APPPHYS225 - Friday 31 October 2008

Quantum-state teleportation

Theory: C. H. Bennett *et al.*, “Teleporting an Unknown Quantum State via Dual Classical and Einstein-Podolsky-Rosen Channels,” Phys. Rev. Lett. **70**, 1895 (1993).

Experiments: A. Furusawa *et al*, “Unconditional quantum teleportation,” Science **282**, 706 (1998); D. Bouwmeester *et al.*, “Experimental quantum teleportation,” Nature **390**, 575 (1997); D. Boschi *et al.*, “Experimental realization of teleporting an unknown pure quantum state via dual classical and Einstein-Podolsky-Rosen channels,” Phys. Rev. Lett. **80**, 1121 (1998); R. Ursin *et al.*, “Quantum teleportation across the Danube,” Nature **430**, 849 (2004).

Say Alice has a two-level quantum system A prepared in initial state

$$|\varphi_a\rangle = a_0|0_a\rangle + a_1|1_a\rangle.$$

She would like Charlie to have a two-level quantum system C prepared in the identical state

$$|\varphi_c\rangle = a_0|0_c\rangle + a_1|1_c\rangle.$$

However, let’s say she cannot directly send system A to Charlie, nor can she communicate the coefficients a_0, a_1 with sufficient precision for their liking. In fact, let’s imagine that Alice does not even *know* these coefficients explicitly – the initial state of system A could have been prepared by a third person who refuses to tell her the coefficients.

The secret of quantum-state teleportation is that Alice and Charlie need to have a shared quantum resource: entanglement. Let’s say that the last time Alice and Charlie got together, they prepared a pair of two-level quantum systems B, C in the entangled joint state

$$|\Psi_{bc}^-\rangle = \frac{1}{\sqrt{2}}(|0_b 1_c\rangle - |1_b 0_c\rangle).$$

A state of this form is often called a “singlet.” Alice keeps system B with her, and Charlie takes C away with him. Recall that C eventually needs to end up in the state $|\varphi_c\rangle$, whose coefficients are unknown to Alice.

There are three two-level systems in the picture, and their joint state lives in the tensor-product Hilbert space $H_A \otimes H_B \otimes H_C$. The initial state of this three-part system is simply

$$|\varphi_a\rangle \otimes |\Psi_{bc}^-\rangle.$$

In order to accomplish quantum-state teleportation, Alice performs the following procedure:

1. Alice brings A together with B, and performs a joint measurement in the “Bell basis”

$$|\Psi^-_{ab}\rangle = \frac{1}{\sqrt{2}}(|0_a1_b\rangle - |1_a0_b\rangle),$$
$$|\Psi^+_{ab}\rangle = \frac{1}{\sqrt{2}}(|0_a1_b\rangle + |1_a0_b\rangle),$$
$$|\Phi^-_{ab}\rangle = \frac{1}{\sqrt{2}}(|0_a0_b\rangle - |1_a1_b\rangle),$$
$$|\Phi^+_{ab}\rangle = \frac{1}{\sqrt{2}}(|0_a0_b\rangle + |1_a1_b\rangle).$$

Note that this is a complete basis for $H_A \otimes H_B$, and that these are all entangled states. The outcome probabilities can be computed explicitly by rewriting the three-part state of A, B, C in terms of the Bell states on A, B:

