# APPPHYS225 - Thursday 2 October 2008

Given an observable $\mathbf{O}$, the spectral decomposition

$$\mathbf{O}|\Psi_i\rangle = \lambda_i|\Psi_i\rangle,$$

$$\mathbf{O} = \sum_{i=1}^{N} \lambda_i \mathbf{\Pi}_i, \quad \mathbf{\Pi}_i = |\Psi_i\rangle\langle\Psi_i|,$$

shows us how we can shift from thinking about eigenvalues and eigenvectors to complete sets of orthogonal projection operators,

$$\{\mathbf{\Pi}_1, \mathbf{\Pi}_2, \ldots, \mathbf{\Pi}_N\}, \quad \sum_{i=1}^{N} \mathbf{\Pi}_i = \mathbf{1}, \quad \mathbf{\Pi}_i \mathbf{\Pi}_j = \mathbf{\Pi}_i \delta_{ij}.$$

Here $N$ must be less than or equal to the dimension of the Hilbert space of the system being measured. We then focus on the probability rule

$$\Pr(i) = \langle\psi|\mathbf{\Pi}_i|\psi\rangle, \quad \Pr(i) = \mathrm{Tr}[\mathbf{\Pi}_i \rho],$$

where the fact that the projectors sum to the identity guaratees that the sum of the $\Pr(i)$ is one. The fact that the projectors are orthogonal means that the outcomes are mutually exclusive - for any possible state we can prepare, if one of the outcomes $i$ has probability one then the others must have probability zero.

The framework of generalized quantum measurement establishes the following fact. Any possible quantum measurement procedure can be specified by a set of operators $\{\mathbf{E}_i\}$, which must satisfy

$$\mathbf{E}_i > 0, \quad \sum_i \mathbf{E}_i = \mathbf{1}.$$

Here the inequality is meant to indicate that each operator $\mathbf{E}_i$ should have real, non-negative eigenvalues, and it is important to note that the number of operators in the set is not bounded by the dimension of the Hilbert space of the system being measured. The probability rule now generalizes to

$$\Pr(i) = \langle\psi|\mathbf{E}_i|\psi\rangle, \quad \Pr(i) = \mathrm{Tr}[\mathbf{E}_i \rho],$$

and again we have normalization of the probabilities by construction. Theory guarantees us not only that any possible measurement procedure can be abstractly represented in this way, but also that any such compete set of positive operators is in principle implementable as a measurement procedure.

As for conditioning, the usual rule of “collapse onto an eigenstate” can be represented by the conditional evolution rule

$$|\psi\rangle \mapsto \frac{\mathbf{\Pi}_i|\psi\rangle}{\sqrt{\langle\psi|\mathbf{\Pi}_i|\psi\rangle}}, \quad \rho \mapsto \frac{\mathbf{\Pi}_i \rho \mathbf{\Pi}_i}{\mathrm{Tr}[\mathbf{\Pi}_i \rho]}.$$

In the more general case, the conditional rule for a given $\{\mathbf{E}_i\}$ depends on the details of exactly how the measurement is implemented. That information can be represented by decomposing each of the operators $\mathbf{E}_i$ further:

$$\mathbf{E}_i = \sum_j (\mathbf{A}_{ij})^\dagger \mathbf{A}_{ij},$$

where the summation can be over any number of elements (often only one) and each $\mathbf{A}_{ij}$ is a positive operator. Given such a decomposition, the conditional evolution rule is

$$\rho \mapsto \frac{\sum_j \mathbf{A}_{ij}\,\rho\,(\mathbf{A}_{ij})^\dagger}{\mathrm{Tr}\left[\sum_j \mathbf{A}_{ij}\,\rho\,(\mathbf{A}_{ij})^\dagger\right]}.$$

That's a lot of notation, but introducing it will allow us now to talk about some very fundamental features of quantum measurement theory.

