APPPHYS225 - Friday 7 November 2008

Before moving on to new material, let's take a quick look at an experimental implementation of a form of quantum cloning:

- W. T. M. Irvine *et al.*, "Optimal Quantum Cloning on a Beam Splitter," Phys. Rev. Lett. **92**, 047902 (2004).

The experiment described in this paper actually implements an approximate form of pure-state cloning, which works in principle for any input state of a two-dimensional quantum system but only has fidelity $2/3$.

In order to explain the approximate cloning scheme, we'll need to start by noting an important property of optical beamsplitters. Recall from our discussion of the 'quantum eraser' scheme (class notes of 11/4) that if α and β are annihilation operators for optical modes coming into an ideal 50/50 beamsplitter, then the annihilation operators of the output modes may be written

$$\gamma = \frac{1}{\sqrt{2}}(\alpha+\beta), \quad \delta = \frac{1}{\sqrt{2}}(\alpha-\beta).$$

Note that $[\alpha,\beta] = 0$ and

$$[\gamma,\delta] = \gamma\delta - \delta\gamma = \frac{1}{2}\{(\alpha+\beta)(\alpha-\beta) - (\alpha-\beta)(\alpha+\beta)\} = 0,$$

$$\left[\gamma^\dagger,\delta^\dagger\right] = \gamma^\dagger\delta^\dagger - \delta^\dagger\gamma^\dagger = -(\gamma\delta-\delta\gamma)^\dagger = 0.$$

The states with two photons in the γ mode, two photons in the δ mode, and one photon in each may thus be written

$$\begin{aligned}
\left(\gamma^\dagger\right)^2|0_\alpha 0_\beta\rangle &= \frac{1}{2}(\alpha^\dagger+\beta^\dagger)(\alpha^\dagger+\beta^\dagger)|0_\alpha 0_\beta\rangle \\
&= \frac{1}{2}(\alpha^\dagger\alpha^\dagger + 2\alpha^\dagger\beta^\dagger + \beta^\dagger\beta^\dagger)|0_\alpha 0_\beta\rangle \\
&= \frac{1}{\sqrt{2}}|2_\alpha 0_\beta\rangle + |1_\alpha 1_\beta\rangle + \frac{1}{\sqrt{2}}|0_\alpha 2_\beta\rangle, \\
|2_\gamma 0_\delta\rangle &= \frac{1}{2}|2_\alpha 0_\beta\rangle + \frac{1}{\sqrt{2}}|1_\alpha 1_\beta\rangle + \frac{1}{2}|0_\alpha 2_\beta\rangle \\
\left(\delta^\dagger\right)^2|0_\alpha 0_\beta\rangle &= \frac{1}{2}(\alpha^\dagger-\beta^\dagger)(\alpha^\dagger-\beta^\dagger)|0_\alpha 0_\beta\rangle \\
&= \frac{1}{2}(\alpha^\dagger\alpha^\dagger - 2\alpha^\dagger\beta^\dagger + \beta^\dagger\beta^\dagger)|0_\alpha 0_\beta\rangle \\
&= \frac{1}{\sqrt{2}}|2_\alpha 0_\beta\rangle - |1_\alpha 1_\beta\rangle + \frac{1}{\sqrt{2}}|0_\alpha 2_\beta\rangle, \\
|0_\gamma 2_\delta\rangle &= \frac{1}{2}|2_\alpha 0_\beta\rangle - \frac{1}{\sqrt{2}}|1_\alpha 1_\beta\rangle + \frac{1}{2}|0_\alpha 2_\beta\rangle
\end{aligned}$$

and

$$\begin{aligned}\boldsymbol{\delta}^\dagger\boldsymbol{\gamma}^\dagger|0_\alpha 0_\beta\rangle &= \frac{1}{2}(\boldsymbol{\alpha}^\dagger - \boldsymbol{\beta}^\dagger)(\boldsymbol{\alpha}^\dagger + \boldsymbol{\beta}^\dagger)|0_\alpha 0_\beta\rangle \\ &= \frac{1}{2}(\boldsymbol{\alpha}^\dagger\boldsymbol{\alpha}^\dagger - \boldsymbol{\beta}^\dagger\boldsymbol{\beta}^\dagger)|0_\alpha 0_\beta\rangle, \\ |1_\gamma 1_\delta\rangle &= \frac{1}{\sqrt{2}}|2_\alpha 0_\beta\rangle - \frac{1}{\sqrt{2}}|0_\alpha 2_\beta\rangle.\end{aligned}$$

