible allocation $(\hat{x}^N, \hat{y}^N)$, define Z as the convex hull of $\cup_{i \in N}[P^i(\hat{x}^i) - Y^i]$.

Suppose $0 \in Z$. Then there must exist a natural number m and, for each $q = 1, 2, \ldots, m$, corresponding convex weights $\alpha_q \in (0, 1]$, agents $i_q \in N$, consumption vectors $x_q \in P^{i_q}(\hat{x}^{i_q})$, and production vectors $y_q \in Y^{i_q}$, such that $\sum_{q=1}^m \alpha_q = 1$ and $0 = \sum_{q=1}^m \alpha_q (x_q - y_q)$. For each $q \in \{1, 2, \ldots, m\}$ and $r = 1, 2, \ldots$, let n_{qr} be the smallest integer that is greater or equal to $r\,\alpha_q$. Then define

$$(\tilde{x}_{qr}, \tilde{y}_{qr}) := \frac{r\,\alpha_q}{n_{qr}}(x_q, y_q) + \left(1 - \frac{r\,\alpha_q}{n_{qr}}\right)(a^{i_q}, a^{i_q})$$

where each $a^{i_q} \in X^{i_q} \cap Y^{i_q}$ is any feasible autarky consumption vector for agent i_q. Because $r\,\alpha_q \leq n_{qr}$ and the sets X^i and Y^i are assumed to be convex for all $i \in N$, the convex combination $(\tilde{x}_{qr}, \tilde{y}_{qr}) \in X^{i_q} \times Y^{i_q}$ for $q = 1, 2, \ldots, m$ and for $r = 1, 2, \ldots$. Moreover,

$$\sum_{q=1}^{m} n_{qr}\,(\tilde{x}_{qr} - \tilde{y}_{qr}) = \sum_{q=1}^{m} r\,\alpha_q\,(x_q - y_q) = 0.$$

Finally, $0 \leq n_{qr} - r\,\alpha_q < 1$ so $r\,\alpha_q / n_{qr} \to 1$ as $r \to \infty$, implying $\tilde{x}_{qr} \to x_q$. Because preferences are continuous, for all sufficiently large r one has $\tilde{x}_{qr} \in P^{i_q}(\hat{x}^{i_q})$ for all $q = 1, 2, \ldots, m$. Now define $Q^i := \{\, q \in \{1, 2, \ldots, m\} \mid i_q = i \,\}$ for each $i \in N$, as well as $s := \sum_{q=1}^m n_{qr}$. Then a coalition which consists of $\sum_{q \in Q^i} n_{qr}$ replicas of each agent $i \in N$ can block the replicated allocation $(\hat{x}^N, \hat{y}^N)^s$ in the replica economy $\mathcal{E}^s$ by allocating $(\tilde{x}_{qr}, \tilde{y}_{qr})$ to n_{qr} replicas of agent i_q, for $q = 1, 2, \ldots, m$. Thus, $(\hat{x}^N, \hat{y}^N)$ is not an Edgeworth equilibrium.

Conversely, if $(\hat{x}^N, \hat{y}^N)$ is an Edgeworth equilibrium, then $0 \notin Z$. Because Z is convex by construction, there exists a price vector $\hat{p} \neq 0$ such that $\hat{p}\,z \geq 0$ for all $z \in Z$, and so for all $z \in \cup_{i \in N}[P^i(\hat{x}^i) - Y^i]$. But preferences are LNS, so $\hat{x}^i \in \operatorname{cl} P^i(\hat{x}^i)$. It follows that $\hat{x}^i - \hat{y}^i \in \operatorname{cl} P^i(\hat{x}^i) - Y^i$, so $\hat{p}\,(\hat{x}^i - \hat{y}^i) \geq 0$ for all $i \in N$. But $\sum_{i \in N}(\hat{x}^i - \hat{y}^i) = 0$ because $(\hat{x}^N, \hat{y}^N)$ is feasible, so $\hat{p}\,(\hat{x}^i - \hat{y}^i) = 0$ for all $i \in N$. Finally, whenever $(x^i, y^i) \in R^i(\hat{x}^i) \times Y^i$, then $x^i \in \operatorname{cl} P^i(\hat{x}^i)$, so $x^i - y^i \in \operatorname{cl} P^i(\hat{x}^i) - Y^i$. Hence, $\hat{p}\,(x^i - y^i) \geq 0$. These results imply that $(\hat{x}^N, \hat{y}^N, \hat{p})$ is a CE. $\square$

Theorem 39 *Suppose agents have convex consumption and production sets for which autarky is feasible, and* $0 \in \operatorname{int} \sum_{i \in N}(X^i - Y^i)$. *Suppose preferences are LNS and continuous, and the economy is directionally irreducible. Then a feasible allocation* $(\hat{x}^N, \hat{y}^N)$ *is an Edgeworth equilibrium if and only if there exists a price vector* $\hat{p} \neq 0$ *such that* $(\hat{x}^N, \hat{y}^N, \hat{p})$ *is a WE.*

PROOF. By Theorem 37, any WE allocation is an Edgeworth equilibrium. Conversely, any Edgeworth equilibrium is a CE allocation, by Theorem 38. Under the stated hypotheses, Theorem 25 guarantees that any CE is a WE. □

8.2 Another limit theorem

Edgeworth equilibria refine the core by requiring that the same allocation, when replicated, belongs to the core of the corresponding replica economy. Nagahisa (1994, Theorem 5) has used alternative refinements of the core in replica economies of different size in order to characterize Walrasian equilibrium. These results, however, rest on rather strong assumptions, including the monotonicity condition considered in Section 7.2, as well as the Pareto indifference axiom used in 8.3.3 below.

8.3 Stability

8.3.1 A Walrasian social choice rule

Let $\mathcal{E} := \langle N, \theta^N \rangle$ denote a typical *economic environment* of pure exchange, with N as the variable finite set of agents, each of whom has a type $\theta^i = (X^i, e^i, R^i)$ described by a consumption set $X^i \subset \mathbb{R}^G_+$, a fixed endowment vector $e^i \in \mathbb{R}^G_{++}$, and a preference ordering R^i. In this section we assume that preferences are strictly monotone and convex.

Given any environment $\mathcal{E}$, let

$$F(\mathcal{E}) := \{\, x^N \in X^N \mid \sum_{i \in N} (x^i - e^i) = 0 \,\} \subset (\mathbb{R}^G)^N$$

denote the set of all feasible consumption allocations in $\mathcal{E}$. Let $\mathrm{WE}(\mathcal{E})$ and $\mathrm{WELT}(\mathcal{E})$ denote the (possibly empty) sets of feasible allocations $\hat{x}^N \in F(\mathcal{E})$ for which there exists an equilibrium price vector $\hat{p} \neq 0$ such that $(\hat{x}^N, \hat{p})$ is, respectively, a WE and a WELT in the environment $\mathcal{E}$. Because preferences are assumed to be strictly monotone, any WE or WELT equilibrium price vector must satisfy $\hat{p} \gg 0$.

Define the respective domains $\mathbf{E}_1$ and $\mathbf{E}_2$ of environments $\mathcal{E}$ so that $\mathrm{WE}(\mathcal{E}) \neq \emptyset$ iff $\mathcal{E} \in \mathbf{E}_1$ and $\mathrm{WELT}(\mathcal{E}) \neq \emptyset$ iff $\mathcal{E} \in \mathbf{E}_2$. Because $\mathrm{WE}(\mathcal{E}) \subset \mathrm{WELT}(\mathcal{E})$ for all $\mathcal{E}$, obviously $\mathbf{E}_1 \subset \mathbf{E}_2$.

Next, define a *social choice rule* (or SCR) as a mapping $\Phi : \mathbf{E} \twoheadrightarrow (\mathbb{R}^G)^N$ which satisfies $\emptyset \neq \Phi(\mathcal{E}) \subset F(\mathcal{E})$ for all environments $\mathcal{E}$ in a specified domain $\mathbf{E}$. Obviously, the *Walrasian social choice rule* $\mathrm{WE}(\cdot)$ and the more general WELT(·) are two such rules, which are defined on the domains $\mathbf{E}_1$ and $\mathbf{E}_2$ respectively and satisfy $\emptyset \neq \mathrm{WE}(\mathcal{E}) \subset \mathrm{WELT}(\mathcal{E})$ for all environments $\mathcal{E} \in \mathbf{E}_1$. Of course, when seeking a characterization of Walrasian equilibrium, it is natural to limit attention to environments in which a Walrasian equilibrium — or at least a WELT — exists. We assume therefore that the domain $\mathbf{E}$ satisfies $\mathbf{E}_1 \subset \mathbf{E} \subset \mathbf{E}_2$.

8.3.2 Stability under non-essential addition

Thomson (1988) in particular introduced axioms relating values $\Phi(\mathcal{E})$ of the SCR in environments with different sets of agents N. He discussed several different SCRs, including Walrasian equilibrium with equal budgets. Here, two of his axioms will be adapted to our more general setting in order to characterize the Walrasian SCR WE(·) as a particular restriction of WELT(·).[25]

First, say that $\mathcal{E} = \langle N, \theta^N \rangle$ is a *sub-economy* of $\tilde{\mathcal{E}} = \langle \tilde{N}, \tilde{\theta}^{\tilde{N}} \rangle$, and write $\mathcal{E} \subset \tilde{\mathcal{E}}$, whenever $N \subset \tilde{N}$ and also $\tilde{\theta}^N = \theta^N$.

Next, say that the SCR Φ is *stable under non-essential addition* if, whenever $\mathcal{E} \in \mathbf{E}$, $\mathcal{E} \subset \tilde{\mathcal{E}}$, and a chosen allocation $\hat{x}^N \in \Phi(\mathcal{E})$ has an extension $\hat{x}^{\tilde{N}} \in F(\tilde{\mathcal{E}})$ which is Pareto efficient in $\tilde{\mathcal{E}}$ and satisfies $\tilde{x}^i = \tilde{e}^i$ for all $i \in \tilde{N} \setminus N$, then $\tilde{\mathcal{E}} \in \mathbf{E}$ and the extended allocation $\hat{x}^{\tilde{N}} \in \Phi(\tilde{\mathcal{E}})$. Thus, if an economy is enlarged by adding new agents to whom allocating a zero net trade vector extends Pareto efficiently a chosen allocation in the original economy $\mathcal{E}$, then that extended allocation should be a possible choice in the enlarged economy $\tilde{\mathcal{E}}$.

Lemma 40 *Suppose* $\mathbf{E} \subset \mathbf{E}_2$*, and the SCR* Φ *is stable under non-essential addition while satisfying* $\Phi(\mathcal{E}) \subset \mathrm{WELT}(\mathcal{E})$ *for each* $\mathcal{E} \in \mathbf{E}$*. Suppose too that* $\hat{x}^N \in \Phi(\mathcal{E}^*) \setminus \mathrm{WE}(\mathcal{E}^*)$ *where* $\mathcal{E}^* \in \mathbf{E}$*. Then there exist* $\tilde{\mathcal{E}} \in \mathbf{E}$ *and* $\tilde{x}^{\tilde{N}} \in \Phi(\tilde{\mathcal{E}})$ *with at least one agent envying another's net trade vector.*

PROOF. By hypothesis, there exists $\hat{p} \gg 0$ at which $\hat{x}^N$ is a WELT in the environment $\mathcal{E}^*$. Construct a new environment $\tilde{\mathcal{E}}$ with $\tilde{N} := N \cup \{0\}$ and $\mathcal{E}^* \subset \tilde{\mathcal{E}}$ by adding to N one extra agent labelled $0 \notin N$ with consumption set $X^0 = \mathbb{R}^G_+$, endowment vector $e^0 \in \mathbb{R}^G_{++}$, and with R^0 represented by the

[25] In order to avoid unnecessary complications because a Pareto efficient allocation may not be a WELT, or even a CELT, we will assume directly that $\Phi(\mathcal{E}) \subset \mathrm{WELT}(\mathcal{E})$ for all $\mathcal{E} \in \mathbf{E}$, rather than that each allocation in $\Phi(\mathcal{E})$ is Pareto efficient. In particular, there is no need to assume that preferences are continuous. A related reason for departing from Thomson's original framework is that the proof of Theorem 5 he offers appears to need modifications and additional assumptions which guarantee that if there is a unique normalized equilibrium price vector, then every Walrasian equilibrium allocation is included in the social choice set.

linear utility function defined by $u^0(x) := \hat{p}\, x$ for all $x \in \mathbb{R}^G_+$. Consider too the extended allocation $\tilde{x}^{\tilde{N}} \in F(\tilde{\mathcal{E}})$ with $\tilde{x}^i = \hat{x}^i$ for all $i \in N$ and $\tilde{x}^0 = e^0$. Then $(\tilde{x}^{\tilde{N}}, \hat{p})$ is a WELT in $\tilde{\mathcal{E}}$. Because preferences are strictly monotone and so LNS, Theorem 15 implies that this allocation is Pareto efficient. Because Φ is stable under non-essential addition, it follows that $\tilde{\mathcal{E}} \in \mathbf{E}$ and $\tilde{x}^{\tilde{N}} \in \Phi(\tilde{\mathcal{E}})$.

By hypothesis, $\hat{x}^N$ is not a WE allocation, so $\hat{p}\,(\hat{x}^i - e^i) \neq 0$ for some $i \in N$. Because $\sum_{i \in N}(\hat{x}^i - e^i) = 0$, there must exist $h \in N$ such that $\hat{p}\,(\hat{x}^h - e^h) > 0$. So agent 0's utility function satisfies

$$u^0(e^0 + \tilde{x}^h - e^h) - u^0(e^0 + 0) = \hat{p}\,(\tilde{x}^h - e^h) > 0.$$

This proves that agent 0 with net trade vector 0 envies agent h with net trade vector $\tilde{x}^h - e^h$. □

8.3.3 Non-discrimination between Pareto indifferent allocations

Gevers (1986, p. 102) introduced an axiom he called "non-discrimination between Pareto indifferent allocations". More concisely, say that the SCR Φ satisfies *Pareto indifference* if whenever the allocations $\tilde{x}^N, \hat{x}^N \in F(\mathcal{E})$ satisfy $\tilde{x}^i \; I^i \; \hat{x}^i$ for all $i \in N$, then $\tilde{x}^N \in \Phi(\mathcal{E}) \iff \hat{x}^N \in \Phi(\mathcal{E})$.

The following result confirms that the Walrasian SCR satisfies Pareto indifference, as does the WELT SCR for a given wealth distribution.

Lemma 41 *Assume preferences are LNS. Suppose that $(\hat{x}^N, \hat{p})$ is a WELT in the environment $\mathbf{E}$, and that $\tilde{x}^N \in F(\mathbf{E})$ satisfies $\tilde{x}^i \; I^i \; \hat{x}^i$ for all $i \in N$. Then $(\tilde{x}^N, \hat{p})$ is also a WELT in $\mathbf{E}$, with $\hat{p}\,\tilde{x}^i = \hat{p}\,\hat{x}^i$ for all $i \in N$.*

PROOF. Because preferences are LNS, Lemma 9 implies that $(\hat{x}^N, \hat{p})$ is a CELT. But $\tilde{x}^i \; I^i \; \hat{x}^i$ for all $i \in N$, so $\hat{p}\,\tilde{x}^i \geq \hat{p}\,\hat{x}^i$ for all $i \in N$. Also feaibility implies that $\sum_{i \in N} \tilde{x}^i = \sum_{i \in N} \hat{x}^i = \sum_{i \in N} e^i$, so $\hat{p}\,\tilde{x}^i = \hat{p}\,\hat{x}^i$ for all $i \in N$. Now, for any $i \in N$, whenever $x^i \in P^i(\tilde{x}^i)$, one has $x^i \in P^i(\hat{x}^i)$ because preferences are transitive, so $\hat{p}\,x^i > \hat{p}\,\hat{x}^i = \hat{p}\,\tilde{x}^i$ because $(\hat{x}^N, \hat{p})$ is a WELT. This confirms that $(\tilde{x}^N, \hat{p})$ is a WELT. □

8.3.4 Stability under non-essential deletion

Next, say that the SCR Φ is *stable under non-essential deletion* if, whenever $\tilde{\mathcal{E}} \in \mathbf{E}$, $\mathcal{E} \subset \tilde{\mathcal{E}}$, and a chosen allocation $\tilde{x}^{\tilde{N}} \in \Phi(\tilde{\mathcal{E}})$ satisfies $\tilde{x}^i = e^i$ for all $i \in \tilde{N} \setminus N$, then $\mathcal{E} \in \mathbf{E}$ and the restricted allocation $\tilde{x}^N \in \Phi(\mathcal{E})$. In other words, ignoring agents whose chosen net trade vectors in $\tilde{\mathcal{E}}$ happen to be zero leaves an allocation to the other agents which is chosen in the subeconomy $\mathcal{E}$.

Lemma 42 *Suppose Φ satisfies Pareto indifference and is stable under non-essential deletion on the domain* $\mathbf{E}$*, where* $\mathbf{E}_1 \subset \mathbf{E}$*. Suppose too that* $\Phi(\mathcal{E}) \subset \mathrm{WE}(\mathcal{E})$ *for each* $\mathcal{E} \in \mathbf{E}$*. Then* $\mathbf{E} = \mathbf{E}_1$ *and* $\Phi(\mathcal{E}) = \mathrm{WE}(\mathcal{E})$ *for all* $\mathcal{E} \in \mathbf{E}$*.*

PROOF. Let $(\hat{x}^N, \hat{p})$ be any WE in the environment $\mathcal{E}$. Construct a new environment $\tilde{\mathcal{E}}$ with $\tilde{N} := N \cup \{0\}$ and $\mathcal{E} \subset \tilde{\mathcal{E}}$ by adding to N one extra agent $0 \notin N$ with consumption set $X^0 = \mathbb{R}^G_+$, preference ordering R^0 represented by the linear utility function $u^0(x) \equiv \hat{p}\, x$, and endowment vector $e^0 \in \mathbb{R}^G_{++}$ whose respective components are

$$e^0_g := \frac{1}{\hat{p}_g} \sum_{h \in G} \sum_{i \in N} \hat{p}_h\, e^i_h \text{ (all } g \in G). \tag{28}$$

Consider too the allocation $\tilde{x}^{\tilde{N}} \in F(\tilde{\mathcal{E}})$ with $\tilde{x}^i = \hat{x}^i$ for all $i \in N$ and with $\tilde{x}^0 = e^0$. Evidently $(\tilde{x}^{\tilde{N}}, \hat{p})$ is a WE in the environment $\tilde{\mathcal{E}}$. In particular $\tilde{\mathcal{E}} \in \mathbf{E}_1$, so $\tilde{\mathcal{E}} \in \mathbf{E}$, which implies that $\Phi(\tilde{\mathcal{E}}) \neq \emptyset$.

Given any alternative price vector $p \gg 0$, define

$$\alpha(p) := \min\{\, p_g/\hat{p}_g \mid g \in G \,\} \quad \text{and} \quad G(p) := \arg\min\{\, p_g/\hat{p}_g \mid g \in G \,\}. \tag{29}$$

Then the Walrasian consumption vector $x^0(p)$ demanded by agent 0 obviously satisfies $x^0_g(p) = 0$ for all $G \setminus G(p)$, so (29) implies that

$$\sum_{g \in G(p)} \hat{p}_g\, x^0_g(p) = \frac{1}{\alpha(p)} \sum_{g \in G(p)} p_g\, x^0_g(p) = \frac{1}{\alpha(p)} \sum_{g \in G} p_g\, e^0_g \geq \sum_{g \in G} \hat{p}_g\, e^0_g. \tag{30}$$

From (30) and (28) it follows that

$$\sum_{g \in G(p)} \hat{p}_g\, [x^0_g(p) - e^0_g] \geq \sum_{g \in G \setminus G(p)} \hat{p}_g\, e^0_g = \sum_{g \in G \setminus G(p)} \sum_{h \in G} \sum_{i \in N} \hat{p}_h\, e^i_h. \tag{31}$$

Whenever p is not proportional to $\hat{p}$ and so $G \setminus G(p) \neq \emptyset$, (31) implies that

$$\sum_{g \in G(p)} \hat{p}_g\, [x^0_g(p) - e^0_g] > \sum_{g \in G(p)} \hat{p}_g \sum_{i \in N} e^i_g$$

and so $x^0_h(p) - e^0_h > \sum_{i \in N} e^i_h$ for at least one $h \in G(p)$. But $x^i_h(p) \geq 0$ for all $i \in N$, so $\sum_{i \in \tilde{N}} [x^i_h(p) - e^i_h] \geq x^0_h(p) - e^0_h - \sum_{i \in N} e^i_h > 0$ for this $h \in G(p)$. We conclude that the only possible Walrasian equilibrium price vectors in the environment $\tilde{\mathcal{E}}$ must be proportional to $\hat{p}$. Because $(\tilde{x}^{\tilde{N}}, \hat{p})$ is a WE, it is easy to see that any other WE allocation $x^{\tilde{N}}$ must satisfy $x^i\ I^i\ \tilde{x}^i$ for all $i \in N$. But $\emptyset \neq \Phi(\tilde{\mathcal{E}}) \subset \mathrm{WE}(\tilde{\mathcal{E}})$ and the SCR is assumed to satisfy Pareto indifference, so $\tilde{x}^{\tilde{N}} \in \Phi(\tilde{\mathcal{E}})$. Because $\tilde{x}^0 = e^0$, stability under non-essential deletion implies that $\mathcal{E} \in \mathbf{E}$ and that $\hat{x}^N = \tilde{x}^N \in \Phi(\mathcal{E})$. Since this is true for any WE $(\hat{x}^N, \hat{p})$, it follows that $\mathrm{WE}(\mathcal{E}) \subset \Phi(\mathcal{E})$ and so, by the hypotheses of the Lemma, that $\mathrm{WE}(\mathcal{E}) = \Phi(\mathcal{E})$. $\square$

Theorem 43 *Suppose that the SCR Φ is defined on a domain $\mathbf{E}$ with $\mathbf{E}_1 \subset \mathbf{E} \subset \mathbf{E}_2$ and $\Phi(\mathcal{E}) \subset \mathrm{WELT}(\mathcal{E})$ for all $\mathcal{E} \in \mathbf{E}$. Suppose too that Φ satisfies Pareto indifference, generates envy-free net trades, and is stable under both non-essential deletion and non-essential addition. Then Φ is the Walrasian social choice rule — i.e., $\Phi(\mathcal{E}) = \mathrm{WE}(\mathcal{E}) \neq \emptyset$ for all $\mathcal{E} \in \mathbf{E}$, where $\mathbf{E} = \mathbf{E}_1$.*

PROOF. Because Φ generates envy-free net trades and is stable under non-essential addition, Lemma 40 implies that $\Phi(\mathcal{E}) \subset \mathrm{WE}(\mathcal{E})$ for all $\mathcal{E} \in \mathbf{E}$. The result then follows from Lemma 42. $\square$

It is worth noting that the above proofs require only that the two stability properties hold when $\#(\tilde{N} \setminus N) = 1$ — i.e., when only one agent at a time is added or deleted from the economic environment.

