Operation elements from indirect measurements

If the initial system state is

$$|\Psi_S\rangle = c_0|0_S\rangle + c_1|1_S\rangle,$$

we have a measurement for which

$$\begin{aligned}\Pr(0) &= |c_0a_0|0_S\rangle + c_1a_1|1_S\rangle|^2 = |c_0a_0|^2 + |c_1a_1|^2,\\ \Pr(1) &= |c_0a_1|0_S\rangle + c_1a_0|1_S\rangle|^2 = |c_0a_1|^2 + |c_1a_0|^2,\end{aligned}$$

and the post-measurement states are

$$\begin{aligned}|\Psi_{\text{post}}\rangle &= \frac{c_0a_0|0_S\rangle + c_1a_1|1_S\rangle}{\sqrt{\Pr(0)}} \qquad i = 0,\\ &= \frac{c_0a_1|0_S\rangle + c_1a_0|1_S\rangle}{\sqrt{\Pr(1)}} \qquad i = 1.\end{aligned}$$

From this we infer that the 'operation elements' $\{\mathbf{A}_0, \mathbf{A}_1\}$ should satisfy

$$\begin{aligned}\mathbf{A}_0|\Psi_S\rangle &= \begin{pmatrix} c_0a_0 \\ c_1a_1 \end{pmatrix} = \begin{pmatrix} a_0 & 0 \\ 0 & a_1 \end{pmatrix}\begin{pmatrix} c_0 \\ c_1 \end{pmatrix},\\ \mathbf{A}_1|\Psi_S\rangle &= \begin{pmatrix} c_0a_1 \\ c_1a_0 \end{pmatrix} = \begin{pmatrix} a_1 & 0 \\ 0 & a_0 \end{pmatrix}\begin{pmatrix} c_0 \\ c_1 \end{pmatrix},\end{aligned}$$

and we double-check that

$$\begin{aligned}\langle\Psi_S|\mathbf{A}_0^\dagger\mathbf{A}_0|\Psi_S\rangle &= \begin{pmatrix} c_0^* & c_1^* \end{pmatrix}\begin{pmatrix} |a_0|^2 & 0 \\ 0 & |a_1|^2 \end{pmatrix}\begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = |c_0a_0|^2 + |c_1a_1|^2 = \Pr(0),\\ \langle\Psi_S|\mathbf{A}_1^\dagger\mathbf{A}_1|\Psi_S\rangle &= \begin{pmatrix} c_0^* & c_1^* \end{pmatrix}\begin{pmatrix} |a_1|^2 & 0 \\ 0 & |a_0|^2 \end{pmatrix}\begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = |c_0a_1|^2 + |c_1a_0|^2 = \Pr(1).\end{aligned}$$

Likewise,

$$\mathbf{A}_0^\dagger\mathbf{A}_0 + \mathbf{A}_1^\dagger\mathbf{A}_1 = \begin{pmatrix} |a_0|^2 + |a_1|^2 & 0 \\ 0 & |a_0|^2 + |a_1|^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

since $|A\rangle = a_0|0_A\rangle + a_1|1_A\rangle$ is a normalized state. Note that for $|a_0|^2 \neq |a_1|^2$, neither $\mathbf{A}_0^\dagger\mathbf{A}_0$ nor $\mathbf{A}_1^\dagger\mathbf{A}_1$ is a projector or even proportional to a projector.

For the example with

$$\begin{aligned}\Pr(0) &= |c_0a_0|^2 + |c_1a_1|^2,\\ \Pr(1) &= |c_0a_1|^2 + |c_1a_0|^2,\end{aligned}$$

but now

$$\left|\Psi_{\text{post}}\right\rangle = \frac{c_1 a_1 |0_S\rangle + c_0 a_0 |1_S\rangle}{\sqrt{\Pr(0)}} \qquad i = 0,$$
$$= \frac{c_1 a_0 |0_S\rangle + c_0 a_1 |1_S\rangle}{\sqrt{\Pr(1)}} \qquad i = 1,$$

Likewise,

$$\mathbf{A}_0^\dagger \mathbf{A}_0 + \mathbf{A}_1^\dagger \mathbf{A}_1 = \begin{pmatrix} |a_0|^2 + |a_1|^2 & 0 \\ 0 & |a_0|^2 + |a_1|^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Comparing these two examples we also see that it should in principle be possible to realize a measurement with

$$\{\mathbf{A}_0, \mathbf{A}_1\} = \left\{ \begin{pmatrix} 0 & a_1 \\ a_0 & 0 \end{pmatrix}, \begin{pmatrix} a_1 & 0 \\ 0 & a_0 \end{pmatrix} \right\},$$

although this was not done in the class notes.

