APPPHYS225 - Thursday/Friday 23-24 October 2008

Ensemble-dependent bounds for accessible information

As we have previously discussed, for an ensemble of quantum states $\{p_i, \rho_i\}$ (where for pure-state ensembles we have $\rho_i = |\Psi_i\rangle\langle\Psi_i|$) we can compute the expectation value of any observable $\mathbf{A}$ from the ensemble density matrix:

$$\langle \mathbf{A} \rangle = \sum_i p_i \,\mathrm{Tr}[\rho_i \mathbf{A}] = \mathrm{Tr}\left[\sum_i p_i \rho_i \mathbf{A}\right] = \mathrm{Tr}[\rho \mathbf{A}], \quad \rho = \sum_i p_i \rho_i.$$

Since this holds for $\mathbf{A}$ an element of a POVM, we see that the 'overall' statistics of any possible measurement performed on states drawn from an ensemble $\{p_i, \rho_i\}$ can be predicted on the basis of the ensemble density matrix alone.

This is not to say that we never need to know the full ensemble specification! To see this we need only review the concept of mutual information (and its relation to channel capacity), *e.g.*, from Wikipedia.

In the paper "Ensemble-Dependent Bounds for Accessible Information in Quantum Mechanics," [C. A. Fuchs and C. M. Caves, Phys. Rev. Lett. **73**, 3047 (1994)] the authors derive ensemble-dependent upper and lower bounds to the accessible information, which is defined as the maximum over all measurements $\{\mathbf{E}_b\}$ of the mutual information

$$I = H(\rho) - \sum_{i=1}^{n} p_i H(\rho_i),$$

where the ensemble of signal states is $\{p_i, \rho_i\}$ and $\rho = \sum_i p_i \rho_i$ is the ensemble density matrix. We have previously considered the case

$$\rho_0 = |\Psi_+(\theta)\rangle\langle\Psi_+(\theta)| = \cos^2\theta|0\rangle\langle 0| + \sin\theta\cos\theta(|0\rangle\langle 1| + |1\rangle\langle 0|) + \sin^2\theta|1\rangle\langle 1|,$$
$$\rho_1 = |\Psi_-(\theta)\rangle\langle\Psi_-(\theta)| = \cos^2\theta|0\rangle\langle 0| - \sin\theta\cos\theta(|0\rangle\langle 1| + |1\rangle\langle 0|) + \sin^2\theta|1\rangle\langle 1|,$$

with $t = \frac{1}{2}$ and therefore

$$\rho = \frac{1}{2}\rho_0 + \frac{1}{2}\rho_1 = \cos^2\theta|0\rangle\langle 0| + \sin^2\theta|1\rangle\langle 1|.$$

The Holevo bound on accessible information is

$$I_{acc} \leq S(\rho) - \frac{1}{2}S(\rho_0) - \frac{1}{2}S(\rho_1) = -2\cos^2\theta\ln(\cos\theta) - 2\sin^2\theta\ln(\sin\theta),$$

where we have computed by inspection

$$S(\rho) = -2\cos^2\theta\ln(\cos\theta) - 2\sin^2\theta\ln(\sin\theta),$$
$$S(\rho_0) = S(\rho_1) = 0,$$

since the signal states are pure and ρ has been given in a diagonal form. Since the signal states are pure states, the Holevo bound depends only on ρ; in the paper it is noted that this is the best bound that can be expressed solely in terms of the ensemble density matrix.

In class we noted that the optimal projective measurement is $\{\mathbf{P}_+, \mathbf{P}_-\}$, where

$$\mathbf{P}_{\pm} = |\pm\rangle\langle\pm|, \quad |\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle).$$

Hence with respect to this measurement

$$\mathrm{Tr}[\rho\mathbf{P}_{+}] = \langle + |\rho| +\rangle = \frac{1}{2}(\langle 0| + \langle 1|)(\cos^2\theta|0\rangle\langle 0| + \sin^2\theta|1\rangle\langle 1|)(|0\rangle + |1\rangle) = \frac{1}{2},$$
$$\mathrm{Tr}[\rho\mathbf{P}_{-}] = \langle - |\rho| -\rangle = \frac{1}{2}(\langle 0| - \langle 1|)(\cos^2\theta|0\rangle\langle 0| + \sin^2\theta|1\rangle\langle 1|)(|0\rangle - |1\rangle) = \frac{1}{2},$$
$$H(\rho) = \ln 2,$$

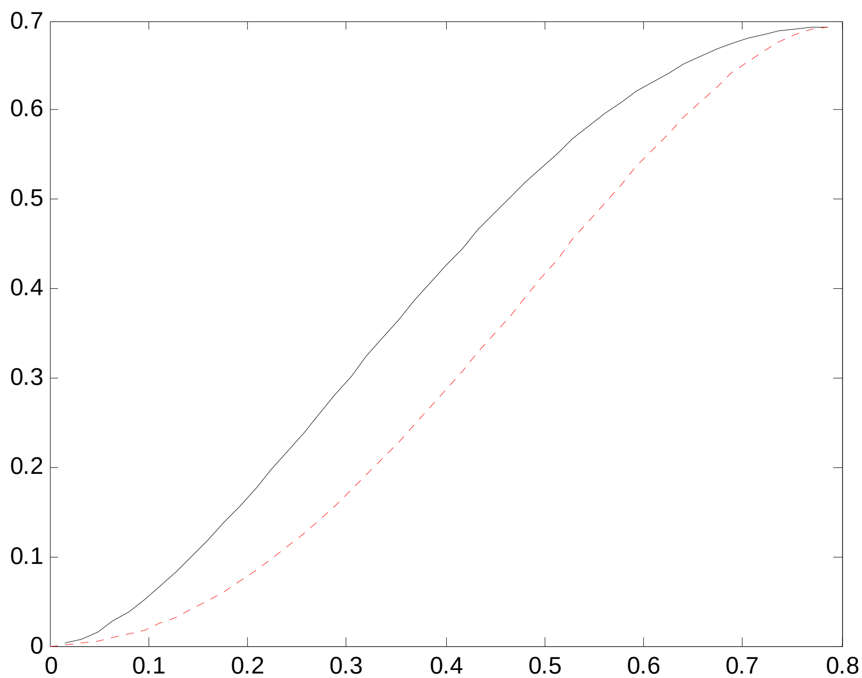
and we previously calculated the entropies

$$H(\rho_0) = H(\rho_1) = \ln 2 - (\cos\theta + \sin\theta)^2 \ln[\cos\theta + \sin\theta] - (\cos\theta - \sin\theta)^2 \ln[\cos\theta - \sin\theta],$$

and therefore

$$I = H(\rho) - \frac{1}{2}H(\rho_0) - \frac{1}{2}H(\rho_1)$$
$$= (\cos\theta + \sin\theta)^2 \ln[\cos\theta + \sin\theta] + (\cos\theta - \sin\theta)^2 \ln[\cos\theta - \sin\theta].$$

Below we plot both the mutual information of the optimal projective measurement (dashed, lower curve) and the Holevo bound.



