APPPHYS225 - Thursday 4 December 2008

Today's notes develop some ideas from the paper by Hardy (homework reading for today). Earlier in the term we have seen that there is a one-to-one correspondence between density matrices ρ and Bloch vectors $\begin{pmatrix} \langle\sigma_x\rangle & \langle\sigma_y\rangle & \langle\sigma_z\rangle \end{pmatrix}$,

$$\rho \leftrightarrow \frac{1}{2}(\mathbf{I} + \langle\sigma_x\rangle\sigma_x + \langle\sigma_y\rangle\sigma_y + \langle\sigma_z\rangle\sigma_z).$$

Since the Pauli matrices are traceless and satisfy

$$\mathrm{Tr}[\sigma_i\sigma_j] = 2\delta_{ij},$$

we easily verify that $\mathrm{Tr}[\rho] = 1$ and that $\mathrm{Tr}[\rho\sigma_x]$ pulls out the 'parameter' $\langle\sigma_x\rangle$, *et cetera*.

From this parameterization we recognize the idea that given many spin-1/2 particles (qubits), all of which have been prepared in the same unknown state, we can determine the density matrix ρ by measuring the x, y and z components of spin. This doesn't quite match up to Hardy's setup, however, as he wants us to measure probabilities of outcomes. We can make the connection simply by noting (using spectral decomposition) the following correspondences:

$$\sigma_x \leftrightarrow \{\Pi_{+x}, \Pi_{-x}\}, \quad \sigma_y \leftrightarrow \{\Pi_{+y}, \Pi_{-y}\}, \quad \sigma_z \leftrightarrow \{\Pi_{+z}, \Pi_{-z}\},$$

$$\langle\sigma_x\rangle = \mathrm{Pr}[+x] - \mathrm{Pr}[-x], \quad \langle\sigma_y\rangle = \mathrm{Pr}[+y] - \mathrm{Pr}[-y], \quad \langle\sigma_z\rangle = \mathrm{Pr}[+z] - \mathrm{Pr}[-z],$$

where Π_{+x} is a projector onto the eigenvector of σ_x with eigenvalue $+1$, *et cetera*. Hence we can think of Hardy's measurement machine with settings x, y and z, with outcomes $+1$ and -1, and we would for example determine the parameter $\langle\sigma_x\rangle$ by making many measurements with the setting x and using frequency to determine the probabilities $\mathrm{Pr}[+x]$ and $\mathrm{Pr}[-x]$. We furthermore note that

$$\Pi_{-x} = \mathbf{1} - \Pi_{+x}, \quad \Pi_{-y} = \mathbf{1} - \Pi_{+y}, \quad \Pi_{-z} = \mathbf{1} - \Pi_{+z},$$

from which it follows that we could completely determine ρ from knowledge of just three probabilities, such as

$$p_1 = \langle\Pi_{+x}\rangle, \quad p_2 = \langle\Pi_{+y}\rangle, \quad p_3 = \langle\Pi_{+z}\rangle.$$

We know that there one can only distinguish two states in a spin-1/2 Hilbert space with zero probability of error, and we have just established that three probabilities are required to determine an abritrary state. Hence in Hardy's notation, $N = 2$ and $K = 3 = N^2 - 1$.

Once thing to note here is that the procedure we have specified for determining the state of the particles with Hardy's machine seems perhaps less clean than it could be. In particular, there are three different settings with two outcomes each involved in this method of state estimate ('tomography'), all for the sake of determining three parameters. Can we do better? One thing we cannot do is construct a POVM that simply reads out a sufficient set of probabilities directly. For example,

$$\Pi_{+x} + \Pi_{+y} + \Pi_{+z} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ i \end{pmatrix}\begin{pmatrix} 1 & -i \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & \frac{1}{2} - \frac{1}{2}i \\ \frac{1}{2} + \frac{1}{2}i & 1 \end{pmatrix},$$

which is clearly not proportional to the identity. Hence we cannot simply construct a POVM with three elements proportional to the $+x$, $+y$ and $+z$ projectors.

