APPPHYS225 - Tuesday 28 October 2008

Quantum dynamics in the Heisenberg picture

In case you haven't seen this before, note that in general in quantum mechanics we can either apply the time-development operator $\mathbf{T}(t,0)$ to the states (Schrödinger picture)

$$|\Psi(t)\rangle = \mathbf{T}(t,0)|\Psi(0)\rangle,$$
$$\rho(t) = \mathbf{T}(t,0)\rho(0)\mathbf{T}^\dagger(t,0),$$

or to the operators (Heisenberg picture),

$$\mathbf{A}(t) = \mathbf{T}^\dagger(t,0)\mathbf{A}(0)\mathbf{T}(t,0).$$

Note that the density matrix/operator maps differently than observables and other operators. Either way we end up computing identical values for measurement probabilities since in general

$$\langle\mathbf{A}\rangle(t) = \mathrm{Tr}[\rho(t)\mathbf{A}] = \mathrm{Tr}\left[\mathbf{T}(t,0)\rho(0)\mathbf{T}^\dagger(t,0)\mathbf{A}\right], \quad \text{Schrödinger},$$
$$\langle\mathbf{A}\rangle(t) = \mathrm{Tr}[\rho\mathbf{A}(t)] = \mathrm{Tr}\left[\rho\mathbf{T}^\dagger(t,0)\mathbf{A}(0)\mathbf{T}(t,0)\right], \quad \text{Heisenberg},$$

and we have cyclic property of the trace. Recall that for closed systems the time-development operator is unitary and can be obtained by exponentiating the Hamiltonian operator. For a Hamiltonian that is constant on the time interval $[0,t]$:

$$i\hbar\frac{d}{dt}|\Psi(t)\rangle = \mathbf{H}|\Psi(t)\rangle,$$
$$\mathbf{T}(t,0) = \exp(-i\mathbf{H}t/\hbar),$$
$$\mathbf{T}^\dagger(t,0) = \exp(i\mathbf{H}t/\hbar) = \mathbf{T}(0,t) = \mathbf{T}^{-1}(t,0).$$

Note that this all applies straightforwardly even when we have a joint system, as for example in the Heisenberg picture

$$\mathbf{A}(t)\otimes\mathbf{B}(t) = \mathbf{T}^\dagger(t,0)\mathbf{A}(0)\otimes\mathbf{B}(0)\mathbf{T}(t,0),$$

where $\mathbf{T}(t,0)$ is here understood to be an operator on the joint Hilbert space $H^A\otimes H^B$, the exponential of a joint Hamiltonian.

To see a very simple example of how this works, even in the classical setting, consider a two-element sample space $\Omega = \{\omega_H,\omega_T\}$ for a coin flip. Let $m(\cdot)$ be the probability distribution function, and let $X(\cdot)$ be a random variable that indexes the result:

$$X(\omega_H) = +1, \quad X(\omega_T) = -1.$$

We know from previous lectures that we can represent $m(\cdot)$ and $X(\cdot)$ as matrices,

$$m(\cdot) \leftrightarrow \begin{pmatrix} \Pr(\omega_H) & 0 \\ 0 & \Pr(\omega_T) \end{pmatrix}, \quad X(\cdot) \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Consider the action of "manually" turning the coin over, so that $\omega_H \mapsto \omega_T$ and $\omega_T \mapsto \omega_H$. We can represent this dynamic with the unitary matrix

$$U = U^\dagger = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad U^2 = 1.$$

Consider a scenario in which we first flip the coin and then manually turn it over without looking at it. In the Schrödinger picture we would compute

$$(m) \mapsto U^\dagger (m) U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \Pr(\omega_H) & 0 \\ 0 & \Pr(\omega_T) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \Pr(\omega_T) & 0 \\ 0 & \Pr(\omega_H) \end{pmatrix},$$

and

$$\langle X \rangle = \mathrm{Tr}\left[U^\dagger (m) U (X) \right] = \mathrm{Tr}\left[\begin{pmatrix} \Pr(\omega_T) & 0 \\ 0 & -\Pr(\omega_H) \end{pmatrix} \right] = \Pr(\omega_T) - \Pr(\omega_H).$$

