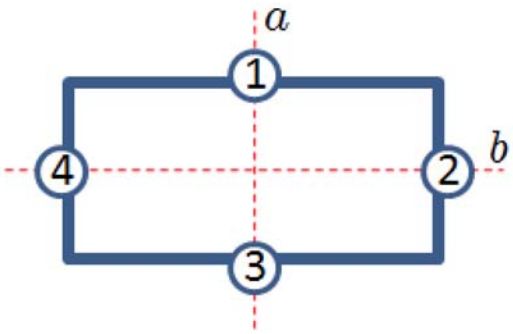
APPPHYS225 - Friday 14 November 2008

The quantum example in today's notes follows methods presented in G. Chiribella *et al.*, "Efficient Use of Quantum Resources for the Transmission of a Reference Frame," Phys. Rev. Lett. **93**, 180503 (2004); and G. Chiribella *et al.*, "Covariant quantum measurements that maximize the likelihood," Phys. Rev. A **70**, 062105 (2004).

Yesterday we briefly examined the dihedral group $\boldsymbol{D}_2$, corresponding to the symmetries of a rectangle. Its elements are $\{e,a,b,c\}$ where a is the reflection through $1-3$, b is the reflection through $2-4$, and c is the rotation by π, and the multiplication table is

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e



Today we begin by introducing a new linear representation motivated by the way that $\boldsymbol{D}_2$ induces permutations of the four points labeled in the above figure. Looking at the diagram we easily infer

$$
\begin{array}{|l|}
\hline
e : 1 \mapsto 1,\ 2 \mapsto 2,\ 3 \mapsto 3,\ 4 \mapsto 4, \\
\hline
a : 1 \mapsto 1,\ 2 \mapsto 4,\ 3 \mapsto 3,\ 4 \mapsto 2, \\
\hline
b : 1 \mapsto 3,\ 2 \mapsto 2,\ 3 \mapsto 1,\ 4 \mapsto 4, \\
\hline
c : 1 \mapsto 3,\ 2 \mapsto 4,\ 3 \mapsto 1,\ 4 \mapsto 2. \\
\hline
\end{array}
$$

If we define four basis vectors

$$
v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad v_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},
$$

it follows that we should have matrix representations

$$
D(a) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad D(b) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad D(c) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},
$$

with $D(e)$ equal to the 4×4 identity. We confirm

$$D(a)D(b) = \begin{pmatrix} 1&0&0&0 \\ 0&0&0&1 \\ 0&0&1&0 \\ 0&1&0&0 \end{pmatrix} \begin{pmatrix} 0&0&1&0 \\ 0&1&0&0 \\ 1&0&0&0 \\ 0&0&0&1 \end{pmatrix} = \begin{pmatrix} 0&0&1&0 \\ 0&0&0&1 \\ 1&0&0&0 \\ 0&1&0&0 \end{pmatrix} = D(c),$$

$$D(a)D(c) = \begin{pmatrix} 1&0&0&0 \\ 0&0&0&1 \\ 0&0&1&0 \\ 0&1&0&0 \end{pmatrix} \begin{pmatrix} 0&0&1&0 \\ 0&0&0&1 \\ 1&0&0&0 \\ 0&1&0&0 \end{pmatrix} = \begin{pmatrix} 0&0&1&0 \\ 0&1&0&0 \\ 1&0&0&0 \\ 0&0&0&1 \end{pmatrix} = D(b),$$

$$D(b)D(c) = \begin{pmatrix} 0&0&1&0 \\ 0&1&0&0 \\ 1&0&0&0 \\ 0&0&0&1 \end{pmatrix} \begin{pmatrix} 0&0&1&0 \\ 0&0&0&1 \\ 1&0&0&0 \\ 0&1&0&0 \end{pmatrix} = \begin{pmatrix} 1&0&0&0 \\ 0&0&0&1 \\ 0&0&1&0 \\ 0&1&0&0 \end{pmatrix} = D(a),$$

and likewise for $D(b)D(c) = D(a)$, $D(c)D(a) = D(b)$, $D(c)D(b) = D(a)$.