Note that points on the upper (solid) curve would actually be ‘achievable’ as the mutual information of the optimal projective measurement for binary ensembles

$$p_0 = \cos^2\theta, \quad \rho_0 = |0\rangle\langle 0|, \quad p_1 = \sin^2\theta, \quad \rho_1 = |1\rangle\langle 1|,$$

which have the same ensemble density matrix as our favorite case with $|\Psi_{\pm}(\theta)\rangle$.

Joint state space for two subsystems

Suppose we have two independent quantum systems. It seems clear that we can separately consider the representation of their physical states in two independent Hilbert spaces. Labelling the systems A and B, we can simply chose state vectors

$$|\Psi_A\rangle \in H_A,$$

and

$$|\Psi_B\rangle \in H_B.$$

What if we need to bring these systems together and let them interact?

The joint state space for two such systems corresponds to the *tensor product* of H_A and H_B, denoted $H_{AB} = H_A \otimes H_B$.

Let N_A be the dimension of H_A, and N_B the dimension of H_B. If $\{|1_A\rangle, |2_A\rangle, |3_A\rangle, \dots\}$ is a complete orthonormal basis for H_A and $\{|1_B\rangle, |2_B\rangle, |3_B\rangle, \dots\}$ is a complete orthonormal basis for H_B, then $H_A \otimes H_B$ is the Hilbert space of dimension $N_{AB} = N_A N_B$ spanned by the vectors of the form $|i_A\rangle \otimes |j_B\rangle$.

Hence arbitrary states in H_{AB} have the form

$$|\Psi_{AB}\rangle = \sum_{i=1}^{N_A} \sum_{j=1}^{N_B} c_{ij} |i_A\rangle \otimes |j_B\rangle.$$

As long as we fix an ordering for the new basis states $|i_A\rangle \otimes |j_B\rangle$, the set of $N_A N_B$ complex coefficients can be used as a vector representation for kets in H_{AB}.

The tensor product operation between vectors has the following properties:

1. Linearity: $(\alpha|\Psi_A\rangle) \otimes |\Psi_B\rangle = \alpha(|\Psi_A\rangle \otimes |\Psi_B\rangle)$, where α is a complex number
2. Distributivity: $|\Psi_A\rangle \otimes (|\Psi_B^1\rangle + |\Psi_B^2\rangle) = |\Psi_A\rangle \otimes |\Psi_B^1\rangle + |\Psi_A\rangle \otimes |\Psi_B^2\rangle$.
3. 'Commutativity': formally, $|\Psi_A\rangle \otimes |\Psi_B\rangle$ is the same as $|\Psi_B\rangle \otimes |\Psi_A\rangle$. In practice however, it is wise to use consistent ordering.
4. Adjoint: $(|\Psi_A\rangle \otimes |\Psi_B\rangle)^\dagger = \langle\Psi_A| \otimes \langle\Psi_B|$.
5. Scalar product: $(\langle\Psi_A^1| \otimes \langle\Psi_B^1|)(|\Psi_A^2\rangle \otimes |\Psi_B^2\rangle) = \langle\Psi_A^1|\Psi_A^2\rangle\langle\Psi_B^1|\Psi_B^2\rangle$.

It is important to note that basis kets $|i_A\rangle \otimes |j_B\rangle \in H_{AB}$ thus inherit orthogonality from their 'factors' in H_A and H_B.

Entanglement

The most profound consequence of this mathematical rule for representation of joint states is that there exist $|\Psi_{AB}\rangle \in H_{AB}$ that cannot be expressed the tensor product of a state $|\Psi_A\rangle \in H_A$ with a state $|\Psi_B\rangle \in H_B$. Such 'nonfactorizable' states are said to be *entangled*.

For example, let's consider two two-dimensional systems. Say we have chosen orthonormal bases $\{|0_A\rangle, |1_A\rangle\}$ for H_A and $\{|0_B\rangle, |1_B\rangle\}$ for H_B. Then H_{AB} is spanned by the four states

$$|0_A\rangle \otimes |0_B\rangle, \quad |0_A\rangle \otimes |1_B\rangle, \quad |1_A\rangle \otimes |0_B\rangle, \quad |1_A\rangle \otimes |1_B\rangle.$$

Factorizable (nonentangled) states in H_{AB} are all of the form

$$\begin{aligned} \left|\Psi_{AB}^{fac}\right\rangle &= (c_0^A|0_A\rangle + c_1^A|1_A\rangle) \otimes (c_0^B|0_B\rangle + c_1^B|1_B\rangle) \\ &= c_0^A c_0^B|0_A\rangle \otimes |0_B\rangle + c_0^A c_1^B|0_A\rangle \otimes |1_B\rangle \\ &\quad + c_1^A c_0^B|1_A\rangle \otimes |0_B\rangle + c_1^A c_1^B|1_A\rangle \otimes |1_B\rangle. \end{aligned}$$

That is, a certain relationship exists between the coefficients of the four basis states in H_{AB}.

A simple example of an entangled state, whose coefficients do not exhibit the above relationship, is

$$\begin{aligned} |\Psi_{AB}\rangle &= \frac{1}{\sqrt{2}}(|0_A\rangle \otimes |0_B\rangle + |1_A\rangle \otimes |1_B\rangle) \\ &\neq |\Psi_A\rangle \otimes |\Psi_B\rangle. \end{aligned}$$

When the joint state of two subsystems is entangled, ***there is no way to assign a pure quantum state to either subsystem alone***. As we shall see below, it is possible to ascribe *mixed* quantum states to each of the subsystems considered alone, but first we'll need to have a look at operators on H_{AB}.