Say we have a quantum system $A$ whose state is described by a vector in a two-dimensional Hilbert space $H_A$,

$$|\Psi_A\rangle \in H_A.$$

We consider a scenario in which someone has prepared $A$ in one of the following two states,

$$|\Psi_+(\theta)\rangle = \cos\theta|0_A\rangle + \sin\theta|1_A\rangle,$$
$$|\Psi_-(\theta)\rangle = \cos\theta|0_A\rangle - \sin\theta|1_A\rangle,$$

where $0 \le \theta < \frac{\pi}{4}$. The inner product between these vectors is easily seen to be

$$\langle\Psi_-(\theta)|\Psi_+(\theta)\rangle = (\cos\theta\langle 0_A| - \sin\theta\langle 1_A|)(\cos\theta|0_A\rangle + \sin\theta|1_A\rangle)$$
$$= \cos^2\theta - \sin^2\theta,$$

which is nonzero for any $\theta$ in the open inverval $[0, \pi/4)$. Hence we know that there is no measurement that can distinguish perfectly between the two alternatives.

It is easy to show that the optimal projective measurement for distinguishing two non-orthogonal vectors consists of

$$\{\mathbf{E}_i\} = \{\mathbf{\Pi}_+ = |+\rangle\langle +|, \mathbf{\Pi}_- = |-\rangle\langle -|\},$$
$$|+\rangle \equiv \frac{1}{\sqrt{2}}(|0_A\rangle + |1_A\rangle),$$
$$|-\rangle \equiv \frac{1}{\sqrt{2}}(|0_A\rangle - |1_A\rangle).$$

The operators $\mathbf{\Pi}_+$ and $\mathbf{\Pi}_-$ project onto an orthogonal pair of vectors that 'straddles' the alternatives $|\Psi_+(\theta)\rangle$ and $|\Psi_-(\theta)\rangle$. The measurement outcome probabilites are easily computed. In the case that $|\Psi_+(\theta)\rangle$ is actually prepared,

$$\Pr(+|\Psi_+(\theta)) \equiv \langle\Psi_+(\theta)|\mathbf{\Pi}_+|\Psi_+(\theta)\rangle$$
$$= \left|(\cos\theta\langle 0_A| + \sin\theta\langle 1_A|)\frac{1}{\sqrt{2}}(|0_A\rangle + |1_A\rangle)\right|^2$$
$$= \frac{1}{2}|\cos\theta + \sin\theta|^2$$
$$= \frac{1}{2}(\cos^2\theta + 2\cos\theta\sin\theta + \sin^2\theta)$$
$$= \frac{1}{2} + \cos\theta\sin\theta,$$

$$\begin{aligned}
\Pr(-\,|\Psi_+(\theta)) &\equiv \langle\Psi_+(\theta)|\Pi_-|\Psi_+(\theta)\rangle \\
&= \left|(\cos\theta\langle 0_A| + \sin\theta\langle 1_A|)\frac{1}{\sqrt{2}}(|0_A\rangle - |1_A\rangle)\right|^2 \\
&= \frac{1}{2}|\cos\theta - \sin\theta|^2 \\
&= \frac{1}{2} - \cos\theta\sin\theta.
\end{aligned}$$

Clearly $\Pr(+\,|\Psi_+(\theta)) + \Pr(-\,|\Psi_+(\theta)) = 1$, and the probability of error here is $\Pr(-\,|\Psi_+(\theta)) > 0$. In the case that $|\Psi_-(\theta)\rangle$ is actually prepared,

$$\begin{aligned}
\Pr(+\,|\Psi_-(\theta)) &\equiv \langle\Psi_-(\theta)|\Pi_+|\Psi_-(\theta)\rangle \\
&= \left|(\cos\theta\langle 0_A| - \sin\theta\langle 1_A|)\frac{1}{\sqrt{2}}(|0_A\rangle + |1_A\rangle)\right|^2 \\
&= \frac{1}{2}|\cos\theta - \sin\theta|^2 \\
&= \frac{1}{2} - \cos\theta\sin\theta,
\end{aligned}$$