Let us define a classical sample space for a pebble that can be placed on any one of the four labeled points of the above rectangle diagram:

$$\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}.$$

Here ω_i corresponds to the configuration in which the pebble is placed on point i. We can represent any probability density function $m(\cdot)$ by a diagonal 4×4 state matrix,

$$\rho = \begin{pmatrix} m(\omega_1) & 0 & 0 & 0 \\ 0 & m(\omega_2) & 0 & 0 \\ 0 & 0 & m(\omega_3) & 0 \\ 0 & 0 & 0 & m(\omega_4) \end{pmatrix},$$

and note that the $\boldsymbol{D}_2$ permutations act on ρ via conjugation. For example,

$$a : \rho \mapsto D(a)\rho D^{\dagger}(a)$$

$$= \begin{pmatrix} 1&0&0&0 \\ 0&0&0&1 \\ 0&0&1&0 \\ 0&1&0&0 \end{pmatrix} \begin{pmatrix} m_1&0&0&0 \\ 0&m_2&0&0 \\ 0&0&m_3&0 \\ 0&0&0&m_4 \end{pmatrix} \begin{pmatrix} 1&0&0&0 \\ 0&0&0&1 \\ 0&0&1&0 \\ 0&1&0&0 \end{pmatrix} = \begin{pmatrix} m_1&0&0&0 \\ 0&m_4&0&0 \\ 0&0&m_3&0 \\ 0&0&0&m_2 \end{pmatrix},$$

where we have introduced the shorthand $m_i \equiv m(\omega_i)$. Likewise the $\boldsymbol{D}_2$ permutations can act on random variables via conjugation. If F is the matrix representation of a function $f(\cdot)$, we have for example $F \mapsto D^{\dagger}(a)FD(a)$. Since the $D(\cdot)$ matrices we are working with are actually all real symmetric and orthogonal, we will drop the conjugation symbols in what follows.

Suppose we are interested in an estimation scenario in which you get to place the pebble on whichever location you like, and then I secretly perform a random $\boldsymbol{D}_2$ transformation with uniform probability

$$\Pr(e) = \Pr(a) = \Pr(b) = \Pr(c) = \frac{1}{4}.$$

Your task is to guess which transformation I performed, and the information you have is the new location of the pebble. Clearly this estimation task cannot be performed perfectly, since according to the permutations we noted above you will have an ambiguity no matter which initial point you choose. For example, if you place the pebble initially on point 1 the possible final points are 1 and 3. If you find it on point 1 after my transformation, you know only that the transformation was either e or a. If you find it on point 3, you know only that the transformation was either b or c. Similar scenarios hold for any other initial point, so overall you will only be able to guess the correct transformation with probability 1/2.

The basic problem here is that the set of transformations we have chosen, the $\boldsymbol{D}_2$ transformations, share too much symmetry with the set of points on which you can initially place the pebble. Note that if you were allowed to place the pebble initially on one of the corners of the rectangle, you could determine the transformation with probability 1.

But for today let's stick with the set of symmetry points indicated on the rectangle diagram. Although it can't actually help the estimation problem to randomize the initial position of the pebble, it is interesting to consider the way that $\boldsymbol{D}_2$ transformations map an initial mixed state such as

$$\rho_{12} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We have

$$D(e)\rho_{12}D(e) = \rho_{12}, \quad D(a)\rho_{12}D(a) = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \equiv \rho_{14},$$

$$D(b)\rho_{12}D(b) = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \equiv \rho_{23}, \quad D(c)\rho_{12}D(c) = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \equiv \rho_{34}.$$

Thus we recover a scenario in which the four D_2 transformations map the initial state ρ_{12} into four distinct final states $\{\rho_{12}, \rho_{14}, \rho_{23}, \rho_{34}\}$. Of course these states are not mutually orthogonal, and as a result it is still not possible to guess the transformation

with probability better than $1/2$. Note that the fact that our initial mixed state 'covered' a corner was important. If we started instead with the initial state ρ_{13} we would actually have final states (extending the notation introduced above)

$$D(e)\rho_{13}D(e) = D(a)\rho_{13}D(a) = D(b)\rho_{13}D(b) = D(c)\rho_{13}D(c) = \rho_{13},$$

hence we would be unable to guess with probability of success better than $1/4$.

The mixed-state calculations suggest that it might be possible to do better in a quantum generalization of this $\boldsymbol{D}_2$ estimation problem. Let us therefore retain exactly the same set of 4×4 representation matrices $\{D(e), D(a), D(b), D(c)\}$ but open up the state to pure states in a four-dimensional complex Hilbert space. Can we find an initial quantum state such that we do better than the $1/2$ probability of success for a classical configuration space?

First let us note that there is no choice of intial state $|\Psi\rangle$ that even enables perfect detection of a non-trivial transformation, let alone perfect identification of the specific transformation. In order to have perfect detection we would need $D(\sigma)|\Psi\rangle$ to be orthogonal to $|\Psi\rangle$ for all $\sigma \neq e$,

$$\langle\Psi|D(a)|\Psi\rangle = \langle\Psi|D(b)|\Psi\rangle = \langle\Psi|D(c)|\Psi\rangle = 0.$$

With a little foresight (drawn from a diagonalization performed below) we can parametrize

$$|\Psi\rangle \leftrightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} w-y \\ x-z \\ w+y \\ x+z \end{pmatrix},$$

and compute

$$\langle D(a)\rangle_\Psi = |w|^2 + |x|^2 + |y|^2 - |z|^2, \quad \langle D(b)\rangle_\Psi = |w|^2 + |x|^2 - |y|^2 + |z|^2, \quad \langle D(c)\rangle_\Psi = |w|^2 + |x|^2 - |y|^2 - |z|^2.$$

Normalization gives us $|w|^2 = 1 - |x|^2 - |y|^2 - |z|^2$, so in order for all of these expectations to vanish we need

$$1 - 2|z|^2 = 0, \quad 1 - 2|y|^2 = 0, \quad 1 - 2|y|^2 - 2|z|^2 = 0.$$

Clearly we cannot satisfy all of these at the same time.