It follows that for a state on the input modes

$$|1_\alpha 1_\beta\rangle = \frac{1}{\sqrt{2}}|2_\gamma 0_\delta\rangle - \frac{1}{\sqrt{2}}|0_\gamma 2_\delta\rangle,$$

we have the interesting fact that it is possible for both photons to go out either the γ mode or the δ mode but we cannot have one photon in each. This type of 'bunching' is a consequence of the fact that photons are bosons, and occurs in practice only when the photons coming in to the beamsplitter have the same 'shape' (waveform) and polarization. Photons that have orthogonal polarizations and/or waveforms will act independently at the beamsplitter.

Now consider a situation where we have two photons (call them A and B), assumed to have identical waveforms, arriving synchronously on the two input ports of an ideal 50/50 (non-polarizing) beamsplitter. The quantum state that we want to clone is the polarization state of the A photon, which we can write

$$|\varphi_A\rangle = c_h|H_A\rangle + c_v|V_A\rangle,$$

where $\{|H_A\rangle, |V_A\rangle\}$ are basis states representing horizontal and vertical polarization. The initial polarization state of the target photon is the completely mixed state,

$$\boldsymbol{\Sigma}_B = \frac{1}{2}|H_B\rangle\langle H_B| + \frac{1}{2}|V_B\rangle\langle V_B|.$$

(In the paper, this completely mixed state is prepared by producing a Bell singlet state of B and an auxilliary photon C, via spontaneous parametric down-conversion, and then tracing out the state of photon C.) Note that we can just as well write

$$\boldsymbol{\Sigma}_B = \frac{1}{2}|\varphi_B\rangle\langle\varphi_B| + \frac{1}{2}|\bar{\varphi}_B\rangle\langle\bar{\varphi}_B|,$$

where

$$|\varphi_B\rangle = c_h|H_B\rangle + c_v|V_B\rangle, \quad |\bar{\varphi}_B\rangle = c_v|H_B\rangle - c_h|V_B\rangle,$$

since the orthonormal pair of states $\{|\varphi_B\rangle, |\bar{\varphi}_B\rangle\}$ is just as good a basis for the B Hilbert space as $\{|H_B\rangle, |V_B\rangle\}$ is. We can now see the basic idea of the probabilistic cloning scheme. We can think of the initial state of the target photon as being either $|\varphi_B\rangle$ or $|\bar{\varphi}_B\rangle$ with probability $1/2$, and thus half the time we expect to observe the bunching effect described above. This will cause both the input and target photons to exit from the same port of the beamsplitter, in the same polarization state $|\varphi_A\rangle \otimes |\varphi_B\rangle$. Thus if we are regarding one specific output port to be the desired output port, we succeed in cloning the polarization state of the input photon with overall probability $1/4$. Now with probability $1/2$ the initial state of the target photon should be regarded as $|\bar{\varphi}_B\rangle$, in which case the input and target photons will behave independently at the beamsplitter. There is thus an overall probability $1/8$ that both photons go into the desired output port (with polarization state $|\varphi_A\rangle \otimes |\bar{\varphi}_B\rangle$), overall probability $1/8$ that both photons go out the wrong output port, and overall probability $1/4$ that one photon goes out each output port.

Summing this all up... If we pay attention only to the set of events in which two photons emerge from the desired output port, there is a relative probability of $2/3$ that cloning succeeds and probability $1/3$ that we instead obtain the orthogonal complement of the input state in the output state, but it is important to note that we have no way of knowing which. Hence we should think of the output as a mixed state with weight $2/3$ for the cloned state and $1/3$ for the orthogonal complement. As described in the introduction of this experimental paper, this is actually the best outcome allowed by quantum mechanics for an approximate cloning scheme that works for arbitrary two-dimensional input states.