$$\begin{aligned}
\Pr(-\,|\Psi_-(\theta)) &\equiv \langle\Psi_-(\theta)|\Pi_-|\Psi_-(\theta)\rangle \\
&= \left|(\cos\theta\langle 0_A| - \sin\theta\langle 1_A|)\frac{1}{\sqrt{2}}(|0_A\rangle - |1_A\rangle)\right|^2 \\
&= \frac{1}{2}|\cos\theta + \sin\theta|^2 \\
&= \frac{1}{2} + \cos\theta\sin\theta.
\end{aligned}$$

Again the probabilities are clearly normalized and the probability of error is the same $\Pr(+\,|\Psi_-(\theta)) > 0$. Recall from last time that

$$\mathrm{PE}(\rho_0,\rho_1) = \frac{1}{2} - \frac{1}{4}\mathrm{Tr}|\rho_0 - \rho_1| = \frac{1}{2} - \frac{1}{4}\sum_j |\lambda_j|,$$

where $\{\lambda_j\}$ are the eigenvalues of the matrix $\Gamma = \rho_0 - \rho_1$. In this case we have

$$\begin{aligned}
\rho_0 &= |\Psi_+\rangle\langle\Psi_+| \to \begin{pmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{pmatrix}, \\
\rho_1 &= |\Psi_-\rangle\langle\Psi_-| \to \begin{pmatrix} \cos^2\theta & -\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \sin^2\theta \end{pmatrix}, \\
\Gamma &= \begin{pmatrix} 0 & 2\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & 0 \end{pmatrix}.
\end{aligned}$$

The eigenvalues of $\Gamma$ are $\pm 2\sin\theta\cos\theta$, so we should indeed have

$$\mathrm{PE}(\rho_0,\rho_1) = \frac{1}{2} - \sin\theta\cos\theta.$$

Since the probabilities of error are greater than zero for the projective measurement strategy, even when $\theta$ is close to $\pi/4$, we can never really be sure in any single trial that we have obtained a 'correct' answer. The particular pair of projectors described above minimizes the probabilities of error for any standard measurement. Now that we have learned about generalized measurements, however, a rather different type of strategy may come to mind. It turns out that we can describe a measurement that yields 'guaranteed' results in the sense that when we get a '+' outcome we may be absolutely sure that the state was $|\Psi_+(\theta)\rangle$, and when we get a '–' result we may be absolutely sure that the state was $|\Psi_-(\theta)\rangle$. The catch is that we sometimes get an *inconclusive* result for the measurement, which must necessarily have more than two outcomes!

Let us try to construct this measurement from scratch. If we want to have a measurement outcome that absolutely excludes the possibility $|\Psi_-(\theta)\rangle$, we should choose a projector onto the vector perpendicular to it:

$$\begin{aligned}
\boldsymbol{\Pi}_{-\perp} &= |-_\perp\rangle\langle -_\perp|,\\
|-_\perp\rangle &\equiv \sin\theta|0_A\rangle + \cos\theta|1_A\rangle,\\
\langle -_\perp|\Psi_-(\theta)\rangle &= (\sin\theta\langle 0_A| + \cos\theta\langle 1_A|)(\cos\theta|0_A\rangle - \sin\theta|1_A\rangle)\\
&= \sin\theta\cos\theta - \cos\theta\sin\theta\\
&= 0.
\end{aligned}$$