$$\begin{aligned}
|\varphi_a\rangle \otimes |\Psi^-_{bc}\rangle &= \frac{1}{\sqrt{2}}(a_0|0_a\rangle + a_1|1_a\rangle) \otimes (|0_b1_c\rangle - |1_b0_c\rangle) \\
&= \frac{1}{\sqrt{2}}(a_0|0_a0_b1_c\rangle - a_0|0_a1_b0_c\rangle + a_1|1_a0_b1_c\rangle - a_1|1_a1_b0_c\rangle) \\
&= \frac{1}{2}\left(\begin{array}{c} a_0(|\Phi^+_{ab}\rangle + |\Phi^-_{ab}\rangle)|1_c\rangle - a_0(|\Psi^+_{ab}\rangle + |\Psi^-_{ab}\rangle)|0_c\rangle \\ +a_1(|\Psi^+_{ab}\rangle - |\Psi^-_{ab}\rangle)|1_c\rangle - a_1(|\Phi^+_{ab}\rangle - |\Phi^-_{ab}\rangle)|0_c\rangle \end{array}\right) \\
&= -\frac{1}{2}\left(\begin{array}{c} |\Psi^-_{ab}\rangle(a_0|0_c\rangle + a_1|1_c\rangle) + |\Psi^+_{ab}\rangle(a_0|0_c\rangle - a_1|1_c\rangle) \\ |\Phi^-_{ab}\rangle(a_1|0_c\rangle + a_0|1_c\rangle) + |\Phi^+_{ab}\rangle(a_1|0_c\rangle - a_0|1_c\rangle) \end{array}\right).
\end{aligned}$$

Since $|a_0|^2 + |a_1|^2 = 1$, we see that all four outcomes have probability $\frac{1}{4}$. Also, we can easily read off the post-measurement states for system C:

$$\Psi^- : \quad a_0|0_c\rangle + a_1|1_c\rangle = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}|\varphi_c\rangle \equiv U_1|\varphi_c\rangle,$$
$$\Psi^+ : \quad a_0|0_c\rangle - a_1|1_c\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}|\varphi_c\rangle \equiv U_2|\varphi_c\rangle,$$
$$\Phi^- : \quad a_1|0_c\rangle + a_0|1_c\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}|\varphi_c\rangle \equiv U_3|\varphi_c\rangle,$$
$$\Phi^+ : \quad a_1|0_c\rangle - a_0|1_c\rangle = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}|\varphi_c\rangle \equiv U_4|\varphi_c\rangle.$$

Note that each of the four transformation operators $U_{1,2,3,4}$ are unitary, and therefore invertible.

2. Alice broadcasts a number from $1..4$ corresponding to the result she actually obtained. Note that this is just two bits of classical information, versus an infinite number of bits that would be needed to transmit two complex coefficients (a_0, a_1) with arbitrary precision.

3. When Charlie learns the result, he knows which of the four post-measurement states C has been left in! To recover $|\varphi_c\rangle$ exactly, he need only apply the appropriate inverse transformation.

So when Alice and Charlie implement this quantum-state teleportation protocol, they can be certain that Charlie will end up with system C left in the state

$$|\varphi_c\rangle = a_0|0_c\rangle + a_1|1_c\rangle,$$

where a_0, a_1 are arbitrary complex coefficients known only to Charlie! The total *resource cost* of the procedure is one entangled pair plus two bits of classical communication.

In quantum information theory, we would say that teleportation demonstrates an equivalence between quantum bits (*qubits*), entanglement "bits" (*e-bits*), and classical bits (*c-bits*):

$$1 \text{ qubit} = 1 \text{ e-bit} + 2 \text{ c-bits}.$$

Here a qubit is implicitly defined as the amount of information represented by the state of a two-level quantum system.

So what is it that actually got "teleported"...?

Teleportation of the state on an algebra

In order to answer this question, let's try to formulate a *classical* version of teleportation, first working with Bayes' Rule to simplify the formalism. Assume we have three systems each with two classical configurations,

$$\Omega_A = \{\omega_{0a}, \omega_{1a}\}, \quad \Omega_B = \{\omega_{0b}, \omega_{1b}\}, \quad \Omega_C = \{\omega_{0c}, \omega_{1c}\},$$