A more formal calculation can be carried out using creation operators, as follows (we will reproduce the calculation shown in the experimental paper, but with a permutation of the photon labels to maintain consistency with our previous cloning discussions). Keeping in mind the extra C photon produced by the parametric down-conversion process, we can write the initial state of all the input modes as

$$|\Psi_{ABC}\rangle = \frac{1}{\sqrt{2}}\left(\mathbf{b}_\varphi^\dagger \mathbf{c}_{\bar\varphi}^\dagger - \mathbf{b}_{\bar\varphi}^\dagger \mathbf{c}_\varphi^\dagger\right)\mathbf{a}_\varphi^\dagger|0\rangle,$$

where $|0\rangle$ denotes the vacuum state of all optical modes, $\left\{\mathbf{a}_\varphi^\dagger, \mathbf{b}_\varphi^\dagger, \mathbf{c}_\varphi^\dagger\right\}$ are creation operators for photons with polarization state $|\varphi\rangle$, and $\left\{\mathbf{a}_{\bar\varphi}^\dagger, \mathbf{b}_{\bar\varphi}^\dagger, \mathbf{c}_{\bar\varphi}^\dagger\right\}$ are creation operators for photons with polarization state $|\bar\varphi\rangle$. Here we take advantage of the fact that the form of a Bell singlet state is invariant under local changes of basis, *i.e.*,

$$\left|\Psi_{BC}^{(-)}\right\rangle = \frac{1}{\sqrt{2}}\left(|H_B V_C\rangle - |V_B H_C\rangle\right) = \frac{1}{\sqrt{2}}\left(|\varphi_B \bar\varphi_C\rangle - |\bar\varphi_B \varphi_C\rangle\right).$$

Then taking into account the beamsplitter transformation,

$$\mathbf{e}_\varphi^\dagger = \frac{1}{\sqrt{2}}\left(\mathbf{a}_\varphi^\dagger + \mathbf{b}_\varphi^\dagger\right), \quad \mathbf{f}_\varphi^\dagger = \frac{1}{\sqrt{2}}\left(\mathbf{a}_\varphi^\dagger - \mathbf{b}_\varphi^\dagger\right),$$

$$\mathbf{a}_\varphi^\dagger = \frac{1}{\sqrt{2}}\left(\mathbf{e}_\varphi^\dagger + \mathbf{f}_\varphi^\dagger\right), \quad \mathbf{b}_\varphi^\dagger = \frac{1}{\sqrt{2}}\left(\mathbf{e}_\varphi^\dagger - \mathbf{f}_\varphi^\dagger\right),$$

and similarly for the $\bar\varphi$ polarization modes, we have

$$\begin{aligned}
|\Psi_{ABC}\rangle &= \frac{1}{2\sqrt{2}}\left(\left(\mathbf{e}_\varphi^\dagger - \mathbf{f}_\varphi^\dagger\right)\mathbf{c}_{\bar\varphi}^\dagger - \left(\mathbf{e}_{\bar\varphi}^\dagger - \mathbf{f}_{\bar\varphi}^\dagger\right)\mathbf{c}_\varphi^\dagger\right)\left(\mathbf{e}_\varphi^\dagger + \mathbf{f}_\varphi^\dagger\right)|0\rangle \\
&= \frac{1}{2\sqrt{2}}\left(\left(\mathbf{e}_\varphi^\dagger - \mathbf{f}_\varphi^\dagger\right)\left(\mathbf{e}_\varphi^\dagger + \mathbf{f}_\varphi^\dagger\right)\mathbf{c}_{\bar\varphi}^\dagger - \left(\mathbf{e}_{\bar\varphi}^\dagger - \mathbf{f}_{\bar\varphi}^\dagger\right)\left(\mathbf{e}_\varphi^\dagger + \mathbf{f}_\varphi^\dagger\right)\mathbf{c}_\varphi^\dagger\right)|0\rangle \\
&= \frac{1}{2\sqrt{2}}\left(\left(\mathbf{e}_\varphi^\dagger \mathbf{e}_\varphi^\dagger - \mathbf{f}_\varphi^\dagger \mathbf{f}_\varphi^\dagger\right)\mathbf{c}_{\bar\varphi}^\dagger - \left(\mathbf{e}_{\bar\varphi}^\dagger \mathbf{e}_\varphi^\dagger + \mathbf{e}_{\bar\varphi}^\dagger \mathbf{f}_\varphi^\dagger - \mathbf{e}_\varphi^\dagger \mathbf{f}_{\bar\varphi}^\dagger - \mathbf{f}_{\bar\varphi}^\dagger \mathbf{f}_\varphi^\dagger\right)\mathbf{c}_\varphi^\dagger\right)|0\rangle.
\end{aligned}$$