Tensor products of operators

If $\mathbf{A}$ is an operator on H_A and $\mathbf{B}$ is an operator on H_B, then

$$\mathbf{A} \otimes \mathbf{B}$$

is a valid operator on H_{AB}. Its action on an arbitrary state

$$|\Psi_{AB}\rangle = \sum_{i,j} c_{ij}|i_A\rangle \otimes |j_B\rangle$$

is defined by

$$(\mathbf{A} \otimes \mathbf{B})|\Psi_{AB}\rangle = \sum_{i,j} c_{ij}(\mathbf{A}|i_A\rangle) \otimes (\mathbf{B}|j_B\rangle).$$

In the case where $\mathbf{A}$ and $\mathbf{B}$ are both normal, we may also write

$$\begin{aligned} \mathbf{A} \otimes \mathbf{B} &= \left(\sum_i \lambda_i^A \mathbf{P}_i^A\right) \otimes \left(\sum_j \lambda_j^B \mathbf{P}_j^B\right) \\ &= \sum_{ij} \lambda_i^A \lambda_j^B \mathbf{P}_i^A \otimes \mathbf{P}_j^B. \end{aligned}$$

Note that the usual relationship holds between projectors on the joint state space and outer-products of joint state vectors:

$$\begin{aligned} (|\Psi_A\rangle \otimes |\Psi_B\rangle)(\langle\Psi_A| \otimes \langle\Psi_B|) &= |\Psi_A\rangle\langle\Psi_A| \otimes |\Psi_B\rangle\langle\Psi_B| \\ &= \mathbf{P}_A \otimes \mathbf{P}_B. \end{aligned}$$

Hence any complete set of joint projectors (summing to the identity operator on H_{AB}) specifies a complete measurement.

As was the case with state vectors, linear combinations of tensor-product operators are also valid opeators on H_{AB}:

$$\mathbf{O}_{AB} = \sum_m c_m \mathbf{A}_m \otimes \mathbf{B}_m.$$

Hence, not all operators on a joint state space are factorizable.

Given subsystem density operators ρ_A and ρ_B, we can form a tensor-product density operator that describes a mixed ensemble of states in H_{AB}:

$$\rho_{AB} = \rho_A \otimes \rho_B.$$

In general, one can form convex combinations of such ρ_{AB} to construct new joint density operators.

One can also construct joint density operators directly from ensembles of pure states in H_{AB}. For instance, the density operator corresponding to the entangled state described above is

$$|\Psi_{AB}\rangle = \frac{1}{\sqrt{2}}[|0_A\rangle \otimes |0_B\rangle + |1_A\rangle \otimes |1_B\rangle]$$

$$\rho_{AB} = |\Psi_{AB}\rangle\langle\Psi_{AB}|$$

$$= \frac{1}{2}\left[\begin{array}{c} |0_A\rangle\langle 0_A| \otimes |0_B\rangle\langle 0_B| + |0_A\rangle\langle 1_A| \otimes |0_B\rangle\langle 1_B| \\ +|1_A\rangle\langle 0_A| \otimes |1_B\rangle\langle 0_B| + |1_A\rangle\langle 1_A| \otimes |1_B\rangle\langle 1_B| \end{array}\right],$$

and in general

$$\rho_{AB} = \sum_i p_i |\Psi^i_{AB}\rangle\langle\Psi^i_{AB}|.$$

Note that operators on a tensor-product space can be expressed as complex matrices o_{kl}:

$$\mathbf{O}_{AB} = \sum_{kl} o_{kl} |k_{AB}\rangle\langle l_{AB}|,$$

where the summations both run over a complete set of N_{AB} basis vectors.

Given matrix representations for subsystem operators $\mathbf{A}$ and $\mathbf{B}$, it is customary to choose an ordering for the basis states of the joint space such that

$$\mathbf{A} \otimes \mathbf{B} \leftrightarrow \begin{pmatrix} a_{11}B & a_{12}B & a_{13}B & \square \\ a_{21}B & a_{22}B & a_{23}B & \cdots \\ a_{31}B & a_{32}B & a_{33}B & \square \\ \square & \vdots & \square & \ddots \end{pmatrix}.$$

For example if $\{|1_A\rangle, |2_A\rangle, \ldots\}$ is the orthnormal basis for H_A used in defining the matrix representation of A, and similarly for H_B, then

$$|1_{AB}\rangle \leftrightarrow |1_A\rangle \otimes |1_B\rangle,$$
$$|2_{AB}\rangle \leftrightarrow |1_A\rangle \otimes |2_B\rangle,$$
$$|3_{AB}\rangle \leftrightarrow |1_A\rangle \otimes |3_B\rangle,$$
$$\vdots$$
$$|(N_B+1)_{AB}\rangle \leftrightarrow |2_A\rangle \otimes |1_B\rangle,$$
$$\vdots$$

As a result, the common class of operators $\mathbf{1}^A \otimes \mathbf{B}$ will have block-diagonal representations.

Working with tensor products

Let's work with our favorite example of two two-dimensional Hilbert spaces H_A and H_B, with given complete orthonormal bases $\{|0_A\rangle, |1_A\rangle\}$ and $\{|0_B\rangle, |1_B\rangle\}$. Let's also choose the simple tensor-product basis for H_{AB}, $\{|0_A0_B\rangle, |0_A1_B\rangle, |1_A0_B\rangle, |1_A1_B\rangle\}$.

Suppose we are given vectors $|\Psi_A\rangle \in H_A$ and $|\Psi_B\rangle \in H_B$:

$$|\Psi_A\rangle = a_0|0_A\rangle + a_1|1_A\rangle \leftrightarrow \begin{pmatrix} a_0 \\ a_1 \end{pmatrix},$$
$$|\Psi_B\rangle = b_0|0_B\rangle + b_1|1_B\rangle \leftrightarrow \begin{pmatrix} b_0 \\ b_1 \end{pmatrix}.$$

Then $|\Psi_A\Psi_B\rangle \in H_{AB}$ has the vector representation

$$|\Psi_A\rangle \otimes |\Psi_B\rangle = (a_0|0_A\rangle + a_1|1_A\rangle) \otimes (b_0|0_B\rangle + b_1|1_B\rangle)$$
$$= a_0b_0\,|0_A0_B\rangle + a_0b_1\,|0_A1_B\rangle + a_1b_0\,|1_A0_B\rangle + a_1b_1\,|1_A1_B\rangle$$
$$\leftrightarrow \begin{pmatrix} a_0b_0 \\ a_0b_1 \\ a_1b_0 \\ a_1b_1 \end{pmatrix}.$$

Likewise,

$$\langle\Psi_A\Psi_B| \leftrightarrow \begin{pmatrix} a_0^*b_0^* & a_0^*b_1^* & a_1^*b_0^* & a_1^*b_1^* \end{pmatrix}.$$

Moving on to operators, let's compute a matrix representation for $\boldsymbol{\sigma}_x^A \otimes \boldsymbol{\sigma}_x^B$, where $\boldsymbol{\sigma}_x = |0\rangle\langle 1| + |1\rangle\langle 0|$. So

$$\begin{aligned}\boldsymbol{\sigma}_x^A \otimes \boldsymbol{\sigma}_x^B &= (|0_A\rangle\langle 1_A| + |1_A\rangle\langle 0_A|) \otimes (|0_B\rangle\langle 1_B| + |1_B\rangle\langle 0_B|) \\ &= |0_A 0_B\rangle\langle 1_A 1_B| + |0_A 1_B\rangle\langle 1_A 0_B| + |1_A 0_B\rangle\langle 0_A 1_B| + |1_A 1_B\rangle\langle 0_A 0_B| \\ &\leftrightarrow \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.\end{aligned}$$