and an initial probability distribution on the A subsystem

$$p_a(\omega_{0a}) = p_{0a}, \quad p_a(\omega_{1a}) = p_{1a}.$$

By analogy to the quantum case we should need to specify a joint state $p_{bc}(\omega_b, \omega_c)$ and a random variable $B_{AB}(\cdot)$ on $\Omega_A \times \Omega_B.$ Again reasoning by analogy, we might expect that depending on the result of the measurement of $B_{AB}(\cdot)$ we may end up with a transformed version of $p_a(\cdot)$ on the C subsystem. Since the only reasonable transformation of a classical two-state probability distribution is a permutation (transposition), we can suppose that $B_{AB}(\cdot)$ has just two possible values and hence two level sets, which we guess to be

$$\Omega_{b1} = \{\omega_{0a} \times \omega_{0b}, \omega_{1a} \times \omega_{1b}\}, \quad \Omega_{b2} = \{\omega_{0a} \times \omega_{1b}, \omega_{1a} \times \omega_{0b}\}.$$

We have the initial probability distribution function

$$p_{abc}(\omega_a, \omega_b, \omega_c) = p_a(\omega_a) p_{bc}(\omega_b, \omega_c).$$

and we are going to start by measuring $B_{AB}(\cdot).$ The joint forward probabilities are

$$\begin{aligned}
\Pr(B_{AB}=1,\omega_c=\omega_{0c}) &= \sum_{\omega_a}\sum_{\omega_b} p_a(\omega_a)p_{bc}(\omega_b,\omega_{0c})\chi_{b1}(\omega_a,\omega_b) \\
&= p_{0a}p_{bc}(\omega_{b0},\omega_{0c}) + p_{1a}p_{bc}(\omega_{b1},\omega_{0c}), \\
\Pr(B_{AB}=2,\omega_c=\omega_{0c}) &= \sum_{\omega_a}\sum_{\omega_b} p_a(\omega_a)p_{bc}(\omega_b,\omega_{0c})\chi_{b2}(\omega_a,\omega_b) \\
&= p_{0a}p_{bc}(\omega_{b1},\omega_{0c}) + p_{1a}p_{bc}(\omega_{b0},\omega_{0c}), \\
\Pr(B_{AB}=1,\omega_c=\omega_{1c}) &= \sum_{\omega_a}\sum_{\omega_b} p_a(\omega_a)p_{bc}(\omega_b,\omega_{1c})\chi_{b1}(\omega_a,\omega_b) \\
&= p_{0a}p_{bc}(\omega_{b0},\omega_{1c}) + p_{1a}p_{bc}(\omega_{b1},\omega_{1c}), \\
\Pr(B_{AB}=2,\omega_c=\omega_{1c}) &= \sum_{\omega_a}\sum_{\omega_b} p_a(\omega_a)p_{bc}(\omega_b,\omega_{1c})\chi_{b2}(\omega_a,\omega_b) \\
&= p_{0a}p_{bc}(\omega_{b1},\omega_{1c}) + p_{1a}p_{bc}(\omega_{b0},\omega_{1c}).
\end{aligned}$$

We also have the marginal probabilities

$$\begin{aligned}
\Pr(B_{AB}=1) &= \Pr(B_{AB}=1,\omega_c=\omega_{0c}) + \Pr(B_{AB}=1,\omega_c=\omega_{1c}) \\
&= p_{0a}p_{bc}(\omega_{b0},\omega_{0c}) + p_{1a}p_{bc}(\omega_{b1},\omega_{0c}) + p_{0a}p_{bc}(\omega_{b0},\omega_{1c}) + p_{1a}p_{bc}(\omega_{b1},\omega_{1c}), \\
\Pr(B_{AB}=2) &= p_{0a}p_{bc}(\omega_{b1},\omega_{0c}) + p_{1a}p_{bc}(\omega_{b0},\omega_{0c}) + p_{0a}p_{bc}(\omega_{b1},\omega_{1c}) + p_{1a}p_{bc}(\omega_{b0},\omega_{1c}).
\end{aligned}$$