Following the spirit of the papers cited at the top of today's notes, we therefore proceed in a more sophisticated fashion by first writing an expression for the average probability of error:

$$\langle\varepsilon\rangle = \sum_{\sigma_* \in \{e,a,b,c\}} \Pr(\sigma_*) \sum_{\sigma \neq \sigma_*} \Pr(\sigma|\sigma_*)\varepsilon(\sigma,\sigma_*).$$

Here the idea is that we will define a POVM $\{\mathbf{E}_e, \mathbf{E}_a, \mathbf{E}_b, \mathbf{E}_c\}$, with all $\mathbf{E}_\sigma \geq 0$ and $\sum \mathbf{E}_\sigma = \mathbf{1}$, so that

$$\Pr(\sigma|\sigma_*) = \mathrm{Tr}[\mathbf{E}_\sigma D(\sigma_*)|\Psi\rangle\langle\Psi|D(\sigma_*)],$$

where $|\Psi\rangle$ is the initial state of the system, and

$$\begin{aligned} \varepsilon(\sigma,\sigma_*) &= 0, \quad \sigma = \sigma_*, \\ &= 1, \quad \sigma \neq \sigma_*. \end{aligned}$$

We note that this simple error function is *covariant* with respect to $\boldsymbol{D}_2$ in that $\varepsilon(h\sigma, h\sigma_*) = \varepsilon(\sigma, \sigma_*)$ for all $h \in \boldsymbol{D}_2$. To see this, just note that if $\sigma = \sigma_*$ then $h\sigma = h\sigma_*$ for any h, and if $\sigma \neq \sigma_*$ then $h\sigma \neq h\sigma_*$ because every h has an inverse. It follows from symmetry considerations that the optimal POVM should have the covariant form

$$\boxed{\mathbf{E}_\sigma = D(\sigma)\Xi D(\sigma), \quad \Xi \geq 0.}$$

Assuming this, we find that

$$\boxed{\Pr(\sigma|\sigma_*) = \mathrm{Tr}\,[\mathbf{E}_\sigma D(\sigma_*)|\Psi\rangle\langle\Psi|D(\sigma_*)] = \mathrm{Tr}\,[D(\sigma)\Xi D(\sigma)D(\sigma_*)|\Psi\rangle\langle\Psi|D(\sigma_*)],}$$

and the probability of success is thus

$$\boxed{\begin{aligned}\Pr(\sigma_*|\sigma_*) &= \mathrm{Tr}\,[D(\sigma_*)\Xi D(\sigma_*)D(\sigma_*)|\Psi\rangle\langle\Psi|D(\sigma_*)] \\ &= \mathrm{Tr}\,[D(\sigma_*)\Xi|\Psi\rangle\langle\Psi|D(\sigma_*)] \\ &= \mathrm{Tr}\,[\Xi|\Psi\rangle\langle\Psi|],\end{aligned}}$$

where we have used the group multiplication property that $D(\sigma_*)D(\sigma_*) = 1$ and cyclic property of trace. We thus see that our task is to find Ξ and $|\Psi\rangle$ that jointly maximize the *likelihood function* $\mathrm{Tr}\,[\Xi|\Psi\rangle\langle\Psi|]$.

Again using clues from the above-cited papers, we begin by decomposing our permutation-motivated representation of $\boldsymbol{D}_2$ into irreps. The abelian nature of the group multiplication table tells us that any one of the three non-diagonal matrices $\{D(a), D(b), D(c)\}$ could be a candidate Casimir operator, and a casual inspection of their eigenvectors suggests that $D(c)$ is the best candidate. Its eigenvectors are

$$\boxed{\left\{\begin{pmatrix}-1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\-1\\0\\1\end{pmatrix}\right\} \leftrightarrow -1, \quad \left\{\begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\1\end{pmatrix}\right\} \leftrightarrow 1,}$$

so the diagonalizing transformation that results is $D(\sigma) \mapsto S^{-1}D(\sigma)S$ with

$$\boxed{S = \frac{1}{\sqrt{2}}\begin{pmatrix}1&0&-1&0\\0&1&0&-1\\1&0&1&0\\0&1&0&1\end{pmatrix}, \quad S^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix}1&0&1&0\\0&1&0&1\\-1&0&1&0\\0&-1&0&1\end{pmatrix}.}$$