In the Heisenberg picture,

$$(X) \mapsto U (X) U^\dagger = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\langle X \rangle = \mathrm{Tr}\left[(m) U (X) U^\dagger \right] = \mathrm{Tr}\left[\begin{pmatrix} -\Pr(\omega_H) & 0 \\ 0 & \Pr(\omega_T) \end{pmatrix} \right] = \Pr(\omega_T) - \Pr(\omega_H).$$

Ramon's problem: the Projection Postulate

The following material was originally outlined by Ramon van Handel. Our goal will be to show that in an indirect implementation of a projective measurement, of the kind we discussed last time, it is actually possible to use the classical rules for conditional expectation to derive the post-measurement *quantum* state of the system. In a sense, we thus make the Projection Postulate appear to be a derived notion rather than an axiom. To simplify the notation we make use of the convention

$$\rho(A) \equiv \mathrm{Tr}[\rho A],$$

where ρ is a density matrix and A is an operator. We will also sometimes use $E(\cdot)$ to denote the expectation value of either a classical random variable or a quantum observable.

We begin with a preliminary reminder of how the classical notion of conditional expectation can be applied to commuting quantum observables.

a. Show that the classical definition of $E(X \mid Y)$ is equivalent to

$$E(X \mid Y) = \sum_i \sum_j x_j \frac{\rho(P_j Q_i)}{\rho(Q_i)} Q_i = \sum_i \frac{\rho(X Q_i)}{\rho(Q_i)} Q_i,$$

for two commuting observables $X = \sum_j x_j P_j$ and $Y = \sum_i y_i Q_i$ in an algebra A with state ρ. Here x_j, y_i are the eigenvalues and P_j, Q_i the eigenprojectors of $X, Y,$ respectively.

The basic idea here is to map the quantum observables into classical random variables using simultaneous diagonalization, apply the conditional expectation, and then map back. Explicitly, since X and Y commute there exists a linear transformation T such that

$$TXT^{-1},\ TYT^{-1} \in M_n$$

are diagonal $n \times n$ matrices with $\{x_j\}$ and $\{y_i\}$ along their diagonals. Now we construct a classical configuration space by associating ω_i with the i^{th} position along the matrix diagonal. Then we can define classical random variables

$$\xi(\omega_j) \equiv x_j, \quad \Upsilon(\omega_i) \equiv y_i,$$

with corresponding level sets such that

$$\xi(\cdot) = \sum_j x_j \chi_{\Omega_j^x}(\cdot), \quad \Upsilon(\cdot) = \sum_i y_i \chi_{\Omega_i^y}(\cdot).$$

In this way we establish a correspondence

$$\chi_{\Omega_j^x}(\cdot) \leftrightarrow P_j, \quad \chi_{\Omega_i^y}(\cdot) \leftrightarrow Q_i.$$

Then according to the usual definition,

$$E\left(\xi \,|\, \Upsilon\right)(\cdot) = \sum_i \sum_j x_j \frac{\Pr(\Omega_j^x \cap \Omega_i^y)}{\Pr(\Omega_i^y)} \chi_{\Omega_i^y}(\cdot)$$

$$= \sum_i \sum_j x_j \frac{E\left(\chi_{\Omega_j^x}\chi_{\Omega_i^y}\right)}{E\left(\chi_{\Omega_i^y}\right)} \chi_{\Omega_i^y}(\cdot),$$

which we can invert through our correspondence to obtain

$$E\left(X \,|\, Y\right) = \sum_i \sum_j x_j \frac{\rho(P_j Q_i)}{\rho(Q_i)} Q_i = \sum_i \frac{\rho(X Q_i)}{\rho(Q_i)} Q_i,$$

where the second equation follows from linearity of the trace.

Now we move on to considering interaction of a system and ancilla ('meter'), in the Heisenberg picture, via maps $j : X \mapsto U^* X U$.

b. Show that $j(\sigma_{x,y,z} \otimes 1)$ commute with $j(1 \otimes \sigma_z)$. Now define $\pi(\sigma_{x,y,z}) = E\left(j(\sigma_{x,y,z} \otimes 1) \,|\, j(1 \otimes \sigma_z)\right)$. Show that $\pi(\sigma_{x,y,z})$ commute with each other and with $j(1 \otimes \sigma_z)$. Argue that we can thus simultaneously infer $\sigma_{x,y,z}$ after interaction with the meter.