8.4 Consistency and converse consistency

An alternative characterization using a variable set of agents is due to van den Nouweland, Peleg and Tijs (1996) — see also Dagan (1995, 1996). Their main results apply to pure exchange economic environments in which agents all have regular smooth preferences as defined in Section 2.5.[26] In particular, each agent's type $\theta^i = (X^i, e^i, R^i)$ is described by a consumption set $X^i = \mathbb{R}^G_+$, an endowment vector $e^i \in \mathbb{R}^G_{++}$, and a smooth preference ordering R^i that is strictly monotone, convex and continuous. But each environment $\mathcal{E} := \langle N, \theta^N, z\rangle$ is a *generalized economy*, with z as an exogenously given aggregate net supply vector. The associated *feasible set* is defined by

$$F(N, \theta^N, z) := \{\, x^N \in (\mathbb{R}^G_+)^N \mid \sum_{i \in N} (x^i - e^i) = z \,\}.$$

Say that $(\hat{x}^N, \hat{p})$ is a WE in the environment $\mathcal{E} = \langle N, \theta^N, z\rangle$ provided that $\hat{x}^N \in F(\mathcal{E})$, and $\hat{x}^i$ is a Walrasian demand at the price vector $\hat{p}$, for each $i \in N$. When preferences are LNS, this implies that $\hat{p}\, z = \sum_{i\in N} \hat{p}\,(\hat{x}^i - \hat{e}^i) = 0$.

[26] This is somewhat imprecise. More exactly, instead of assuming that there is a $\mathcal{C}^1$ utility function, they require only the existence of a unique normalized supporting price vector at each $x \in \mathbb{R}^G_{++}$.

A *solution* or *social choice rule* (SCR) on a *domain* $\mathbf{E}$ of environments $\mathcal{E}$ is a correspondence $\Phi : \mathbf{E} \mapsto\!\!\!\to (\mathbb{R}^G_+)^N$ satisfying $\emptyset \neq \Phi(\mathcal{E}) \subset F(\mathcal{E})$ for all $\mathcal{E} \in \mathbf{E}$.[27]

Given any environment $\mathcal{E} = \langle N, \theta^N, z \rangle$ with $\#N \geq 2$, any proper subset K of N, and any consumption allocation $x^N \in F(N, \theta^N, z)$, define the *reduced environment*

$$\mathcal{E}^K(x^N) := \langle K, \theta^K, z - \sum_{i \in N \setminus K} (x^i - e^i) \rangle.$$

The SCR Φ on the domain $\mathbf{E}$ is said to be *consistent* if, given any $\mathcal{E} = \langle N, \theta^N, z \rangle \in \mathbf{E}$, any proper subset K of N, and any consumption allocation $x^N \in \Phi(\mathcal{E})$, one has $\mathcal{E}^K(x^N) \in \mathbf{E}$ and $x^K \in \Phi(\mathcal{E}^K(x^N))$.

On the other hand, the SCR Φ on the domain $\mathbf{E}$ is said to be *converse consistent* if, given any $\mathcal{E} = \langle N, \theta^N, z \rangle \in \mathbf{E}$, and any consumption allocation x^N which is Pareto efficient in the environment $\mathcal{E}$, then $x^N \in \Phi(\mathcal{E})$ provided that $\mathcal{E}^K(x^N) \in \mathbf{E}$ and $x^K \in \Phi(\mathcal{E}^K(x^N))$ for every proper subset K of N.

Consistency and converse consistency are essentially strengthenings of Thomson's (1988) conditions of stability under non-essential deletion and addition, respectively, as described in Section 8.3. The strengthenings allow the no envy condition of Theorem 43 to be dropped. However, one other condition of "Pareto efficiency in two-agent environments" is still needed in the following characterization result.

Theorem 44 *Suppose that the SCR Φ is defined on the restricted domain $\mathbf{E}$ of all regular smooth generalized economic environments in which a Walrasian equilibrium exists. Then Φ is the Walrasian rule if and only if it satisfies both consistency and converse consistency, and in addition, whenever $\#N = 2$, then every $x^N \in \Phi(\mathcal{E})$ is Pareto efficient relative to the feasible set $F(\mathcal{E})$.*

8.5 Consistency and converse consistency with interactive opportunity sets

Serrano and Volij (1998) provide an alternative characterization using a different notion of reduced environment. Also, instead of considering generalized environments that each include an exogenous net supply vector z, they use the more standard framework of Section 3.1 in which each agent $i \in N$ has a consumption set X^i, a production set Y^i, and a preference ordering R^i.

Another important difference is that the concept of a reduced environment involves a modified version of Serrano and Volij's (2000) multilateral interactive opportunity sets whose construction was discussed in Section 6.3. Specifically,

[27] Van den Nouweland, Peleg and Tijs (1996) allow $\Phi(\mathcal{E})$ to be empty in some generalized economies. It is simpler to exclude this possibility.

the sequences $Z_S^{i,m}(x^N)$ $(m = 0, 1, 2, \ldots)$ are defined for each (non-empty) coalition $S \subset N$, each agent $i \in S$, and each consumption allocation $x^N \in X^N$. The construction starts as before with $Z_S^{i,0}(x^N) := Y^i$ when $m = 0$. Thereafter, given any fixed m and the sets $Z_S^{i,m}(x^N)$ (all $i \in S$), the next set for agent i is

$$Z_S^{i,m+1}(x^N) := Z_S^{i,m}(x^N) - \sum_{h \in S \setminus \{i\}} \Big([R^h(x^h) - Z_S^{h,m}(x^N)] \cup \{0\} \Big) .$$

This is an obvious modification of (26). As before, $Z_S^{i,m}(x^N) \subset Z_S^{i,m+1}(x^N)$ for $m = 0, 1, 2, \ldots$, so the limit set $Z_S^i(x^N) := \cup_{m=0}^{\infty} Z_S^{i,m}(x^N)$ is well defined.

Next, given the environment $\mathcal{E} = \langle N, \theta^N \rangle$, the coalition $K \subset N$, and the consumption allocation $x^N \in X^N$, define the *reduced environment* $\mathcal{E}^K(x^N) := \langle K, \theta^K(x^N) \rangle$ where $\theta^i(x^N) := \langle X^i, Y_K^i(x^N), R^i \rangle$ for all $i \in K$, with $Y_K^i(x^N) := \cup_{S \subset N \setminus K} Z_{\{i\} \cup S}^i(x^N)$. Thus agent i's opportunity set $Y_K^i(x^N)$ when interacting with coalition K in the reduced environment $\mathcal{E}^K(x^N)$ reflects the possibilities for trade outside this coalition, as represented by the sets $Z_{\{i\} \cup S}^i(x^N)$ for $S \subset N \setminus K$. The corresponding definitions of *consistency* and of *converse consistency with interactive opportunity sets* are exactly the same as the definitions in Section 8.4, though they apply to different sets $\mathcal{E}$ and $\mathcal{E}^K(x^N)$.

The following characterization result uses the set A^i of optimal autarky allocations for each agent $i \in N$, as defined in Section 5.1:

Theorem 45 *The SCR Φ is the Walrasian rule if and only if it satisfies both consistency and converse consistency with interactive opportunity sets, and in addition $\Phi(\mathcal{E}) \subset A^i$ whenever $\# N = \{i\}$.*

8.6 Minimal message spaces

Minimal message spaces were briefly mentioned at the end of Section 7.1 for economies with a fixed set of agents but a variable type profile. Earlier, Sonnenschein (1974) used a similar idea, requiring that there should be no redundant messages, in order to characterize the Walrasian mechanism as a rule which selects core allocations while satisfying other axioms. Amongst these the most prominent is the "swamping" axiom S: Given any allowable message and any finite economy, that message is in the equilibrium message set for some larger finite economy that extends the original finite economy. Thus, given any message, the presence of any fixed set of agents in the economy does not preclude that message occurring in equilibrium when a large enough number of other agents are added to the economy.

9 Statistical continuum economies

9.1 Continuum economies

Let Θ denote the (metric) space of agent types, as defined in Section 2.7. In Section 3.1 a *finite economy* was defined implicitly in the obvious way as a mapping $i \mapsto \theta^i$ from the finite set N to Θ. Following Aumann (1964), the standard model of a *continuum economy* involves the set of agents $N = [0, 1]$, and a mapping $i \mapsto \theta^i$ from N to Θ. Not every mapping makes economic sense, however. Instead, it is usual to assume that N is given its *Borel σ-field* — defined as the smallest family $\mathcal{B}$ of subsets of N that includes:

(1) all relatively open sets;
(2) the complement $N \setminus B$ of any Borel set $B \in \mathcal{B}$;
(3) the union $\cup_{n=1}^{\infty} B_n$ of any countable family of Borel sets $B_n \in \mathcal{B}$ $(n = 1, 2, \ldots)$.

We also define *Lebesgue measure* λ on the Borel σ-field $\mathcal{B}$. It is the unique mapping $\lambda : \mathcal{B} \to [0, 1]$ such that $\lambda([a, b]) = b - a$ whenever $[a, b]$ is an interval with $0 \leq a \leq b \leq 1$, and which is *countably additive* in the sense that $\lambda(B) = \sum_{n=1}^{\infty} \lambda(B_n)$ whenever B is the union $\cup_{n=1}^{\infty} B_n$ of the countable family of pairwise disjoint Borel sets $B_n \in \mathcal{B}$ $(n = 1, 2, \ldots)$.

As a metric space, Θ also has a Borel σ-field, which we denote by $\mathcal{F}$. It has then been usual to assume, following Hildenbrand (1974), that the mapping $i \mapsto \theta^i$ from N to Θ is *measurable* — i.e., for each Borel subset $K \in \mathcal{F}$, the set $\theta^{-1}(K) := \{ i \in N \mid \theta^i \in K \}$ should belong to $\mathcal{B}$.

Such measurability is highly restrictive, however. To see why, suppose Θ is an *n-parameter domain* of agent types for which there is a homeomorphism between Θ and a subset of $\mathbb{R}^n$. Then any measurable mapping $i \mapsto \theta^i$ from N to Θ must be "nearly continuous" — specifically, given any $\epsilon > 0$, there exists a Borel measurable set $K_\epsilon \subset \Theta$ with $\lambda(K_\epsilon) > 1 - \epsilon$ and a continuous mapping $\theta_\epsilon : K_\epsilon \to \Theta$ such that $\theta_\epsilon^i = \theta^i$ for all $i \in K_\epsilon$. Indeed, this is a direct application of Lusin's Theorem, a well-known result in measure theory.[28]

[28] See Aliprantis and Border (1999), who show that the same property would hold whenever the metric space Θ is second countable — or *a fortiori*, separable.

9.2 *Statistical economies*

Usually all that matters about agents' types θ^i $(i \in N)$ is their *distribution*. This is represented by a (probability) measure μ on Θ, with $\mu(K) \geq 0$ as the proportion of agents having $\theta \in K$, for each Borel set K. This measure must satisfy the standard conditions that $\mu(\Theta) = 1$, and also $\mu(\cup_{n=1}^{\infty} K_n) = \sum_{n=1}^{\infty} \mu(K_n)$ whenever the sets K_n $(n = 1, 2, \ldots)$ are pairwise disjoint.

9.3 *Statistical continuum economies*

Although the distribution μ on Θ captures most important features of agents' types, it need not represent the fact that there are many agents. For example, if there happen to be two types θ' and θ'' such that the distribution satisfies $\mu(\{\theta'\}) = \mu(\{\theta'\}) = \frac{1}{2}$, this could be because there are only two agents, or because there is a continuum of agents of whom exactly half have each of these two types.

Also, in some contexts it is important to allow asymmetric allocations in which some agents of the same type receive different net trade vectors. Indeed, in non-convex environments, this may be essential in any Walrasian equilibrium. Obviously, this requires a mathematical framework rich enough to allow the net trade vector t^i of each agent $i \in N$ to depend not only on i's type θ^i, but possibly also directly on i.

For these reasons, the formulation used here involves the entire Cartesian product $N \times \Theta$ of label–type pairs or "potential" agents (i, θ). Then N is the set of actual agents i who each have a type θ^i that may vary. We do not assume that the mapping $i \mapsto \theta^i$ is measurable. Instead, we assume that the economy is described by the joint distribution of pairs (i, θ). This involves considering the *product σ-field* $\mathcal{B} \otimes \mathcal{F}$, defined as the smallest σ-field that contains all *measurable rectangles* of the form $B \times F$ with $B \in \mathcal{B}$ and $F \in \mathcal{F}$. Then the *joint distribution* of (i, θ) is a probability measure ν defined on the measurable sets in $\mathcal{B} \otimes \mathcal{F}$.

An important restriction on the probability measure ν which describes the statistical continuum economy is that it should *conform* with λ on $\mathcal{B}$, meaning that $\nu([a, b] \times \Theta) = \lambda([a, b]) = b - a$ whenever $0 \leq a \leq b \leq 1$. More generally, the marginal measure $\text{marg}_N \nu$ of ν on N should be the Lebesgue measure λ — i.e.,

$$\nu(B \times \Theta) = \lambda(B) \text{ whenever } B \in \mathcal{B}. \tag{32}$$

Let $\mathcal{M}_\lambda(N \times \Theta)$ denote the set of probability measures on the product σ-field $\mathcal{B} \otimes \mathcal{F}$ of $N \times \Theta$ that satisfy (32).

It is worth noting that each continuum economy, described by a measurable mapping $i \mapsto \theta^i$, is included as a special case. Indeed, such a continuum economy is fully described by the measure ν restricted to the graph

$$\Gamma := \{ (i, \theta) \in N \times \Theta \mid \theta = \theta^i \}$$

of the measurable mapping $i \mapsto \theta^i$, with $\nu((B \times \Theta) \cap \Gamma) = \lambda(B)$ for all $B \in \mathcal{B}$.

The joint measure ν can be interpreted as a theoretical limiting distribution when n pairs (i, θ) are randomly drawn from $N \times \Theta$, with:

- identifiers i drawn as i.i.d. random variables from the uniform distribution described by Lebesgue measure λ on $N = [0, 1]$;
- the type θ of each agent $i \in N$ drawn from that agent's conditional distribution $\nu(\cdot|i)$ on Θ.

Then the pairs (i, θ) are mutually independent with identical distribution $\nu \in \mathcal{M}_\lambda(N \times \Theta)$. As $n \to \infty$, the law of large numbers implies that the empirical distributions of i and of (i, θ) converge almost surely to λ and ν, respectively.

9.4 Allocation mechanisms

An *allocation mechanism* given ν is a mapping $(i, \theta) \mapsto t^i_\theta$ from $N \times \Theta$ to $\mathbb{R}^G$ which is *measurable* in the sense that the set $\{ (i, \theta) \in N \times \Theta \mid t^i_\theta \in S \}$ belongs to $\mathcal{B} \otimes \mathcal{F}$ for every Borel set S of the Euclidean space $\mathbb{R}^G$, and also *feasible* in the sense that $t^i_\theta \in T_\theta$ for ν-a.e. $(i, \theta) \in N \times \Theta$, while the *mean net trade vector* defined by the integral $\int_{N \times \Theta} t^i_\theta \, \mathrm{d}\nu$ is equal to 0. Such an allocation mechanism will typically be denoted by $t^{N \times \Theta}$.

It is obviously natural to consider such allocation mechanisms in a statistical economy described by a distribution ν over $N \times \Theta$. But even when the economy is described by a measurable mapping $i \mapsto \theta^i$ from $N = [0, 1]$ to Θ, some characterizations of Walrasian equilibrium require specifying what net trade vector t^i_θ agent i would receive in the counterfactual event that i's type were to change to some arbitrary type $\theta \neq \theta^i$. In particular, this is important when considering strategyproofness.

9.5 Pareto efficiency

Say that the particular allocation mechanism $\hat{t}^{N \times \Theta}$ is *weakly Pareto efficient* if there is no alternative allocation mechanism $t^{N \times \Theta}$ with

$$\nu \left(\{ (i, \theta) \in N \times \Theta \mid t^i_\theta \in P_\theta(\hat{t}^i_\theta) \} \right) = 1.$$

Say that the allocation mechanism $\hat{t}^{N\times\Theta}$ is *Pareto efficient* if there is no alternative allocation mechanism $t^{N\times\Theta}$ with

$$\begin{aligned} &\nu\left(\{\,(i,\theta)\in N\times\Theta \mid t^i_\theta \in R_\theta(\hat{t}^i_\theta)\,\}\right) = 1, \\ \text{while}\quad &\nu\left(\{\,(i,\theta)\in N\times\Theta \mid t^i_\theta \in P_\theta(\hat{t}^i_\theta)\,\}\right) > 0. \end{aligned}$$

9.6 Walrasian and compensated equilibria

Without any lump-sum wealth redistribution, an agent with type θ has a corresponding *Walrasian budget set* at a given price vector $p \in \mathbb{R}^G \setminus \{0\}$ defined by

$$B_\theta(p) := \{\, t \in T_\theta \mid p\,t \leq 0\,\},$$

as well as a *Walrasian demand set* defined by

$$\xi_\theta(p) := \{\, t \in B_\theta(p) \mid t' \in P_\theta(t) \implies p\,t' > 0\,\}.$$

The corresponding *compensated demand set*, on the other hand, is defined by

$$\xi^C_\theta(p) := \{t \in B_\theta(p) \mid t' \in R_\theta(t) \implies p\,t' \geq 0\}.$$

Suppose that $\hat{t}^{N\times\Theta}$ is a feasible allocation mechanism, and $\hat{p} \neq 0$ a price vector. Then the pair $(\hat{t}^{N\times\Theta}, \hat{p})$ is a Walrasian equilibrium (or WE) if $\hat{t}^i_\theta \in \xi_\theta(\hat{p})$ for ν-a.e. pair $(i,\theta) \in N\times\Theta$. The same pair is a compensated equilibrium (or CE) if $\hat{t}^i_\theta \in \xi^C_\theta(\hat{p})$ for ν-a.e. pair $(i,\theta) \in N\times\Theta$.

9.7 Lump-sum wealth redistribution

Wealth distribution rules for finite economies, along with associated Walrasian equilibria, were defined in Section 3.3. In the statistical continuum economy described by the distribution ν on $N\times\Theta$, a *wealth distribution rule* $w^{N\times\Theta}(p)$ is a real-valued function $(i,\theta,p) \mapsto w^i_\theta(p)$ defined on $N\times\Theta\times(\mathbb{R}^G\setminus\{0\})$ that is measurable w.r.t. (i,θ), continuous and homogeneous of degree one w.r.t. p, while satisfying $\int_{N\times\Theta} w^i_\theta(p)\,\mathrm{d}\nu = 0$ for each $p \neq 0$. With this as the wealth distribution rule, the definitions in Section 9.6 of Walrasian budget set, Walrasian demand set, and compensated demand set change in the obvious way to become

$$\begin{aligned} B^i_\theta(p) &:= \{t \in T_\theta \mid p\,t \leq w^i_\theta(p)\} \\ \xi^i_\theta(p) &:= \{t \in B^i_\theta(p) \mid t' \in P_\theta(t) \implies p\,t' > w^i_\theta(p)\} \\ \xi^{iC}_\theta(p) &:= \{t \in B^i_\theta(p) \mid t' \in R_\theta(t) \implies p\,t' \geq w^i_\theta(p)\} \end{aligned}$$

respectively. These three sets depend on i as well as θ only because $w^i_\theta(p)$ is allowed to depend on i.

Relative to the wealth distribution rule $w^{N\times\Theta}(p)$, the pair $(\hat{t}^{N\times\Theta}, \hat{p})$ consisting of the feasible allocation mechanism $\hat{t}^{N\times\Theta}$ and the price vector $\hat{p} \neq 0$ is a WELT (respectively, CELT) if $\hat{t}^i_\theta \in \xi^i_\theta(\hat{p})$ (respectively, $\hat{t}^i_\theta \in \xi^{iC}_\theta(\hat{p})$) for ν-a.e. pair $(i, \theta) \in N \times \Theta$.

9.8 Walrasian equilibrium with equal budgets

The analysis of Section 3.6 is easily extended to the present continuum economy setting. Thus, results which characterize WE can be used to characterize WEEB instead by applying them to equivalent equal opportunity economies in which $T_\theta = X_\theta - Y$ for each type $\theta \in \Theta$, where X_θ is the consumption set, and Y is the common opportunity set.

10 Efficiency theorems in continuum economies

10.1 First efficiency theorem

The first efficiency theorem is a routine extension of Theorem 15.