Any time we obtain the measurement outcome corresponding to this projector, we may be absolutely sure that the state was not prepared in state $|\Psi_-(\theta)\rangle$, hence in our scenario the preparation must have been $|\Psi_+(\theta)\rangle$. Likewise, we can easily find a projector that excludes $|\Psi_+(\theta)\rangle$,

$$\begin{aligned}
\boldsymbol{\Pi}_{+\perp} &= |+_\perp\rangle\langle +_\perp|,\\
|+_\perp\rangle &\equiv \sin\theta|0_A\rangle - \cos\theta|1_A\rangle,\\
\langle +_\perp|\Psi_+(\theta)\rangle &= (\sin\theta\langle 0_A| - \cos\theta\langle 1_A|)(\cos\theta|0_A\rangle + \sin\theta|1_A\rangle)\\
&= \sin\theta\cos\theta - \cos\theta\sin\theta\\
&= 0.
\end{aligned}$$

Now the difficulty is that

$$\begin{aligned}
\boldsymbol{\Pi}_{-\perp} + \boldsymbol{\Pi}_{+\perp} &\leftrightarrow \begin{pmatrix}\sin\theta\\ \cos\theta\end{pmatrix}\begin{pmatrix}\sin\theta & \cos\theta\end{pmatrix} + \begin{pmatrix}\sin\theta\\ -\cos\theta\end{pmatrix}\begin{pmatrix}\sin\theta & -\cos\theta\end{pmatrix}\\
&= \begin{pmatrix}\sin^2\theta & \sin\theta\cos\theta\\ \sin\theta\cos\theta & \cos^2\theta\end{pmatrix} + \begin{pmatrix}\sin^2\theta & -\sin\theta\cos\theta\\ -\sin\theta\cos\theta & \cos^2\theta\end{pmatrix}\\
&= \begin{pmatrix}2\sin^2\theta & 0\\ 0 & 2\cos^2\theta\end{pmatrix}
\end{aligned}$$

is clearly not equal to the identity operator, hence this particular pair of projectors is not complete.

We can try to salvage our strategy by noting that the addition of a *third* operator

$$2\mathbf{E}_? \leftrightarrow \begin{pmatrix} 2\cos^2\theta & 0 \\ 0 & 2\sin^2\theta \end{pmatrix}$$

would make the set at least proportional to the identity,

$$\begin{aligned} \mathbf{\Pi}_{-\perp} + \mathbf{\Pi}_{+\perp} + 2\mathbf{E}_? &\leftrightarrow \begin{pmatrix} 2\sin^2\theta & 0 \\ 0 & 2\cos^2\theta \end{pmatrix} + \begin{pmatrix} 2\cos^2\theta & 0 \\ 0 & 2\sin^2\theta \end{pmatrix} \\ &= \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}. \end{aligned}$$

The operator $\mathbf{E}_?$ is clearly positive since it is diagonal and its diagonal elements are the squares of real numbers, hence the set $\{\mathbf{E}_+ \equiv \frac{1}{2}\mathbf{\Pi}_{-\perp}, \mathbf{E}_- \equiv \frac{1}{2}\mathbf{\Pi}_{+\perp}, \mathbf{E}_?\}$ forms a valid POVM! So in principle, we know that there is a tri-valued measurement whose outcome probabilities are given by:

| □ | $\Psi_+(\theta)$ | $\Psi_-(\theta)$ |
|---|---|---|
| $+$ | $\frac{1}{2}\langle \Psi_+(\theta) \| \mathbf{\Pi}_{-\perp} \| \Psi_+(\theta) \rangle$ | $0$ |
| $-$ | $0$ | $\frac{1}{2}\langle \Psi_-(\theta) \| \mathbf{\Pi}_{+\perp} \| \Psi_-(\theta) \rangle$ |
| $?$ | $\langle \Psi_+(\theta) \| \mathbf{E}_? \| \Psi_+(\theta) \rangle$ | $\langle \Psi_-(\theta) \| \mathbf{E}_? \| \Psi_-(\theta) \rangle$ |

.