At this point let us try inserting

$$p_{bc}(\omega_{0b},\omega_{0c}) = \frac{1}{2},\quad p_{bc}(\omega_{0b},\omega_{1c}) = 0,\quad p_{bc}(\omega_{1b},\omega_{0c}) = 0,\quad p_{bc}(\omega_{1b},\omega_{1c}) = \frac{1}{2},$$

which is a classically correlated state of B and C. Then from Bayes' Rule,

$$\begin{aligned}
\Pr(\omega_c=\omega_{0c}|B_{AB}=1) &= \frac{\Pr(B_{AB}=1,\omega_c=\omega_{0c})}{\Pr(B_{AB}=1)} = \frac{p_{0a}}{2}\frac{1}{\frac{p_{0a}}{2}+\frac{p_{1a}}{2}} = p_{0a}, \\
\Pr(\omega_c=\omega_{1c}|B_{AB}=1) &= \frac{\Pr(B_{AB}=1,\omega_c=\omega_{1c})}{\Pr(B_{AB}=1)} = p_{1a},
\end{aligned}$$

and

$$\begin{aligned}
\Pr(\omega_c=\omega_{0c}|B_{AB}=2) &= \frac{\Pr(B_{AB}=2,\omega_c=\omega_{0c})}{\Pr(B_{AB}=2)} = p_{1a}, \\
\Pr(\omega_c=\omega_{1c}|B_{AB}=2) &= \frac{\Pr(B_{AB}=2,\omega_c=\omega_{1c})}{\Pr(B_{AB}=2)} = p_{0a}.
\end{aligned}$$

Hence we see that, as hoped, with $B_{AB}=1$ we transfer the $p_a(\bullet)$ distribution to subsystem $C,$ while if we obtain $B_{AB}=2$ the distribution is transfered but transposed.

Let us now try to rework this example in the algebra-of-random variables setting, first in the classical case and then in the quantum. Intuitively, it seems that we should be trying to show that for any random variable Q defined on both the A and C subsystems, by which we mean

$$Q_A(\omega_{0a}) = Q_C(\omega_{0c}),\quad Q_A(\omega_{1a}) = Q_C(\omega_{1c}),$$

we would like

$$E(Q_C|B_{AB}=1)=E(Q_A),$$
$$E(j_2(Q_C)|B_{AB}=2)=E(Q_A),$$

where

$$j_2(Q_C)(\omega_{0c})=Q_C(\omega_{1c}),\quad j_2(Q_C)(\omega_{1c})=Q_C(\omega_{0c}).$$

In the classical setting we have the usual definition of conditional expectation (assuming from here on that all random variables are ampliated to $\Omega_A\times\Omega_B\times\Omega_C$)

$$E(Q_C|B_{AB})(\bullet)=\frac{E(\chi_{b1}\times Q_C)}{E(\chi_{b1})}\chi_{b1}(\bullet)+\frac{E(\chi_{b2}\times Q_C)}{E(\chi_{b2})}\chi_{b2}(\bullet),$$

and

$$\begin{aligned}E(E(Q_C|B_{AB}))&=\frac{E(\chi_{b1}\times Q_C)}{E(\chi_{b1})}E(\chi_{b1})+\frac{E(\chi_{b2}\times Q_C)}{E(\chi_{b2})}E(\chi_{b2})\\&=\frac{E(\chi_{b1}\times Q_C)}{E(\chi_{b1})}\Pr(B_{AB}=1)+\frac{E(\chi_{b2}\times Q_C)}{E(\chi_{b2})}\Pr(B_{AB}=2).\end{aligned}$$