Theorem 46 *Any WELT is weakly Pareto efficient, and is Pareto efficient if all agents have LNS preferences.*

PROOF. Let $(\hat{t}^{N\times\Theta}, \hat{p})$ be any WELT. Suppose that the mapping $(i, \theta) \mapsto t^i_\theta$ is measurable w.r.t. the product σ-field on $N \times \Theta$, while satisfying $t^i_\theta \in P_\theta(\hat{t}^i_\theta)$ for ν-a.e. $(i, \theta) \in N \times \Theta$. Because $\hat{t}^i_\theta \in \xi^i_\theta(\hat{p})$ for ν-a.e. (i, θ), it follows that $\hat{p}\, t^i_\theta > w^i_\theta(\hat{p})$ for ν-a.e. (i, θ). Integrating this inequality over $N \times \Theta$ gives

$$\int_{N\times\Theta} \hat{p}\, t^i_\theta \,\mathrm{d}\nu > \int_{N\times\Theta} w^i_\theta(p) \,\mathrm{d}\nu = 0. \tag{33}$$

This implies that $\int_{N\times\Theta} t^i_\theta \,\mathrm{d}\nu \neq 0$, so the mapping $(i, \theta) \mapsto t^i_\theta$ does not give a feasible allocation. Conversely, no feasible allocation $t^{N\times\Theta}$ can satisfy $t^i_\theta \in P_\theta(\hat{t}^i_\theta)$ for ν-a.e. $(i, \theta) \in N \times \Theta$.

In the case when $\succsim_\theta$ is LNS for ν-a.e. (i, θ), Lemma 9 implies that the WELT is a CELT. So if the mapping $(i, \theta) \mapsto t^i_\theta$ is measurable w.r.t. the product σ-field on $N \times \Theta$, while satisfying $t^i_\theta \in R_\theta(\hat{t}^i_\theta)$ for ν-a.e. $(i, \theta) \in N \times \Theta$, then

$\hat{p}\, t^i_\theta \geq w^i_\theta(\hat{p})$ for ν-a.e. (i, θ). And if $t^i_\theta \in P_\theta(\hat{t}^i_\theta)$ for a non-null set of pairs (i, θ), then $\hat{p}\, t^i_\theta > w^i_\theta(\hat{p})$ in that set, implying that (33) holds as before. □

10.2 Second efficiency theorem

The following result is the obvious counterpart of Theorem 16, bearing in mind that a continuum of (potential) agents allows one to relax the assumption that preferences are convex. On the other hand, we assume preferences are continuous in order to allow the measure ν to be constructed on $N \times \Theta$. The proof is an obvious adaptation of that found in Hildenbrand (1974, p. 232).

Theorem 47 *Suppose that agents' preferences are LNS and continuous. Then any weakly Pareto efficient allocation is a CELT.*

PROOF. Suppose that the feasible allocation $\hat{t}^{N \times \Theta}$ is weakly Pareto efficient. Recall from Section 2.7 that each agent's type θ is identified with the closed graph of the preference ordering $\succsim_\theta$. Together with the measurability of the mapping $(i, \theta) \mapsto \hat{t}^i_\theta$, this ensures that the correspondence $(i, \theta) \mapsto\!\!\!\to P_\theta(\hat{t}^i_\theta)$ has a measurable graph. Define the set

$$Z := \int_{N \times \Theta} P_\theta(\hat{t}^i_\theta)\; \mathrm{d}\nu \tag{34}$$

of all possible integrals of measurable selections $(i, \theta) \mapsto t^i_\theta \in P_\theta(\hat{t}^i_\theta)$. Because preferences are LNS, one has $\hat{t}^i_\theta \in \operatorname{cl} P_\theta(\hat{t}^i_\theta)$ for all $(i, \theta) \in N \times \Theta$. But $\int_{N \times \Theta} \hat{t}^i_\theta \, \mathrm{d}\nu = 0$, so there must exist a measurable and bounded function $(i, \theta) \mapsto a^i_\theta$ such that $\hat{t}^i_\theta + a^i_\theta \in P_\theta(\hat{t}^i_\theta)$ for ν-a.e. $(i, \theta) \in N \times \Theta$. Then $(i, \theta) \mapsto \hat{t}^i_\theta + a^i_\theta$ is an integrable selection from the correspondence $(i, \theta) \mapsto\!\!\!\to P_\theta(\hat{t}^i_\theta)$. This implies that $\int_{N \times \Theta} (\hat{t}^i_\theta + a^i_\theta)\, \mathrm{d}\nu \in Z$, so Z is certainly non-empty. Because the measure ν is non-atomic, it is well known that Z must be convex — see, for example, Hildenbrand (1974, p. 62, Theorem 3).

Weak Pareto efficiency of $\hat{t}^{N \times \Theta}$ implies that $0 \notin Z$. By the separating hyperplane theorem, there exists $\hat{p} \neq 0$ such that

$$0 \leq \inf \hat{p}\, Z = \int_{N \times \Theta} \inf \hat{p}\, P_\theta(\hat{t}^i_\theta)\; \mathrm{d}\nu, \tag{35}$$

where the equality is implied by (34) and Hildenbrand (1974, p. 63, Prop. 6). But $\hat{t}^i_\theta \in \operatorname{cl} P_\theta(\hat{t}^i_\theta)$ because preferences are LNS, so

$$\inf \hat{p}\, P_\theta(\hat{t}^i_\theta) \leq \hat{p}\, \hat{t}^i_\theta \text{ for all } (i, \theta) \in N \times \Theta. \tag{36}$$

Because $0 = \int_{N\times\Theta} \hat{t}^i_\theta \, \mathrm{d}\nu$, it follows from (35) and (36) that

$$0 \leq \int_{N\times\Theta} \inf \hat{p}\, P_\theta(\hat{t}^i_\theta) \, \mathrm{d}\nu \leq \int_{N\times\Theta} \hat{p}\, \hat{t}^i_\theta \, \mathrm{d}\nu = 0. \tag{37}$$

Together (36) and (37) imply that $\inf \hat{p}\, P_\theta(\hat{t}^i_\theta) = \hat{p}\, \hat{t}^i_\theta$ for ν-a.e. $(i,\theta) \in N \times \Theta$. In particular, because preferences are LNS, for ν-a.e. (i,θ) one has $\hat{p}\, t \geq \hat{p}\, \hat{t}^i_\theta$ whenever $t \in R_\theta(\hat{t}^i_\theta)$. This implies that $(\hat{t}^{N\times\Theta}, \hat{p})$ is a CELT relative to the specific wealth distribution rule $w^{N\times\Theta}(p)$ defined by $w^i_\theta(p) := p\, \hat{t}^i_\theta$ for all $i \in N$, $\theta \in \Theta$ and $p \in \mathbb{R}^G \setminus \{0\}$. □

10.3 Non-oligarchic allocations

Convex preferences were not needed to prove Theorem 47. Convex feasible sets, however, are assumed in the following argument concerning sufficient conditions for a CELT to be a WELT. When each T_θ is convex, the definition and results of Section 4.4 are fairly easily extended to continuum economies. Specifically, given the weakly Pareto efficient allocation $\hat{t}^{N\times\Theta}$, the set $K \subset N \times \Theta$ with $0 < \nu(K) < 1$ is said to be an *oligarchy* if there is no alternative feasible allocation $t^{N\times\Theta}$ satisfying $t^i_\theta \succ_\theta \hat{t}^i_\theta$ for ν-a.e. $(i,\theta) \in K$.

Theorem 48 *Assume agents' preferences are LNS and continuous, and each feasible set T_θ is convex. Assume too that $0 \in \operatorname{int} \int_{N\times\Theta} T_\theta \, \mathrm{d}\nu$. Then any non-oligarchic weakly Pareto efficient (NOWPE) allocation is a WELT.*

PROOF. Let $\hat{t}^{N\times\Theta}$ be any weakly Pareto efficient allocation. By Theorem 47, this allocation is a CELT at some price vector $\hat{p} \neq 0$. Let K be the set of all potential agents $(i,\theta) \in N \times \Theta$ with cheaper points $\underline{t}^i_\theta \in T_\theta$ satisfying $\hat{p}\, \underline{t}^i_\theta < \hat{p}\, \hat{t}^i_\theta$. By the definitions of a statistical continuum economy and of an allocation mechanism, the set K is measurable. Because $0 \in \operatorname{int} \int_{N\times\Theta} T_\theta \, \mathrm{d}\nu$, there exists an integrable mapping $(i,\theta) \mapsto \bar{t}^i_\theta \in T_\theta$ on $N \times \Theta$ such that $\hat{p} \int_{N\times\Theta} \bar{t}^i_\theta \, \mathrm{d}\nu < 0$. Because $\hat{p} \int_{N\times\Theta} \hat{t}^i_\theta \, \mathrm{d}\nu = 0$, there must exist a non-null measurable set $H \subset N \times \Theta$ such that $\hat{p}\, \bar{t}^i_\theta < \hat{p}\, \hat{t}^i_\theta$ for all $(i,\theta) \in H$. So $H \subset K$, implying that K is non-null.

Consider any measurable mapping $(i,\theta) \mapsto t^i_\theta$ satisfying both $t^i_\theta \in T_\theta$ for ν-a.e. $(i,\theta) \in N \times \Theta$ and $t^i_\theta \succ_\theta \hat{t}^i_\theta$ for ν-a.e. $(i,\theta) \in K$. Because each agent $i \in K$ has a cheaper point, Lemma 12 implies that $\hat{p}\, t^i_\theta > \hat{p}\, \hat{t}^i_\theta$ for ν-a.e. $(i,\theta) \in K$. But no agent outside K has a cheaper point, so $\hat{p}\, t^i_\theta \geq \hat{p}\, \hat{t}^i_\theta$ for ν-a.e. $(i,\theta) \in (N\times\Theta)\setminus K$. Because K must be non-null, it follows that

$$\hat{p} \int_{N\times\Theta} t^i_\theta \, \mathrm{d}\nu > \hat{p} \int_{N\times\Theta} \hat{t}^i_\theta \, \mathrm{d}\nu = 0.$$

Hence $\int_{N\times\Theta} t^i_\theta \, d\nu \neq 0$, so $t^{N\times\Theta}$ cannot be a feasible allocation. Except when $\nu(K) = 1$, it follows that $\hat{t}^{N\times\Theta}$ is oligarchic, with K as an oligarchy.

Conversely, if $\hat{t}^{N\times\Theta}$ is a NOWPE allocation, then $\nu(K) = 1$, so almost all agents have cheaper points. So the CELT $(\hat{t}^{N\times\Theta}, \hat{p})$ is actually a WELT. □

10.4 Individual non-convexities

When the individual feasible sets T_θ are non-convex for a non-null set of potential agents (i, θ), the cheaper point Lemma 12 cannot be applied, and the conclusion of Theorem 48 may not hold. Examples illustrating this possibility can be found in Dasgupta and Ray (1986) as well as Coles and Hammond (1995). Additional assumptions will be presented here which do guarantee that a weakly Pareto efficient allocation is a WELT.

Any individual feasible set T_θ is said to be *piecewise convex* if there exists a countable (i.e., finite or countably infinite) collection T^m_θ $(m \in M_\theta \subset \mathbb{N})$ of closed convex sets such that $T_\theta = \cup_{m\in M_\theta} T^m_\theta$. That is, even if T_θ is not convex, it must be the union of a countable collection of *convex components* or "pieces". These components may be disjoint, but they may also intersect.

Piecewise convexity excludes some non-convex feasible sets such as

$$\{\, t \in \mathbb{R}^2 \mid t \geqq (-1,-1),\ (t_1+1)^2 + (t_2+1)^2 \geq 1 \,\}.$$

But it allows many forms of setup cost. It also allows indivisible goods, which are consistent with a feasible set such as $\{\, t \in \mathbb{R}^D \times \mathbb{Z}^H \mid t \geqq \underline{t} \,\}$ for some fixed lower bound $\underline{t} \in \mathbb{R}^D \times \mathbb{Z}^H$, where D is the set of divisible goods, and $H := G \setminus D$ is the set of indivisible goods, with $\mathbb{Z}$ denoting the set of integers.

Given any agent type $\theta \in \Theta$ and any net trade vector $t \in T_\theta$, define the set

$$M_\theta(t) := \{\, m \in M_\theta \mid P_\theta(t) \cap T^m_\theta \neq \emptyset \,\} \tag{38}$$

of natural numbers $m \in \mathbb{N}$ which index those convex components T^m_θ that intersect the strict preference set $P_\theta(t)$. Given a price vector $p \neq 0$, let

$$W_\theta(p,t) := \{\, \inf p\, T^m_\theta \mid m \in M_\theta(t) \,\} \tag{39}$$

be the associated countable set of *critical* wealth levels at which a type θ agent can just afford to reach one of the convex components T^m_θ $(m \in M_\theta(t))$. Extending the second efficiency theorem to allow individual non-convexities will rely on the following generalization of the cheaper point Lemma 12.

Lemma 49 *Let $(i,\theta) \in N \times \Theta$ be any potential agent with a piecewise convex feasible set T_θ and continuous preferences. Suppose that the net trade vector*

$\hat{t}^i_\theta$ is any CELT for (i,θ) at the price vector $p \neq 0$ — i.e., suppose

$$t \in R_\theta(\hat{t}^i_\theta) \Longrightarrow p\,t \geq p\,\hat{t}^i_\theta. \tag{40}$$

Suppose too that $p\,\hat{t}^i_\theta \notin W_\theta(p,\hat{t}^i_\theta)$. Then $\hat{t}^i_\theta$ is a WELT for (i,θ) — i.e.,

$$t \in P_\theta(\hat{t}^i_\theta) \Longrightarrow p\,t > p\,\hat{t}^i_\theta.$$

PROOF. Consider any $t \in P_\theta(\hat{t}^i_\theta)$. Piecewise convexity of T_θ implies that $t \in T^m_\theta$ for some $m \in M_\theta(\hat{t}^i_\theta)$. The hypothesis $p\,\hat{t}^i_\theta \notin W_\theta(p,\hat{t}^i_\theta)$ implies that $p\,\hat{t}^i_\theta \neq \inf p\,T^m_\theta$. One possibility is that $p\,\hat{t}^i_\theta < \inf p\,T^m_\theta$, in which case $p\,t > p\,\hat{t}^i_\theta$, as required.

Alternatively $p\,\hat{t}^i_\theta > \inf p\,T^m_\theta$, and so there exists $\underline{t} \in T^m_\theta$ such that $p\,\underline{t} < p\,\hat{t}^i_\theta$. In this case, define $\tilde{t} := (1-\alpha)t + \alpha\,\underline{t}$ where $0 < \alpha < 1$. Because T^m_θ is convex, $t \in P_\theta(\hat{t}^i_\theta)$, and preferences are continuous, one has $\tilde{t} \in R_\theta(\hat{t}^i_\theta)$ for a suitably small α. So (40) implies that $p\,\tilde{t} \geq p\,\hat{t}^i_\theta$. But $p\,\underline{t} < p\,\hat{t}^i_\theta$ and $0 < \alpha < 1$, so

$$p\,t = \frac{1}{1-\alpha}(p\,\tilde{t} - \alpha\,p\,\underline{t}) > \frac{1}{1-\alpha}(p\,\hat{t}^i_\theta - \alpha\,p\,\hat{t}^i_\theta) = p\,\hat{t}^i_\theta.$$

Because this argument works for all $t \in P_\theta(\hat{t}^i_\theta)$, the result follows. $\square$

To reduce notation, let $W^*_\theta(p,\hat{t}^i_\theta)$ denote the set $W_\theta(p,\hat{t}^i_\theta) \setminus \{\inf p\,T_\theta\}$. Then, to elaborate the basic idea used by Mas-Colell (1977), Yamazaki (1978, 1981) and Coles and Hammond (1995), say that the CELT $(\hat{t}^{N\times\Theta}, p)$ is *dispersed* if

$$\nu(\{\,(i,\theta) \in N \times \Theta \mid p\,\hat{t}^i_\theta \in W^*_\theta(p,\hat{t}^i_\theta)\,\}) = 0. \tag{41}$$

In other words, the set of potential agents who have both cheaper points and critical wealth levels should be of measure zero.

Theorem 50 *Suppose each agent's feasible set T_θ is piecewise convex, with $0 \in \operatorname{int} \int_{N\times\Theta} T_\theta \,\mathrm{d}\nu$. Suppose too that preferences are LNS and continuous. Then any CELT which is non-oligarchic and dispersed must be a WELT.*

PROOF. Let $(\hat{t}^{N\times\Theta}, p)$ be a CELT. As in the proof of Theorem 48, let K be the set of all $(i,\theta) \in N \times \Theta$ for which there is a cheaper point $\underline{t}^i_\theta \in T_\theta$ satisfying $p\,\underline{t}^i_\theta < p\,\hat{t}^i_\theta$. Because $0 \in \operatorname{int} \int_{N\times\Theta} T_\theta \,\mathrm{d}\nu$, the set K must be non-null.

Consider any measurable mapping $(i,\theta) \mapsto t^i_\theta$ satisfying both $t^i_\theta \in T_\theta$ for ν-a.e. $(i,\theta) \in N \times \Theta$ and $t^i_\theta \succ_\theta \hat{t}^i_\theta$ for ν-a.e. $(i,\theta) \in K$. Because the CELT is assumed to be dispersed and each agent $(i,\theta) \in K$ has a cheaper point, for ν-a.e. $(i,\theta) \in K$ one has $p\,\hat{t}^i_\theta \notin W_\theta(p,\hat{t}^i_\theta)$ and so, by Lemma 49, $\hat{p}\,t^i_\theta > \hat{p}\,\hat{t}^i_\theta$. But no agent outside K has a cheaper point, so $\hat{p}\, t^i_\theta \geq \hat{p}\, \hat{t}^i_\theta$ for ν-a.e. $(i,\theta) \in (N\times\Theta)\backslash K$. Because K must be non-null, it follows that

$$\hat{p} \int_{N\times\Theta} t^i_\theta \, \mathrm{d}\nu > \hat{p} \int_{N\times\Theta} \hat{t}^i_\theta \, \mathrm{d}\nu = 0.$$

Hence, $t^{N\times\Theta}$ cannot be a feasible allocation. Unless $\nu(K) = 1$, it follows that K is an oligarchy.

Conversely, if $\hat{t}^{N\times\Theta}$ is non-oligarchic, then $\nu(K) = 1$, implying that almost all agents have cheaper points. Then dispersion implies $p\, \hat{t}^i_\theta \notin W_\theta(p, \hat{t}^i_\theta)$ for ν-a.e. $(i,\theta) \in N \times \Theta$. By Lemma 49 holds, the CELT is actually a WELT. □

A sufficient condition for (41) to hold is that $p, \hat{t}^i_\theta \notin W^*_\theta := \cup_{m\in M_\theta}\{\inf p\, T^m_\theta\} \setminus \{\inf p\, T_\theta\}$ for ν-a.e. $(i,\theta) \in N \times \Theta$. Because W^*_θ is a countable subset of $\mathbb{R}$, this may not be very restrictive. It is not restrictive at all when each T_θ is convex, in which case we can regard each T_θ as consisting of a single convex component, implying that W^*_θ is empty.

11 Statistical continuum economies: existence theorems

11.1 Integrably bounded gains from trade sets

The compensated equilibrium existence theorem for finite economies presented in Section 5.3 does not extend immediately to a continuum economy. With infinitely many agents, the set of feasible allocations and the gains from trade set are both subsets of an infinite-dimensional space, which creates technical difficulties. In particular, the collective gains from trade set defined in Section 5.3 is unlikely to be bounded because, as the number of agents tends to infinity, one or more of those agents may be able to extract unbounded quantities of some goods from other agents even if all of them remain no worse off than under autarky. Even if one imposes "equal treatment", requiring all agents of the same type to receive an identical allocation, the collective gains from trade set becomes unbounded as the number of different agent types tends to infinity.

To overcome these obstacles to existence, we restrict attention to agent types $\theta \in \Theta$ for which the feasible sets of net trade vectors T_θ allow autarky. We also assume that the weak gains from trade sets $R_\theta(0)$ are collectively *integrably bounded below* — i.e., there exist lower bounds $\underline{t}_\theta$ (all $\theta \in \Theta$) such that $t \geqq \underline{t}_\theta$ for all $t \in R_\theta(0)$; moreover each lower bound $\underline{t}_\theta \leqq 0$ can be selected so that the mapping $\theta \mapsto \underline{t}_\theta$ is ν-integrable (meaning that $\int_{N\times\Theta} |\underline{t}_{\theta g}| \, \mathrm{d}\nu < \infty$ for each $g \in G$), with $\int_{N\times\Theta} \underline{t}_\theta \,\mathrm{d}\nu \ll 0$. In Section 11.5 it is also assumed that preferences are weakly monotone, and in Section 11.6, that individual agents' feasible sets are convex. Convexity of preferences, however, is not required.

Eventually, Sections 11.7 and 11.8 even relax convexity of the feasible sets of net trades in order to allow indivisible goods. The appropriately modified existence proof uses dispersion ideas similar to those of Section 10.4.

11.2 Continuity of the budget and demand correspondences

The existence proofs in this Section will rely on important continuity properties of the compensated demand correspondence when the space of agents' continuous types Θ is given the topology described in Section 2.7.

The first result concerns variations in θ when p is fixed.

Lemma 51 *For each fixed $p \neq 0$, the correspondence $\theta \mapsto\!\!\!\to \xi^C_\theta(p)$ has a closed graph in $\Theta \times \mathbb{R}^G$.*

PROOF. Let $(\theta_n, t_n)_{n=1}^\infty$ be any sequence of points which belong to the graph of $\theta \mapsto\!\!\!\to \xi^C_\theta(p)$ because $t_n \in \xi^C_{\theta_n}(p)$ for $n = 1, 2, \ldots$. Suppose too that the sequence converges to $(\bar\theta, \bar t)$ as $n \to \infty$. Because $t_n \in T_{\theta_n}$ for each n, it follows from part (1) of Lemma 5 that $\bar t \in T_{\bar\theta}$. Also $p\,t_n \leq 0$ for each n. Taking the limit as $n \to \infty$ gives $p\,\bar t \leq 0$, so $\bar t \in B_{\bar\theta}(p)$.