This yields

$$\boxed{D(a) \mapsto \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}, \quad D(b) \mapsto \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&1\end{pmatrix}, \quad D(c) \mapsto \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}.}$$

It follows that our original 'permutation' representation decomposes into the direct sum of four one-dimensional representations. We have two copies of the trivial representation, one copy of the representation $\{e \mapsto 1, a \mapsto 1, b \mapsto -1, c \mapsto -1\}$, and one copy of $\{e \mapsto 1, a \mapsto -1, b \mapsto 1, c \mapsto -1\}$. Note that the latter two representations are not

equivalent. The methods of Chiribella *et al.* then suggest that we should take $|\Psi\rangle$ as a coherent superposition of states from each of the *distinct* irreps. In the original vector basis,

$$|\Psi\rangle \leftrightarrow \frac{1}{\sqrt{6}}\begin{pmatrix}1\\0\\1\\0\end{pmatrix} + \frac{1}{\sqrt{6}}\begin{pmatrix}-1\\0\\1\\0\end{pmatrix} + \frac{1}{\sqrt{6}}\begin{pmatrix}0\\-1\\0\\1\end{pmatrix} = \frac{1}{\sqrt{6}}\begin{pmatrix}0\\-1\\2\\1\end{pmatrix},$$

where the three vectors left of the equality are eigenvectors of $D(c)$ corresponding to the first, third and fourth irreps. We take equal coefficients in this superposition as there is nothing *a priori* to distinguish the three one-dimensional irreps. It is perhaps relevant to note that in the parametrization introduced above, this state corresponds to

$$w = y = z = \frac{1}{\sqrt{3}}, \quad x = 0,$$

which actually minimizes the quantity

$$|\langle D(a)\rangle|^2 + |\langle D(b)\rangle|^2 + |\langle D(c)\rangle|^2 = 3 - 16\gamma + 24\gamma^2,$$

which is proportional to the average overlap of the transformed states with the initial state, where we have written $\gamma \equiv |y|^2 = |z|^2$ since y and z appear symmetrically. We have

$$\frac{d}{d\gamma} \to -16 + 48\gamma, \quad \gamma \to \frac{1}{3},$$

from which $w = y = z = 1/\sqrt{3}$ immediately follows.

Now we need to determine $\boldsymbol{\Xi}$. Again symmetry considerations from the papers suggest $\boldsymbol{\Xi} \propto |\eta\rangle\langle\eta| + |\Phi\rangle\langle\Phi|$, where

$$|\eta\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\\0\end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\0\\1\\0\end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix}0\\-1\\0\\1\end{pmatrix} = \sqrt{3}\,|\Psi\rangle$$

is an unnormalized coherent superposition of the eigenstates of $D(c)$ corresponding to the three included irreps and $|\Phi\rangle$ is the eigenstate corresponding to the unused irrep. The proportionality will be used to enforce

$$\boldsymbol{\Xi} + D(a)\boldsymbol{\Xi}D(a) + D(b)\boldsymbol{\Xi}D(b) + D(c)\boldsymbol{\Xi}D(c) = \mathbf{1}.$$

We find by straightforward calculation that the required constant is $1/4$, leaving us with

$$\boldsymbol{\Xi} = \frac{1}{8}\begin{pmatrix}0\\-1\\2\\1\end{pmatrix}\begin{pmatrix}0&-1&2&1\end{pmatrix} + \frac{1}{8}\begin{pmatrix}0\\1\\0\\1\end{pmatrix}\begin{pmatrix}0&1&0&1\end{pmatrix} = \frac{1}{4}\begin{pmatrix}0&0&0&0\\0&1&-1&0\\0&-1&2&1\\0&0&1&1\end{pmatrix},$$

which is positive and furthermore satisfies the POVM sum rule. As a result, we have $\mathbf{E}_e = \boldsymbol{\Xi}$ and

$$\mathbf{E}_a = \frac{1}{4}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}, \quad \mathbf{E}_b = \frac{1}{4}\begin{pmatrix} 2 & -1 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{E}_c = \frac{1}{4}\begin{pmatrix} 2 & 1 & 0 & -1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}.$$

We can then immediately verify that

$$\Pr(\sigma_*|\sigma_*) = \frac{1}{24}\mathrm{Tr}\left[\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & -1 \\ 0 & -2 & 4 & 2 \\ 0 & -1 & 2 & 1 \end{pmatrix}\right] = \frac{3}{4},$$

and the probabilites of error are all 1/12. Hence we do in fact find that a higher probability of success is possible in the quantum generalization of our model. The covariant structure of the optimal POVM provided essential guidance for our calculation.