But perhaps we can find a different sort of POVM to do the trick with a minimal number of possible outcomes. Note that if we want to determine three independent parameters we actually need a POVM with four possible outcomes, because of the normalization constraint—a three-outcome POVM only has two independent probabilities. Generally speaking there do exist a variety of such 'informationally-complete' POVM's for quantum systems living in Hilbert spaces of arbitrary dimension, but recent attention has focused on a class of so-called 'symmetric informationally complete' (SIC) POVM's. It will be interesting for us to have a look at the SIC POVM's for dimension 2, as the notion of covariance pops up again in a nice way. In what follows we will largely follow the discussion of D. M. Appleby, H. B. Dang and C. A. Fuchs in their paper "Physical Significance of Symmetric Informationally-Complete Sets of Quantum States" [arXiv:0707.2071v1].

In at least some dimensions, SIC POVM's can be constructed as covariant POVM's with respect to representations of dsicrete Weyl-Heisenberg groups. While there doesn't seem to be a proof that Weyl-Heisenberg SIC POVM's can be constructed in any dimension, examples have been discovered numerically in dimension up to 45 [J. M. Rennes *et al.*, "Symmetric informationally complete quantum measurements," J. Math. Phys. **45**, 2171 (2004)], and it is conjectured that they exist in every dimension.

The Weyl-Heisenberg transformations are defined by

$$\mathbf{D}_r = \tau^{r_1 r_2} \mathbf{X}^{r_1} \mathbf{Z}^{r_2},$$

where in a Hilbert space of dimension d,

$$\tau = -\exp(\pi i/d),$$
$$\mathbf{Z}|j\rangle = \omega^j |j\rangle, \quad \omega \equiv \exp(2\pi i/d),$$
$$\mathbf{X}|j\rangle = |j + 1 \bmod d\rangle.$$

In dimension $d = 2$ we thus have

$$\tau \to -i,$$
$$\mathbf{Z}|j\rangle \to (-1)^j |j\rangle = \boldsymbol{\sigma}_z |j\rangle,$$
$$\mathbf{X}|j\rangle \to |j + 1 \bmod 2\rangle = \boldsymbol{\sigma}_x |j\rangle.$$

Hence, with $r_1 \in \{0,1\}$ and $r_2 \in \{0,1\}$ we have

$$\mathbf{D}_{00} = \mathbf{1}, \quad \mathbf{D}_{01} = \boldsymbol{\sigma}_z, \quad \mathbf{D}_{10} = \boldsymbol{\sigma}_x, \quad \mathbf{D}_{11} = -i\boldsymbol{\sigma}_x\boldsymbol{\sigma}_z = \boldsymbol{\sigma}_y.$$

Using the representation rule we have learned previously,

$$(\hat{n},\varphi) \mapsto \exp\left(\frac{-i}{\hbar}\hat{n}\cdot\vec{\mathbf{J}}\varphi\right) \leftrightarrow \begin{pmatrix} \cos\frac{\varphi}{2} - in_z\sin\frac{\varphi}{2} & (-in_x - n_y)\sin\frac{\varphi}{2} \\ (-in_x + n_y)\sin\frac{\varphi}{2} & \cos\frac{\varphi}{2} + in_z\sin\frac{\varphi}{2} \end{pmatrix},$$

we see that

$$\mathbf{D}_{01} = \boldsymbol{\sigma}_z \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = i\exp\left(\frac{-i}{\hbar}\hat{z}\cdot\vec{\mathbf{J}}\pi\right),$$

$$\mathbf{D}_{10} = \boldsymbol{\sigma}_x \leftrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = i\exp\left(\frac{-i}{\hbar}\hat{x}\cdot\vec{\mathbf{J}}\pi\right),$$

$$\mathbf{D}_{11} = \boldsymbol{\sigma}_y \leftrightarrow \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = i\exp\left(\frac{-i}{\hbar}\hat{y}\cdot\vec{\mathbf{J}}\pi\right).$$

Hence the Weyl-Heisenberg transformations correspond (up to an overall phase) to doing nothing, rotation by π around the z-axis, rotation by π around the x-axis, and rotation by π around the y-axis.