Computing these explicitly,

$$\begin{aligned} \frac{1}{2}\langle \Psi_+(\theta) | \mathbf{\Pi}_{-\perp} | \Psi_+(\theta) \rangle &= \frac{1}{2}|(\cos\theta\langle 0_A | + \sin\theta\langle 1_A |)(\sin\theta|0_A\rangle + \cos\theta|1_A\rangle)|^2 \\ &= \frac{1}{2}|\cos\theta\sin\theta + \sin\theta\cos\theta|^2 \\ &= 2\sin^2\theta\cos^2\theta, \\ \frac{1}{2}\langle \Psi_-(\theta) | \mathbf{\Pi}_{+\perp} | \Psi_-(\theta) \rangle &= \frac{1}{2}|(\cos\theta\langle 0_A | - \sin\theta\langle 1_A |)(\sin\theta|0_A\rangle - \cos\theta|1_A\rangle)|^2 \\ &= \frac{1}{2}|\cos\theta\sin\theta + \sin\theta\cos\theta|^2 \\ &= 2\sin^2\theta\cos^2\theta, \end{aligned}$$

$$\langle \Psi_+(\theta) | \mathbf{E}_? | \Psi_+(\theta) \rangle = \begin{pmatrix} \cos\theta & \sin\theta \end{pmatrix} \begin{pmatrix} \cos^2\theta & 0 \\ 0 & \sin^2\theta \end{pmatrix} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & \sin\theta \end{pmatrix} \begin{pmatrix} \cos^3\theta \\ \sin^3\theta \end{pmatrix}$$

$$= \cos^4\theta + \sin^4\theta,$$

$$\langle \Psi_-(\theta) | \mathbf{E}_? | \Psi_-(\theta) \rangle = \begin{pmatrix} \cos\theta & -\sin\theta \end{pmatrix} \begin{pmatrix} \cos^2\theta & 0 \\ 0 & \sin^2\theta \end{pmatrix} \begin{pmatrix} \cos\theta \\ -\sin\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & -\sin\theta \end{pmatrix} \begin{pmatrix} \cos^3\theta \\ -\sin^3\theta \end{pmatrix}$$

$$= \cos^4\theta + \sin^4\theta.$$

Hence our table is

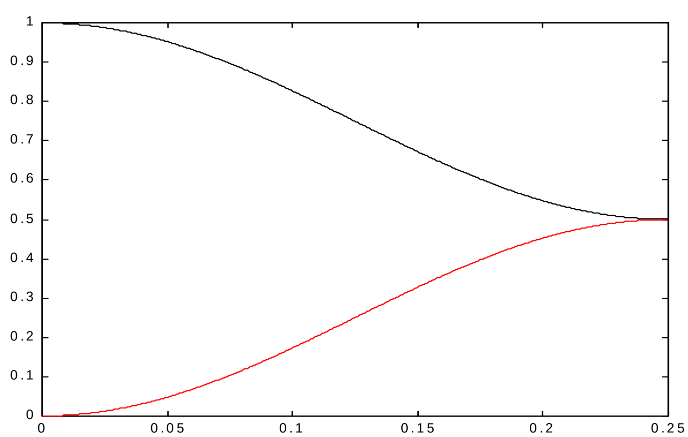
| | $\Psi_+(\theta)$ | $\Psi_-(\theta)$ |
|---|---|---|
| $+$ | $2\sin^2\theta\cos^2\theta$ | $0$ |
| $-$ | $0$ | $2\sin^2\theta\cos^2\theta$ |
| $?$ | $\cos^4\theta + \sin^4\theta$ | $\cos^4\theta + \sin^4\theta$ |

.