From the second line we can infer that

$$E(Q_C|B_{AB}=1)=\frac{E(\chi_{b1}\times Q_C)}{E(\chi_{b1})},\quad E(j_2(Q_C)|B_{AB}=2)=\frac{E(\chi_{b_2}\times j_2(Q_C))}{E(\chi_{b_2})},$$

and hence we need to compute

$$\begin{aligned}\frac{E(\chi_{b1}\times Q_C)}{E(\chi_{b1})}&=\frac{\sum_{\omega_a,\omega_b,\omega_c}p_a(\omega_a)p_{bc}(\omega_b,\omega_c)\chi_{b1}(\omega_a,\omega_b)Q_C(\omega_c)}{\sum_{\omega_a,\omega_b,\omega_c}p_a(\omega_a)p_{bc}(\omega_b,\omega_c)\chi_{b1}(\omega_a,\omega_b)}\\&=\frac{\sum_{\omega_c}\{p_{0a}p_{bc}(\omega_{0b},\omega_c)+p_{1a}p_{bc}(\omega_{1b},\omega_c)\}Q_C(\omega_c)}{\sum_{\omega_c}\{p_{0a}p_{bc}(\omega_{0b},\omega_c)+p_{1a}p_{bc}(\omega_{1b},\omega_c)\}}\\&=\frac{p_{0a}p_{bc}(\omega_{0b},\omega_{0c})Q_C(\omega_{0c})+p_{1a}p_{bc}(\omega_{1b},\omega_{1c})Q_C(\omega_{1c})}{p_{0a}p_{bc}(\omega_{0b},\omega_{0c})+p_{1a}p_{bc}(\omega_{0b},\omega_{0c})}\\&=p_{0a}Q_C(\omega_{0c})+p_{1a}Q_C(\omega_{1c}),\end{aligned}$$

and similarly

$$\begin{aligned}\frac{E(\chi_{b_2}\times j_2(Q_C))}{E(\chi_{b_2})}&=\frac{\sum_{\omega_a,\omega_b,\omega_c}p_a(\omega_a)p_{bc}(\omega_b,\omega_c)\chi_{b2}(\omega_a,\omega_b)j_2(Q_C)(\omega_c)}{\sum_{\omega_a,\omega_b,\omega_c}p_a(\omega_a)p_{bc}(\omega_b,\omega_c)\chi_{b2}(\omega_a,\omega_b)}\\&=\frac{p_{0a}p_{bc}(\omega_{1b},\omega_{1c})j_2(Q_C)(\omega_{1c})+p_{1a}p_{bc}(\omega_{0b},\omega_{0c})j_2(Q_C)(\omega_{0c})}{p_{0a}p_{bc}(\omega_{1b},\omega_{1c})+p_{1a}p_{bc}(\omega_{0b},\omega_{1c})}\\&=p_{0a}\,Q_C(\omega_{0c})+p_{1a}\,Q_C(\omega_{1c}).\end{aligned}$$

Since by assumption $Q_C(\omega_{0c})=Q_A(\omega_{0a})$ and $Q_C(\omega_{1c})=Q_A(\omega_{1a})$, we thus have

$$E(Q_C|B_{AB}=1)=E(j_2(Q_C)|B_{AB}=2)=p_{0a}Q_A(\omega_{0a})+p_{1a}Q_A(\omega_{1a})=E(Q_A),$$

as desired.

Hence we can say that at least in the classical case, we can characterize the effect of teleportation as transferring the assignment of expectation values of $Q_A(\bullet)$ (the state on the algebra of random variables on Ω_A) to an equivalent assignment of expectation values of $Q_C(\bullet)$. So in a sense, it is our "predictions" that get transferred from one subsystem to another.

Moving on to the quantum generalization of this algebra-of-observables version of the calculation, let's start by reminding ourselves of the notation. Let the particle whose state is to be teleported be particle A with initial density matrix ρ_A, which we take to be a pure state so that we can recycle calculational results from the first section of today's class notes:

$$\rho_A = |a_0|^2|0_A\rangle\langle 0_A| + a_0 a_1^*|0_A\rangle\langle 1_A| + a_0^* a_1|1_A\rangle\langle 0_A| + |a_1|^2|1_A\rangle\langle 1_A|.$$