Suppose $\tilde t \in P_{\bar\theta}(\bar t)$. Parts (2) and (3) of Lemma 5 imply that there exists a sequence $(\tilde t_n)_{n=1}^\infty$ converging to $\tilde t$ whose terms satisfy $\tilde t_n \in P_{\theta_n}(t_n)$ for all large n. Because $t_n \in \xi^C_{\theta_n}(p)$ for each n, it follows that $p\,\tilde t_n \geq 0$ for all large n, and so $p\,\tilde t \geq 0$ in the limit as $n \to \infty$.

Finally, because agents' preferences are LNS and transitive, any $t' \in R_{\bar\theta}(\bar t)$ is the limit of a sequence $(t'_n)_{n=1}^\infty$ in $P_{\bar\theta}(\bar t)$ that converges to t'. The previous paragraph showed that $p\,t'_n \geq 0$ for $n = 1, 2, \ldots$, so $p\,t' \geq 0$ in the limit. Hence $\bar t \in \xi^C_{\bar\theta}(p)$, thus confirming that the graph of $\theta \mapsto\!\!\!\to \xi^C_\theta(p)$ is closed. □

The second result concerns variations in p when θ is fixed.

Lemma 52 *For each fixed θ, the correspondence $p \mapsto\!\!\!\to \xi^C_\theta(p)$ has a relatively closed graph in $(\mathbb{R}^G \setminus \{0\}) \times \mathbb{R}^G$.*

PROOF. Suppose that $(p_n, t_n)_{n=1}^\infty$ is any sequence of points which are in the graph of $p \mapsto\!\!\!\to B_\theta(p)$ because $p_n \neq 0$ and $t_n \in B_\theta(p_n)$ for $n = 1, 2, \ldots$. Suppose too that the sequence converges to $(\bar{p}, \bar{t})$ as $n \to \infty$, where $\bar{p} \neq 0$. Then $\bar{t} \in T_\theta$ because T_θ is closed. Also $p_n t_n \leq 0$ for each n, so taking the limit as $n \to \infty$ implies that $\bar{p}\,\bar{t} \leq 0$. This confirms that $\bar{t} \in B_\theta(\bar{p})$.

Suppose in addition that each point of the convergent sequence $(p_n, t_n)_{n=1}^\infty$ is in the graph of $p \mapsto\!\!\!\rightarrow \xi_\theta^C(p)$. Consider any $t' \in P_\theta(\bar{t})$. Because preferences are continuous and $t_n \to \bar{t}$ as $n \to \infty$, it follows that $t' \succ_\theta t_n$ for large enough n. But each $t_n \in \xi_\theta^C(p_n)$, so $p_n t' \geq 0$ for all large n. Taking the limit as $n \to \infty$ implies that $\bar{p}\, t' \geq 0$. Because preferences are LNS, the same must be true whenever $t' \in R_\theta(\bar{t})$. This proves that $\bar{t} \in \xi_\theta^C(\bar{p})$, thus confirming that the graph of $p \mapsto\!\!\!\rightarrow \xi_\theta^C(p)$ is closed. □

One reason for using the compensated demand correspondence $\xi_\theta^C(\cdot)$ is precisely that the usual Walrasian demand correspondence $\xi_\theta(\cdot)$ may not have a closed graph near any price vector p at which there is no cheaper point $t \in T_\theta$ satisfying $p\,t < 0$.

11.3 Integrably bounded restricted budget and demand correspondences

Any Walrasian net trade vector $\hat{t}_\theta^i \in \xi_\theta(p)$ satisfies $0 \notin P_\theta(\hat{t}_\theta^i)$ and so, because preferences are complete, $\hat{t}_\theta^i \in R_\theta(0)$. Consequently, any WE $(\hat{t}^{N\times\Theta}, \hat{p})$ satisfies weak gains from trade. A CE need not, however, unless each $\xi_\theta^C(p)$ is replaced by the *restricted compensated demand set*

$$\bar{\xi}_\theta^C(p) := \xi_\theta^C(p) \cap R_\theta(0) = \{\, t \in \bar{B}_\theta(p) \mid t' \in R_\theta(t) \Longrightarrow p\,t' \geq 0 \,\},$$

where $\bar{B}_\theta(p) := B_\theta(p) \cap R_\theta(0) = \{\, t \in R_\theta(0) \mid p\,t \leq 0 \,\}$ is the *restricted budget set*. The following proofs demonstrate existence of a *restricted* CE $(\hat{t}^{N\times\Theta}, \hat{p})$ satisfying weak gains from trade because $\hat{t}_\theta^i \in \bar{\xi}_\theta^C(\hat{p})$ for ν-a.e. $(i,\theta) \in N \times \Theta$.

Say that a correspondence $F : N \times \Theta \twoheadrightarrow \mathbb{R}^G$ is *integrably bounded* if there exist integrable functions $(i,\theta) \mapsto a_\theta^i$ and $(i,\theta) \mapsto b_\theta^i$ such that, for ν-a.e. $(i,\theta) \in N \times \Theta$, one has $a_\theta^i \leqq t_\theta^i \leqq b_\theta^i$ whenever $t_\theta^i \in F_\theta^i$.

Lemma 53 *Suppose the feasible sets T_θ allow autarky, and the weak gains from trade sets $R_\theta(0)$ are bounded below by the ν-integrable function $\theta \mapsto \underline{t}_\theta$. Suppose preferences are LNS and continuous. Then for each fixed $p \gg 0$ the restricted compensated demand correspondence $(i,\theta) \mapsto\!\!\!\rightarrow \bar{\xi}_\theta^C(p)$ has non-empty compact values which are integrably bounded, as well as a closed graph.*

PROOF. By definition, each $t \in \bar{B}_\theta(p)$ satisfies $t \in R_\theta(0)$ and so $t \geqq \underline{t}_\theta$. Because $p \gg 0$, the argument used to prove Lemma 6 shows that any $t \in \bar{B}_\theta(p)$

also satisfies $t \leqq \bar{t}_\theta(p)$, where $\bar{t}_\theta(p)$ is the vector with components defined by

$$\bar{t}_{\theta g}(p) := - \sum_{h \in G \setminus \{g\}} p_h \, \underline{t}_{\theta h} / p_g \quad (\text{all } g \in G). \tag{42}$$

Hence $\bar{B}_\theta(p)$ is bounded. Also, because $\underline{t}_\theta$ is ν-integrable, (42) implies that so is $\bar{t}_\theta(p)$. Because $\bar{B}_\theta(p)$ is evidently a closed set, it must be compact. Obviously $0 \in \bar{B}_\theta(p)$ for each p and θ, so it is non-empty.

As discussed at the start of this section, $\xi_\theta(p) \subset R_\theta(0)$. So $\xi_\theta(p)$ consists of net trade vectors t that maximize $\succsim_\theta$ subject to $t \in \bar{B}_\theta(p)$. But the restricted budget set $\bar{B}_\theta(p)$ is compact. Because preferences are continuous, it follows that the Walrasian demand set $\xi_\theta(p)$ is non-empty. Also, because preferences are LNS, the proof of Lemma 9 shows that $\xi_\theta(p) \subset \xi_\theta^C(p)$. Hence $\xi_\theta(p) \subset R_\theta(0) \cap \xi_\theta^C(p) = \bar{\xi}_\theta^C(p)$, which implies that $\bar{\xi}_\theta^C(p)$ is non-empty.

Finally, let G_R and $G_C(p)$ denote the graphs of $\theta \mapsto\!\!\!\rightarrow R_\theta(0)$ and $\theta \mapsto\!\!\!\rightarrow \xi_\theta^C(p)$ respectively. Then G_R and $G_C(p)$ are both closed, by part (1) of Lemma 5 and Lemma 51 respectively. So therefore is $N \times [G_R \cap G_C(p)]$, which is the graph of $(i, \theta) \mapsto\!\!\!\rightarrow \bar{\xi}_\theta^C(p)$. In particular $\bar{\xi}_\theta^C(p)$ must be a closed set, for each fixed θ and p. As a closed subset of the compact set $\bar{B}_\theta(p)$, it must be compact. □

11.4 Existence of compensated equilibrium with free disposal

We begin by using arguments due to Khan and Yamazaki (1981) and Yamazaki (1981) to prove existence of a CE *with free disposal* $(\hat{t}^{N \times \Theta}, \hat{p})$. This is defined as a pair satisfying $\hat{t}_\theta^i \in \xi_\theta^C(\hat{p})$ for ν-a.e. $(i, \theta) \in N \times \Theta$, as well as $\hat{z} := \int_{N \times \Theta} \hat{t}_\theta^i \, d\nu \leqq 0$, $\hat{p} > 0$, and $\hat{p}\, \hat{z} = 0$. Because $\hat{p} > 0$ whereas $\hat{z} \leqq 0$, it follows that $\hat{p}_g \, \hat{z}_g = 0$ for all $g \in G$, implying the *rule of free goods* — if $\hat{z}_g < 0$ then $\hat{p}_g = 0$, whereas $\hat{z}_g = 0$ if $\hat{p}_g > 0$. Furthermore, because $\hat{p}\, \hat{z} = 0$ while $\hat{p}_g \, \hat{t}_\theta^i \leq 0$ for ν-a.e. $(i, \theta) \in N \times \Theta$, one has $\hat{p}_g \, \hat{t}_\theta^i = 0$ for ν-a.e. $(i, \theta) \in N \times \Theta$.

Lemma 54 *Suppose agents' feasible sets T_θ allow autarky, and the weak gains from trade sets $R_\theta(0)$ are bounded below by the integrable function $\theta \mapsto \underline{t}_\theta$. Suppose too that preferences are LNS and continuous. Then there exists a compensated equilibrium with free disposal satisfying weak gains from trade.*

PROOF. Let $\Delta := \{\, p \in \mathbb{R}_+^G \mid \sum_{g \in G} p_g = 1 \,\}$ denote the unit simplex of normalized non-negative price vectors, with interior Δ^0 of normalized strictly positive price vectors. For each $n = 1, 2, \ldots$, define the non-empty domain

$$D_n := \{\, p \in \Delta \mid p_g \geq 1/(\#G + n) \ (\text{all } g \in G) \,\}.$$

Note that, for each fixed $\theta \in \Theta$, whenever $t_\theta \in \bar{B}_\theta(p)$ and $p \in \Delta$, then

$$p_g \, t_{\theta g} = - \sum_{h \in G \setminus \{g\}} p_h \, t_{\theta h} \leq - \sum_{h \in G \setminus \{g\}} p_h \, \underline{t}_{\theta h} \leq \max\{ -\underline{t}_{\theta h} \mid h \in G \}.$$

Hence, whenever $p \in D_n$ and $t_\theta \in \bar{B}_\theta(p)$, then $\underline{t}_\theta \leqq t_\theta \leqq \bar{t}_\theta$ where $\underline{t}_\theta \leqq 0$ and the respective components of $\bar{t}_\theta$ are given by

$$\bar{t}_{\theta g} := (\#G + n) \max\{ -\underline{t}_{\theta h} \mid h \in G \} \quad (\text{all } g \in G). \tag{43}$$

That is, every $t_\theta \in \bar{B}_\theta(p)$ satisfies $\underline{t}_\theta \leqq t_\theta \leqq \bar{t}_\theta$ uniformly for all $p \in D_n$.

Next, for each $n = 1, 2, \ldots$, define the aggregate excess compensated demand correspondence $\zeta_n : D_n \twoheadrightarrow \mathbb{R}^G$ by

$$p \mapsto\!\!\!\rightarrow \zeta_n(p) := \int_{N \times \Theta} \bar{\xi}^C_\theta(p) \; \mathrm{d}\nu.$$

By Lemma 53, the correspondence $(i, \theta) \mapsto\!\!\!\rightarrow \bar{\xi}^C_\theta(p)$ has non-empty values and is integrably bounded, while its graph is closed and so measurable. So for each fixed $p \in D_n$, there exists an integrable selection from $(i, \theta) \mapsto\!\!\!\rightarrow \bar{\xi}^C_\theta(p)$, implying that $\zeta_n(p)$ is non-empty. Also, $\zeta_n(p)$ is convex because ν is non-atomic. But preferences are LNS, so for all $\theta \in \Theta$ one has $p\,t = 0$ whenever $t \in \xi^C_\theta(p)$ and so whenever $t \in \bar{\xi}^C_\theta(p) = \xi^C_\theta(p) \cap R_\theta(0)$. It follows that $p\,z = 0$ whenever $z \in \zeta_n(p)$. Moreover, because $\underline{t}_\theta \leqq t_\theta \leqq \bar{t}_\theta$ for all $t_\theta \in \bar{\xi}^C_\theta(p)$ where $\theta \mapsto \underline{t}_\theta$ is ν-integrable, it follows from (43) that $\theta \mapsto \bar{t}_\theta$ is also ν-integrable, and that

$$\int_{N \times \Theta} \underline{t}_\theta \; \mathrm{d}\nu \leqq z \leqq \int_{N \times \Theta} \bar{t}_\theta \; \mathrm{d}\nu \tag{44}$$

for all $z \in \zeta_n(p)$ and all $p \in D_n$. Hence, the arguments in Hildenbrand (1974, p. 73, Props. 7 and 8) establish: (i) because each set $\bar{\xi}^C_\theta(p)$ is closed and (44) holds, the correspondence ζ_n is compact-valued; (ii) the graph of $\zeta_n : D_n \twoheadrightarrow \mathbb{R}^G$ is closed. Using (44) once again, it follows that the graph of ζ_n is compact.

For $n = 1, 2, \ldots$, one can find a compact convex set $Z_n \subset \mathbb{R}^G$ large enough so that the graph of ζ_n is a subset of $D_n \times Z_n$. Then define $\phi_n(z) := \arg\max\{ p\,z \mid p \in D_n \}$ for each $z \in Z_n$. Obviously $\phi_n(z)$ is non-empty and convex.

As in the proof of Theorem 24, one can show that the graph of the correspondence ϕ_n is closed, and so compact as a closed subset of the compact set $Z_n \times D_n$. Consider next the correspondence $\psi_n : D_n \times Z_n \twoheadrightarrow D_n \times Z_n$ which is defined for each $n = 1, 2, \ldots$ by $\psi_n(p, z) := \phi_n(z) \times \zeta_n(p)$. This correspondence has a convex domain and non-empty convex values. Its graph is easily seen to be the Cartesian product of the graph of ϕ_n with the graph of ζ_n, and so compact as the Cartesian product of two compact sets. Hence, Kakutani's theorem can be applied to demonstrate the existence of a fixed point $(p_n, z_n) \in D_n \times Z_n$ for each $n = 1, 2, \ldots$. This fixed point satisfies

$(p_n, z_n) \in \psi_n(p_n, z_n)$ and so $p_n \in \phi_n(z_n)$, $z_n \in \zeta_n(p_n)$. In particular, for all $p \in D_n$ one has $p\, z_n \leq p_n\, z_n = 0$.

Because the vector $(\#G)^{-1}(1, 1, \ldots, 1) \in D_n$, this result and (44) imply that

$$z_n \in Z^* := \{\, z \in \mathbb{R}^G \mid z \geqq \int_{N\times\Theta} \underline{t}_\theta \, \mathrm{d}\nu;\ \frac{1}{\#G} \sum_{g\in G} z_{ng} \leq 0 \,\}.$$

So the sequence $(p_n, z_n)_{n=1}^\infty$ lies in the compact subset $\Delta \times Z^*$ of $\mathbb{R}^G \times \mathbb{R}^G$, and must have a convergent subsequence. Retaining only the terms of this subsequence, we can assume that (p_n, z_n) converges to some pair $(\hat{p}, z^*) \in \Delta \times \mathbb{R}^G$.

Next, any $p \in \Delta^0$ satisfies $p \in D_n$ for all large n, so $p\, z_n \leq p_n\, z_n = 0$. Taking limits yields $p\, z^* \leq 0$ for all $p \in \Delta^0$, so $z^* \leqq 0$.

By definition of ζ_n, for $n = 1, 2, \ldots$ one has $z_n = \int_{N\times\Theta} t^i_{\theta n} \, \mathrm{d}\nu \geqq \int_{N\times\Theta} \underline{t}_\theta \, \mathrm{d}\nu$ where $t^i_{\theta n} \in \bar{\xi}^C_\theta(p_n)$ for ν-a.e. $(i, \theta) \in N \times \Theta$. Now we apply "Fatou's lemma in several dimensions" due to Schmeidler (1970) — see also Hildenbrand (1974, p. 69, Lemma 3). Because $(i, \theta) \mapsto v^i_{\theta n} := t^i_{\theta n} - \underline{t}_\theta$ $(n = 1, 2, \ldots)$ is a sequence of ν-integrable functions from $N \times \Theta$ into $\mathbb{R}^G_+$, and because

$$\int_{N\times\Theta} v^i_{\theta n} \, \mathrm{d}\nu = z_n - \int_{N\times\Theta} \underline{t}_\theta \, \mathrm{d}\nu \to \hat{v} := z^* - \int_{N\times\Theta} \underline{t}_\theta \, \mathrm{d}\nu,$$

as $n \to \infty$, there exists an integrable function $(i, \theta) \mapsto v^i_\theta \in \mathbb{R}^G_+$ such that v^i_θ is an accumulation point of the sequence $v^i_{\theta n}$ for ν-a.e. $(i, \theta) \in N \times \Theta$, and also $\int_{N\times\Theta} v^i_\theta \, \mathrm{d}\nu \leqq \hat{v}$. Next, define $\hat{t}^i_\theta := v^i_\theta + \underline{t}_\theta$ for all $(i, \theta) \in N \times \Theta$. For ν-a.e. $(i, \theta) \in N \times \Theta$, this implies that $\hat{t}^i_\theta$ is an accumulation point of the sequence $t^i_{\theta n}$. Because $t^i_{\theta n} \in \bar{\xi}^C_\theta(p_n)$, while $p_n \to \hat{p}$ and $\bar{\xi}^C_\theta$ has a closed graph, it follows that $\hat{t}^i_\theta \in \bar{\xi}^C_\theta(\hat{p})$. Furthermore

$$\hat{z} := \int_{N\times\Theta} \hat{t}^i_\theta \, \mathrm{d}\nu = \int_{N\times\Theta} (v^i_\theta + \underline{t}_\theta) \, \mathrm{d}\nu \leqq \hat{v} + \int_{N\times\Theta} \underline{t}_\theta \, \mathrm{d}\nu = z^* \leqq 0.$$

Finally, because preferences are LNS, one has $\hat{p}\, \hat{t}^i_\theta = 0$ for ν-a.e. $(i, \theta) \in N \times \Theta$, so $\hat{p}\, \hat{z} = \int_{N\times\Theta} \hat{p}\, \hat{t}^i_\theta \, \mathrm{d}\nu = 0$. This completes the proof that $(\hat{t}^{N\times\Theta}, \hat{p})$ is a CE with free disposal that satisfies weak gains from trade. □

11.5 Monotone preferences and existence of compensated equilibrium

When preferences are weakly monotone, a CE without free disposal exists:

Lemma 55 *Suppose agents' preferences are weakly monotone, and $(\hat{t}^{N\times\Theta}, \hat{p})$ is a CE with free disposal. Then there exists a CE $(\tilde{t}^{N\times\Theta}, \hat{p})$ with the same equilibrium price vector and with $\tilde{t}^i_\theta \geqq \hat{t}^i_\theta$ for ν-a.e. $(i, \theta) \in N \times \Theta$.*

PROOF. First, let $\hat{z} := \int_{N\times\Theta} \hat{t}^i_\theta \, \mathrm{d}\nu$. Then $\hat{z} \leqq 0$ and $\hat{p}\,\hat{z} = 0$ by definition of CE with free disposal. Second, define $\tilde{t}^i_\theta := \hat{t}^i_\theta - \hat{z}$ for all (i,θ). This definition implies that $\int_{N\times\Theta} \tilde{t}^i_\theta \, \mathrm{d}\nu = \int_{N\times\Theta} \hat{t}^i_\theta \, \mathrm{d}\nu - \hat{z} = 0$, and also that $\hat{p}\,\tilde{t}^i_\theta = \hat{p}\,(\hat{t}^i_\theta - \hat{z}) = \hat{p}\,\hat{t}^i_\theta \leq 0$ for ν-a.e. $(i,\theta) \in N \times \Theta$. Weak monotonicity of preferences then implies that $\tilde{t}^i_\theta \in R_\theta(\hat{t}^i_\theta)$ for ν-a.e. $(i,\theta) \in N \times \Theta$. Since preferences are transitive, one must have $\hat{p}\,t \geq 0$ whenever $t \succsim_\theta \tilde{t}^i_\theta$, because the same is true whenever $t \succsim_\theta \hat{t}^i_\theta$. Hence $\tilde{t}^i_\theta \in \xi^C_\theta(\hat{p})$ for ν-a.e. $(i,\theta) \in N \times \Theta$. This confirms that $(\tilde{t}^{N\times\Theta}, \hat{p})$ is a CE satisfying $\tilde{t}^i_\theta \geqq \hat{t}^i_\theta$ for ν-a.e. $(i,\theta) \in N \times \Theta$. □

Obviously, when the equilibrium allocation $\hat{t}^{N\times\Theta}$ satisfies weak gains from trade, so does $\tilde{t}^{N\times\Theta}$, because preferences are transitive and weakly monotone.

11.6 Existence of Walrasian equilibrium

Convexity of preferences was not required to prove Lemma 54. Convexity of each feasible set T_θ, however, *is* needed for the following argument showing that a CE is a WE provided that $0 \in \operatorname{int} \int_{N\times\Theta} T_\theta \, \mathrm{d}\nu$ and the economy satisfies a suitable irreducibility assumption. This assumption will adapt the condition used in Section 5.4 for a continuum economy in much the same way as Hildenbrand (1972, p. 85) adapted McKenzie's original assumption.