The last two-outcome example from the class notes is

$$\Pr(0) = |c_0 a_0|^2 + |c_1 a_0|^2 = |a_0|^2,$$
$$\Pr(1) = |c_0 a_1|^2 + |c_1 a_1|^2 = |a_1|^2,$$

with

$$\left|\Psi_{\text{post}}\right\rangle = \frac{c_0 a_0 |0_S\rangle + c_1 a_0 |1_S\rangle}{\sqrt{\Pr(0)}} = c_0 |0_S\rangle + c_1 |1_S\rangle \qquad i = 0,$$
$$= \frac{c_1 a_1 |0_S\rangle + c_0 a_1 |1_S\rangle}{\sqrt{\Pr(1)}} = c_0 |1_S\rangle + c_1 |0_S\rangle \qquad i = 1.$$

We infer that the $\{\mathbf{A}_0, \mathbf{A}_1\}$ should satisfy

$$\mathbf{A}_0 |\Psi_S\rangle = \begin{pmatrix} c_0 a_0 \\ c_1 a_0 \end{pmatrix} = \begin{pmatrix} a_0 & 0 \\ 0 & a_0 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix},$$
$$\mathbf{A}_1 |\Psi_S\rangle = \begin{pmatrix} c_1 a_1 \\ c_0 a_1 \end{pmatrix} = \begin{pmatrix} 0 & a_1 \\ a_1 & 0 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix},$$

and we double-check that

$$\left\langle \Psi_S \middle| \mathbf{A}_0^\dagger \mathbf{A}_0 \middle| \Psi_S \right\rangle = \begin{pmatrix} c_0^* & c_1^* \end{pmatrix} \begin{pmatrix} |a_0|^2 & 0 \\ 0 & |a_0|^2 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = |c_0 a_0|^2 + |c_1 a_0|^2 = \Pr(0),$$
$$\left\langle \Psi_S \middle| \mathbf{A}_1^\dagger \mathbf{A}_1 \middle| \Psi_S \right\rangle = \begin{pmatrix} c_0^* & c_1^* \end{pmatrix} \begin{pmatrix} 0 & a_1^* \\ a_1^* & 0 \end{pmatrix} \begin{pmatrix} 0 & a_1 \\ a_1 & 0 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix}$$
$$= \begin{pmatrix} c_0^* & c_1^* \end{pmatrix} \begin{pmatrix} |a_1|^2 & 0 \\ 0 & |a_1|^2 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = |c_0 a_1|^2 + |c_1 a_1|^2 = \Pr(1).$$

The normalization is clearly okay here.

Turning finally to the three-outcome example, we have

$$
\begin{aligned}
\Pr(0) &= |c_0 a_0|^2 + |c_1 a_2|^2, \\
\Pr(1) &= |c_0 a_1|^2 + |c_1 a_0|^2, \\
\Pr(2) &= |c_0 a_2|^2 + |c_1 a_1|^2,
\end{aligned}
$$

with post-measurement states

$$
\begin{aligned}
\left|\Psi_{\text{post}}\right\rangle &= \frac{c_0 a_0 |0_S\rangle + c_1 a_2 |1_S\rangle}{\sqrt{\Pr(0)}} \qquad i = 0, \\
&= \frac{c_0 a_1 |0_S\rangle + c_1 a_0 |1_S\rangle}{\sqrt{\Pr(1)}} \qquad i = 1, \\
&= \frac{c_0 a_2 |0_S\rangle + c_1 a_1 |1_S\rangle}{\sqrt{\Pr(2)}} \qquad i = 2.
\end{aligned}
$$