The possible fiducial vectors for $d = 2$ are [J. M. Rennes *et al.*], in the basis of eigenvectors of $\boldsymbol{\sigma}_z$,

$$\left\{ \frac{1}{\sqrt{6}}\begin{pmatrix} \sqrt{3+\sqrt{3}} \\ e^{i\pi/4}\sqrt{3-\sqrt{3}} \end{pmatrix}, \frac{1}{\sqrt{6}}\begin{pmatrix} -\sqrt{3-\sqrt{3}} \\ e^{i\pi/4}\sqrt{3+\sqrt{3}} \end{pmatrix} \right\}.$$

We can calculate the corresponding Bloch vectors. For the first fiducial vector,

$$\langle\boldsymbol{\sigma}_x\rangle = \frac{1}{6}\begin{pmatrix} \sqrt{3+\sqrt{3}} & e^{-i\pi/4}\sqrt{3-\sqrt{3}} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \sqrt{3+\sqrt{3}} \\ e^{i\pi/4}\sqrt{3-\sqrt{3}} \end{pmatrix}$$
$$= \frac{1}{6}e^{i\pi/4}\sqrt{\left(3+\sqrt{3}\right)\left(3-\sqrt{3}\right)} + \frac{1}{6}e^{-i\pi/4}\sqrt{\left(3+\sqrt{3}\right)\left(3-\sqrt{3}\right)} = \frac{1}{3}\cos\frac{\pi}{4}\sqrt{6} = \frac{1}{\sqrt{3}},$$

$$\langle\boldsymbol{\sigma}_y\rangle = \frac{1}{6}\begin{pmatrix} \sqrt{3+\sqrt{3}} & e^{-i\pi/4}\sqrt{3-\sqrt{3}} \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} \sqrt{3+\sqrt{3}} \\ e^{i\pi/4}\sqrt{3-\sqrt{3}} \end{pmatrix}$$
$$= -\frac{i}{6}e^{i\pi/4}\sqrt{\left(3+\sqrt{3}\right)\left(3-\sqrt{3}\right)} + \frac{i}{6}e^{-i\pi/4}\sqrt{\left(3+\sqrt{3}\right)\left(3-\sqrt{3}\right)} = \frac{1}{3}\sin\frac{\pi}{4}\sqrt{6} = \frac{1}{\sqrt{3}},$$

$$\langle\boldsymbol{\sigma}_z\rangle = \frac{1}{6}\begin{pmatrix} \sqrt{3+\sqrt{3}} & e^{-i\pi/4}\sqrt{3-\sqrt{3}} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \sqrt{3+\sqrt{3}} \\ e^{i\pi/4}\sqrt{3-\sqrt{3}} \end{pmatrix}$$
$$= \frac{1}{6}\left(3+\sqrt{3}\right) - \frac{1}{6}\left(3-\sqrt{3}\right) = \frac{1}{\sqrt{3}}.$$

We thus see that this is a normalized Bloch vector along the $(1,1,1)$ direction. Applying the WH transformations, we clearly generate

$$\left\{\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}, \frac{1}{\sqrt{3}}\begin{pmatrix}-1\\-1\\1\end{pmatrix}, \frac{1}{\sqrt{3}}\begin{pmatrix}1\\-1\\-1\end{pmatrix}, \frac{1}{\sqrt{3}}\begin{pmatrix}-1\\1\\-1\end{pmatrix}\right\}.$$