If we regard the $\mathbf{e}$ mode as the desired output mode, then we can group the relevant terms in which two photons actually are present in that mode:

$$\begin{aligned}
|\Psi_{ABC}\rangle &= \frac{1}{2\sqrt{2}}\left(\mathbf{e}_\varphi^\dagger \mathbf{e}_\varphi^\dagger \mathbf{c}_{\bar\varphi}^\dagger - \mathbf{e}_{\bar\varphi}^\dagger \mathbf{e}_\varphi^\dagger \mathbf{c}_\varphi^\dagger\right)|0\rangle - \frac{1}{2\sqrt{2}}\left(\mathbf{f}_\varphi^\dagger \mathbf{f}_\varphi^\dagger \mathbf{c}_{\bar\varphi}^\dagger + \left(\mathbf{e}_{\bar\varphi}^\dagger \mathbf{f}_\varphi^\dagger - \mathbf{e}_\varphi^\dagger \mathbf{f}_{\bar\varphi}^\dagger - \mathbf{f}_{\bar\varphi}^\dagger \mathbf{f}_\varphi^\dagger\right)\mathbf{c}_\varphi^\dagger\right)|0\rangle \\
&= \frac{1}{2\sqrt{2}}\left(\sqrt{2}\,|2_e 0_{\bar e} 0_c 1_{\bar c}\rangle - |1_e 1_{\bar e} 1_c 0_{\bar c}\rangle\right) - \frac{1}{2\sqrt{2}}\left(|0_e 1_{\bar e} 1_f 0_{\bar f} 1_c 0_{\bar c}\rangle - |1_e 0_{\bar e} 1_f 0_{\bar f} 1_c 0_{\bar c}\rangle\right) \\
&\quad - \frac{1}{2\sqrt{2}}\left(\sqrt{2}\,|2_f 0_{\bar f} 0_c 1_{\bar c}\rangle - |1_f 1_{\bar f} 1_c 0_{\bar c}\rangle\right).
\end{aligned}$$

The notation here is such that, for example, $|2_e 0_{\bar{e}} 0_c 1_{\bar{c}}\rangle$ indicates a state with two photons in the $\mathbf{e}_\varphi$ mode and one in the $\mathbf{c}_{\bar{\varphi}}$ mode. Note that factors of $\sqrt{2}$ have appeared in two places as a result of the repeated action of a creation operator. The overall probability of the terms with two photons in the $\mathbf{e}$ output channel is easily seen to be $1/4 + 1/8 = 3/8$. If we post-select the subset of events in which two photons emerge in the desired output channel, and trace over the polarization state of the C photon, we end up with

$$\boldsymbol{\rho}_B = \frac{2}{3}|\varphi_B\rangle\langle\varphi_B| + \frac{1}{3}|\bar{\varphi}_B\rangle\langle\bar{\varphi}_B|$$

for the polarization state of the B photon. As promised this has fidelity $2/3$ with respect to the desired 'clone' state of $|\varphi_B\rangle$.

A comprehensive survey of quantum cloning work, as of a few years ago, can be found in:

- V. Scarani *et al.*, "Quantum cloning," Rev. Mod. Phys. **77**, 1225 (2005).

Nonlocality without entanglement

In this section we discuss results from the following paper:

- C. H. Bennett *et al.*, "Quantum nonlocality without entanglement," Phys. Rev. A **59**, 1070 (1999).

We have discussed on many occasions in this class the fact that if we have a quantum probability scenario in which all the state matrices and observables that we happen to be interested in commute, the quantum model can be mapped to an entirely equivalent classical probability model by switching to a basis in which all matrices are simultaneously diagonalized. For example, if we are considering an ensemble of mutually orthogonal pure states and we wish only to consider measurements that can distinguish between these states, the state matrices and the relevant projection operators can all be diagonalized in the basis specified by the pure states (as vectors) themselves. We then associate each quantum pure basis state with a distinct configuration of the classical model and consider the ensemble to represent an essentially classical set of alternatives because they can be perfectly distinguished by measurement of a single system.