Given the particular feasible allocation $\hat{t}^{N\times\Theta}$ and any $(i,\theta) \in N \times \Theta$, define $U^i_\theta(\hat{t}^i_\theta)$ as the cone of vectors $\alpha(t - \hat{t}^i_\theta)$ with $\alpha > 0$ and $t \in P_\theta(\hat{t}^i_\theta)$. Similarly, define $V^i_\theta(\hat{t}^i_\theta)$ as the closed cone of vectors $\alpha(t - \hat{t}^i_\theta)$ with $\alpha \geq 0$ and $t \in T_\theta$. Next, define W_θ as the cone of vectors $-\beta\, t$ with $\beta \geq 0$ and $t \in T_\theta$. Finally, define Z^i_θ as the cone of vectors $-\gamma\, \hat{t}^i_\theta$ with $\gamma \geq 0$. Then the economy is said to be *directionally irreducible* provided that, for any subset $K \subset N \times \Theta$ with $0 < \nu(K) < 1$ and any feasible allocation $\hat{t}^{N\times\Theta}$, the two sets

$$\int_K U^i_\theta(\hat{t}^i_\theta) \, \mathrm{d}\nu + \int_{(N\times\Theta)\setminus K} V^i_\theta(\hat{t}^i_\theta) \, \mathrm{d}\nu \quad \text{and} \quad \int_{(N\times\Theta)\setminus K} W_\theta \, \mathrm{d}\nu + \int_{N\times\Theta} Z^i_\theta \, \mathrm{d}\nu$$

intersect.

Theorem 56 *Suppose that agents' feasible sets T_θ are convex, and that $0 \in \operatorname{int} \int_{N\times\Theta} T_\theta \, \mathrm{d}\nu$. Suppose too that agents' preferences are LNS and continuous, and the economy is directionally irreducible. Then any CE is a WE.*

PROOF. Let $(\hat{t}^{N\times\Theta}, \hat{p})$ be a CE. Because $0 \in \operatorname{int} \int_{N\times\Theta} T_\theta \, \mathrm{d}\nu$, a non-null set of agents $(i,\theta) \in N \times \Theta$ have "cheaper points" $\underline{t}^i_\theta \in T_\theta$ satisfying $\hat{p}\,\underline{t}^i_\theta < 0$.

Let K be any measurable subset of $N \times \Theta$ satisfying $0 < \nu(K) < 1$ whose members all have such cheaper points. By directional irreducibility, there exist measurable selections:

(1) $(i, \theta) \mapsto t^i_\theta \in T_\theta$ and $(i, \theta) \mapsto \alpha^i_\theta \geq 0$ (for ν-a.e. $(i, \theta) \in N \times \Theta$) with $t^i_\theta \succ_\theta \hat{t}^i_\theta$ and $\alpha^i_\theta > 0$ for ν-a.e. $(i, \theta) \in K$;
(2) $(i, \theta) \mapsto \bar{t}^i_\theta \in T_\theta$ and $(i, \theta) \mapsto \beta^i_\theta \geq 0$ (for ν-a.e. $(i, \theta) \in (N \times \Theta) \setminus K$);
(3) $(i, \theta) \mapsto \gamma^i_\theta \geq 0$ (for ν-a.e. $(i, \theta) \in N \times \Theta$);

such that

$$\int_K \alpha^i_\theta \, (t^i_\theta - \hat{t}^i_\theta) \, \mathrm{d}\nu + \int_{(N \times \Theta) \setminus K} [\alpha^i_\theta \, (t^i_\theta - \hat{t}^i_\theta) + \beta^i_\theta \, \bar{t}^i_\theta] \, \mathrm{d}\nu + \int_{N \times \Theta} \gamma^i_\theta \, \hat{t}^i_\theta \, \mathrm{d}\nu = 0$$

and so

$$\int_{N \times \Theta} \alpha^i_\theta \, t^i_\theta \, \mathrm{d}\nu + \int_{(N \times \Theta) \setminus K} \beta^i_\theta \, \bar{t}^i_\theta \, \mathrm{d}\nu = \int_{N \times \Theta} (\alpha^i_\theta - \gamma^i_\theta) \, \hat{t}^i_\theta \, \mathrm{d}\nu. \tag{45}$$

Because $(\hat{t}^{N \times \Theta}, \hat{p})$ is a CE and preferences are LNS, one has $\hat{p} \, \hat{t}^i_\theta = 0$ for ν-a.e. $(i, \theta) \in N \times \Theta$. Also, Lemma 12 and the definition of K imply that $\hat{p} \, t^i_\theta > 0$ for ν-a.e. $(i, \theta) \in K$. From (45), it follows that

$$\int_{(N \times \Theta) \setminus K} \hat{p} \, (\alpha^i_\theta \, t^i_\theta + \beta^i_\theta \, \bar{t}^i_\theta) \, \mathrm{d}\nu = - \int_K \alpha^i_\theta \, \hat{p} \, t^i_\theta \, \mathrm{d}\nu - \int_{(N \times \Theta} \gamma^i_\theta \, \hat{p} \, \hat{t}^i_\theta < 0.$$

So there is a non-null subset $K' \subset (N \times \Theta) \setminus K$ such that either $\alpha^i_\theta \, \hat{p} \, t^i_\theta < 0$ or $\beta^i_\theta \, \hat{p} \, \bar{t}^i_\theta < 0$ for all $(i, \theta) \in K'$. Because $\alpha^i \geq 0$ for ν-a.e. $(i, \theta) \in N \times \Theta$ and $\beta^i \geq 0$ for ν-a.e. $(i, \theta) \in (N \times \Theta) \setminus K$, there is a non-null set of $(i, \theta) \in (N \times \Theta) \setminus K$ for whom either $\hat{p} \, t^i_\theta < 0$ or $\hat{p} \, \bar{t}^i_\theta < 0$, so $(i, \theta$ has a cheaper point. Hence, no subset K of $N \times \Theta$ with $\nu(K) < 1$ can include almost all pairs (i, θ) with cheaper points. The only other possibility is that almost all (i, θ) must have cheaper points. By Lemma 12, this implies that the CE is actually a WE. $\square$

11.7 Indivisible goods and constrained monotone preferences

Because monotone preferences require each feasible set T_θ to allow free disposal, they obviously rule out indivisible goods. So does the assumption in Section 11.6 that each T_θ is a convex set. Nevertheless, many results for continuum economies discussed here can be extended to indivisible goods, using methods such as those discussed in Mas-Colell (1977) and Yamazaki (1978, 1981). One has to weaken the monotone preferences assumption, and also assume sufficient dispersion in the marginal distribution of agents' feasible sets T_θ induced by distribution measure $\nu \in \mathcal{M}_\lambda(N \times \Theta)$.

Assume that G can be partitioned into the sets D of *divisible* and H of *indivisible* goods. Instead of $\mathbb{R}^G$, the natural commodity space becomes $\mathbb{R}^D \times \mathbb{Z}^H$, where $\mathbb{Z}^H$ is the Cartesian product of a suitable number of copies of $\mathbb{Z}$, the set of all integers. Obviously we assume that $T_\theta \subset \mathbb{R}^D \times \mathbb{Z}^H$ for all $\theta \in \Theta$. Note that preferences can be LNS only if D is non-empty.

Within this restricted commodity space, say that preferences are *constrained weakly monotone* if $t' \in R_\theta(t)$ whenever $t \in T_\theta$ and $t' \geqq t$ with $t' \in \mathbb{R}^D \times \mathbb{Z}^H$.

Lemma 57 *Suppose agents' preferences are constrained weakly monotone, and $(\hat{t}^{N\times\Theta}, \hat{p})$ is a CE with free disposal. Then there exists a CE $(\tilde{t}^{N\times\Theta}, \hat{p})$ with the same equilibrium prices and with $\tilde{t}^i_\theta \geqq \hat{t}^i_\theta$ for ν-a.e. $(i,\theta) \in N \times \Theta$.*

PROOF. As in the proof of Lemma 54, let $\hat{z} := \int_{N\times\Theta} \hat{t}^i_\theta \, d\nu$. Then $\hat{z} \leqq 0$ and $\hat{p}\,\hat{z} = 0$ by definition of CE with free disposal. For each indivisible good $g \in H$, define $z^*_g \in \mathbb{Z}_-$ as the largest integer that does not exceed $\hat{z}_g \leq 0$, and let $Z^*_g := \int_{N\times\Theta} \{\, z^*_g, 0 \,\} \, d\nu$. Because the measure ν is non-atomic, the set Z^*_g is convex, so it contains the whole interval $[z^*_g, 0]$, including $\hat{z}_g$. Hence, for each $g \in H$ there is a measurable selection $(i,\theta) \mapsto z^i_{\theta g}$ from the correspondence $(i,\theta) \twoheadrightarrow \{\, z^*_g, 0 \,\} \subset \mathbb{Z}_-$ such that $\hat{z}_g = \int_{N\times\Theta} z^i_{\theta g} \, d\nu$.

To include divisible goods as well, given any $(i,\theta) \in N \times \Theta$ simply define $z^i_{\theta g} := \hat{z}_g$ for all $g \in D$ and let z^i_θ be the vector $(z^i_{\theta g})_{g\in G} \in \mathbb{R}^G_-$. Then $\hat{z} = \int_{N\times\Theta} z^i_\theta \, d\nu$ for the measurable function $(i,\theta) \mapsto z^i_\theta \in \mathbb{R}^D_- \times \mathbb{Z}^H_-$. Now define $\tilde{t}^i_\theta := \hat{t}^i_\theta - z^i_\theta$ for all $(i,\theta) \in N \times \Theta$. Because both $\hat{t}^i_\theta$ and z^i_θ always belong to $\mathbb{R}^D \times \mathbb{Z}^H$, so does each $\tilde{t}^i_\theta$. Also $\tilde{t}^i_\theta \geqq \hat{t}^i_\theta$ for all $(i,\theta) \in N \times \Theta$ because $z^i_\theta \leqq 0$. Finally, the rule of free goods stated in Section 11.4 implies that $z^i_{\theta g} = 0$ unless $\hat{p}_g = 0$, so $\hat{p}\,\tilde{t}^i_\theta = \hat{p}\,\hat{t}^i_\theta \leq 0$ for ν-a.e. $(i,\theta) \in N \times \Theta$.

The rest of the proof closely follows that of Lemma 55, so will be omitted. □

11.8 Dispersion and existence of Walrasian equilibrium

As in Section 10.4, assume that each agent's feasible set T_θ is piecewise convex — i.e., that $T_\theta = \cup_{m\in M_\theta} T^m_\theta$ where $M_\theta \subset \mathbb{N}$. Now adapt the definitions (38), (39), and (41) in Section 10.4. Given any $\theta \in \Theta$, first define

$$\hat{M}_\theta := \{\, m \in M_\theta \mid P_\theta(0) \cap T^m_\theta \neq \emptyset \,\}$$

where $P_\theta(0)$ is the set of net trade vectors that are strictly preferred to autarky. Second, given a price vector $p \neq 0$, let

$$\hat{W}_\theta(p) := \{\, \inf p\, T^m_\theta \mid m \in \hat{M}_\theta \,\}$$

be the associated countable set of critical wealth levels. Finally, say that agents have *dispersed feasible sets* if for all $p > 0$ one has

$$\nu(\{ (i,\theta) \in N \times \Theta \mid 0 \in \hat{W}_\theta(p) \setminus \{\inf p\, T_\theta\} \}) = 0.$$

Lemma 58 *Suppose agents' feasible sets T_θ are piecewise convex and dispersed, while $0 \in \operatorname{int} \int_{N\times\Theta} T_\theta \, d\nu$. Suppose agents' preferences are LNS and continuous, and that the economy is directionally irreducible. Then any CE satisfying weak gains from trade is a WE.*

PROOF. Suppose $(\hat{t}^{N\times\Theta}, \hat{p})$ is a CE satisfying weak gains from trade. Consider any $(i,\theta) \in N \times \Theta$ who has a cheaper point because $\inf \hat{p}\, T_\theta < 0$. Any $t \in P_\theta(\hat{t}^i_\theta)$ satisfies $t \in P_\theta(0)$ because preferences are transitive, so $t \in T^m_\theta$ for some $m \in \hat{M}_\theta$. Because agents have dispersed feasible sets, it follows that $\hat{p}\, \hat{t}^i_\theta \notin \hat{W}_\theta(\hat{p})$. So almost any potential agent $(i,\theta) \in N \times \Theta$ with a cheaper point meets the conditions required for Lemma 49 to show that any $t \in P_\theta(\hat{t}^i_\theta)$ satisfies $\hat{p}\, t > 0$. This observation implies that all the arguments used to prove Theorem 56 still apply, so the CE is a WE. □

Theorem 59 *Suppose agents' feasible sets T_θ allow autarky, are piecewise convex and dispersed, while satisfying $0 \in \operatorname{int} \int_{N\times\Theta} T_\theta \, d\nu$. Suppose their weak gains from trade sets are integrably bounded below. Suppose preferences are LNS, continuous, and constrained weakly monotone. Suppose finally that the economy is directionally irreducible. Then there exists a WE.*

PROOF. This follows by combining the results of Lemmas 54, 57, and 58. □

12 Equivalence Theorems for the Core and f-Core

12.1 The core and f-core

The feasible allocation mechanism $\hat{t}^{N\times\Theta}$ is in the *core* if there is no *blocking coalition* $K \subset N \times \Theta$ satisfying $\nu(K) > 0$ with a *blocking mechanism* t^K in the form of a measurable mapping $(i,\theta) \mapsto t^i_\theta$ from K to $\mathbb{R}^G$ that satisfies $t^i_\theta \in P_\theta(\hat{t}^i_\theta)$ for ν-a.e. $(i,\theta) \in K$, as well as $\int_K t^i_\theta \, d\nu = 0$ — i.e., feasibility within K. This is an obvious extension of the earlier definition of the core for a finite economy in Section 8.1. It also generalizes Aumann's (1964) original definition for a continuum economy, as well as Hildenbrand (1974).

The feasible allocation mechanism $\hat{t}^{N\times\Theta}$ is in the *f-core* if there is no finite family of pairwise disjoint non-null sets $K_1, \ldots, K_m \subset N \times \Theta$ and corresponding natural numbers $n_1, \ldots, n_m \in \mathbb{N}$ such that, for some $r > 0$, there is a blocking coalition $K \subset \cup_{j=1}^m K_j$ with $\nu(K \cap K_j) = r\, n_j$ $(j = 1, \ldots, m)$ and so $\nu(K) = r \sum_{j=1}^m n_j$, while the blocking mechanism t^K satisfies $t_\theta^i = \bar{t}_j$ for all $(i, \theta) \in K_j$ for some finite collection $\bar{t}_1, \ldots, \bar{t}_m \in \mathbb{R}^G$ with $\sum_{j=1}^m n_j\, \bar{t}_j = 0$.[29]

Thus an allocation is not in the f-core only if some blocking coalition K can be decomposed into a continuum of finite sub-coalitions, each consisting of $\sum_{j=1}^m n_j$ members, with each sub-coalition choosing exactly the same pattern of net trade vectors for its members. In particular, blocking is possible even when trade is restricted to take place separately within finite sub-coalitions. Obviously, therefore, any allocation in the core also belongs to the f-core.

12.2 Walrasian equilibria belong to the core

This section provides sufficient conditions for both the core and the f-core to coincide with the set of WE allocations. The first part of this equivalence result is easy:

Theorem 60 *If $(\hat{t}^{N\times\Theta}, \hat{p})$ is any WE, then $\hat{t}^{N\times\Theta}$ belongs to the core.*

PROOF. Let $K \subset N \times \Theta$ be any non-null set. Suppose the measurable mapping $t : K \to \mathbb{R}^G$ satisfies $t_\theta^i \in P_\theta(\hat{t}_\theta^i)$ for ν-a.e. $(i, \theta) \in K$. Because $\hat{t}_\theta^i \in \xi_\theta(\hat{p})$, it follows that $\hat{p}\, t_\theta^i > 0$ for ν-a.e. $(i, \theta) \in K$. This implies that $\int_K \hat{p}\, t_\theta^i \, d\nu > 0$, so $\int_K t_\theta^i \, d\nu \neq 0$. Thus, t^K cannot be a blocking strategy, so no blocking strategy exists. □

12.3 f-core allocations are compensated equilibria

First, we introduce a new assumption on agents' preferences. To do so, let $\mathbb{Q} \subset \mathbb{R}$ denote the set of *rational numbers* — i.e., ratios of integers m/n where $m, n \in \mathbb{Z}$. Of course, $\mathbb{Q}$ is countable. Then let $\mathbb{Q}^G \subset \mathbb{R}^G$ denote the collection of vectors whose co-ordinates are all rational; it is also countable.

Next, say that the preferences described by the pair $(T_\theta, \succsim_\theta)$ are *locally non-satiated in rational net trade vectors* (or LNS in $\mathbb{Q}^G$) if, for any $t \in T_\theta$ and any neighbourhood V of t in $\mathbb{R}^G$, there exists $t' \in V \cap \mathbb{Q}^G$ such that $t' \in P_\theta(t)$. This

[29] The above definition of the f-core is based on Kaneko and Wooders (1986) and Hammond, Kaneko and Wooders (1989) — see also Hammond (1999a). The first two of these papers also show that any feasible allocation mechanism $\hat{t}^{N\times\Theta}$ can be achieved as the limit of a sequence of mechanisms in which finite coalitions form to trade amongst their members.

obviously strengthens the LNS assumption that has been used so often in this chapter. Indeed, suppose that $G = \{1, 2\}$ and that there exists a scalar $\xi > 0$ for which $T_\theta = \{ t \in \mathbb{R}^G_+ \mid t_2 = \xi\, t_1 \}$. Suppose too that, given any $t, t' \in T_\theta$, one has $t \succsim_\theta t' \iff t_1 \geq t'_1$. These preferences are obviously LNS for all $\xi > 0$. But $T_\theta \cap \mathbb{Q}^G = \{(0, 0)\}$ when ξ is irrational, so preferences are LNS in $\mathbb{Q}^G$ if and only if ξ is rational.

Nevertheless, the assumption is still weak enough to be satisfied by any monotone preference relation.

Lemma 61 *Monotone preferences are LNS in $\mathbb{Q}^G$.*

PROOF. Suppose that the preferences described by the pair $(T_\theta, \succsim_\theta)$ are monotone. Given any $t \in T_\theta$ and any neighbourhood V of t in $\mathbb{R}^G$, there exists $t' \gg t$ such that $t' \in V \cap \mathbb{Q}^G$, and then $t' \in P_\theta(t)$. $\square$

The following result does not rely on convexity, continuity, or montonicity of preferences. The proof relies on a key idea introduced in Aumann (1964).

Theorem 62 *Suppose preferences are LNS in $\mathbb{Q}^G$. Then any allocation $t^{N\times\Theta}$ in the f-core is a CE at some price vector $p \neq 0$.*

PROOF. By definition, the mapping $(i, \theta) \mapsto t^i_\theta$ associated with the allocation $t^{N\times\Theta}$ must be measurable w.r.t. the product σ-field on $N \times \Theta$. For each $t \in \mathbb{Q}^G$, define $\hat{K}(t) := \{ (i, \theta) \in N \times \Theta \mid t \in P_\theta(t^i_\theta) \}$, which is also measurable when $N \times \Theta$ is given the product σ-field.

Let $K_0 := \cup\{ \hat{K}(t) \mid t \in \mathbb{Q}^G,\ \nu(\hat{K}(t)) = 0 \}$. As the union of a countable family of null sets, the set K_0 is measurable and $\nu(K_0) = 0$. Then let $K' := (N \times \Theta) \setminus K_0$ and define C as the convex hull of the set $\cup_{(i,\theta)\in K'}[\mathbb{Q}^G \cap P_\theta(t^i_\theta)]$.

Suppose $0 \in C$. Then there is a natural number m and a collection $(i_j, \theta_j, t_j, r_j)$ $(j = 1, \ldots, m)$ of points in $K' \times \mathbb{Q}^G \times [0, 1]$ such that $t_j \in P_{\theta_j}(t^{i_j}_{\theta_j})$ (each j), whereas the positive real numbers r_j $(j = 1, \ldots, m)$ are convex weights satisfying

$$\sum_{j=1}^{m} r_j = 1 \quad \text{and also} \quad \sum_{j=1}^{m} r_j\, t_j = 0. \tag{46}$$

After excluding any $(i_j, \theta_j, t_j, r_j)$ for which $r_j = 0$, it loses no generality to assume that $r_j > 0$ for $j = 1, \ldots, m$.

Because each $t_j \in \mathbb{Q}^G$, every coefficient in the system (46) of simultaneous equations in the convex weights r_j $(j = 1, \ldots, m)$ must be rational. Moreover, the system can be solved by pivoting operations or Gaussian elimination. Since

(46) has at least one solution, there must be a *rational* solution. Multiplying this solution by the least common denominator of the rational fractions r_j yields natural numbers $n_j \in \mathbb{N}$ ($j = 1, \ldots, m$) such that $0 = \sum_{j=1}^m n_j t_j$.

Because each $(i_j, \theta_j) \notin K_0$, by definition there is no $t'_j \in \mathbb{Q}^G$ with $(i_j, \theta_j) \in \hat{K}(t'_j)$ and $\nu(\hat{K}(t'_j)) = 0$. It follows that $\nu(\hat{K}(t_j)) > 0$ ($j = 1, \ldots, m$).