We infer that the $\{\mathbf{A}_0, \mathbf{A}_1, \mathbf{A}_2\}$ should satisfy

$$
\begin{aligned}
\mathbf{A}_0 |\Psi_S\rangle &= \begin{pmatrix} c_0 a_0 \\ c_1 a_2 \end{pmatrix} = \begin{pmatrix} a_0 & 0 \\ 0 & a_2 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix}, \\
\mathbf{A}_1 |\Psi_S\rangle &= \begin{pmatrix} c_0 a_1 \\ c_1 a_0 \end{pmatrix} = \begin{pmatrix} a_1 & 0 \\ 0 & a_0 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix}, \\
\mathbf{A}_2 |\Psi_S\rangle &= \begin{pmatrix} c_0 a_2 \\ c_1 a_1 \end{pmatrix} = \begin{pmatrix} a_2 & 0 \\ 0 & a_1 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix},
\end{aligned}
$$

and we double-check that

$$
\begin{aligned}
\left\langle \Psi_S | \mathbf{A}_0^\dagger \mathbf{A}_0 | \Psi_S \right\rangle &= \begin{pmatrix} c_0^* & c_1^* \end{pmatrix} \begin{pmatrix} |a_0|^2 & 0 \\ 0 & |a_2|^2 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = |c_0 a_0|^2 + |c_1 a_2|^2 = \Pr(0), \\
\left\langle \Psi_S | \mathbf{A}_1^\dagger \mathbf{A}_1 | \Psi_S \right\rangle &= \begin{pmatrix} c_0^* & c_1^* \end{pmatrix} \begin{pmatrix} |a_1|^2 & 0 \\ 0 & |a_0|^2 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = |c_0 a_1|^2 + |c_1 a_0|^2 = \Pr(1), \\
\left\langle \Psi_S | \mathbf{A}_2^\dagger \mathbf{A}_2 | \Psi_S \right\rangle &= \begin{pmatrix} c_0^* & c_1^* \end{pmatrix} \begin{pmatrix} |a_2|^2 & 0 \\ 0 & |a_1|^2 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = |c_0 a_2|^2 + |c_1 a_1|^2 = \Pr(1),
\end{aligned}
$$

and verify

$$
\mathbf{A}_0^\dagger \mathbf{A}_0 + \mathbf{A}_1^\dagger \mathbf{A}_1 + \mathbf{A}_2^\dagger \mathbf{A}_2 = \begin{pmatrix} |a_0|^2 + |a_1|^2 + |a_2|^2 & 0 \\ 0 & |a_0|^2 + |a_1|^2 + |a_2|^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
$$

Partial trace example

Just to practice taking partial traces, let's look at an entangled state that appeared at an intermediate stage of one of our indirect measurement examples. Use of the coupling transformation

$$\mathbf{C}_{AS} \equiv \mathbf{1}^S \otimes |0_A\rangle\langle 0_A| + (|1_S\rangle\langle 0_S| + |0_S\rangle\langle 1_S|) \otimes |1_A\rangle\langle 1_A|$$

led to

$$\begin{aligned} |\Psi_S\rangle &\mapsto |\Psi_S\rangle \otimes (a_0|0_A\rangle + a_1|1_A\rangle) \\ &\mapsto \mathbf{C}_{AS}|\Psi_S\rangle \otimes (a_0|0_A\rangle + a_1|1_A\rangle) \\ &= c_0 a_0|0_S\rangle \otimes |0_A\rangle + c_0 a_1|1_S\rangle \otimes |1_A\rangle + c_1 a_0|1_S\rangle \otimes |0_A\rangle + c_1 a_1|0_S\rangle \otimes |1_A\rangle. \end{aligned}$$

The corresponding density matrix is

$$\begin{aligned} \boldsymbol{\rho} = {} & |c_0 a_0|^2|0_S 0_A\rangle\langle 0_S 0_A| + |c_0|^2 a_0^* a_1|1_S 1_A\rangle\langle 0_S 0_A| + |a_0|^2 c_0^* c_1|1_S 0_A\rangle\langle 0_S 0_A| + a_0^* a_1 c_0^* c_1|0_S 1_A\rangle\langle 0_S 0_A| \\ & + |c_0|^2 a_0 a_1^*|0_S 0_A\rangle\langle 1_S 1_A| + |c_0 a_1|^2|1_S 1_A\rangle\langle 1_S 1_A| + a_0 a_1^* c_0^* c_1|1_S 0_A\rangle\langle 1_S 1_A| + |a_1|^2 c_0^* c_1|0_S 1_A\rangle\langle 1_S 1_A| \\ & + c_0 c_1^*|a_0|^2|0_S 0_A\rangle\langle 1_S 0_A| + c_0 a_1 a_0^* c_1^*|1_S 1_A\rangle\langle 1_S 0_A| + |c_1 a_0|^2|1_S 0_A\rangle\langle 1_S 0_A| + |c_1|^2 a_0^* a_1|0_S 1_A\rangle\langle 1_S 0_A| \\ & + c_0 c_1^* a_0 a_1^*|0_S 0_A\rangle\langle 0_S 1_A| + c_0 c_1^*|a_1|^2|1_S 1_A\rangle\langle 0_S 1_A| + |c_1|^2 a_0 a_1^*|1_S 0_A\rangle\langle 0_S 1_A| + |c_1 a_1|^2|0_S 1_A\rangle\langle 0_S 1_A|. \end{aligned}$$