We clearly see that the columns are normalized, as required,

$$2\sin^2\theta\cos^2\theta + \cos^4\theta + \sin^4\theta = (\cos^2\theta + \sin^2\theta)^2 = 1.$$

Our probabilites of 'making the wrong guess' are zero, our probabilites of getting the correct answer (and knowing it!) are $2\sin^2\theta\cos^2\theta$ for either preparation, and our probabilites of getting an inconclusive result '$?$' are $\cos^4\theta + \sin^4\theta.$ You may find it enlightening to look at a plot of these quantities:




Here the x-axis is $\theta$ (in units of $\pi$), the red curve is the probability of an unambiguous correct result, and the black curve is the probability of an inconclusive result. At $\theta = 0$

the two states $|\Psi_{\pm}(\theta)\rangle$ are identical so we can only get inconclusive results, and at $\theta = \pi/4$ the two states are orthogonal. There it is clear that the tri-valued measurement is a bad idea (since the probability of success is $0.5$ instead of $1$), but in between it seems fair to say that we have gained something since we can sometimes obtain the correct answer without ever making an incorrect guess!

But have we actually made an *optimal* choice for the third positive operator $\mathbf{E}_?$, for this general measurement scheme? Note that we could have also considered any POVM of the form

$$\mathbf{E}_+ = \frac{p}{2}\mathbf{\Pi}_{-\perp},$$
$$\mathbf{E}_- = \frac{p}{2}\mathbf{\Pi}_{+\perp},$$
$$\mathbf{E}_? = \mathbf{1} - (\mathbf{E}_+ + \mathbf{E}_-),$$

where $0 \le p \le 1$ is an adjustable real parameter. Then the probabilities of error are still clearly zero, so to optimize $p$ it suffices to *minimize* the probability of inconclusive outcome,

$$\Pr(?) = \langle\Psi_+(\theta)|\mathbf{E}_?|\Psi_+(\theta)\rangle = \langle\Psi_-(\theta)|\mathbf{E}_?|\Psi_-(\theta)\rangle,$$

where the two expectation values are equated by symmetry. Going back to a matrix representation, we may compute

$$\begin{aligned}\mathbf{E}_? &= \mathbf{1} - (\mathbf{E}_+ + \mathbf{E}_-)\\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix} - \frac{p}{2}\begin{pmatrix} 2\sin^2\theta & 0 \\ 0 & 2\cos^2\theta\end{pmatrix}\\ &= \begin{pmatrix} 1 - p\sin^2\theta & 0 \\ 0 & 1 - p\cos^2\theta\end{pmatrix},\end{aligned}$$

$$\begin{aligned}\langle\mathbf{E}_?\rangle &= \begin{pmatrix} \cos\theta & \sin\theta\end{pmatrix}\begin{pmatrix} 1 - p\sin^2\theta & 0 \\ 0 & 1 - p\cos^2\theta\end{pmatrix}\begin{pmatrix} \cos\theta \\ \sin\theta\end{pmatrix}\\ &= \begin{pmatrix} \cos\theta & \sin\theta\end{pmatrix}\begin{pmatrix} (1 - p\sin^2\theta)\cos\theta \\ (1 - p\cos^2\theta)\sin\theta\end{pmatrix}\\ &= (1 - p\sin^2\theta)\cos^2\theta + (1 - p\cos^2\theta)\sin^2\theta\\ &= 1 - 2p\sin^2\theta\cos^2\theta.\end{aligned}$$

We note that since $0 \le \sin^2\theta\cos^2\theta \le 1/4$ for $\theta \in [0, \pi/4]$, we should make $p$ as large as possible to minimize $\Pr(?)$. However, we must take care to ensure that $\mathbf{E}_?$ remains a positive operator! Since it is diagonal this boils down to making sure that its diagonal elements are non-negative, so we have