We first explicity check that

$$j(\sigma_{x,y,z} \otimes 1) = U^*(\sigma_{x,y,z} \otimes 1) U$$

commute with

$$j(1 \otimes \sigma_z) = U^*(1 \otimes \sigma_z) U.$$

Straightforwardly,

$$
\begin{aligned}
j(\sigma_{x,y,z} \otimes 1)j(1 \otimes \sigma_z) &= U^*(\sigma_{x,y,z} \otimes 1)UU^*(1 \otimes \sigma_z)U \\
&= U^*(\sigma_{x,y,z} \otimes 1)(1 \otimes \sigma_z)U \\
&= U^*(\sigma_{x,y,z} \otimes \sigma_z)U, \\
j(1 \otimes \sigma_z)j(\sigma_{x,y,z} \otimes 1) &= U^*(1 \otimes \sigma_z)UU^*(\sigma_{x,y,z} \otimes 1)U \\
&= U^*(1 \otimes \sigma_z)(\sigma_{x,y,z} \otimes 1)U \\
&= U^*(\sigma_{x,y,z} \otimes \sigma_z)U,
\end{aligned}
$$

where we are using the usual definition of product on $A \otimes A$. Since we have shown that $j(\sigma_{x,y,z} \otimes 1)$ and $j(1 \otimes \sigma_z)$ commute, we can define

$$
\begin{aligned}
\pi(\sigma_{x,y,z}) &\equiv E\big(j(\sigma_{x,y,z} \otimes 1)\,|j(1 \otimes \sigma_z)\big) \\
&= \sum_i \frac{\rho(U^*(\sigma_{x,y,z} \otimes 1)UQ_i)}{\rho(Q_i)} Q_i,
\end{aligned}
$$

where

$$
j(1 \otimes \sigma_z) = U^*(1 \otimes \sigma_z)U = \sum_i y_i Q_i.
$$

Noting that the $\rho(U^*(\sigma_{x,y,z} \otimes 1)UQ_i)/\rho(Q_i)$ are just numbers, it is easy to see that these conditional expectations commute. For example,

$$
\begin{aligned}
\pi(\sigma_x)\pi(\sigma_y) &= \sum_i \frac{\rho(U^*(\sigma_x \otimes 1)UQ_i)}{\rho(Q_i)} Q_i \sum_j \frac{\rho(U^*(\sigma_y \otimes 1)UQ_j)}{\rho(Q_j)} Q_j \\
&= \sum_i \sum_j \frac{\rho(U^*(\sigma_x \otimes 1)UQ_i)}{\rho(Q_i)} \frac{\rho(U^*(\sigma_y \otimes 1)UQ_j)}{\rho(Q_j)} Q_i Q_j \\
&= \sum_i \sum_j \frac{\rho(U^*(\sigma_x \otimes 1)UQ_i)}{\rho(Q_i)} \frac{\rho(U^*(\sigma_y \otimes 1)UQ_j)}{\rho(Q_j)} \delta_{ij} Q_i \\
&= \sum_i \sum_j \frac{\rho(U^*(\sigma_y \otimes 1)UQ_j)}{\rho(Q_j)} \frac{\rho(U^*(\sigma_x \otimes 1)UQ_i)}{\rho(Q_i)} Q_j Q_i \\
&= \pi(\sigma_y)\pi(\sigma_x).
\end{aligned}
$$

Likewise,

$$
\begin{aligned}
\pi(\sigma_x)j(1\otimes\sigma_z) &= \sum_i \frac{\rho(U^*(\sigma_x\otimes 1)UQ_i)}{\rho(Q_i)} Q_i \sum_j y_j Q_j \\
&= \sum_i \sum_j \frac{\rho(U^*(\sigma_x\otimes 1)UQ_i)}{\rho(Q_i)} y_j Q_i Q_j \\
&= \sum_i \sum_j \frac{\rho(U^*(\sigma_x\otimes 1)UQ_i)}{\rho(Q_i)} y_j \delta_{ij} Q_i \\
&= \sum_i \sum_j \frac{\rho(U^*(\sigma_x\otimes 1)UQ_i)}{\rho(Q_i)} y_j Q_j Q_i \\
&= j(1\otimes\sigma_z)\pi(\sigma_x).
\end{aligned}
$$