Then the two particles prepared in the Bell singlet are particles B and C, with initial density matrix

$$\begin{aligned}
|\psi_{BC}^-\rangle &= \frac{1}{\sqrt{2}}(|0_B\rangle \otimes |1_C\rangle - |1_B\rangle \otimes |0_C\rangle), \\
\rho_{BC} &= |\psi_{BC}^-\rangle\langle\psi_{BC}^-| \\
&= \frac{1}{2}(|0_B 1_C\rangle\langle 0_B 1_C| - |0_B 1_C\rangle\langle 1_B 0_C| - |1_B 0_C\rangle\langle 0_B 1_C| + |1_B 0_C\rangle\langle 1_B 0_C|).
\end{aligned}$$

If Q_A and Q_C are equivalent observables on the A and C subsystems, then we expect that

$$\begin{aligned}
&\mathrm{Tr}[(\rho_{00}|0_a\rangle\langle 0_a| + \rho_{01}|0_a\rangle\langle 1_a| + \rho_{10}|1_a\rangle\langle 0_a| + \rho_{11}|1_a\rangle\langle 1_a|)Q_A] \\
&= \mathrm{Tr}[(\rho_{00}|0_c\rangle\langle 0_c| + \rho_{01}|0_c\rangle\langle 1_c| + \rho_{10}|1_c\rangle\langle 0_c| + \rho_{11}|1_c\rangle\langle 1_c|)Q_C].
\end{aligned}$$

Let us also note the general rule

$$\mathrm{Tr}[A \otimes B] = \mathrm{Tr}\left[\begin{pmatrix} a_{11}B & \square & \square \\ \square & \ddots & \square \\ \square & \square & a_{nn}B \end{pmatrix}\right] = \sum_i (a_{ii}\mathrm{Tr}[B]) = \mathrm{Tr}[A]\mathrm{Tr}[B].$$

Our initial joint state is

$$\rho_{ABC} = \rho_A \otimes \rho_{BC} = \rho_A \otimes |\psi_{BC}^-\rangle\langle\psi_{BC}^-|,$$

and initially

$$E(Q_A \otimes 1_{BC}) = \mathrm{Tr}[(Q_A \otimes 1_{BC})(\rho_A \otimes \rho_{BC})] = \mathrm{Tr}[\rho_A Q_A]\mathrm{Tr}[\rho_{BC}] = \mathrm{Tr}[\rho_A Q_A].$$

After performing a measurement of the observable $B_{AB} \otimes 1_C$, where

$$B_{AB} = P_{\psi_{AB}^-} + 2P_{\psi_{AB}^+} + 3P_{\phi_{AB}^-} + 4P_{\phi_{AB}^+}$$

is an observable on the AB subsystem whose eigenstates are the Bell states, we would like to have

$$E(1_{AB} \otimes U_i^* Q_C U_i | B_{AB} \otimes 1_C = i) = \mathrm{Tr}[\rho_A Q_A],$$

where the U_i were defined in our initial Schrodinger-picture discussion of the teleportation protocol. We can check that $E(1_A \otimes 1_B \otimes Q_C)$ is initially independent of ρ_A :

$$
\begin{aligned}
E(1_A \otimes 1_B \otimes Q_C) &= \mathrm{Tr}[(1_A \otimes 1_B \otimes Q_C)(\rho_A \otimes \rho_{BC})] \\
&= \mathrm{Tr}[\rho_A]\,\mathrm{Tr}[(1_B \otimes Q_C)\rho_{BC}] \\
&= \frac{1}{2}\mathrm{Tr}[(1_B \otimes Q_C)|0_B 1_C\rangle\langle 0_B 1_C|] - \frac{1}{2}\mathrm{Tr}[(1_B \otimes Q_C)|0_B 1_C\rangle\langle 1_B 0_C|] \\
&\quad - \frac{1}{2}\mathrm{Tr}[(1_B \otimes Q_C)|1_B 0_C\rangle\langle 0_B 1_C|] + \frac{1}{2}\mathrm{Tr}[(1_B \otimes Q_C)|1_B 0_C\rangle\langle 1_B 0_C|] \\
&= \frac{1}{2}\mathrm{Tr}[|0_B\rangle\langle 0_B| \otimes Q_C|1_C\rangle\langle 1_C|] - \frac{1}{2}\mathrm{Tr}[|0_B\rangle\langle 1_B| \otimes Q_C|1_C\rangle\langle 0_C|] \\
&\quad - \frac{1}{2}\mathrm{Tr}[|1_B\rangle\langle 0_B| \otimes Q_C|0_C\rangle\langle 1_C|] + \frac{1}{2}\mathrm{Tr}[|1_B\rangle\langle 1_B| \otimes Q_C|0_C\rangle\langle 0_C|] \\
&= \frac{1}{2}\mathrm{Tr}[Q_C|1_C\rangle\langle 1_C|] + \frac{1}{2}\mathrm{Tr}[Q_C|0_C\rangle\langle 0_C|] \\
&= \frac{1}{2}\mathrm{Tr}[Q_C].
\end{aligned}
$$