$$p^{-1} = \max(\sin^2\theta, \cos^2\theta).$$

Since $\theta \in [0, \pi/4]$, we can set $p = \cos^{-2}\theta$ and thus

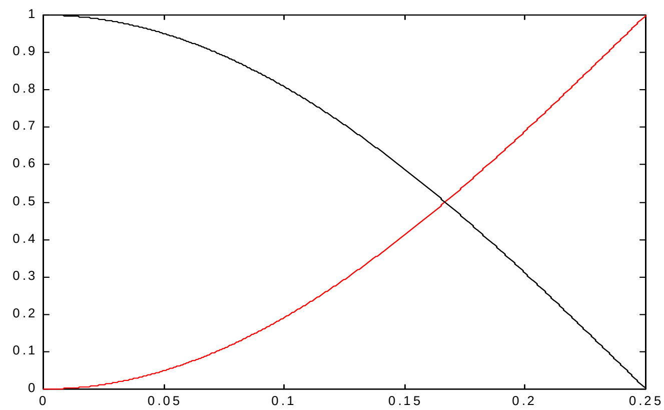
$$\mathbf{E}_? = \begin{pmatrix} 1-\tan^2\theta & 0 \\ 0 & 0 \end{pmatrix},$$

$$\Pr(?) = \langle \mathbf{E}_? \rangle = 1 - 2\sin^2\theta.$$

With this new choice of POVM, our outcome probability table looks like

| | $\Psi_+(\theta)$ | $\Psi_-(\theta)$ |
|---|---|---|
| $+$ | $2\sin^2\theta$ | $0$ |
| $-$ | $0$ | $2\sin^2\theta$ |
| $?$ | $1-2\sin^2\theta$ | $1-2\sin^2\theta$ |

with graph:




Here again the x-axis is $\theta$ in units of $\pi$, the red curve is the probability of getting the correct unambiguous answer, and the black curve is the probability of getting the inconclusive answer. This looks much better since $\Pr(?)$ now goes to zero at $\theta = \pi/4$.

So what is this optimal $\mathbf{E}_?$? Looking at its matrix form

$$\mathbf{E}_? = \begin{pmatrix} 1-\tan^2\theta & 0 \\ 0 & 0 \end{pmatrix}$$

we see that *it is proportional to a projector*,

$$\mathbf{E}_? = (1-\tan^2\theta)|0_A\rangle\langle 0_A| \equiv (1-\tan^2\theta)\,\mathbf{\Pi}_0.$$

Hence we find that we may think of the optimal POVM,

$$\left\{\frac{1}{2}\cos^{-2}\theta\,\mathbf{\Pi}_{-\perp},\ \frac{1}{2}\cos^{-2}\theta\,\mathbf{\Pi}_{+\perp},\ (1-\tan^2\theta)\mathbf{\Pi}_0\right\},$$

as projections onto a set of three vectors $\{|-_\perp\rangle, |+_\perp\rangle, |0_A\rangle\}$ with optimized 'weights.'

It is interesting to note that for general $p$, our third operator

$$\mathbf{E}_? = \begin{pmatrix} 1-p\sin^2\theta & 0 \\ 0 & 1-p\cos^2\theta \end{pmatrix}$$

is not proportional to a projector since

$$(\mathbf{E}_?)^2 = \begin{pmatrix} 1 - 2p\sin^2\theta + p^2\sin^4\theta & 0 \\ 0 & 1 - 2p\cos^2\theta + p^2\cos^4\theta \end{pmatrix}$$
$$\neq \alpha\mathbf{E}_?$$

for any scalar $\alpha$. Looking at the original $p = 1$ case for example,

$$\mathbf{E}_? = \begin{pmatrix} 1 - \sin^2\theta & 0 \\ 0 & 1 - \cos^2\theta \end{pmatrix} = \begin{pmatrix} \cos^2\theta & 0 \\ 0 & \sin^2\theta \end{pmatrix},$$
$$(\mathbf{E}_?)^2 = \begin{pmatrix} \cos^4\theta & 0 \\ 0 & \sin^4\theta \end{pmatrix}.$$

The only way to have proportionality is if

$$\cos^2\theta = \sin^2\theta,$$

or $\theta = \pi/4$ which is a boundary of the interval we consider. Hence we have our first example of a ‘useful’ Effect that is neither a projector nor proportional to a projector.

B. Huttner *et al.*, “Unambiguous quantum measurement of nonorthogonal states,” Phys. Rev. A **54**, 3783 (1996).