It is interesting and important to note, however, that subtleties can arise if we are talking about joint systems. In particular, in scenarios as elementary as a pair of three-dimensional Hilbert spaces or a triple of two-dimensional Hilbert spaces it is possible to find ensembles of mutually orthogonal pure states that cannot be perfectly distinguished by measurements performed on a single system, if we require those measurements to be *local*. That is, if we imagine that (in a $H_A \otimes H_B$ scenario) Alice and Bob each have one of the subsystems in their laboratory, and allow them only to perform POVM's of the form $\{\mathbf{1}^A \otimes \mathbf{E}_i^B\}$ and $\{\mathbf{F}_j^A \otimes \mathbf{1}^B\}$, then Alice and Bob cannot perfectly distinguish among the mutually orthogonal pure states even if we allow them to perform multiple local measurements combined with unlimited classical communication. Of course, if Alice and Bob were to bring the subsystems back together they could simply perform the projective measurement $\{|\Psi_i\rangle\langle\Psi_i|\}$ to determine which joint state they have. In this sense one feels that there is something non-local about such an ensemble of quantum states, even though there is no entanglement and therefore no possibility of violating a CHSH inequality or performing teleportation.

The canonical example of a locally-immeasurable set of orthogonal product states requires two 3-dimensional Hilbert spaces, $H_A = \mathrm{span}\{|0_A\rangle, |1_A\rangle, |2_A\rangle\}$ and $H_B = \mathrm{span}\{|0_B\rangle, |1_B\rangle, |2_B\rangle\}$. Consider the following orthonormal set of nine basis states for $H_A \otimes H_B$, each of which is a product state:

$$|\psi_1\rangle = |1_A\rangle \otimes |1_B\rangle, \quad |\psi_2\rangle = \frac{1}{\sqrt{2}}|0_A\rangle \otimes (|0_B\rangle + |1_B\rangle), \quad |\psi_3\rangle = \frac{1}{\sqrt{2}}|0_A\rangle \otimes (|0_B\rangle - |1_B\rangle),$$

$$|\psi_4\rangle = \frac{1}{\sqrt{2}}|2_A\rangle \otimes (|1_B\rangle + |2_B\rangle), \quad |\psi_5\rangle = \frac{1}{\sqrt{2}}|2_A\rangle \otimes (|1_B\rangle - |2_B\rangle),$$

$$|\psi_6\rangle = \frac{1}{\sqrt{2}}(|1_A\rangle + |2_A\rangle) \otimes |0_B\rangle, \quad |\psi_7\rangle = \frac{1}{\sqrt{2}}(|1_A\rangle - |2_A\rangle) \otimes |0_B\rangle,$$

$$|\psi_8\rangle = \frac{1}{\sqrt{2}}(|0_A\rangle + |1_A\rangle) \otimes |2_B\rangle, \quad |\psi_9\rangle = \frac{1}{\sqrt{2}}(|0_A\rangle - |1_A\rangle) \otimes |2_B\rangle.$$

The way that these states ‘cover’ $H_A \otimes H_B$ is depicted graphically in Figure 1 of the paper cited above.

To illustrate in a very loose way why this orthogonal set of product states is not locally measurable, we start by examining an LOCC (local operations with classical communication) procedure that is capable of perfectly distinguishing among eight of the nine states listed above, with $|\psi_4\rangle$ removed. It turns out that this can be accomplished by Bob and Alice alternately and ‘adaptively’ making local projective measurements, communicating the results to each other after each measurement. can make locally. The procedure is specified in Figure 3 of the paper, and is perhaps best understood with reference to the graphical depiction of Figure 1 or 2.

In fact this exact same procedure can be applied in the full nine-state case, but it will fail to distinguish between states 4 and $5,$ although it will correctly distinguish cases in which the state is either 4 or 5 from cases in which it is in $\{1,2,3,6,7,8,9\}$. If we assume that the states are presented with equal probability this is not so bad, as the average information gain will be close to the full $\log_2(9) \approx 3.1699$ bits (the actual value for this procedure is 2.9477 bits). The procedure can be patched up to do a bit better, by replacing Bob’s opening measurement of $\{\mathbf{\Pi}_{01B}, \mathbf{\Pi}_{2B}\}$, where

$$\mathbf{\Pi}_{01B} = |0_B\rangle\langle 0_B| + |1_B\rangle\langle 1_B|, \quad \mathbf{\Pi}_{2B} = |2_B\rangle\langle 2_B|,$$

with a POVM having operation elements $\{\mathbf{B}_{1r1}, \mathbf{B}_{2r1}\}$, where

$$\mathbf{B}_{1r1} = |0_B\rangle\langle 0_B| + \frac{1}{\sqrt{2}}|1_B\rangle\langle 1_B|, \quad \mathbf{B}_{2r1} = \frac{1}{\sqrt{2}}|1_B\rangle\langle 1_B| + |2_B\rangle\langle 2_B|.$$