Given the ordering we have chosen for the basis of H_{AB}, we could have also used

$$\mathbf{A} \otimes \mathbf{B} \leftrightarrow \begin{pmatrix} a_{00}B & a_{01}B \\ a_{10}B & a_{11}B \end{pmatrix},$$

where in this case

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\mathbf{A} \otimes \mathbf{B} \leftrightarrow \begin{pmatrix} 0\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & 1\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ 1\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & 0\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Partial projections

A particularly useful class of tensor-product operators are the partial projectors,

$$\mathbf{1}^A \otimes \mathbf{P}_j^B$$

and

$$\mathbf{P}_i^A \otimes \mathbf{1}^B,$$

where $\mathbf{P}_j^B$ is a projector onto some state in H_B and likewise for $\mathbf{P}_i^A$. Note that such operators are themselves projectors according to the usual definition, since

$$(\mathbf{A}_1 \otimes \mathbf{B}_1)(\mathbf{A}_2 \otimes \mathbf{B}_2) = \mathbf{A}_1\mathbf{A}_2 \otimes \mathbf{B}_1\mathbf{B}_2.$$

Clearly, observables such as

$$\mathbf{O}_q^A \otimes \mathbf{1}^B$$

can be spectrally decomposed using partial projectors.

If $\mathbf{P}_k^B = |k_B\rangle\langle k_B|$ (where $|k_B\rangle$ is a basis vector), then

$$(\mathbf{1}^A \otimes \mathbf{P}_k^B)|\Psi_{AB}\rangle = (\mathbf{1}^A \otimes \mathbf{P}_k^B)\sum_{ij} c_{ij}|i_A\rangle \otimes |j_B\rangle$$
$$= \sum_{ij} c_{ij}|i_A\rangle \otimes \mathbf{P}_k^B|j_B\rangle = \sum_i c_{ik}|i_A\rangle \otimes |k_B\rangle = |\Psi_A^k\rangle \otimes |k_B\rangle.$$

Hence the effect of a partial projector on a joint state in H_{AB} is to knock out all terms in the superposition that are not consistent with subsystem B being in the k^{th} basis state.

It is very important to appreciate that the action of a partial projector will in general 'affect' the state of both subsystems, *unless* the joint state is factorizable. For example, if

$$|\Psi_{AB}\rangle = |\Psi_A\rangle \otimes |\Psi_B\rangle,$$
$$|\Psi_B\rangle = \sum_{j=1}^{N_B} c_j^B|j_B\rangle,$$

then under $\mathbf{1}^A \otimes \mathbf{P}_k^B$

$$|\Psi_{AB}\rangle \mapsto |\Psi_A\rangle \otimes c_k^B|k_B\rangle.$$

If on the other hand $|\Psi_{AB}\rangle$ is *entangled*, e.g.

$$|\Psi_{AB}\rangle = c_1|\Psi_A^1\rangle \otimes |1_B\rangle + c_2|\Psi_A^2\rangle \otimes |2_B\rangle,$$
$$\langle\Psi_A^1|\Psi_A^2\rangle \neq 1,$$

then

$$\mathbf{1}^A \otimes \mathbf{P}_2^B|\Psi_{AB}\rangle = c_2|\Psi_A^2\rangle \otimes |2_B\rangle.$$

Hence even quantities such as $\langle\mathbf{O}_q^A \otimes \mathbf{1}^B\rangle$ will be changed.

Note that if $\sum_j \mathbf{P}_j^B = \mathbf{1}^B$ (and likewise for the $\{\mathbf{P}_i^A\}$)

$$\sum_{j=1}^{N_B} \mathbf{1}^A \otimes \mathbf{P}_j^B = \mathbf{1}^A \otimes \mathbf{1}^B = \mathbf{1}^{AB},$$
$$\sum_{i=1}^{N_A} \mathbf{P}_i^A \otimes \mathbf{1}^B = \mathbf{1}^A \otimes \mathbf{1}^B.$$

Hence one can speak of a 'complete' set of *partial* projectors (with respect to either H_A or H_B), given by

$$\{\mathbf{1}^A \otimes \mathbf{P}_0^B, \mathbf{1}^A \otimes \mathbf{P}_1^B, \ldots\}$$

or

$$\{\mathbf{P}_0^A \otimes \mathbf{1}^B, \mathbf{P}_1^A \otimes \mathbf{1}^B, \ldots\}.$$

Such sets of operators specify standard measurements on H_{AB} – the projectors in the set are mutually orthogonal and sum to the identity. In essence, this type of measurement probes the state of one subsystem without regard for the other:

$$\Pr(j) = \langle\mathbf{1}^A \otimes \mathbf{P}_j^B\rangle,$$

or

$$\Pr(i) = \langle \mathbf{P}_i^A \otimes \mathbf{1}^B \rangle.$$

But as noted above, the post-measurement state of both subsystems will generally be affected by the outcome, since (for conditioning via the projection postulate)

$$|\Psi_{AB}\rangle \mapsto \frac{\mathbf{1}^A \otimes \mathbf{P}_j^B |\Psi_{AB}\rangle}{\sqrt{\langle \mathbf{1}^A \otimes \mathbf{P}_j^B \rangle}}$$

or

$$|\Psi_{AB}\rangle \mapsto \frac{\mathbf{P}_i^A \otimes \mathbf{1}^B |\Psi_{AB}\rangle}{\sqrt{\langle \mathbf{P}_i^A \otimes \mathbf{1}^B \rangle}}.$$

The usual generalization holds for joint density operators.

Exercise: Suppose systems A and B are initially prepared in the joint pure state

$$|\Psi_{AB}\rangle = \frac{1}{\sqrt{2}}(|0_A\rangle \otimes |0_B\rangle + |1_A\rangle \otimes |1_B\rangle),$$

and we perform a measurement of the A-system observable

$$\mathbf{S}_{xA} \otimes \mathbf{1}^B = \frac{\hbar}{2}\mathbf{P}_{x+}^A \otimes \mathbf{1}^B - \frac{\hbar}{2}\mathbf{P}_{x-}^A \otimes \mathbf{1}^B,$$

$$\mathbf{P}_{x+} \equiv |x_+\rangle\langle x_+|, \quad |x_+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle),$$

$$\mathbf{P}_{x-} \equiv |x_-\rangle\langle x_-|, \quad |x_-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle).$$

What are the possible post-measurement states? What if the initial preparation is the following mixed state?

$$\rho_{AB} = \frac{1}{2}(|0_A 0_B\rangle\langle 0_A 0_B| + |1_A 1_B\rangle\langle 1_A 1_B|).$$

Indirect measurements

We can build more general types of measurements by coupling our system of interest to an ancillary quantum system prepared in a known initial state

$$|A\rangle \in H_A,$$

where H_A is an N_A –dimensional Hilbert space, and then performing a direct measurement on the ancilla.