Thus $j(1\otimes\sigma_z)$ and $\pi(\sigma_{x,y,z})$ are equivalent to a set of classical random variables, and nothing stops us from performing simultaneous inference in the usual manner.

c. Calculate explicit matrix representations for U, $j(\sigma_{x,y,z}\otimes 1)$, $j(1\otimes\sigma_z)$ and $\pi(\sigma_{x,y,z})$.

First recall the usual representations

$$
\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
$$

(I am using a convention where

$$
\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \otimes \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} & a_{12}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \\ a_{21}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} & a_{22}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \end{pmatrix},
$$

which will appear over and over again below.) Then we have

$$
U = U^* = |0\rangle\langle 0| \otimes 1 + |1\rangle\langle 1| \otimes \sigma_x = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},
$$

$$
1\otimes\sigma_z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad j(1\otimes\sigma_z) = U^*(1\otimes\sigma_z)U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
$$

Likewise,

$$\sigma_x \otimes 1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad j(\sigma_x \otimes 1) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$\sigma_y \otimes 1 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \quad j(\sigma_y \otimes 1) = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix},$$

$$\sigma_z \otimes 1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad j(\sigma_z \otimes 1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Next we compute the eigenvectors of $j(1 \otimes \sigma_z)$:

$$\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\} \leftrightarrow -1, \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\} \leftrightarrow 1,$$

hence

$$j(1 \otimes \sigma_z) = (+1) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + (-1) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
$$\equiv y_1 Q_1 + y_2 Q_2.$$

Then finally, assuming a state

$$\rho \otimes |0\rangle\langle 0| = \begin{pmatrix} \rho_{11} & 0 & \rho_{12} & 0 \\ 0 & 0 & 0 & 0 \\ \rho_{21} & 0 & \rho_{22} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

we compute

$$\pi(\sigma_x) = \sum_i \frac{\rho(U^*(\sigma_x \otimes 1)UQ_i)}{\rho(Q_i)} Q_i = 0.$$

Similarly,

$$\pi(\sigma_y) = \sum_i \frac{\rho(U^*(\sigma_y \otimes 1)UQ_i)}{\rho(Q_i)} Q_i = 0,$$

and

$$\pi(\sigma_z) = \sum_i \frac{\rho(U^*(\sigma_z \otimes 1)UQ_i)}{\rho(Q_i)} Q_i = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

d. As $\pi(\sigma_{x,y,z})$ and $j(1 \otimes \sigma_z)$ (and 1) all commute, they generate a commutative subalgebra of $A \otimes A$... Construct explicitly a classical sample space Ω and state $p(\omega)$, and use these to express $\pi(\sigma_{x,y,z})$ and $j(1 \otimes \sigma_z)$ as classical random variables.

Clearly we can just use positions along the diagonal of the matrix representations we found above. Hence,

$$\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\},$$

$$\pi(\sigma_x) \mapsto X : \Omega \to R, \quad X(\omega_1) = X(\omega_2) = X(\omega_3) = X(\omega_4) = 0,$$

$$\pi(\sigma_y) \mapsto Y : \Omega \to R, \quad Y(\omega_1) = Y(\omega_2) = Y(\omega_3) = Y(\omega_4) = 0,$$

$$\pi(\sigma_z) \mapsto Z : \Omega \to R, \quad Z(\omega_1) = Z(\omega_4) = 1,\ Z(\omega_2) = Z(\omega_3) = -1,$$

$$j(1 \otimes \sigma_z) \mapsto M : \Omega \to R, \quad M(\omega_1) = M(\omega_4) = 1,\ M(\omega_2) = M(\omega_3) = -1.$$