Define $r := \min_j \{\nu(\hat{K}(t_j))\}/\sum_{j=1}^m n_j$. Then $r > 0$. For each $j = 1, 2, \ldots, m$, let K_j be any subset of $\hat{K}(t_j)$ having measure $n_j r$, and suppose that the different sets K_j are pairwise disjoint. Even if all the sets $\hat{K}(t_j)$ are equal, this will still be possible. Let $K := \cup_{j=1}^m K_j$. Then define t^K so that each potential agent $(i_j, \theta_j) \in K_j$ gets t_j instead of $t^{i_j}_{\theta_j}$. Because $\sum_{j=1}^m n_j t_j = 0$, this t^K is a blocking allocation. So $t^{N \times \Theta}$ does not belong to the f-core.

Conversely, if $t^{N \times \Theta}$ does belong to the f-core, it has just been proved that $0 \notin C$. By definition, C is convex, so there must exist $p \neq 0$ such that $p\,z \geq 0$ for all $z \in C$. In particular, whenever $(i, \theta) \in K'$ and $t \in \mathbb{Q}^G \cap P_\theta(t^i_\theta)$, then $t \in C$ and so $p\,t \geq 0$. Because $\mathbb{Q}^G$ is dense in $\mathbb{R}^G$, it follows that $p\,t \geq 0$ whenever there exists $(i, \theta) \in K'$ such that $t \in P_\theta(t^i_\theta)$, and so, by LNS, such that $t \in R_\theta(t^i_\theta)$. In particular, $p\,t^i_\theta \geq 0$ for all $(i, \theta) \in K'$. But $\nu(K') = 1$ and $\int_{N \times \Theta} t^i_\theta \, d\nu = 0$. So for ν-a.e. (i, θ) one has $p\,t^i_\theta = 0$, as well as $p\,t \geq 0$ whenever $t \in R_\theta(t^i_\theta)$. This proves that $(t^{N \times \Theta}, p)$ is a CE. □

Corollary 63 *Suppose agents' preferences are LNS in $\mathbb{Q}^G$. Then WE $\subset$ Core $\subset$ f-Core $\subset$ CE.*

PROOF. In Section 12.1 it was noted that Core $\subset$ f-Core. The result follows immediately from Theorems 60 and 62. □

The following equivalence theorem provides two characterizations of WE:

Theorem 64 *Suppose that agents' feasible sets T_θ are piecewise convex and dispersed, while $0 \in \operatorname{int} \int_{N \times \Theta} T_\theta \, d\nu$. Suppose agents' preferences are LNS in $\mathbb{Q}^G$ as well as continuous, and that the economy is directionally irreducible. Then WE = Core = f-Core.*

PROOF. This is obvious from Lemma 58 and Corollary 63. □

Of course a special case of Theorem 64 is when each T_θ is convex instead of piecewise convex; then dispersion is automatically satisfied.

Varian (1974) defines a feasible allocation $\hat{t}^{N\times\Theta}$ as *c-fair* if, whenever $K, L \subset N \times \Theta$ with $0 < \nu(L) \leq \nu(K) \leq 1$, there is no measurable mapping $(i,\theta) \mapsto t^i_\theta \in T_\theta$ from L to $\mathbb{R}^G$ satisfying $\int_L t^i_\theta \, \mathrm{d}\nu = \int_K \hat{t}^i_\theta \, \mathrm{d}\nu$ with $t^i_\theta \in R_\theta(\hat{t}^i_\theta)$ for ν-a.e. $(i,\theta) \in L$, as well as $\nu(\{ (i,\theta) \in L \mid t^i_\theta \in P_\theta(\hat{t}^i_\theta) \}) > 0$.[30] Thus, no smaller coalition can do better by sharing an equal aggregate net trade vector between its members. Varian also defines $\hat{t}^{N\times\Theta}$ as *c′-fair* if the condition $\int_L t^i_\theta \, \mathrm{d}\nu = \int_K \hat{t}^i_\theta \, \mathrm{d}\nu$ is replaced by $[1/\nu(L)] \int_L t^i_\theta \, \mathrm{d}\nu = [1/\nu(K)] \int_K \hat{t}^i_\theta \, \mathrm{d}\nu$. Thus, no smaller coalition can do better with an equal mean net trade vector.

Theorem 65 *Provided agents have LNS preferences, any WE allocation is both c-fair and c′-fair.*

PROOF. Suppose that $(\hat{t}^{N\times\Theta}, \hat{p})$ is a WE. Suppose too that $K, L \subset N \times \Theta$ are measurable with $0 < \nu(L) \leq \nu(K) \leq 1$, and that the measurable mapping $(i,\theta) \mapsto t^i_\theta \in T_\theta$ from L to $\mathbb{R}^G$ satisfies $t^i_\theta \in R_\theta(\hat{t}^i_\theta)$ for ν-a.e. $(i,\theta) \in L$, as well as $\nu(\{ (i,\theta) \in L \mid t^i_\theta \in P_\theta(\hat{t}^i_\theta) \}) > 0$. Because preferences are LNS, $\hat{p}\, t^i_\theta \geq \hat{p}\, \hat{t}^i_\theta = 0$ for ν-a.e. $(i,\theta) \in L$, with strict inequality on a non-null subset of L. It follows that

$$\int_L \hat{p}\, t^i_\theta \, \mathrm{d}\nu > 0 = \int_L \hat{p}\, \hat{t}^i_\theta \, \mathrm{d}\nu = \int_K \hat{p}\, \hat{t}^i_\theta \, \mathrm{d}\nu.$$

This contradicts both $\int_L t^i_\theta \, \mathrm{d}\nu = \int_K \hat{t}^i_\theta \, \mathrm{d}\nu$ and $\int_L t^i_\theta \, \mathrm{d}\nu/\nu(L) = \int_K \hat{t}^i_\theta \, \mathrm{d}\nu/\nu(K)$. Hence, the equilibrium allocation $\hat{t}^{N\times\Theta}$ is both c-fair and c'-fair. □

Conversely:

Theorem 66 *If the feasible allocation $\hat{t}^{N\times\Theta}$ is c-fair, or c′-fair, then it belongs to the core.*

PROOF. Suppose, on the contrary, that the feasible allocation $\hat{t}^{N\times\Theta}$ is not in the core. Then there exists a blocking coalition $L \subset N \times \Theta$ with an allocation t^L to its members such that $\int_L t^i_\theta \, \mathrm{d}\nu = 0$ and $t^i_\theta \in P_\theta(\hat{t}^i_\theta)$ for ν-a.e. $(i,\theta) \in L$. Because feasibility requires that $\int_{N\times\Theta} \hat{t}^i_\theta \, \mathrm{d}\nu = 0$, taking $K = N \times \Theta$ in the above definitions implies that $\hat{t}^{N\times\Theta}$ is neither c-fair nor c'-fair. □

Again, under the hypotheses of Theorem 64, Theorems 65 and 66 show that Walrasian equilibria can be characterized as c-fair (or c'-fair) allocations.

[30] Varian acknowledges being inspired by Karl Vind's unpublished lecture notes.

Mas-Colell (1989) has developed an interesting coarsening of the core, similar to the bargaining set in a cooperative game. It offers an alternative characterization of Walrasian equilibrium for continuum economies — at least for a standard pure exchange economy in which the feasible set of each agent type $\theta \in \Theta$ is $T_\theta = \{ t \in \mathbb{R}^G \mid t + e_\theta \geqq 0 \}$, where $\theta \mapsto e_\theta \in \mathbb{R}^G_{++}$ is measurable. It will also be assumed that preferences for net trades are strictly monotone.

Given any feasible allocation $\hat{t}^{N\times\Theta}$, say that (K, t^K) is an *objection* if:

(1) K is a non-null measurable subset of $N \times \Theta$;
(2) t^K is an integrable mapping $(i,\theta) \mapsto t^i_\theta$ from K to $\mathbb{R}^G$ satisfying:
  (a) $\int_K t^i_\theta \, \mathrm{d}\nu = 0$;
  (b) $t^i_\theta \in R_\theta(\hat{t}^i_\theta)$ for ν-a.e. $(i,\theta) \in K$;
  (c) $\nu(\{ (i,\theta) \in K \mid t^i_\theta \succ_\theta \hat{t}^i_\theta \}) > 0$.

This is similar to when K is a blocking coalition, except that a non-null subset of K may only weakly prefer the new allocation.

Given any feasible allocation $\hat{t}^{N\times\Theta}$ and any objection (K, t^K), say that the pair $(\tilde{K}, \tilde{t}^{\tilde{K}})$ is a *counter-objection* to the objection if:

(1) $\tilde{K}$ is a non-null measurable subset of $N \times \Theta$;
(2) $\tilde{t}^{\tilde{K}}$ is an integrable mapping $(i,\theta) \mapsto \tilde{t}^i_\theta$ from $\tilde{K}$ to $\mathbb{R}^G$ satisfying:
  (a) $\int_{\tilde{K}} \tilde{t}^i_\theta \, \mathrm{d}\nu = 0$;
  (b) $\tilde{t}^i_\theta \in P_\theta(t^i_\theta)$ for ν-a.e. $(i,\theta) \in \tilde{K} \cap K$;
  (c) $\tilde{t}^i_\theta \in P_\theta(\hat{t}^i_\theta)$ for ν-a.e. $(i,\theta) \in \tilde{K} \setminus K$.

Next, given any feasible allocation $\hat{t}^{N\times\Theta}$ and the objection (K, t^K), say that the objection is *justified* if there is no counter-objection. Finally, the *bargaining set* is defined as the set of all allocations for which there is no justified objection. That is, every objection is subject to a counter-objection.

This is Mas-Colell's equivalence theorem:

Theorem 67 *Suppose that for each $\theta \in \Theta$ one has $T_\theta = \{ t \in \mathbb{R}^G \mid t + e_\theta \geqq 0 \}$ where $\theta \mapsto e_\theta \in \mathbb{R}^G_{++}$ is measurable. Suppose too that each $\succsim_\theta$ is continuous and strictly monotone. Then feasible allocation $\hat{t}^{N\times\Theta}$ belongs to the bargaining set if and only if there exists $\hat{p} \gg 0$ such that $(\hat{t}^{N\times\Theta}, \hat{p})$ is a WE.*

No formal proof will be given here. However, it is easy to see that, given LNS preferences, any WE allocation allows no objection whatsoever, just as an WE allocation cannot be blocked and so belongs to the core. Conversely, to see that any allocation in the bargaining set is a WE, an important step is to realize that only *Walrasian objections* need be considered. These are defined as objections (K, t^K) to the feasible allocation $\hat{t}^{N\times\Theta}$ for which there exists a price vector $p > 0$ such that, for ν-a.e. $(i,\theta) \in N \times \Theta$, whenever $\tilde{t} \in T_\theta$:

(1) $(i,\theta) \in K$ and $\tilde{t} \in R_\theta(t^i_\theta)$ together imply $p\,\tilde{t} \geq 0$;
(2) $(i,\theta) \in (N \times \Theta) \setminus K$ and $\tilde{t} \in R_\theta(\hat{t}^i_\theta)$ together imply $p\,\tilde{t} \geq 0$.

No attempt has been made to weaken the hypotheses of Theorem 67 to those under which core equivalence has been proved, though one suspects some steps could be made in this direction — see, for example, Yamazaki (1995).

Extending this concept, as well as Ray's (1989) work on credible coalitions, Dutta *et al.* (1989) consider a "consistent" bargaining set for general cooperative games in characteristic function form. This set is based on chains of successive objections, each of which is an objection to the immediately preceding objection. A further objection to any such chain is *valid* if there is no valid counter-objection to this extended chain; the further objection is *invalid* if there is a valid counter-objection. This is a circular definition, but the circularity can be circumvented — see, for example, Greenberg (1990). Note that, in the case of an exchange economy, any allocation in this consistent bargaining set must be in the bargaining set, because consistency makes it harder to find a valid counter-objection to any given objection. On the other hand, any allocation in the core must be in the consistent bargaining set. This is because any allocation in the core is not blocked, so there can be no objection at all, let alone a valid objection. Of course, when the equivalence theorem holds, and both bargaining set and core consist of the set of Walrasian equilibrium allocations, then the same set is also the consistent bargaining set as well.

13 Envy-free mechanisms

13.1 Full f-Pareto efficiency

The main result of this section is considerably simplified if one imposes a stronger form of Pareto efficiency. The weakly Pareto efficient allocation $\hat{t}^{N\times\Theta}$ is said to be *fully f-Pareto efficient* if no finite coalition $C \subset N \times \Theta$ of potential agents can find net trade vectors $t^i_\theta \in T_\theta$ (all $(i,\theta) \in C$) satisfying both $\sum_{(i,\theta)\in C} t^i_\theta = \sum_{(i,\theta)\in C} \hat{t}^i_\theta$ and $t^i_\theta \in P_\theta(\hat{t}^i_\theta)$ for all $(i,\theta) \in C$.

This definition is somewhat similar to that of the f-core of an economy in which each individual $(i,\theta) \in N \times \Theta$ has initial endowment $\hat{t}^i_\theta$ instead of 0. One key difference, however, is that the coalition C can be drawn from *all* of $N \times \Theta$, not just from agents outside some null set.

It is important to realize that fully f-Pareto efficient allocations do exist. Indeed, say that the WELT $(\hat{t}^{N\times\Theta}, \hat{p})$ is *full* if the allocation satisfies the extra condition that $\hat{t}^i_\theta \in \xi^i_\theta(\hat{p})$ for *all* $(i,\theta) \in N \times \Theta$ — not merely for ν-a.e. $(i,\theta) \in N \times \Theta$. Then it is easy to show that any full WELT allocation is fully f-Pareto efficient, using an argument similar to the proof of the first efficiency Theorem 15 for a finite economy. Moreover, the Walrasian demand set $\xi^i_\theta(\hat{p})$ will be non-empty for all $(i,\theta) \in N \times \Theta$ when every preference ordering $\succsim_\theta$ is continuous and the Walrasian budget set $B^i_\theta(\hat{p}) = \{\, t \in T_\theta \mid \hat{p}\, t \leq w^i_\theta \,\}$ is compact for all $(i,\theta) \in N \times \Theta$ — because, for example, $\hat{p} \gg 0$ and every set T_θ is closed and bounded below. Then any WELT can be converted into a full WELT by changing the allocations to at most a null set of potential agents (i,θ) so that they satisfy $\hat{t}^i_\theta \in \xi^i_\theta(\hat{p})$. Such changes, of course, have no effect on the mean net trade vector $\int_{N\times\Theta} \hat{t}^i_\theta \, \mathrm{d}\nu = 0$ or on the equilibrium price vector $\hat{p}$.

On the other hand, with smooth preferences such as those satisfying conditions (a) and (b) in Section 2.8, any fully f-Pareto efficient and interior allocation $\hat{t}^{N\times\Theta}$ must equate all agents' (positive) marginal rates of substitution between any pair of goods. Hence, there must exist a common suitably normalized price vector $\hat{p} \gg 0$ such that $(\hat{t}^{N\times\Theta}, \hat{p})$ is a full WELT.

13.2 Self-selective allocations

The allocation $\hat{t}^{N\times\Theta}$ is said to be *self-selective* if $\hat{t}^i_\theta \succsim_\theta \hat{t}^i_\eta$ whenever $(i,\theta,\eta) \in N \times \Theta \times \Theta$ satisfies $\hat{t}^i_\eta \in T_\theta$. Thus, each potential agent $(i,\theta) \in N \times \Theta$ weakly prefers the allocated net trade vector $\hat{t}^i_\theta$ to any feasible $\hat{t}^i_\eta \in T_\theta$ allocated to an alternative type $\eta \in \Theta$.

The following Lemma plays an important role in subsequent proofs.

Lemma 68 *Suppose there is a smooth type domain Θ, as defined in Section 2.8. Let $(\hat{t}^{N\times\Theta}, p)$ be any full WELT with $p \gg 0$ and with $\hat{t}^i_\theta = h_\theta(p, w^i_\theta)$ where $w^i_\theta = p\,\hat{t}^i_\theta > \underline{w}_\theta(p) = \inf p\, T_\theta$ for all $(i,\theta) \in N \times \Theta$. Then the allocation $\hat{t}^{N\times\Theta}$ is self-selective if and only if $w^i_\theta \equiv w^i$, independent of θ, for all $(i,\theta) \in N \times \Theta$.*

PROOF. Suppose $w^i_\theta \equiv w^i$, independent of θ. Then given any $(i,\theta,\eta) \in N \times \Theta \times \Theta$, one has $p\,\hat{t}^i_\eta = w^i$. So the definition of a full WELT implies that $\hat{t}^i_\theta = h_\theta(p, w^i_\theta) \succsim_\theta \hat{t}^i_\eta$ whenever $\hat{t}^i_\eta \in T_\theta$. Hence, the allocation $\hat{t}^{N\times\Theta}$ is self-selective.

Conversely, suppose $(\hat{t}^{N\times\Theta}, \hat{p})$ is a full WELT in which the allocation $\hat{t}^{N\times\Theta}$ is self-selective. Consider any fixed $i \in N$. Let $(\theta_n)_{n=1}^\infty$ be any convergent sequence with limit θ. By smoothness condition (c) in Section 2.8, the WELT net trade vectors $h_{\theta_n}(p, w^i_{\theta_n})$ and $h_\theta(p, w^i_\theta)$ are interior points of T_{θ_n} and T_θ

respectively. By part (2) of Lemma 5, one has $h_{\theta_n}(p, w^i_{\theta_n}) \in T_\theta$ and $h_\theta(p, w^i_\theta) \in T_{\theta_n}$ for all large n, so self-selection implies that

$$v_{\theta_n}(p, w^i_{\theta_n}) = u_{\theta_n}(h_{\theta_n}(p, w^i_{\theta_n})) \geq u_{\theta_n}(h_\theta(p, w^i_\theta)) \tag{47}$$

$$\text{and} \quad v_\theta(p, w^i_\theta) = u_\theta(h_\theta(p, w^i_\theta)) \geq u_\theta(h_{\theta_n}(p, w^i_{\theta_n})). \tag{48}$$

Because $w^i_\theta > \underline{w}_\theta(p)$, part (2) of Lemma 6 implies $w^i_\theta > \underline{w}_{\theta_n}(p)$ for all large n. Also, by smoothness conditions (b) and (d) of Section 2.8, the functions u_θ and v_θ are continuous w.r.t. θ. So given any small enough $\epsilon > 0$ and any sufficiently large n, (47) implies that

$$\begin{aligned} v_\theta(p, w^i_{\theta_n}) + \epsilon > v_{\theta_n}(p, w^i_{\theta_n}) &\geq u_{\theta_n}(h_\theta(p, w^i_\theta)) \\ &> u_\theta(h_\theta(p, w^i_\theta)) - \epsilon = v_\theta(p, w^i_\theta) - \epsilon, \end{aligned} \tag{49}$$

and so $v_\theta(p, w^i_{\theta_n}) \geq v_\theta(p, w^i_\theta)$. But $v_\theta(p, w)$ is strictly increasing in w, and so $w^i_{\theta_n} \geq w^i_\theta$. Therefore $w_* := \liminf_{n\to\infty} w^i_{\theta_n}$ exists and is finite, with $w_* \geq w^i_\theta$.

Also, because of smoothness condition (e), the constraint (48) implies that $w^i_{\theta_n}$ is bounded above. Hence, $w^* := \limsup_{n\to\infty} w^i_{\theta_n}$ exists and is finite, with

$$w^* \geq w_* \geq w^i_\theta > \underline{w}_\theta(p). \tag{50}$$

But some subsequence of $(\theta_n, w^i_{\theta_n})$ must converge to (θ, w^*). By smoothness condition (c), the corresponding subsequence of $h_{\theta_n}(p, w^i_{\theta_n})$ converges to $h_\theta(p, w^*)$. Taking the limit of (48) for this subsequence implies that

$$v_\theta(p, w^i_\theta) \geq u_\theta(h_\theta(p, w^*)) = v_\theta(p, w^*).$$

Because $v_\theta(p, w)$ is strictly increasing w.r.t. w, it follows that $w^i_\theta \geq w^*$. In combination with (50), this shows that $w_* = w^i_\theta = w^*$. By definition of w_* and w^*, it follows that $w^i_{\theta_n} \to w^i_\theta$. Because θ_n was an arbitrary convergent sequence, this proves that w^i_θ is a continuous function of θ.