The general situation is as follows. We first take the system in its pre-measurement state ρ_{pre} and combine it with the ancilla, such that their joint state can be written

$$\rho_{\text{pre}} \otimes |A\rangle\langle A| \in H_S \otimes H_A.$$

The joint system then evolves under some (possibly time-dependent) interaction Hamiltonian $\mathbf{H}_{\text{int}}(t)$ for a fixed time interval, yielding

$$\mathbf{U}_t \left(\rho_{\text{pre}} \otimes |A\rangle\langle A|\right) \mathbf{U}_t^\dagger,$$

where

$$\mathbf{U}_t = T\left\{\exp\left[\frac{-i}{\hbar}\int_0^t dt'\,\mathbf{H}_{\text{int}}(t')\right]\right\}.$$

At this point the interaction is turned off, and in principle the ancilla can be taken away from the system. A projective measurement on the ancilla, whose statistics are specified by some set of partial projectors $\{\mathbf{\Pi}_i^A \otimes \mathbf{1}^S\}$, will lead to

$$\Pr(i) = \text{Tr}\left[\mathbf{\Pi}_i^A \otimes \mathbf{1}^S\,\mathbf{U}_t\left(\rho_{\text{pre}} \otimes |A\rangle\langle A|\right)\mathbf{U}_t^\dagger\mathbf{\Pi}_i^A \otimes \mathbf{1}^S\right],$$

and we expect to have post-measurement states

$$\rho_{\text{post}} = \frac{\mathbf{\Pi}_i^A \otimes \mathbf{1}^S\,\mathbf{U}_t\left(\rho_{\text{pre}} \otimes |A\rangle\langle A|\right)\mathbf{U}_t^\dagger\,\mathbf{\Pi}_i^A \otimes \mathbf{1}^S}{\Pr(i)}.$$

Note that the set $\{\mathbf{\Pi}_i^A \otimes \mathbf{1}^S\}$ qualifies as a complete set on the joint state space $H_S \otimes H_A$ since

$$\begin{aligned}\sum_i(\mathbf{\Pi}_i^A \otimes \mathbf{1}^S) &= \left(\sum_i \mathbf{\Pi}_i^A\right) \otimes \mathbf{1}^S \\ &= \mathbf{1}^A \otimes \mathbf{1}^S \\ &= \mathbf{1}^{A\otimes S},\end{aligned}$$

while the separability of these operators clearly indicates that this measurement can be performed by actions involving the ancilla only.

The essential idea here is that the interaction $\mathbf{U}_t$ should generate entanglement between the system and ancilla, such that their states become correlated. One common example of such an interaction is

$$\mathbf{U}_t = \mathbf{C}_{SA} \equiv |0_S\rangle\langle 0_S| \otimes \mathbf{1}^A + |1_S\rangle\langle 1_S| \otimes (|1_A\rangle\langle 0_A| + |0_A\rangle\langle 1_A|),$$

the controlled-not interaction between two two-dimensional quantum systems. If the initial system state is

$$|\Psi_S\rangle = c_0|0_S\rangle + c_1|1_S\rangle,$$

and we choose $|A\rangle = |0_A\rangle$, then we have the sequence

$$\begin{aligned}|\Psi_S\rangle &\mapsto |\Psi_S\rangle \otimes |0_A\rangle \\ &\mapsto \mathbf{C}_{SA}|\Psi_S\rangle \otimes |0_A\rangle \\ &= c_0|0_S\rangle \otimes |0_A\rangle + c_1|1_S\rangle \otimes |1_A\rangle,\end{aligned}$$

or in density operator notation:

$$\begin{aligned}\rho_{\text{pre}} &= |\Psi_S\rangle\langle\Psi_S| \\ &\mapsto |\Psi_S\rangle\langle\Psi_S| \otimes |0_A\rangle\langle 0_A| \\ &\mapsto \mathbf{C}_{SA}|\Psi_S\rangle\langle\Psi_S| \otimes |0_A\rangle\langle 0_A|\mathbf{C}_{SA}^\dagger \\ &= (c_0|0_S\rangle \otimes |0_A\rangle + c_1|1_S\rangle \otimes |1_A\rangle)(c_0^*\langle 0_S| \otimes \langle 0_A| + c_1^*\langle 0_S| \otimes \langle 0_A|).\end{aligned}$$

If now we measure the ancilla (projectively) in its $\{|0_A\rangle, |1_A\rangle\}$ basis, the probabilities will be

$$\begin{aligned}\Pr(0) &= |c_0|^2, \\ \Pr(1) &= |c_1|^2.\end{aligned}$$

It can furthermore be seen that the post-measurement states are given by

$$\left|\Psi_{\text{post}}\right\rangle = |0_S\rangle \qquad i = 0,$$
$$= |1_S\rangle \qquad i = 1.$$

We thus find that this controlled-not procedure leads to an indirect measurement whose statistics and post-measurement states are identical to those of a projective measurement of the $\{|0_S\rangle, |1_S\rangle\}$ basis.

Next let us consider a similar procedure, but with

$$|A\rangle = a_0|0_A\rangle + a_1|1_A\rangle.$$

Then

$$|\Psi_S\rangle \mapsto |\Psi_S\rangle \otimes (a_0|0_A\rangle + a_1|1_A\rangle)$$
$$\mapsto \mathbf{C}_{SA}(c_0a_0|0_S\rangle \otimes |0_A\rangle + c_0a_1|0_S\rangle \otimes |1_A\rangle + c_1a_0|1_S\rangle \otimes |0_A\rangle + c_1a_1|1_S\rangle \otimes |1_A\rangle)$$
$$= c_0a_0|0_S\rangle \otimes |0_A\rangle + c_0a_1|0_S\rangle \otimes |1_A\rangle + c_1a_0|1_S\rangle \otimes |1_A\rangle + c_1a_1|1_S\rangle \otimes |0_A\rangle$$
$$= (c_0a_0|0_S\rangle + c_1a_1|1_S\rangle) \otimes |0_A\rangle + (c_0a_1|0_S\rangle + c_1a_0|1_S\rangle) \otimes |1_A\rangle,$$

and

$$\Pr(0) = |c_0a_0|0_S\rangle + c_1a_1|1_S\rangle|^2$$
$$= |c_0a_0|^2 + |c_1a_1|^2,$$
$$\Pr(1) = |c_0a_1|0_S\rangle + c_1a_0|1_S\rangle|^2$$
$$= |c_0a_1|^2 + |c_1a_0|^2.$$