If we compute the projectors in this "SIC" POVM,

$$|\psi\rangle\langle\psi| \leftrightarrow \frac{1}{6}\begin{pmatrix}\sqrt{3+\sqrt{3}}\\ e^{i\pi/4}\sqrt{3-\sqrt{3}}\end{pmatrix}\begin{pmatrix}\sqrt{3+\sqrt{3}} & e^{-i\pi/4}\sqrt{3-\sqrt{3}}\end{pmatrix} = \frac{1}{2\sqrt{3}}\begin{pmatrix}\sqrt{3}+1 & 1-i\\ 1+i & \sqrt{3}-1\end{pmatrix},$$

$$\boldsymbol{\sigma}_z|\psi\rangle\langle\psi|\boldsymbol{\sigma}_z \leftrightarrow \frac{1}{2\sqrt{3}}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\sqrt{3}+1 & 1-i\\ 1+i & \sqrt{3}-1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = \frac{1}{2\sqrt{3}}\begin{pmatrix}\sqrt{3}+1 & -1+i\\ -1-i & \sqrt{3}-1\end{pmatrix},$$

$$\boldsymbol{\sigma}_x|\psi\rangle\langle\psi|\boldsymbol{\sigma}_x \leftrightarrow \frac{1}{2\sqrt{3}}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\sqrt{3}+1 & 1-i\\ 1+i & \sqrt{3}-1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \frac{1}{2\sqrt{3}}\begin{pmatrix}\sqrt{3}-1 & 1+i\\ 1-i & \sqrt{3}+1\end{pmatrix},$$

$$\boldsymbol{\sigma}_y|\psi\rangle\langle\psi|\boldsymbol{\sigma}_y \leftrightarrow \frac{1}{2\sqrt{3}}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}\sqrt{3}+1 & 1-i\\ 1+i & \sqrt{3}-1\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} = \frac{1}{2\sqrt{3}}\begin{pmatrix}\sqrt{3}-1 & -1-i\\ -1+i & \sqrt{3}+1\end{pmatrix}.$$

Adding these together,

$$\begin{pmatrix}\sqrt{3}+1 & 1-i\\ 1+i & \sqrt{3}-1\end{pmatrix} + \begin{pmatrix}\sqrt{3}+1 & -1+i\\ -1-i & \sqrt{3}-1\end{pmatrix} + \begin{pmatrix}\sqrt{3}-1 & 1+i\\ 1-i & \sqrt{3}+1\end{pmatrix} + \begin{pmatrix}\sqrt{3}-1 & -1-i\\ -1+i & \sqrt{3}+1\end{pmatrix}$$
$$= \begin{pmatrix}4\sqrt{3} & 0\\ 0 & 4\sqrt{3}\end{pmatrix},$$

$$|\psi\rangle\langle\psi| + \boldsymbol{\sigma}_z|\psi\rangle\langle\psi|\boldsymbol{\sigma}_z + \boldsymbol{\sigma}_x|\psi\rangle\langle\psi|\boldsymbol{\sigma}_x + \boldsymbol{\sigma}_y|\psi\rangle\langle\psi|\boldsymbol{\sigma}_y \leftrightarrow \frac{1}{2\sqrt{3}}\begin{pmatrix}4\sqrt{3} & 0\\ 0 & 4\sqrt{3}\end{pmatrix} = 2\mathbf{I}.$$

It thus follows that the set $\{\frac{1}{2}|\psi\rangle\langle\psi|, \frac{1}{2}\boldsymbol{\sigma}_z|\psi\rangle\langle\psi|\boldsymbol{\sigma}_z, \frac{1}{2}\boldsymbol{\sigma}_x|\psi\rangle\langle\psi|\boldsymbol{\sigma}_x, \frac{1}{2}\boldsymbol{\sigma}_y|\psi\rangle\langle\psi|\boldsymbol{\sigma}_y\}$ constitutes a valid POVM (we are also using the straightforward observation that each operator in this set has eigenvalues $\frac{1}{2}$ and 0).