Next, consider any piecewise $\mathcal{C}^1$-arc $s \mapsto \theta(s)$ mapping $[0, 1]$ to Θ. Suppose that $s \mapsto \theta(s)$ is $\mathcal{C}^1$ in the open interval $(\underline{s}, \bar{s}) \subset [0, 1]$. Consider any disjoint pair $s, s' \in (\underline{s}, \bar{s})$. Because $v_{\theta(s)}(p, w)$ and $v_{\theta(s')}(p, w)$ are both $\mathcal{C}^1$ functions of w, there must exist w, w' in the interval between $w^i_{\theta(s)}$ and $w^i_{\theta(s')}$ such that

$$v_{\theta(s)}(p, w^i_{\theta(s')}) - v_{\theta(s)}(p, w^i_{\theta(s)}) = (w^i_{\theta(s')} - w^i_{\theta(s)})\,(v_{\theta(s)})'_w(p, w) \tag{51}$$

$$\text{and } v_{\theta(s')}(p, w^i_{\theta(s')}) - v_{\theta(s')}(p, w^i_{\theta(s)}) = (w^i_{\theta(s')} - w^i_{\theta(s)})\,(v_{\theta(s')})'_w(p, w'). \tag{52}$$

Because w^i_θ is continuous in θ, smoothness condition (d) implies that $v_\theta(p, w^i_\theta)$ is also continuous in θ. Then smoothness condition (c) and Lemma 5 together imply that $h_{\theta(s')}(p, w^i_{\theta(s')}) \in T_{\theta(s)}$ and $h_{\theta(s)}(p, w^i_{\theta(s)}) \in T_{\theta(s')}$ whenever s' is sufficiently close to s. Consequently the self-selection constraints and the definitions of $v_{\theta(s)}$, $v_{\theta(s')}$ imply that

$$v_{\theta(s)}(p, w^i_{\theta(s)}) = u_{\theta(s)}(h_{\theta(s)}(p, w^i_{\theta(s)})) \geq u_{\theta(s)}(h_{\theta(s')}(p, w^i_{\theta(s')})) \tag{53}$$

$$\text{and} \quad v_{\theta(s')}(p, w^i_{\theta(s')}) = u_{\theta(s')}(h_{\theta(s')}(p, w^i_{\theta(s')})) \geq u_{\theta(s')}(h_{\theta(s)}(p, w^i_{\theta(s)})). \tag{54}$$

Dividing (52) by the positive number $(v_{\theta(s)})'_w(p, w')$ and (51) by $(v_{\theta(s')})'_w(p, w)$, then using (54) and (53) respectively, one can derive

$$\begin{aligned} &\frac{u_{\theta(s')}(h_{\theta(s)}(p, w^i_{\theta(s)})) - u_{\theta(s')}(h_{\theta(s')}(p, w^i_{\theta(s)}))}{(v_{\theta(s')})'_w(p, w')} \leq w^i_{\theta(s')} - w^i_{\theta(s)} \\ \leq\ &\frac{u_{\theta(s)}(h_{\theta(s)}(p, w^i_{\theta(s')})) - u_{\theta(s)}(h_{\theta(s')}(p, w^i_{\theta(s')}))}{(v_{\theta(s)})'_w(p, w)}. \end{aligned}$$

Next, divide each term of these inequalities by $s' - s$ (when $s' \neq s$) and then take the limit as $s' \to s$. Because then $\theta(s') \to \theta(s)$, the smoothness conditions of Section 2.8 imply that the derivative $\frac{\mathrm{d}}{\mathrm{d}s} w^i_{\theta(s)}$ exists and also that

$$\begin{aligned} (v_\theta)'_w(p, w^i) \frac{\mathrm{d}}{\mathrm{d}s} w^i_{\theta(s)} &= -\frac{\mathrm{d}}{\mathrm{d}s} u_\theta(h_{\theta(s)}(p, w^i)) \\ &= -\sum_{g \in G} (u_\theta)'_g(h_{\theta(s)}(p, w^i)) \frac{\mathrm{d}}{\mathrm{d}s} h_{\theta(s)g}(p, w^i) \end{aligned} \tag{55}$$

where $\theta = \theta(s)$ and $w^i = w^i_{\theta(s)}$ with s fixed, and where $(u_\theta)'_g$ denotes $\partial u_\theta / \partial t_g$. But the usual first-order conditions for utility maximization subject to the constraint $p\,t \leq w^i$ are $(u_\theta)'_g = (v_\theta)'_w\, p_g$ for each $g \in G$. So (55) simplifies to

$$\frac{\mathrm{d}}{\mathrm{d}s} w^i_{\theta(s)} = -\sum_{g \in G} p_g \frac{\mathrm{d}}{\mathrm{d}s} h_{\theta(s)g}(p, w^i). \tag{56}$$

Differentiating the budget identity $w^i \equiv \sum_{g \in G} p_g\, h_{\theta(s)g}(p, w^i)$ w.r.t. s shows, however, that the right-hand side of (56) must equal zero. Hence, $\frac{\mathrm{d}}{\mathrm{d}s} w^i_{\theta(s)} = 0$ along any $\mathcal{C}^1$ path $s \mapsto \theta(s)$ in Θ, implying that $w^i_{\theta(s)}$ is constant throughout the interval $(\underline{s}, \bar{s})$. By continuity, w^i_θ must also be constant along any piecewise $\mathcal{C}^1$ arc in the type domain Θ. So the hypothesis in Section 2.8 that Θ is piecewise $\mathcal{C}^1$-arc connected implies that w^i_θ is independent of θ everywhere. □

The following example shows that Lemma 68 may not hold if the type domain Θ is not piecewise $\mathcal{C}^1$-arc connected, as required by the smoothness assumption of Section 2.8.

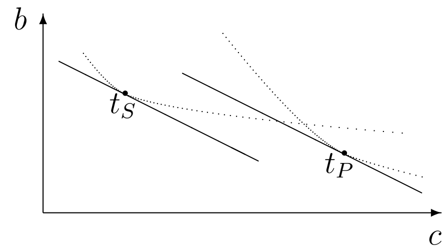

Fig. 2. Self-Selective Allocation of Books and Conspicuous Consumption

Example 69 Suppose there are two goods — books (b) and a second good (c) which we will call "conspicuous consumption", following Veblen. There are two types of agents, the scholarly (S) and the prodigal (P). All agents of the same type receive the same allocation of books and of conspicuous consumption. The consumption vectors of the two types are t_S and t_P respectively, as illustrated in Fig. 2. In particular, the scholarly receive relatively more books and the prodigal receive relatively more conspicuous consumption. This allocation is self-selective and also a WELT, given that the agents of each type have their common allocations supported by parallel budget lines touching their appropriate convex indifference curves, as indicated in Fig. 2. But the allocation is not a WE. In fact, there is redistribution of wealth from agents of type S to those of type P, with $w^i_P = w_P > w_S = w^i_S$ for all $i \in N$.

13.4 Fully fair allocations

It is easy to extend Foley's (1967) definition of fairness to a continuum economy. First, say that the allocation $\hat{t}^{N\times\Theta}$ is *envy free* if

$$(\nu \times \nu)\Big(\{\,(i,\theta,j,\eta) \in N \times \Theta \times N \times \Theta \mid \hat{t}^j_\eta \in P_\theta(\hat{t}^i_\theta)\,\}\Big) = 0.$$

Following Varian (1974), say that the allocation is *fair* if it is Pareto efficient and envy free.

As with Lemma 68, it is easier to work with strengthened definitions. Thus, say that the allocation $\hat{t}^{N\times\Theta}$ is *fully envy free* if $\hat{t}^i_\theta \succsim_\theta \hat{t}^j_\eta$ for all combinations (i,j,θ,η) such that $i \neq j$ and $\hat{t}^j_\eta \in T_\theta$. And that the allocation is *fully fair* if it is fully f-Pareto efficient and fully envy free.

The following simplifies the main result of Champsaur and Laroque (1981):

Theorem 70 *Suppose the type domain Θ is smooth. Any interior allocation $\hat{t}^{N\times\Theta}$ is fully fair if and only if it is a full WE for a suitable price vector p.*

PROOF. Following the discussion in Section 13.1, it is obvious that any full WE allocation is fully f-Pareto efficient and fully envy free, so fully fair.

To prove the converse, note that because Θ is smooth and the interior allocation $\hat{t}^{N\times\Theta}$ is fully f-Pareto efficient, the argument that concluded Section 13.1 implies that there must exist a price vector $p \gg 0$ such that $(\hat{t}^{N\times\Theta}, p)$ is a full WELT. There is an associated wealth distribution rule $w^{N\times\Theta}$ specified by the measurable function $(i,\theta) \mapsto w^i_\theta$ where $w^i_\theta := p\,\hat{t}^i_\theta > \underline{w}_\theta(p)$ for all $(i,\theta) \in N \times \Theta$, and also $\int_{N\times\Theta} w^i_\theta \, \mathrm{d}\nu = 0$.

In addition, because the allocation is fully envy-free, for each fixed $\theta \in \Theta$ it must satisfy $\hat{t}^i_\theta \succsim_\theta \hat{t}^j_\theta$ for all pairs $i, j \in N$. Now consider the unit interval N as a type domain. Given the fixed $\theta \in \Theta$, the set N trivially satisfies the assumptions of Section 2.8 needed to make it a smooth type domain. So Lemma 68 can be applied with N instead of Θ as the type domain. It follows that $w^i_\theta \equiv w_\theta$ and so $\hat{t}^i_\theta = h_\theta(p, w_\theta) =: \bar{t}_\theta$, both independent of i.

Again, because the allocation is fully envy-free, it follows that $\bar{t}_\theta \succsim_\theta \bar{t}_\eta$ for all $(\theta, \eta) \in \Theta \times \Theta$ such that $\bar{t}_\eta \in T_\theta$. Now Lemma 68 can be applied once more to the self-selective allocation $(i,\theta) \mapsto \bar{t}_\theta$ with Θ as the smooth type domain. It implies that $w_\theta \equiv w$, independent of θ. But $\int_{N\times\Theta} w^i_\theta \, \mathrm{d}\nu = 0$, so $w = 0$. It follows that $(\hat{t}^{N\times\Theta}, p)$ is a full WE. □

14 Strategyproof mechanisms in a continuum economy

14.1 Individual strategyproofness

Section 7.4.1 briefly considered strategyproof allocation mechanisms in finite economies, and cited the negative results of Serizawa and Weymark (2003) in particular. In continuum economies, statistical or not, the picture is much more rosy (Hammond, 1979).

When characterizing WE allocations, it is natural to consider a restricted domain of statistical continuum economies for which a WE exists. Accordingly, let $\mathcal{D} \subset \mathcal{M}_\lambda(N \times \Theta)$ denote a domain of measures ν with $\mathrm{marg}_N\, \nu = \lambda$. Given this domain, define an *allocation mechanism* as a mapping $f : \mathcal{D} \times N \times \Theta \to \mathbb{R}^G$ satisfying $f(\nu, i, \theta) \in T_\theta$ for ν-a.e. (i,θ) in $N \times \Theta$ which, for each fixed $\nu \in \mathcal{D}$, also has the property that the mapping $(i,\theta) \mapsto f(\nu, i, \theta)$ is ν-integrable with

$\int_{N\times\Theta} f(\nu,i,\theta)\,\mathrm{d}\nu = 0$. Thus, for each $\nu \in \mathcal{D}$, the mapping $(i,\theta) \mapsto t^i_\theta(\nu) := f(\nu,i,\theta)$ determines a feasible allocation $t^{N\times\Theta}(\nu)$.

Such an allocation mechanism is said to be (individually) *strategyproof* if, for all $\nu \in \mathcal{D}$ and all $(i,\theta) \in N \times \Theta$, it satisfies the *incentive constraint*

$$f(\nu,i,\theta) \succsim_\theta f(\nu,i,\tilde{\theta}) \text{ for all } \tilde{\theta} \text{ such that } f(\nu,i,\tilde{\theta}) \in T_\theta. \tag{57}$$

This formulation reflects the negligible influence each individual in a continuum economy has on the joint distribution ν of labels $i \in N$ and apparent types $\theta \in \Theta$. In fact, the mechanism f is strategyproof if and only if, for each $\nu \in \mathcal{D}$, the allocation $t^{N\times\Theta}(\nu)$ it generates satisfies the self-selection constraints $t^i_\theta \succsim_\theta t^i_\eta$ whenever $(i,\theta,\eta) \in N\times\Theta\times\Theta$ satisfy $t^i_\eta \in T_\theta$. This convenient property is generally false in finite economies.

Note that the incentive constraint (57) is required to hold for *all* $(i,\theta) \in N \times \Theta$ without exception, not merely for ν-a.e. (i,θ). This strong definition simplifies later results. The strengthening is essentially harmless because the decentralization theorem that follows gives weak sufficient conditions allowing a mechanism that satisfies (57) for ν-a.e. $(i,\theta) \in N \times \Theta$ to be adapted easily by choosing an appropriate value of $f(\nu,i,\theta)$ on the exceptional null set so that the new mechanism satisfies (57) for all (i,θ). In this connection, it is instructive to compare the contrasting approaches and results of Mas-Colell and Vives (1993) with those of Guesnerie (1995).

Several later results specifically concern *anonymous* mechanisms, defined as satisfying the symmetry requirement that f be independent of i.[31]

14.2 Decentralization theorem

A *budget correspondence* is a mapping $B : \mathcal{D} \times N \twoheadrightarrow \mathbb{R}^G$ that specifies each individual's budget set $B^i(\nu)$ in $\mathbb{R}^G$ as a function of $i \in N$ and $\nu \in \mathcal{D}$, with the property that the set $\{\, (i,t) \in N \times \mathbb{R}^G \mid t \in B^i(\nu) \,\}$ should be measurable for every $\nu \in \mathcal{D}$. Note that the set $B^i(\nu)$ is required to be *independent* of $\theta \in \Theta$. It can depend on the identifier i, however, which is observable by definition. For example, $B^i(\nu)$ can depend on agent i's observable characteristics, such as date of birth or other officially recorded demographic events. Indeed, most social security systems and other pension schemes have exactly this feature.

[31] Formally, as Guesnerie (1995) in particular points out, anonymity requires not only *recipient anonymity*, with $f(\nu,i,\theta)$ independent of i, but also the *anonymity in influence*, with $f(\nu,i,\theta) \equiv \phi(\mu,i,\theta)$ where μ is the marginal distribution on Θ induced by the joint distribution ν on $N\times\Theta$. This distinction makes little difference to the following results, however.

The budget correspondence $(\nu, i) \mapsto\!\!\!\to B^i(\nu)$ is said to *decentralize* the mechanism $f(\nu, i, \theta)$ provided that for all $\nu \in \mathcal{D}$ and all $(i, \theta) \in N \times \Theta$, one has $f(\nu, i, \theta) \in B^i(\nu)$ and also

$$t \in B^i(\nu) \cap T_\theta \implies f(\nu, i, \theta) \succsim_\theta t.$$

The following very simple characterization is taken from Hammond (1979).

Theorem 71 *The mechanism $f(\nu, i, \theta)$ is strategyproof if and only if it can be decentralized.*

PROOF. First, suppose $B^i(\nu)$ decentralizes f. Because $f(\nu, i, \tilde{\theta}) \in B^i(\nu)$, obviously $f(\nu, i, \theta) \succsim_\theta f(\nu, i, \tilde{\theta})$ whenever $f(\nu, i, \tilde{\theta}) \in T_\theta$. So the incentive constraints (57) are satisfied.

Conversely, construct the set

$$B^i(\nu) := f(\nu, i, \Theta) := \{\, t \in \mathbb{R}^G \mid \exists \theta \in \Theta : t = f(\nu, i, \theta) \,\} \tag{58}$$

as the range of f as θ varies over Θ, with (ν, i) fixed. Strategyproofness implies that this must be a decentralization. □

Guesnerie (1995) has a very similar result, which he calls the "taxation principle" because it implies, for instance, that an allocation mechanism which is used to redistribute wealth and/or to finance the provision of public goods will be strategyproof if and only if it presents all indistinguishable agents with the same (generally non-linear) budget constraint after the effects of all taxes and subsidies are taken into account.

Obviously many different budget correspondences are possible, giving rise to many different strategyproof mechanisms. Indeed, suppose preferences are continous and the budget correspondence has values which make $B^i(\nu) \cap T_\theta$ a non-empty compact subset of $\mathbb{R}^G$ throughout the domain $\mathcal{D} \times N \times \Theta$. Then Theorem 71 implies that there exists a strategyproof mechanism satisfying

$$f(\nu, i, \theta) \in \{\, t \in B^i(\nu) \cap T_\theta \mid t' \in B^i(\nu) \cap T_\theta \Longrightarrow t \succsim_\theta t' \,\}$$

for all $(\nu, i, \theta) \in \mathcal{D} \times N \times \Theta$. In particular, there is the *Walrasian* budget correspondence with lump-sum transfers. This is defined by

$$B^i(\nu) := \{t \in \mathbb{R}^G \mid p(\nu)\, t \leq w^i(p(\nu), \nu)\}$$

where $p(\nu)$ should be chosen to clear markets given the wealth distribution rule $(i, p, \nu) \mapsto w^i(p, \nu)$ — which must be ν-integrable and satisfy $\int_N w^i(p, \nu)\, d\lambda =$

0 for all (p, ν). An obvious corollary of Theorem 71 is that the associated mechanism which generates WELT allocations must be strategyproof.

Suppose that agents' preferences are LNS, continuous and convex, and that the allocation mechanism $f(\nu, i, \theta)$ is anonymous and strategyproof, while also generating Pareto efficient interior allocations (which must therefore be WELT allocations). Then Theorem 71 implies the existence of a decentralization $B(\nu)$, independent of i and θ. Nevertheless, the mechanism need not generate WE allocations, without lump-sum transfers, as Example 69 in Section 13.3 illustrates. Note that the decentralization in that example could be the two-point set $B(\nu) = \{ t_S, t_P \}$, or some obvious variation including these points. It does not have to be, and actually cannot be, a Walrasian budget decentralization with one common budget line facing both types of agent. This example can be avoided, however, under the smooth type domain hypothesis set out in Section 2.8.

14.3 Limits to redistribution

The following Lemma is based on extensions and corrections by Champsaur and Laroque (1981, 1982) of a result stated by Hammond (1979) and also partly corrected in the appendix to Hammond (1987).

Lemma 72 *Suppose that Θ is a smooth type domain. Then the mechanism $f : \mathcal{D} \times N \times \Theta$ is strategyproof and yields fully f-Pareto efficient interior allocations if and only if, for each $\nu \in \mathcal{D}$ it generates a fully WELT allocation relative to a wealth distribution rule $w^{N\times\Theta}(\nu)$ satisfying $w^i_\theta(\nu) \equiv w^i(\nu)$, independent of θ, where $w^i(\nu) > \underline{w}_\theta(p(\nu)) := \inf p(\nu)\, T_\theta$ for all $(\nu, i, \theta) \in \mathcal{D} \times N \times \Theta$.*

PROOF. Sufficiency follows from Theorem 71, with $B^i(\nu)$ as the Walrasian budget set $\{ t \in \mathbb{R}^G \mid p(\nu)\, t \leq w^i(p(\nu), \nu) \}$ for each $\nu \in \mathcal{D}$ and $i \in N$, where $w^N(p, \nu)$ is the wealth distribution rule, independent of θ, and where $p = p(\nu)$ denotes any WELT price vector that solves $\int_{N\times\Theta} h_\theta(p, w^i(p, \nu))\, d\nu = 0$. Then smoothness condition (c) in Section 2.8 ensures that the fully WELT allocation is interior.

Conversely, fix any $\nu \in \mathcal{D}$ and suppose the allocation $\hat{t}^{N\times\Theta}(\nu)$ defined by $\hat{t}^i_\theta(\nu) := f(\nu, i, \theta)$ is fully f-Pareto efficient and interior. By the argument that concludes Section 13.1, it must generate a fully WELT allocation which is decentralizable by budget constraints of the form $p(\nu)\, t \leq w^i_\theta(\nu)$ for all $(i, \theta) \in N \times \Theta$. Because of interiority, note that $w^i_\theta(\nu) > \underline{w}_\theta(p(\nu))$ for all $(i, \theta) \in N \times \Theta$. Also, strategyproofness implies that the allocation $\hat{t}^{N\times\Theta}(\nu)$ must be self-selective, as defined in Section 13.2. The result follows from Lemma 68. □

14.4 Anonymity

The limits to redistribution presented in Section 14.3 make it easy to derive our main characterization of WE allocation mechanisms on smooth type domains.

Theorem 73 *Suppose Θ is a smooth type domain, and the allocation mechanism $f(\nu, i, \theta)$ is anonymous and strategyproof. Suppose too that the resulting allocation $f(\nu, i, \theta)$ is interior and fully f-Pareto efficient for each $\nu \in \mathcal{D}$. Then the allocation defined by $f(\nu, i, \theta)$ must be a full WE for each $\nu \in \mathcal{D}$.*

PROOF. Given the wealth distribution rule $w^i(\nu)$ of Lemma 72, one has $f(\nu, i, \theta) = h_\theta(p(\nu), w^i(\nu))$ for all (i, θ). For each pair $i, j \in N$, anonymity implies that $h_\theta(p(\nu), w^i(\nu)) = h_\theta(p(\nu), w^j(\nu))$, so

$$w^i(\nu) = p(\nu)\, h_\theta(p(\nu), w^i(\nu)) = p(\nu)\, h_\theta(p(\nu), w^j(\nu)) = w^j(\nu).$$

Hence $w^i(\nu) = w(\nu)$, independent of i, for all $i \in N$. But $\int_{N \times \Theta} w^i(\nu)\, \mathrm{d}\nu = 0$, so $w(\nu) = 0$. The allocation must therefore be a full WE. $\square$

There is an obvious converse of Theorem 73 — with a smooth type domain, any mechanism generating a full WE allocation for each $\nu \in \mathcal{D}$ must be anonymous. Indeed, this is true whenever preferences are merely strictly convex.

14.5 Weak gains from trade with an inclusive type domain

The second characterization of a WE allocation mechanism for a smooth type domain Θ relies on two additional conditions.

Say that the mechanism $f(\nu, i, \theta)$ satisfies *full weak gains from trade* provided that $f(\nu, i, \theta) \succsim_\theta 0$ for all $(\nu, i, \theta) \in \mathcal{D} \times N \times \Theta$.

Say that the smooth type domain Θ is *inclusive* if

$$\{\, u'_\theta(0)/\|u'_\theta(0)\| \mid \theta \in \Theta \,\} = \Delta^0$$

— i.e.., if the domain of normalized gradient vectors at the autarky allocation to the utility functions u_θ is equal to the relative interior of the unit simplex Δ in $\mathbb{R}^G$. Equivalently, given any $p \gg 0$, there must exist $\theta \in \Theta$ such that $h_\theta(p, 0) = 0$.