We can verify that

$$\Pr(0) + \Pr(1) = |c_0a_0|^2 + |c_1a_1|^2 + |c_0a_1|^2 + |c_1a_0|^2$$
$$= |c_0|^2(|a_0|^2 + |a_1|^2) + |c_1|^2(|a_0|^2 + |a_1|^2)$$
$$= 1.$$

The post-measurement states are now

$$\left|\Psi_{\text{post}}\right\rangle = \frac{c_0a_0|0_S\rangle + c_1a_1|1_S\rangle}{\sqrt{\Pr(0)}} \qquad i = 0,$$
$$= \frac{c_0a_1|0_S\rangle + c_1a_0|1_S\rangle}{\sqrt{\Pr(1)}} \qquad i = 1.$$

Note that if $a_0 = a_1 = 1/\sqrt{2}$ the outcome probabilities are equal and independent of $|\Psi_S\rangle$, and both of the post-measurement states are equal to $\left|\Psi_{\text{pre}}\right\rangle$. Of course, our previous case of the equivalent-to-projective measurement was a special case of this one with $a_0 = 1,\ a_1 = 0$.

For our next trick, consider the modified interaction

$$\tilde{\mathbf{C}}_{SA} \equiv |1_S\rangle\langle 0_S| \otimes \mathbf{1}^A + |0_S\rangle\langle 1_S| \otimes (|1_A\rangle\langle 0_A| + |0_A\rangle\langle 1_A|).$$

It may be verified that this interaction is still unitary. With the general ancilla preparation

$$|A\rangle = a_0|0_A\rangle + a_1|1_A\rangle$$

this leads to

$$
\begin{aligned}
|\Psi_S\rangle &\mapsto |\Psi_S\rangle \otimes (a_0|0_A\rangle + a_1|1_A\rangle) \\
&\mapsto \tilde{\mathbf{C}}_{SA}(c_0a_0|0_S\rangle \otimes |0_A\rangle + c_0a_1|0_S\rangle \otimes |1_A\rangle + c_1a_0|1_S\rangle \otimes |0_A\rangle + c_1a_1|1_S\rangle \otimes |1_A\rangle) \\
&= c_0a_0|1_S\rangle \otimes |0_A\rangle + c_0a_1|1_S\rangle \otimes |1_A\rangle + c_1a_0|0_S\rangle \otimes |1_A\rangle + c_1a_1|0_S\rangle \otimes |0_A\rangle, \\
&= (c_1a_1|0_S\rangle + c_0a_0|1_S\rangle) \otimes |0_A\rangle + (c_1a_0|0_S\rangle + c_0a_1|1_S\rangle) \otimes |1_A\rangle,
\end{aligned}
$$

so that we still have

$$
\begin{aligned}
\Pr(0) &= |c_0a_0|^2 + |c_1a_1|^2, \\
\Pr(1) &= |c_0a_1|^2 + |c_1a_0|^2,
\end{aligned}
$$

but now

$$
\begin{aligned}
|\Psi_{\text{post}}\rangle &= \frac{c_1a_1|0_S\rangle + c_0a_0|1_S\rangle}{\sqrt{\Pr(0)}} \qquad i = 0, \\
&= \frac{c_1a_0|0_S\rangle + c_0a_1|1_S\rangle}{\sqrt{\Pr(1)}} \qquad i = 1.
\end{aligned}
$$

If we now set $a_0 = 1,\ a_1 = 0,$ this reproduces like the statistics of a projective measurement but with the opposite mapping of measurement result to post-measurement system state!

Finally, let us consider

$$
\mathbf{C}_{AS} \equiv \mathbf{1}^S \otimes |0_A\rangle\langle 0_A| + (|1_S\rangle\langle 0_S| + |0_S\rangle\langle 1_S|) \otimes |1_A\rangle\langle 1_A|.
$$

Then keeping $|A\rangle = a_0|0_A\rangle + a_1|1_A\rangle$ we have

$$
\begin{aligned}
|\Psi_S\rangle &\mapsto |\Psi_S\rangle \otimes (a_0|0_A\rangle + a_1|1_A\rangle) \\
&\mapsto \mathbf{C}_{AS}(c_0a_0|0_S\rangle \otimes |0_A\rangle + c_0a_1|0_S\rangle \otimes |1_A\rangle + c_1a_0|1_S\rangle \otimes |0_A\rangle + c_1a_1|1_S\rangle \otimes |1_A\rangle) \\
&= c_0a_0|0_S\rangle \otimes |0_A\rangle + c_0a_1|1_S\rangle \otimes |1_A\rangle + c_1a_0|1_S\rangle \otimes |0_A\rangle + c_1a_1|0_S\rangle \otimes |1_A\rangle, \\
&= (c_0a_0|0_S\rangle + c_1a_0|1_S\rangle) \otimes |0_A\rangle + (c_1a_1|0_S\rangle + c_0a_1|1_S\rangle) \otimes |1_A\rangle,
\end{aligned}
$$

so that

$$
\begin{aligned}
\Pr(0) &= |c_0a_0|^2 + |c_1a_0|^2 \\
&= |a_0|^2, \\
\Pr(1) &= |c_0a_1|^2 + |c_1a_1|^2 \\
&= |a_1|^2,
\end{aligned}
$$

and

$$
\begin{aligned}
|\Psi_{\text{post}}\rangle &= \frac{c_0a_0|0_S\rangle + c_1a_0|1_S\rangle}{\sqrt{\Pr(0)}} = c_0|0_S\rangle + c_1|1_S\rangle \qquad i = 0, \\
&= \frac{c_1a_1|0_S\rangle + c_0a_1|1_S\rangle}{\sqrt{\Pr(1)}} = c_0|1_S\rangle + c_1|0_S\rangle \qquad i = 1.
\end{aligned}
$$

We thus have a situation where the statistics of the measurement are independent of $|\Psi_S\rangle,$ but depending on the measurement outcome the post-measurement system state is either equal to the pre-measurement state or ‘flipped’ by the transformation

$$
|0_S\rangle \mapsto |1_S\rangle, \quad |1_S\rangle \mapsto |0_S\rangle.
$$

Our choice of the amplitudes a_0 and a_1 independently sets the relative likelihood of the two outcomes!