Before proceding let us quickly recall some basic expressions for working with operation elements $\{\mathbf{A}_i\}$ in the pure state case:

$$\Pr(i) = \langle\Psi|\mathbf{A}_i^\dagger\mathbf{A}_i|\Psi\rangle = |\mathbf{A}_i|\Psi\rangle|^2, \quad |\Psi\rangle \mapsto_i \frac{\mathbf{A}_i|\Psi\rangle}{\sqrt{\langle\Psi|\mathbf{A}_i^\dagger\mathbf{A}_i|\Psi\rangle}}.$$

If Bob obtains the result corresponding to $\mathbf{B}_{1r1}$, then states $|\psi_4\rangle$ and $|\psi_5\rangle$ would evolve according to

$$|\psi_4\rangle \mapsto \mathbf{1}^A \otimes \left(|0_B\rangle\langle 0_B| + \frac{1}{\sqrt{2}}|1_B\rangle\langle 1_B| \right) \frac{1}{\sqrt{2}}|2_A\rangle \otimes (|1_B\rangle + |2_B\rangle) = \frac{1}{2}|2_A\rangle \otimes |1_B\rangle,$$

$$|\psi_5\rangle \mapsto \mathbf{1}^A \otimes \left(|0_B\rangle\langle 0_B| + \frac{1}{\sqrt{2}}|1_B\rangle\langle 1_B| \right) \frac{1}{\sqrt{2}}|2_A\rangle \otimes (|1_B\rangle - |2_B\rangle) = \frac{1}{2}|2_A\rangle \otimes |1_B\rangle,$$

(note that these are left in an unnormalized form $\sim \mathbf{A}_i|\Psi\rangle$ such that $|\mathbf{A}_i|\Psi\rangle|^2 = \Pr(i)$) and would therefore be left indistinguishable, but if Bob obtains the result corresponding to $\mathbf{B}_{2r1}$,

$$|\psi_4\rangle \mapsto \mathbf{1}^A \otimes \left(\frac{1}{\sqrt{2}}|1_B\rangle\langle 1_B| + |2_B\rangle\langle 2_B| \right) \frac{1}{\sqrt{2}}|2_A\rangle \otimes (|1_B\rangle + |2_B\rangle) = |2_A\rangle \otimes \left(\frac{1}{2}|1_B\rangle + \frac{1}{\sqrt{2}}|2_B\rangle \right),$$

$$|\psi_5\rangle \mapsto \mathbf{1}^A \otimes \left(\frac{1}{\sqrt{2}}|1_B\rangle\langle 1_B| + |2_B\rangle\langle 2_B| \right) \frac{1}{\sqrt{2}}|2_A\rangle \otimes (|1_B\rangle - |2_B\rangle) = |2_A\rangle \otimes \left(\frac{1}{2}|1_B\rangle - \frac{1}{\sqrt{2}}|2_B\rangle \right),$$

which are not indistinguishable but also not orthogonal:

$$\langle\psi_4|\psi_4\rangle = \langle\psi_5|\psi_5\rangle = \frac{3}{4},$$

$$\frac{|\langle\psi_4|\psi_5\rangle|}{\sqrt{\langle\psi_4|\psi_4\rangle\langle\psi_5|\psi_5\rangle}} = \frac{\left|\frac{1}{4} - \frac{1}{2}\right|}{\frac{3}{4}} = \frac{1}{3}.$$

We see from these calculations that $\Pr(\mathbf{B}_{1r1}) = 1/4$ and $\Pr(\mathbf{B}_{2r1}) = 3/4$ for initial state $|\psi_4\rangle$ or $|\psi_5\rangle$, and so we expect to have some improvement in the overall performance of the procedure. The remainder of the procedure must be further modified to take advantage of this modified initial step, but in the paper it is reported that the resulting average information gain is 2.9964 bits.

A 'best known' strategy for Alice and Bob in the full nine-state scenario is described in the paper, starting at the bottom of page 1078, and results in an average information gain of 3.0125 bits, where $\log_2(9) \approx 3.1699$. The paper contains a long and involved proof that it is not possible for Alice and Bob to distinguish all nine states perfectly using only local operations and classical communication.