Having established that this is a symmetric POVM, let us finish by showing that it is indeed informationally complete. For an arbitrary density matrix

$$\boldsymbol{\rho} \leftrightarrow \frac{1}{2}(\mathbf{I} + \langle\boldsymbol{\sigma}_x\rangle\boldsymbol{\sigma}_x + \langle\boldsymbol{\sigma}_y\rangle\boldsymbol{\sigma}_y + \langle\boldsymbol{\sigma}_z\rangle\boldsymbol{\sigma}_z),$$

and adopting the notation $\mathbf{A}_{r_1r_2} = \frac{1}{2}\mathbf{D}_{r_1r_2}|\psi\rangle\langle\psi|\mathbf{D}_{r_1r_2}$ for the POVM elements, we have

$$
\begin{aligned}
\Pr[\mathbf{A}_{00}] &= \mathrm{Tr}\left[\frac{1}{2}|\psi\rangle\langle\psi|\frac{1}{2}(\mathbf{I}+\langle\boldsymbol{\sigma}_x\rangle\boldsymbol{\sigma}_x+\langle\boldsymbol{\sigma}_y\rangle\boldsymbol{\sigma}_y+\langle\boldsymbol{\sigma}_z\rangle\boldsymbol{\sigma}_z)\right] \\
&= \frac{1}{4}(\mathrm{Tr}[|\psi\rangle\langle\psi|]+\langle\boldsymbol{\sigma}_x\rangle\mathrm{Tr}[|\psi\rangle\langle\psi|\boldsymbol{\sigma}_x]+\langle\boldsymbol{\sigma}_y\rangle\mathrm{Tr}[|\psi\rangle\langle\psi|\boldsymbol{\sigma}_y]+\langle\boldsymbol{\sigma}_z\rangle\mathrm{Tr}[|\psi\rangle\langle\psi|\boldsymbol{\sigma}_z]) \\
&= \frac{1}{4}\left(1+\langle\boldsymbol{\sigma}_x\rangle/\sqrt{3}+\langle\boldsymbol{\sigma}_y\rangle/\sqrt{3}+\langle\boldsymbol{\sigma}_z\rangle/\sqrt{3}\right),
\end{aligned}
$$

where we use results regarding the Bloch vector for $|\psi\rangle$ from above. Similarly,

$$
\begin{aligned}
\Pr[\mathbf{A}_{01}] &= \mathrm{Tr}\left[\frac{1}{2}\boldsymbol{\sigma}_z|\psi\rangle\langle\psi|\boldsymbol{\sigma}_z\frac{1}{2}(\mathbf{I}+\langle\boldsymbol{\sigma}_x\rangle\boldsymbol{\sigma}_x+\langle\boldsymbol{\sigma}_y\rangle\boldsymbol{\sigma}_y+\langle\boldsymbol{\sigma}_z\rangle\boldsymbol{\sigma}_z)\right] \\
&= \frac{1}{4}(\mathrm{Tr}[\boldsymbol{\sigma}_z|\psi\rangle\langle\psi|\boldsymbol{\sigma}_z]+\langle\boldsymbol{\sigma}_x\rangle\mathrm{Tr}[\boldsymbol{\sigma}_z|\psi\rangle\langle\psi|\boldsymbol{\sigma}_z\boldsymbol{\sigma}_x]+\langle\boldsymbol{\sigma}_y\rangle\mathrm{Tr}[\boldsymbol{\sigma}_z|\psi\rangle\langle\psi|\boldsymbol{\sigma}_z\boldsymbol{\sigma}_y]+\langle\boldsymbol{\sigma}_z\rangle\mathrm{Tr}[\boldsymbol{\sigma}_z|\psi\rangle\langle\psi|\boldsymbol{\sigma}_z\boldsymbol{\sigma}_z]) \\
&= \frac{1}{4}\left(1-\langle\boldsymbol{\sigma}_x\rangle/\sqrt{3}-\langle\boldsymbol{\sigma}_y\rangle/\sqrt{3}+\langle\boldsymbol{\sigma}_z\rangle/\sqrt{3}\right),
\end{aligned}
$$