In summary, we see that indirect measurements can be used to ‘mimic’ projective measurements, but can also be used to construct measurement procedures in which the transformation from pre- to post-measurement system states is more general than a projection and can be decoupled from the information obtained about $\boldsymbol{\rho}_{\text{pre}}$.

Higher-dimensional ancillas

It is crucial to note that the ancillary system in an indirect measurement can have arbitrary dimension – that is, N_A can be much larger than N_S leading to a measurement procedure on an N_S –dimensional system with more than N_S outcomes!

For example, still considering a two-dimensional system of interest, we could use a three-dimensional ancilla to generate an indirect measurement with three distinct outcomes. One possible interaction operator is

$$\mathbf{C}_{\text{permute}} \equiv |0_S\rangle\langle 0_S| \otimes \mathbf{1}^A + |1_S\rangle\langle 1_S| \otimes \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^A,$$

which performs the permutation on H_A

$$|0_A\rangle \mapsto |1_A\rangle, \quad |1_A\rangle \mapsto |2_A\rangle, \quad |2_A\rangle \mapsto |0_A\rangle,$$

if and only if the system is in state $|1_S\rangle$. With

$$|A\rangle = a_0|0_A\rangle + a_1|1_A\rangle + a_2|2_A\rangle,$$

we thus have

$$\begin{aligned}
|\Psi_S\rangle &\mapsto |\Psi_S\rangle \otimes (a_0|0_A\rangle + a_1|1_A\rangle + a_2|2_A\rangle) \\
&\mapsto \mathbf{C}_{\text{permute}} \begin{pmatrix} c_0a_0|0_S\rangle \otimes |0_A\rangle + c_0a_1|0_S\rangle \otimes |1_A\rangle + c_0a_2|0_S\rangle \otimes |2_A\rangle \\ +c_1a_0|1_S\rangle \otimes |0_A\rangle + c_1a_1|1_S\rangle \otimes |1_A\rangle + c_1a_2|1_S\rangle \otimes |2_A\rangle \end{pmatrix} \\
&= c_0a_0|0_S\rangle \otimes |0_A\rangle + c_0a_1|0_S\rangle \otimes |1_A\rangle + c_0a_2|0_S\rangle \otimes |2_A\rangle \\
&\quad + c_1a_0|1_S\rangle \otimes |1_A\rangle + c_1a_1|1_S\rangle \otimes |2_A\rangle + c_1a_2|1_S\rangle \otimes |0_A\rangle, \\
&= (c_0a_0|0_S\rangle + c_1a_2|1_S\rangle) \otimes |0_A\rangle + (c_0a_1|0_S\rangle + c_1a_0|1_S\rangle) \otimes |1_A\rangle \\
&\quad + (c_0a_2|0_S\rangle + c_1a_1|1_S\rangle) \otimes |2_A\rangle.
\end{aligned}$$

So the outcome probabilites are

$$\begin{aligned}
\Pr(0) &= |c_0a_0|^2 + |c_1a_2|^2, \\
\Pr(1) &= |c_0a_1|^2 + |c_1a_0|^2, \\
\Pr(2) &= |c_0a_2|^2 + |c_1a_1|^2,
\end{aligned}$$

with post-measurement states

$$\left|\Psi_{\text{post}}\right\rangle = \frac{c_0 a_0|0_S\rangle + c_1 a_2|1_S\rangle}{\sqrt{\Pr(0)}} \qquad i = 0,$$

$$= \frac{c_0 a_1|0_S\rangle + c_1 a_0|1_S\rangle}{\sqrt{\Pr(1)}} \qquad i = 1,$$

$$= \frac{c_0 a_2|0_S\rangle + c_1 a_1|1_S\rangle}{\sqrt{\Pr(2)}} \qquad i = 2.$$

With appropriate choices for $a_0 \neq a_1 \neq a_2$ such that $|a_0|^2 + |a_1|^2 + |a_2|^2 = 1$ (for example $a_0 = \sqrt{1/2},\ a_1 = \sqrt{1/3},\ a_2 = \sqrt{1/6}$), we see that the probabilities and post-measurement states associated with the three possible outcomes are all distinct. It is important to note at this point, however, that the 'amount' of information obtained in a measurement with $N_A > N_S$ outcomes can never be greater than that which could be obtained in an optimal N_S –dimensional measurement. As we have seen previously, one can use them to access different 'kinds' of information and to play different strategies in the inference-disturbance tradeoff.

Partial trace and reduced density operators

Having defined partial projectors, we can now define the partial trace operation. Let ρ_{AB} be a density operator on H_{AB}:

$$\rho_{AB} = \sum_{ijkl} \rho_{ijkl}|i_A\rangle \otimes |j_B\rangle\langle k_A| \otimes \langle l_B|,$$

where the summations are take over orthonormal bases for H_A and H_B. Consider the sum of partial projections,

$$\sum_{m=1}^{N_B} (\mathbf{1}^A \otimes \mathbf{P}_m^B)\rho_{AB}(\mathbf{1}^A \otimes \mathbf{P}_m^B)$$

$$= \sum_{m=1}^{N_B} (\mathbf{1}^A \otimes \mathbf{P}_m^B)\left(\sum_{ijkl} \rho_{ijkl}|i_A\rangle \otimes |j_B\rangle\langle k_A| \otimes \langle l_B|\right)(\mathbf{1}^A \otimes \mathbf{P}_m^B)$$

$$= \sum_{m=1}^{N_B}\sum_{i,k=1}^{N_A} \rho_{imkm}|i_A\rangle \otimes |m_B\rangle\langle k_A| \otimes \langle m_B|$$

$$= \sum_{m=1}^{N_B} |m_B\rangle\langle m_B| \otimes \sum_{i,k=1}^{N_A} \rho_{imkm}|i_A\rangle\langle k_A|.$$

We define the partial trace of ρ_{AB} over the B subsystem to be

$$\tilde{\rho}_A \equiv \mathrm{Tr}_B[\rho_{AB}] = \sum_{m=1}^{N_B}\sum_{i,k=1}^{N_A} \rho_{imkm}|i_A\rangle\langle k_A| = \sum_{i,k=1}^{N_A}\left(\sum_{m=1}^{N_B} \rho_{imkm}\right)|i_A\rangle\langle k_A|$$

Here $\tilde{\rho}_A$ is called the 'reduced density operator' for subsystem A. It provides the best possible representation of subsystem A within H_A, when the joint state of A and B is entangled/nonfactorizable.