It is easy to see that $[B_{AB} \otimes 1_C, 1_A \otimes 1_B \otimes U_i^* Q_C U_i] = 0$ for any Q_C. Hence we can use

$$
E\big(X\,|\,Y\big) = \sum_i \sum_j x_j \frac{\rho(P_j Q_i)}{\rho(Q_i)} Q_i = \sum_i \frac{\rho(X Q_i)}{\rho(Q_i)} Q_i,
$$

to form the conditional expectation of $1_A \otimes 1_B \otimes Q_C \equiv \hat{Q}_C$ given $\hat{B}_{AB} \equiv B_{AB} \otimes 1_C$. We obtain

$$
\begin{aligned}
E\big(\hat{Q}_C|\hat{B}_{AB}\big) &= \frac{\rho\big(\hat{Q}_C \hat{P}_{\psi^-_{AB}}\big)}{\rho\big(\hat{P}_{\psi^-_{AB}}\big)}\hat{P}_{\psi^-_{AB}} + \frac{\rho\big(\hat{Q}_C \hat{P}_{\psi^+_{AB}}\big)}{\rho\big(\hat{P}_{\psi^+_{AB}}\big)}\hat{P}_{\psi^+_{AB}} + \frac{\rho\big(\hat{Q}_C \hat{P}_{\phi^-_{AB}}\big)}{\rho\big(\hat{P}_{\phi^-_{AB}}\big)}\hat{P}_{\phi^-_{AB}} + \frac{\rho\big(\hat{Q}_C \hat{P}_{\phi^+_{AB}}\big)}{\rho\big(\hat{P}_{\phi^+_{AB}}\big)}\hat{P}_{\phi^+_{AB}} \\
&= 4\rho(P_{\psi^-_{AB}} \otimes Q_C)\hat{P}_{\psi^-_{AB}} + 4\rho(P_{\psi^+_{AB}} \otimes Q_C)\hat{P}_{\psi^+_{AB}} + 4\rho(P_{\phi^-_{AB}} \otimes Q_C)\hat{P}_{\phi^-_{AB}} + 4\rho(P_{\phi^+_{AB}} \otimes Q_C)\hat{P}_{\phi^+_{AB}},
\end{aligned}
$$

where we recall from above that all four results of the Bell-basis measurement are equiprobable. Reasoning as in the classical case we note that

$$
\begin{aligned}
E\big(E\big(\hat{Q}_C|\hat{B}_{AB}\big)\big) &= 4\rho(P_{\psi^-_{AB}} \otimes Q_C)\rho\big(\hat{P}_{\psi^-_{AB}}\big) + 4\rho(P_{\psi^+_{AB}} \otimes Q_C)\rho\big(\hat{P}_{\psi^+_{AB}}\big) \\
&\quad + 4\rho(P_{\phi^-_{AB}} \otimes Q_C)\rho\big(\hat{P}_{\phi^-_{AB}}\big) + 4\rho(P_{\phi^+_{AB}} \otimes Q_C)\rho\big(\hat{P}_{\phi^+_{AB}}\big) \\
&= 4\rho(P_{\psi^-_{AB}} \otimes Q_C)\,\mathrm{Pr}(B_{AB} = 1) + 4\rho(P_{\psi^+_{AB}} \otimes Q_C)\,\mathrm{Pr}(B_{AB} = 2) \\
&\quad + 4\rho(P_{\phi^-_{AB}} \otimes Q_C)\,\mathrm{Pr}(B_{AB} = 3) + 4\rho(P_{\phi^+_{AB}} \otimes Q_C)\,\mathrm{Pr}(B_{AB} = 4),
\end{aligned}
$$