The state we want can be found by taking

$$\mathrm{Tr}\left[\begin{pmatrix} \omega_1 & 0 & 0 & 0 \\ 0 & \omega_2 & 0 & 0 \\ 0 & 0 & \omega_3 & 0 \\ 0 & 0 & 0 & \omega_4 \end{pmatrix}\begin{pmatrix} \rho_{11} & 0 & \rho_{12} & 0 \\ 0 & 0 & 0 & 0 \\ \rho_{21} & 0 & \rho_{22} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right] = \omega_1\rho_{11} + \omega_3\rho_{22},$$

meaning that

$$p(\omega_1) = \rho_{11}, \quad p(\omega_2) = 0, \quad p(\omega_3) = \rho_{22}, \quad p(\omega_4) = 0.$$

e. The conditional expectations $\pi(\sigma_{x,y,z})$ are very similar to ordinary expectations—only they are random variables. For now, just by analogy, consider defining a "conditional density matrix" as a random 2×2 matrix $\tilde{\rho}(\omega)$ such that $\pi(\sigma_{x,y,z})(\omega) = \mathrm{Tr}[\tilde{\rho}(\omega)\sigma_{x,y,z}]$. Find an explicit expression for $\tilde{\rho}(\omega)$. Interpret the result in terms of what you learned about quantum measurement in previous quantum courses.

We want

$$\mathrm{Tr}[\tilde{\rho}(\omega_1)\sigma_x] = 0,\ \mathrm{Tr}[\tilde{\rho}(\omega_1)\sigma_y] = 0,\ \mathrm{Tr}[\tilde{\rho}(\omega_1)\sigma_z] = 1,$$
$$\mathrm{Tr}[\tilde{\rho}(\omega_2)\sigma_x] = 0,\ \mathrm{Tr}[\tilde{\rho}(\omega_2)\sigma_y] = 0,\ \mathrm{Tr}[\tilde{\rho}(\omega_2)\sigma_z] = -1,$$
$$\mathrm{Tr}[\tilde{\rho}(\omega_3)\sigma_x] = 0,\ \mathrm{Tr}[\tilde{\rho}(\omega_3)\sigma_y] = 0,\ \mathrm{Tr}[\tilde{\rho}(\omega_3)\sigma_z] = -1,$$
$$\mathrm{Tr}[\tilde{\rho}(\omega_4)\sigma_x] = 0,\ \mathrm{Tr}[\tilde{\rho}(\omega_4)\sigma_y] = 0,\ \mathrm{Tr}[\tilde{\rho}(\omega_4)\sigma_z] = 1.$$

Hence we can conclude that $\tilde{\rho}(\omega_1) = \tilde{\rho}(\omega_4)$ and determine the matrix via

$$\mathrm{Tr}\left[\begin{pmatrix}\tilde{\rho}_{11} & \tilde{\rho}_{12}\\ \tilde{\rho}_{21} & \tilde{\rho}_{22}\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right] = \tilde{\rho}_{12} + \tilde{\rho}_{21} = 0,$$
$$\mathrm{Tr}\left[\begin{pmatrix}\tilde{\rho}_{11} & \tilde{\rho}_{12}\\ \tilde{\rho}_{21} & \tilde{\rho}_{22}\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\right] = i\tilde{\rho}_{12} - i\tilde{\rho}_{21} = 0,$$
$$\mathrm{Tr}\left[\begin{pmatrix}\tilde{\rho}_{11} & \tilde{\rho}_{12}\\ \tilde{\rho}_{21} & \tilde{\rho}_{22}\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right] = \tilde{\rho}_{11} - \tilde{\rho}_{22} = 1,$$
$$\mathrm{Tr}\left[\begin{pmatrix}\tilde{\rho}_{11} & \tilde{\rho}_{12}\\ \tilde{\rho}_{21} & \tilde{\rho}_{22}\end{pmatrix}\right] = \tilde{\rho}_{11} + \tilde{\rho}_{22} = 1,$$
$$\Rightarrow \tilde{\rho}(\omega_1) = \tilde{\rho}(\omega_4) = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix},$$