Theorem 74 *Suppose there is an inclusive smooth type domain Θ. Suppose too that $f(\nu, i, \theta)$ is an interior, strategyproof, and fully f-Pareto efficient* *mechanism that satisfies full weak gains from trade. Then the allocation defined by $f(\nu, i, \theta)$ must be a WE for each $\nu \in \mathcal{D}$.*

PROOF. By Lemma 72, given any fixed $\nu \in \mathcal{D}$, the allocation $t^{N\times\Theta}(\nu)$ defined by $t^i_\theta(\nu) := f(\nu, i, \theta)$ for all $(i, \theta) \in N \times \Theta$ must be a full WELT at some price vector $p(\nu) \gg 0$ relative to some wealth distribution rule $w^N(\nu)$ that does not depend on θ. Full weak gains from trade imply that $f(\nu, i, \theta) = h_\theta(p(\nu), w^i(\nu)) \succsim_\theta 0$ for all $(i, \theta) \in N\times\Theta$. Because the type domain is inclusive, there exists $\theta \in \Theta$ such that $h_\theta(p(\nu), 0) = 0 \precsim_\theta h_\theta(p(\nu), w^i(\nu))$ and so $w^i(\nu) \geq 0$ for all $i \in N$. But the wealth distribution rule must satisfy $\int_N w^i(\nu)\, \mathrm{d}\lambda = 0$, so $w^i(\nu) = 0$ for λ-a.e. $i \in N$. Hence, each full WELT must be a WE, without lump-sum transfers. □

14.6 Strategyproof mechanisms of maximal dimension

Say that the mechanism $f(\nu, i, \theta)$ has *dimension* n at ν if for ν-a.e. $(i, \theta) \in N \times \Theta$ there exists a set V^i_θ which is homeomorphic to an open ball in $\mathbb{R}^n$ such that $f(\nu, i, \theta) \in V^i_\theta \subset f(\nu, i, \Theta)$, where $f(\nu, i, \Theta)$ is the range set defined in (58). Because a Walrasian mechanism satisfies $p(\nu)f(\nu, i, \theta) = 0$ for ν-a.e. (i, θ), it has dimension $n \leq \#G - 1$ at each ν in its domain. The main result of Dubey, Mas-Colell and Shubik (1980) suggests the following: [32]

Theorem 75 *Suppose that agents' preferences are monotone and semi-strictly convex. Suppose too that f is an anonymous strategyproof mechanism whose dimension at ν is $n \geq \#G - 1$. Then the corresponding allocation $t^{N\times\Theta}(\nu)$ must be a symmetric full WE.*

[32] Dubey, Mas-Colell and Shubik consider a *strategic market game* in which each player faces a strategy space S, assumed to be a subset of some separable Banach space. Four axioms called anonymity, continuity, convexity, and aggregation together guarantee that S is convex, and that the outcome of the market game is a feasible allocation $t^{N\times\Theta}$ with the property that almost every potential agent's net trade vector t^i_θ can be expressed as a jointly continuous function $F_{\bar{s}}(s^i_\theta)$ of their own strategy choice s^i_θ and of the mean strategy choice $\bar{s} := \int_{N\times\Theta} s^i_\theta\, \mathrm{d}\nu$. Theorem 75 below relies on considerably weaker versions of these assumptions. In particular, the aggregation axiom is dispensed with entirely. Nor is it assumed that $F_{\bar{s}}(s^i_\theta)$ is continuous w.r.t. $\bar{s}$. The latter generalization is important because there may be no continuous selection from the Walrasian equilibrium correspondence. On the other hand, we limit attention to strategyproof direct revelation mechanisms instead of general strategic market games.

PROOF. Fix the distribution ν and write $t^{N\times\Theta}$ instead of $t^{N\times\Theta}(\nu)$. Anonymity implies that $t^i_\theta \equiv f(\nu,\theta)$ for all $(i,\theta) \in N \times \Theta$. Let L denote the linear space spanned by the range set $f(\nu,\Theta)$. Evidently $t^i_\theta \in L$ for all $(i,\theta) \in N\times\Theta$.

Define K_1 and K_2 as the sets of potential agents $(i,\theta) \in N\times\Theta$ for whom there exist, respectively: (i) a vector $v \in L \cap P_\theta(t^i_\theta)$; (ii) a set V homeomorphic to an open ball in $\mathbb{R}^n$ such that $t^i_\theta \in V \subset f(\nu,\Theta)$. For any $(i,\theta) \in K_1 \cap K_2$ and any $\alpha \in (0,1)$ sufficiently small, the convex combination $\tilde{t}^i_\theta := (1-\alpha)t^i_\theta + \alpha v \in V$. Because preferences are semi-strictly convex, it follows that $\tilde{t}^i_\theta \in P_\theta(t^i_\theta) \cap f(\nu,\Theta)$. Strategyproofness implies, therefore, that $K_1 \cap K_2$ must be empty.

Consider any $(i,\theta) \in K_2$. Then $(i,\theta) \notin K_1$, implying that $t^i_\theta \succsim_\theta v$ whenever $v \in L\cap T_\theta$. But the mechanism has dimension n, so $\nu(K_2) = 1$. Because preferences are monotone and so non-satiated, it follows that $L \neq \mathbb{R}^G$. Therefore $n = \#G-1$. So there exists $p \neq 0$ such that L is the hyperplane $\{\, v \in \mathbb{R}^G \mid p\,v = 0\,\}$.

Now, for each $(i,\theta) \in K_2$, the two convex sets L and $P_\theta(t^i_\theta)$ must be disjoint, as well as non-empty, with $t^i_\theta \in L$. Because preferences are monotone, $\{t^i_\theta\} + \mathbb{R}^G_{++} \subset P_\theta(t^i_\theta)$ so t^i_θ is a boundary point of $P_\theta(t^i_\theta)$. It follows that the hyperplane L itself must separate L from $P_\theta(t^i_\theta)$. Also, after a suitable choice of the sign of p, one has $p\,t \geq 0$ for all $t \in \mathbb{R}^G_{++}$, so $p > 0$ and also $p\,t \geq 0$ for all $t \in P_\theta(t^i_\theta)$. Because L and $P_\theta(t^i_\theta)$ are disjoint, in fact $p\,t > 0$ for all $t \in P_\theta(t^i_\theta)$. But $\nu(K_2) = 1$, so $(t^{N\times\Theta}, p)$ is a WE. □

15 Manipulation by Finite Coalitions

15.1 Multilateral strategy proofness

In Example 69 of Section 13.3, though the allocation (t_S, t_P) is strategyproof, it could be manipulated by agents exchanging books for conspicuous consumption "on the side", after at least one scholar in a finite coalition has claimed to be prodigal. With this in mind, say that the allocation mechanism $f(\nu,i,\theta)$ is *multilaterally strategyproof* if, for each $\nu \in \mathcal{D}$, no finite coalition $C \subset N$ with (potential) types $\theta^i \in \Theta$ can find "manipulative" types $\tilde{\theta}^i \in \Theta$ and net trade vectors $t^i \in \mathbb{R}^G$ $(i \in C)$ satisfying $\sum_{i\in C} t^i = 0$ as well as $f(\nu,i,\tilde{\theta}^i) + t^i \in P_\theta\left(f(\nu,i,\theta^i)\right)$ for all $i \in C$.[33]

Considering the case when $\#C = 1$ and so $t^i = 0$, it is obvious that multilateral strategyproofness entails (individual) strategyproofness.

[33] Similar ideas have been applied in different contexts by Gale (1980, 1982), Guesnerie (1981, 1995), Jacklin (1987), Blackorby and Donaldson (1988), Haubrich (1988), Haubrich and King (1990), and Hammond (1987, 1999b).

Following Guesnerie (1981, 1995) in particular, a useful distinction can be made between:

(1) "non-exchangeable" goods for which non-linear pricing is possible;
(2) "exchangeable" goods for which trade on the side cannot be prevented, so only linear pricing is multilaterally strategyproof.

In order to characterize Walrasian equilibrium, the following results concentrate on the case when all goods are exchangeable.

Theorem 76 *If $f(\nu, i, \theta)$ selects a full WE allocation for each $\nu \in \mathcal{D}$, then f is multilaterally strategyproof.*

PROOF. For each $\nu \in \mathcal{D}$, there exists a WE price vector $p = p(\nu) \neq 0$ such that, for all $(i, \theta) \in N \times \Theta$, one has $pf(\nu, i, \theta) \leq 0$ and also $\hat{t} \succ_\theta f(\nu, i, \theta) \implies p\,\hat{t} > 0$. Suppose $C \subset N$ and, for all $i \in C$, the actual types $\theta^i \in \Theta$, potential types $\tilde{\theta}^i \in \Theta$, and net trade vectors $\tilde{t}^i \in \mathbb{R}^G$ together satisfy $f(\nu, i, \tilde{\theta}^i) + \tilde{t}^i \in P_{\theta^i}(f(\nu, i, \theta^i))$. Then $p\,[f(\nu, i, \tilde{\theta}^i) + \tilde{t}^i] > 0 \geq p\,f(\nu, i, \tilde{\theta}^i)$ for all $i \in C$, implying that $\sum_{i \in C} p\,\tilde{t}^i > 0$. Thus $\sum_{i \in C} \tilde{t}^i = 0$ is impossible. □

Theorem 77 *Suppose preferences are LNS in $\mathbb{Q}^G$, as defined in Section 12.3. Suppose the mechanism $f(\nu, i, \theta)$ is multilaterally strategyproof for all $\nu \in \mathcal{D}$. Then each allocation $(i, \theta) \mapsto f(\nu, i, \theta)$ is a CELT w.r.t. a wealth distribution rule satisfying $w^i_\theta(\nu) = w^i(\nu)$, independent of θ, with $\int_N w^i(\nu)\,\mathrm{d}\lambda = 0$.*

PROOF. Given any fixed $\nu \in \mathcal{D}$, we adapt the proof of Theorem 62 showing that any f-core allocation is a CE. To do so, first define

$$M := \{ (i, \theta, t) \in N \times \Theta \times \mathbb{R}^G \mid \exists \tilde{\theta} \in \Theta : f(\nu, i, \tilde{\theta}) + t \succ_\theta f(\nu, i, \tilde{\theta}) \}.$$

For each (i, θ, t), define the associated sections

$$\begin{aligned} M^i_\theta &:= \{ t' \in \mathbb{Q}^G \mid (i, \theta, t') \in M \} \\ \text{and} \quad M(t) &:= \{ (i', \theta') \in N \times \Theta \mid (i', \theta', t) \in M \} \end{aligned}$$

of the set M. Next, define

$$K' := \cup \{ M(t) \mid t \in \mathbb{Q}^G,\ \nu(M(t)) > 0 \},$$

which must satisfy $\nu(K') = 1$. Then let C be the convex hull of $\cup_{(i,\theta) \in K'} M^i_\theta$.

Arguing as in the proof of Theorem 62, multilateral strategyproofness implies that $0 \notin C$. Hence, for each fixed $\nu \in \mathcal{D}$, there exists a separating price vector

$p(\nu) \neq 0$ such that, for all $(i, \theta) \in K'$, one has $p(\nu)\, t \geq 0$ whenever $t \in M^i_\theta$ because $t \in \mathbb{Q}^G$ and there exists $\tilde{\theta} \in \Theta$ such that $f(\nu, i, \tilde{\theta}) + t \succsim_\theta f(\nu, i, \theta)$.

In particular, putting $t = f(\nu, i, \theta) - f(\nu, i, \tilde{\theta})$ implies that $p(\nu)\, t \geq 0$ and so $p(\nu)\, f(\nu, i, \theta) \geq p(\nu)\, f(\nu, i, \tilde{\theta})$ whenever $(i, \theta), (i, \tilde{\theta}) \in K'$. Because θ and $\tilde{\theta}$ can be interchanged, this makes it possible to define $w^i(\nu)$ for each $i \in N$ so that $p(\nu)\, f(\nu, i, \theta) = w^i(\nu)$, independent of θ, for all $(i, \theta) \in K'$. Also, when $\tilde{\theta} = \theta$, putting $\tilde{t} = f(\nu, i, \theta) + t$ implies that, for all $(i, \theta) \in K'$, whenever $\tilde{t} \succsim_\theta f(\nu, i, \theta)$, then $p(\nu)\, t \geq 0$ and so $p(\nu)\, \tilde{t} \geq p(\nu)\, f(\nu, i, \theta)$. Because $\nu(K') = 1$, this shows that $f(\nu, i, \theta)$ is a CELT at prices $p(\nu)$, as claimed. □

Corollary 78 *Suppose preferences are LNS in $\mathbb{Q}^G$. If the anonymous mechanism $f(\nu, \theta)$ is multilaterally strategyproof, then it produces CE allocations.*

PROOF. By Theorem 77, for each fixed $\nu \in \mathcal{D}$ the allocation generated by $f(\nu, \theta)$ must be a CELT w.r.t. a wealth distribution rule satisfying $w^i_\theta(\nu) \equiv w(\nu)$, independent of both i and θ, for ν-a.e. $(i, \theta) \in N \times \Theta$. But then $\int_{N \times \Theta} w^i_\theta(\nu)\, d\nu = 0$, so $w(\nu) = 0$. □

The conditions discussed in Sections 11.6 and 11.8 can be used to ensure that the CE allocations are WE.

15.2 Arbitrage-free allocations

Manipulation by finite coalitions can be regarded as one form of arbitrage. For finite economies, a second form of arbitrage was considered in Section 6.3. Yet another form is the subject of Makowski and Ostroy (1998).

Consider a particular feasible allocation $\hat{t}^{N \times \Theta}$. Let $\mathcal{F}$ denote the family of all finite subsets of $N \times \Theta$, and define the "arbitrage opportunity set"

$$\hat{Z} := \bigcup_{K \in \mathcal{F}} \sum_{(i,\theta) \in K} -[R_\theta(\hat{t}^i_\theta) - \{\hat{t}^i_\theta\}]. \tag{59}$$

Thus $z \in \hat{Z}$ if and only if there is a finite coalition $K \subset N \times \Theta$ of potential agents, together with a collection $a^i_\theta \in \mathbb{R}^G$ $((i, \theta) \in K)$ of "potential arbitrage" net trade vectors satisfying $\hat{t}^i_\theta + a^i_\theta \in R_\theta(\hat{t}^i_\theta)$ for all $(i, \theta) \in K$, such that $z = -\sum_{(i,\theta) \in K} a^i_\theta$. Strengthening and also considerably simplifying Makowski and Ostroy's key definition, say that the allocation $\hat{t}^{N \times \Theta}$ is a *full arbitrage* *equilibrium* if $\hat{t}^i_\theta \succsim_\theta z$ for all $z \in \hat{Z} \cap T_\theta$ and all $(i, \theta) \in N \times \Theta$.[34] In such an equilibrium, no agent can benefit from potential arbitrage net trades.

Next, say that the allocation $\hat{t}^{N \times \Theta}$ is *fully perfectly competitive* if it is a full arbitrage equilibrium, and if there exists $p \neq 0$ such that the half-space

$$H^-(p) := \{ z \in \mathbb{R}^G \mid p\, z \leq 0 \} \tag{60}$$

is the closure of the set $\hat{Z}$. Makowski and Ostroy impose assumptions which imply in particular a *boundary assumption* requiring the strict preference set $P_\theta(t)$ to be open, for all $\theta \in \Theta$ and $t \in T_\theta$. Their major result is that, for generic continuum economies, the "flattening effect of large numbers" of agents ensures that the closure of the set $\hat{Z}$ is indeed such a half-space. So, if the allocation $\hat{t}^{N \times \Theta}$ is an arbitrage equilibrium, in generic economies it is also perfectly competitive. This leads to a characterization of full Walrasian equilibrium because of a result which, in the simplified framework considered here, takes the form:

Theorem 79 *(1) Suppose that preferences are LNS and continuous, and that they satisfy the boundary assumption. Then any fully perfectly competitive allocation is a WE. (2) Provided that preferences are LNS, any full WE is a full arbitrage equilibrium.*

PROOF. (1) Suppose the allocation $\hat{t}^{N \times \Theta}$ is fully perfectly competitive. Let $p \neq 0$ be a price vector such that the half-space $H^-(p)$ defined in (60) is the closure of the set $\hat{Z}$ defined in (59).

Consider any $(i, \theta) \in N \times \Theta$. Suppose that $t \in P_\theta(\hat{t}^i_\theta)$. By the boundary assumption, there exists an open set V in $\mathbb{R}^G$ such that $t \in V \subset P_\theta(\hat{t}^i_\theta)$. But $\hat{t}^{N \times \Theta}$ is fully perfectly competitive, so $P_\theta(\hat{t}^i_\theta) \subset T_\theta \setminus \hat{Z}$. It follows that $\hat{Z} \subset T_\theta \setminus V$. But $H^-(p)$ is the closure of $\hat{Z}$ and $T_\theta \setminus V$ is closed, so $H^-(p) \subset T_\theta \setminus V$. Because $V \subset T_\theta$, this implies that $p\, t' > 0$ for each $t' \in V$, including t.

Because preferences are LNS, the previous paragraph shows that $p\, \hat{t}^i_\theta \geq 0$. This is true for all $(i, \theta) \in N \times \Theta$. But the allocation $\hat{t}^{N \times \Theta}$ is feasible, so $p \int_{N \times \Theta} \hat{t}^i_\theta \, d\nu = 0$, which is only possible when $p\, \hat{t}^i_\theta = 0$ for ν-a.e. $(i, \theta) \in N \times \Theta$. Hence, $(\hat{t}^{N \times \Theta}, p)$ must be a WE.[35]

[34] Makowski and Ostroy's (1998) actual definition of an "arbitrage equilibrium" in effect requires this condition to hold for almost all $(i, \theta) \in N \times \Theta$. It also limits the finite coalitions K in a way that depends on the support of the distribution on $\mathbb{R}^G \times \Theta$ that is induced by the allocation $\hat{t}^{N \times \Theta}$.

[35] It may not be a full WE because one could have $p\, \hat{t}^i_\theta > 0$ on a null subset of $N \times \Theta$.

(2) Suppose $(\hat{t}^{N\times\Theta}, p)$ is a full WE. Because preferences are LNS, it is a full CE also. So for any $(i,\theta) \in N \times \Theta$, one has $p\,t \geq 0 = p\,\hat{t}^i_\theta$ whenever $t \in R_\theta(\hat{t}^i_\theta)$. Hence $p\,z \leq 0$ whenever $z \in -[R_\theta(\hat{t}^i_\theta) - \{\hat{t}^i_\theta\}]$, and so whenever $z \in \hat{Z} \cap T_\theta$. It follows that $\hat{t}^{N\times\Theta}$ is a full arbitrage equilibrium. □

16 Other environments

16.1 Public goods

Competitive market mechanisms are unlikely to perform well when there are public goods. Nevertheless, versions of the efficiency theorems in Sections 4 do hold. Indeed, Foley (1970) and Milleron (1972) suggest how an economy with public goods could be regarded as a special kind of economy with private goods that include "personalized" copies of each public good. Because these copies have to be produced in equal amounts for all consumers, the resulting private good economy cannot have even weakly monotone preferences. Yet most of the results in Sections 4 and 5 for Walrasian equilibria in finite economies with private goods do not rely on preferences being monotone. So these results extend immediately to corresponding results for Lindahl or ratio equilibria (Kaneko, 1977) in economies with both public and private goods. So may some of the later results in Sections 6 and 7, though this remains to be investigated.

One complication that does arise is that, as the number of agents increases, so does the number of dimensions in the relevant commodity space, including the personalized copies of each public good. For this reason, results like those in Section 7 are not easy to extend to public goods. In particular, the core of a replica economy with public goods may not shrink fast enough to exclude lump-sum transfers, even in the limit. Nevertheless van den Nouweland, Tijs and Wooders (2002) do find a counterpart to the main result of Section 8.4.

When there is a continuum of agents, there is also a continuum of personalized copies of each public good, so the commodity space becomes infinite-dimensional. More specifically, Muench (1972) in particular has shown how core equivalence is lost. Strategyproofness, however, can be satisfied, at least formally, as discussed in Hammond (1979). Indeed, a generalization of Theorem 73 could be used to characterize Lindahl equilibria. Though formally strategyproof, however, these mechanisms for continuum economies lack counterparts that are approximately strategyproof in large finite economies — the free-rider problem is hard to overcome.

Other extensions to public goods of the results presented in this chapter are likely to be even less straightforward.

16.2 Externalities

In principle externalities can be treated rather like public goods, by adding dimensions to the commodity space. Appropriate Pigou taxes or subsidies on private activities that create each externality can then be used to help steer the economy toward a Pareto efficient allocation. The definition of an appropriately modified Walrasian equilibrium, however, may not be so straightforward, because the "property rights" which determine agents' endowments need to be specified. Do polluters have the right to create as much pollution as suits them, or should they be required to pay for all the damage they create?

A special case is when each agent's feasible set and preferences are determined by one or more aggregate externalities that get created by all agents together. Each such externality is effectively a public good (or public bad) for which Lindahl prices are appropriate. The right (or duty) to create such an externality, however, should be allocated by prices which can be regarded as per unit Pigou taxes or subsidies, the same for all agents. A detailed analysis is presented in Hammond (1998). A major complication pointed out by Starrett (1972) is that negative externalities of this kind are incompatible with convex production possibilities, so the usual second efficiency theorem is inapplicable.

Similar extensions to a continuum economy are fairly straightforward, with the aggregate externalities becoming "widespread externalities" of the kind considered by Kaneko and Wooders (1986, 1989, 1994), Hammond, Kaneko and Wooders (1989), and Hammond (1995, 1999a).

Though the allocations generated by competitive market mechanisms are unlikely to be Pareto efficient in the presence of externalities, they may nevertheless meet some weaker criterion of constrained Pareto efficiency. Grossman (1977) and Repullo (1988) considered forms of constrained Pareto efficiency that apply when markets are incomplete. Somewhat similar ideas are applied to economies with a continuum of agents and widespread externalities in Hammond (1995). Particularly appealing may be the main result of Hammond, Kaneko and Wooders (1989) — as well as the corresponding result in Hammond (1999a) — that characterizes f-core allocations as "Nash–Walrasian equilibria" in which each agent treats the aggregate externalities as fixed. This is consistent with using Pigou pricing to allocate efficiently agents' rights to contribute to those externalities. Some particular widespread externalities that arise in sequential environments with policy feedback are considered in Hammond (1999c).

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