and hence conclude that

$$
\begin{aligned}
&E\big(\hat{Q}_C|\hat{B}_{AB} = 1\big) = 4\rho(P_{\psi^-_{AB}} \otimes Q_C), \quad E(Q_C|B_{AB} = 2) = 4\rho(P_{\psi^+_{AB}} \otimes Q_C), \\
&E\big(\hat{Q}_C|\hat{B}_{AB} = 3\big) = 4\rho(P_{\phi^-_{AB}} \otimes Q_C), \quad E(Q_C|B_{AB} = 4) = 4\rho(P_{\phi^+_{AB}} \otimes Q_C).
\end{aligned}
$$

Computing first for the $B_{AB} = 1$ case,

$$
\begin{aligned}
E\big(\hat{Q}_C|\hat{B}_{AB} = 1\big) &= 4\rho(P_{\psi^-_{AB}} \otimes Q_C) \\
&= 4\mathrm{Tr}[(P_{\psi^-_{AB}} \otimes Q_C)(\rho_A \otimes \rho_{BC})] \\
&= 4\mathrm{Tr}[(P_{\psi^-_{AB}} \otimes 1_C)(1_{AB} \otimes Q_C)(P_{\psi^-_{AB}} \otimes 1_C)(\rho_A \otimes \rho_{BC})] \\
&= 4\mathrm{Tr}[(1_{AB} \otimes Q_C)(P_{\psi^-_{AB}} \otimes 1_C)(\rho_A \otimes \rho_{BC})(P_{\psi^-_{AB}} \otimes 1_C)],
\end{aligned}
$$

at which point we recall that above we have already computed

$$(P_{\psi^-_{AB}} \otimes 1_C)(\rho_A \otimes \rho_{BC})(P_{\psi^-_{AB}} \otimes 1_C) = \frac{1}{4}|\Psi^-_{ab}\rangle\langle\Psi^-_{ab}| \otimes (a_0|0_c\rangle + a_1|1_c\rangle)(a_0^*\langle 0_c| + a_1^*\langle 1_c|).$$

Hence,

$$\begin{aligned} E\left(\hat{Q}_C|\hat{B}_{AB} = 1\right) &= \mathrm{Tr}\left[(1_{AB} \otimes Q_C)|\Psi^-_{ab}\rangle\langle\Psi^-_{ab}| \otimes (a_0|0_c\rangle + a_1|1_c\rangle)(a_0^*\langle 0_c| + a_1^*\langle 1_c|)\right] \\ &= \mathrm{Tr}\left[Q_C(a_0|0_c\rangle + a_1|1_c\rangle)(a_0^*\langle 0_c| + a_1^*\langle 1_c|)\right] \\ &= \mathrm{Tr}\left[\rho_A Q_A\right], \end{aligned}$$

as we have been trying to show. Analogous reults for $\hat{B}_{AB} = 2,3,4$ presumably follow straightforwardly.

Hence we see that also in the quantum case we can say that “what is teleported” is the assignment of expectation values from observables on spin A to observables on spin C, that is, *predictions*. We also see that the heart of the calculation involves an application of the classical probability rule for conditional expectation.

So is there anything uniquely *quantum* about teleportation? Perhaps it would be better to say that there are classical and quantum versions of teleportation, and that entangled states are required for the latter, in the same sense that there are both classical and quantum versions of uncertainty and correlation.