$$
\begin{aligned}
\Pr[\mathbf{A}_{10}] &= \mathrm{Tr}\left[\frac{1}{2}\boldsymbol{\sigma}_x|\psi\rangle\langle\psi|\boldsymbol{\sigma}_x\frac{1}{2}(\mathbf{I}+\langle\boldsymbol{\sigma}_x\rangle\boldsymbol{\sigma}_x+\langle\boldsymbol{\sigma}_y\rangle\boldsymbol{\sigma}_y+\langle\boldsymbol{\sigma}_z\rangle\boldsymbol{\sigma}_z)\right] \\
&= \frac{1}{4}\left(1+\langle\boldsymbol{\sigma}_x\rangle/\sqrt{3}-\langle\boldsymbol{\sigma}_y\rangle/\sqrt{3}-\langle\boldsymbol{\sigma}_z\rangle/\sqrt{3}\right),
\end{aligned}
$$

$$
\begin{aligned}
\Pr[\mathbf{A}_{11}] &= \mathrm{Tr}\left[\frac{1}{2}\boldsymbol{\sigma}_y|\psi\rangle\langle\psi|\boldsymbol{\sigma}_y\frac{1}{2}(\mathbf{I}+\langle\boldsymbol{\sigma}_x\rangle\boldsymbol{\sigma}_x+\langle\boldsymbol{\sigma}_y\rangle\boldsymbol{\sigma}_y+\langle\boldsymbol{\sigma}_z\rangle\boldsymbol{\sigma}_z)\right] \\
&= \frac{1}{4}\left(1-\langle\boldsymbol{\sigma}_x\rangle/\sqrt{3}+\langle\boldsymbol{\sigma}_y\rangle/\sqrt{3}-\langle\boldsymbol{\sigma}_z\rangle/\sqrt{3}\right).
\end{aligned}
$$

We thus have

$$
\begin{aligned}
\langle\boldsymbol{\sigma}_x\rangle &= 2\sqrt{3}\left(\Pr[\mathbf{A}_{00}]+\Pr[\mathbf{A}_{11}]-\frac{1}{2}\right), \\
\langle\boldsymbol{\sigma}_y\rangle &= 2\sqrt{3}\left(\Pr[\mathbf{A}_{00}]+\Pr[\mathbf{A}_{10}]-\frac{1}{2}\right), \\
\langle\boldsymbol{\sigma}_z\rangle &= 2\sqrt{3}\left(\Pr[\mathbf{A}_{00}]+\Pr[\mathbf{A}_{01}]-\frac{1}{2}\right).
\end{aligned}
$$

Hence our POVM is symmetric by construction, and informationally complete.

In general in d dimensions with SIC POVM $\left\{\mathbf{E}_i = \frac{1}{d}\boldsymbol{\Pi}_i\right\}$ there is a decomposition [Appleby, Dang and Fuchs]

$$
\boldsymbol{\rho} = \sum_i\left((d+1)p_i-\frac{1}{d}\right)\Pi_i, \quad p_i \equiv \mathrm{Tr}[\boldsymbol{\rho}\mathbf{E}_i].
$$

This general correspondence shows that we can represent an arbitrary quantum state ($d \times d$ density matrix) by a set of d^2measurement probabilities p_i, but one should be careful to note that not every normalized set of p_i actually generates a valid (non-negative) density matrix. As discussed in the paper, one can actually find a pair of formulae that must be satisfied by pure states (rank-1 projectors) in this representation, and the full set of quantum states can then be generated by convex combination.

Appleby, Dang and Fuchs discuss an interesting geometric significance of SIC POVM's as the closest approximation that can be constructed in a quantum state space to projective measurement in an informationally-complete orthnormal basis (note that a true orthonormal basis measurement has only $N-1$ independent parameters, not $K = N^2-1$). In the case of classical probability we have $K = N-1$ and therefore we can have informationally complete orthonormal measurements; states thus appear naturally as probability distributions over the elementary configurations, and pure states have definite measurement outcomes. In the quantum case the elements of the IC POVM are not mutually orthogonal, meaning that there are no pure states that definite outcomes.