When would we need such a representation? Suppose systems A and B are allowed to interact, and as a result end up in some entangled state $|\Psi_{AB}^{ent}\rangle$. Then, however, someone comes and removes subsystem B from our lab. Once B becomes unavailable to us, we can only make measurements of the form

$$\{\mathbf{P}_0^A \otimes \mathbf{1}^B, \mathbf{P}_1^A \otimes \mathbf{1}^B, \ldots\}.$$

The statistics of all such measurements are predicted by the reduced density operator:

$$\begin{aligned}
\Pr(i) &= \mathrm{Tr}[\boldsymbol{\rho}_{AB}\mathbf{P}_i^A \otimes \mathbf{1}^B] \\
&= \mathrm{Tr}\left[\left(\sum_{j=1}^{N_B}(\mathbf{1}^A \otimes \mathbf{P}_j^B)\right)\boldsymbol{\rho}_{AB}\left(\sum_{k=1}^{N_B}(\mathbf{1}^A \otimes \mathbf{P}_k^B)\right)\mathbf{P}_i^A \otimes \mathbf{1}^B\right] \\
&= \mathrm{Tr}\left[\left(\sum_{j=1}^{N_B}(\mathbf{1}^A \otimes \mathbf{P}_j^B)\boldsymbol{\rho}_{AB}(\mathbf{1}^A \otimes \mathbf{P}_j^B)\right)\mathbf{P}_i^A \otimes \mathbf{1}^B\right] \\
&= \mathrm{Tr}\left[\left(\sum_{j=1}^{N_B}|j_B\rangle\langle j_B| \otimes \sum_{k,l=1}^{N_A}\rho_{kjlj}|k_A\rangle\langle l_A|\right)(\mathbf{P}_i^A \otimes \mathbf{1}^B)^2\right] \\
&= \mathrm{Tr}\left[\left(\sum_{j=1}^{N_B}|j_B\rangle\langle j_B| \otimes \sum_{k,l=1}^{N_A}\rho_{kjlj}\mathbf{P}_i^A|k_A\rangle\langle l_A|\mathbf{P}_i^A\right)\right] \\
&= \mathrm{Tr}\left[\left(\sum_{j=1}^{N_B}|j_B\rangle\langle j_B| \otimes \rho_{ijij}|i_A\rangle\langle i_A|\right)\right] \\
&= \sum_{k=1}^{N_B}\sum_{l=1}^{N_A}\langle l_A| \otimes \langle k_B|\left(\sum_{j=1}^{N_B}|j_B\rangle\langle j_B| \otimes \rho_{ijij}|i_A\rangle\langle i_A|\right)|l_A\rangle \otimes |k_B\rangle \\
&= \sum_{j=1}^{N_B}\rho_{ijij}.
\end{aligned}$$

Likewise,

$$\begin{aligned}
\mathrm{Tr}[\tilde{\boldsymbol{\rho}}_A\mathbf{P}_i^A] &= \mathrm{Tr}\left[\sum_{k,l=1}^{N_A}\left(\sum_{j=1}^{N_B}\rho_{ijkj}\right)|k_A\rangle\langle l_A|\mathbf{P}_i^A\right] \\
&= \mathrm{Tr}\left[\sum_{k=1}^{N_A}\left(\sum_{j=1}^{N_B}\rho_{ijkj}\right)|k_A\rangle\langle i_A|\right] \\
&= \sum_{m=1}^{N_A}\langle m_A|\sum_{k=1}^{N_A}\left(\sum_{j=1}^{N_B}\rho_{ijkj}\right)|k_A\rangle\langle i_A||m_A\rangle
\end{aligned}$$

$$= \sum_{m=1}^{N_A} \sum_{k=1}^{N_A} \left(\sum_{j=1}^{N_B} \rho_{ijkj} \right) \delta_{mk} \delta_{im}$$

$$= \sum_{j=1}^{N_B} \rho_{ijij}.$$

A notationally more convenient, but mathematically less precise way of computing the partial trace is as follows:

$$\mathrm{Tr}_B[\boldsymbol{\rho}_{AB}] = \sum_{m=1}^{N_B} \langle m_B | \boldsymbol{\rho}_{AB} | m_B \rangle = \sum_{m=1}^{N_B} \langle m_B | \left(\sum_{ijkl} \rho_{ijkl} | i_A \rangle \otimes | j_B \rangle \langle k_A | \otimes \langle l_B | \right) | m_B \rangle$$

$$= \sum_{m=1}^{N_B} \sum_{i,k=1}^{N_A} \rho_{imkm} | i_A \rangle \langle k_A |.$$

It is perhaps useful to see a few examples of this type of manipulation.

Open quantum systems

Suppose we have a composite system $H_{AB} = H_A \otimes H_B$ with Hamiltonian $\mathbf{H}_{AB}$. We know that the overall dynamics is described by the SE

$$i\hbar \frac{d}{dt} |\Psi_{AB}(t)\rangle = \mathbf{H}_{AB} |\Psi_{AB}(t)\rangle.$$

Let's say, however, that H_A corresponds to a small, 'compact' physical system that we are trying to study in the lab, whereas H_B actually represents the degrees of freedom of some environmental reservoir. If we are unable (as is always the case) to completely isolate the system from the environment, then the Hamiltonian will not separate: $\mathbf{H}_{AB} \neq \mathbf{H}_A \otimes \mathbf{1}_B + \mathbf{1}_A \otimes \mathbf{H}_B,\ \mathbf{T}_{AB}(t,0) \neq \mathbf{T}_A(t,0) \otimes \mathbf{T}_B(t,0).$ Hence, even for pure initial states of the system $|\Psi_A(0)\rangle \in H_A$, the above SE *may* (for some initial states) induce evolution into entangled states of the system and environment.

In general we will be unable to perform complete measurements on the joint Hilbert space H_{AB}, because reservoirs are usually infinite-dimensional (hence H_{AB} will be also). So limiting our attention to the system H_A, it appears that we must settle for a density-operator description obtained by tracing over the environmental degrees of freedom:

$$\tilde{\rho}_A(t) = \mathrm{Tr}_B[|\Psi_{AB}(t)\rangle\langle\Psi_{AB}(t)|].$$

From what we have learned about entanglement in previous lectures, we may expect that this type of evolution (formation of entanglements with an unobservable reservoir) will lead to loss of purity for the system state. Such phenomena are generally referred to as 'decoherence.'

Under certain assumptions about the nature of $\mathbf{H}_{AB}$ and of the environment H_B, it is sometimes possible to derive a closed-form evolution equation for the reduced density operator $\tilde{\rho}_A(t)$. In a 'Master Equation' of this type, operators and states for H_B do not appear explicitly because they have been analytically traced-out of the equations of motion. This type of approach is particularly useful for understanding things like dissipation and thermal fluctuations in a quantum-mechanical setting, and the overall field of studying these things has come to be known as the theory of open quantum systems.