Partial trace over the ancilla state yields the following terms:

$$\begin{aligned} \langle 0_A|\boldsymbol{\rho}|0_A\rangle &= |c_0 a_0|^2|0_S\rangle\langle 0_S| + |a_0|^2 c_0^* c_1|1_S\rangle\langle 0_S| + c_0 c_1^*|a_0|^2|0_S\rangle\langle 1_S| + |c_1 a_0|^2|1_S\rangle\langle 1_S|, \\ \langle 1_A|\boldsymbol{\rho}|1_A\rangle &= |c_0 a_1|^2|1_S\rangle\langle 1_S| + |a_1|^2 c_0^* c_1|0_S\rangle\langle 1_S| + c_0 c_1^*|a_1|^2|1_S\rangle\langle 0_S| + |c_1 a_1|^2|0_S\rangle\langle 0_S|, \end{aligned}$$

and thus

$$\begin{aligned} \mathrm{Tr}_A[\boldsymbol{\rho}] = {} & (|c_0 a_0|^2 + |c_1 a_1|^2)|0_S\rangle\langle 0_S| + (c_0 c_1^*|a_0|^2 + c_0^* c_1|a_1|^2)|0_S\rangle\langle 1_S| \\ & + (c_0^* c_1|a_0|^2 + c_0 c_1^*|a_1|^2)|1_S\rangle\langle 0_S| + (|c_0 a_1|^2 + |c_1 a_0|^2)|1_S\rangle\langle 1_S|. \end{aligned}$$

Partial trace over the system state yields the following terms:

$$\begin{aligned} \langle 0_S|\boldsymbol{\rho}|0_S\rangle &= |c_0 a_0|^2|0_A\rangle\langle 0_A| + a_0^* a_1 c_0^* c_1|1_A\rangle\langle 0_A| + c_0 c_1^* a_0 a_1^*|0_A\rangle\langle 1_A| + |c_1 a_1|^2|1_A\rangle\langle 1_A|, \\ \langle 1_S|\boldsymbol{\rho}|1_S\rangle &= |c_0 a_1|^2|1_A\rangle\langle 1_A| + a_0 a_1^* c_0^* c_1|0_A\rangle\langle 1_A| + c_0 a_1 a_0^* c_1^*|1_A\rangle\langle 0_A| + |c_1 a_0|^2|0_A\rangle\langle 0_A|, \end{aligned}$$

and thus

$$\begin{aligned} \mathrm{Tr}_S[\boldsymbol{\rho}] = {} & (|c_0 a_0|^2 + |c_1 a_0|^2)|0_A\rangle\langle 0_A| + (c_0 c_1^* a_0 a_1^* + a_0 a_1^* c_0^* c_1)|0_A\rangle\langle 1_A| \\ & + (a_0^* a_1 c_0^* c_1 + c_0 a_1 a_0^* c_1^*)|1_A\rangle\langle 0_A| + (|c_0 a_1|^2 + |c_1 a_1|^2)|1_A\rangle\langle 1_A| \\ = {} & |a_0|^2|0_A\rangle\langle 0_A| + a_0 a_1^*(c_0 c_1^* + c_0^* c_1)|0_A\rangle\langle 1_A| + a_0^* a_1(c_0^* c_1 + c_0 c_1^*)|1_A\rangle\langle 0_A| + |a_1|^2|1_A\rangle\langle 1_A|. \end{aligned}$$

We note that the two different partial traces yield different results.