(where we additionally invoke hermiticity and positive-semidefiniteness). Likewise,

$$\mathrm{Tr}\left[\begin{pmatrix}\tilde{\rho}_{11} & \tilde{\rho}_{12}\\ \tilde{\rho}_{21} & \tilde{\rho}_{22}\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right] = \tilde{\rho}_{12} + \tilde{\rho}_{21} = 0,$$
$$\mathrm{Tr}\left[\begin{pmatrix}\tilde{\rho}_{11} & \tilde{\rho}_{12}\\ \tilde{\rho}_{21} & \tilde{\rho}_{22}\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\right] = i\tilde{\rho}_{12} - i\tilde{\rho}_{21} = 0,$$
$$\mathrm{Tr}\left[\begin{pmatrix}\tilde{\rho}_{11} & \tilde{\rho}_{12}\\ \tilde{\rho}_{21} & \tilde{\rho}_{22}\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right] = \tilde{\rho}_{11} - \tilde{\rho}_{22} = -1,$$
$$\mathrm{Tr}\left[\begin{pmatrix}\tilde{\rho}_{11} & \tilde{\rho}_{12}\\ \tilde{\rho}_{21} & \tilde{\rho}_{22}\end{pmatrix}\right] = \tilde{\rho}_{11} + \tilde{\rho}_{22} = 1,$$
$$\Rightarrow \tilde{\rho}(\omega_2) = \tilde{\rho}(\omega_3) = \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}.$$

This is of course exactly what we would expect, as now

$$p(\omega_1) = \rho_{11}, \quad j(1\otimes\sigma_z)(\omega_1) = 1, \quad \tilde{\rho}(\omega_1) = |0\rangle\langle 0|,$$
$$p(\omega_3) = \rho_{22}, \quad j(1\otimes\sigma_z)(\omega_3) = 1, \quad \tilde{\rho}(\omega_3) = |1\rangle\langle 1|,$$
$$p(\omega_1) + p(\omega_3) = 1,$$
$$p(\omega_2) = p(\omega_4) = 0.$$

f. Show that $\rho\big(E(X\,|\,Y)\big) = \rho(X)$ for any commuting X, Y. Use this to show that the random density matrix $\tilde{\rho}(\omega)$ together with the classical state $p(\omega)$ form a non-redundant representation of the state $\rho\otimes\rho_0$ restricted to the (noncommutative) subalgebra of $A\otimes A$ generated by $j(\sigma_{x,y,z}\otimes 1)$, $j(1\otimes\sigma_z)$, and 1.

We note that

$$E\big(X\,|\,Y\big) \equiv \sum_i \frac{\rho(XQ_i)}{\rho(Q_i)} Q_i,$$
$$Y = \sum_j y_j Q_j,$$

for any commuting X, Y. Hence

$$\rho\big(E(X\,|\,Y)\big) = \rho\left(\sum_i \frac{\rho(XQ_i)}{\rho(Q_i)} Q_i\right) = \sum_i \frac{\rho(XQ_i)}{\rho(Q_i)} \rho(Q_i)$$
$$= \sum_i \rho(XQ_i) = \rho\left(X\sum_i Q_i\right) = \rho(X),$$

assuming Y is self-adjoint and thus has a spanning set of eigenvectors, so $\sum_i Q_i = 1$.

Even without using this, we can prove the desired fact by brute force. Looking at

$$j(1\otimes\sigma_z) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad j(\sigma_x\otimes 1) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$
$$j(\sigma_y\otimes 1) = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \quad j(\sigma_z\otimes 1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

we can read off

$$j(1\otimes\sigma_z) = \sigma_z\otimes\sigma_z, \quad j(\sigma_x\otimes 1) = \sigma_x\otimes\sigma_x,$$
$$j(\sigma_y\otimes 1) = \sigma_y\otimes\sigma_x, \quad j(\sigma_z\otimes 1) = \sigma_z\otimes 1.$$

Let's see what these generate:

$$
\begin{aligned}
&(\sigma_z \otimes \sigma_z)(\sigma_z \otimes \sigma_z) = 1 \otimes 1, \quad (\sigma_z \otimes \sigma_z)(\sigma_x \otimes \sigma_x) = i\sigma_y \otimes i\sigma_y,\\
&(\sigma_z \otimes \sigma_z)(\sigma_y \otimes \sigma_x) = -i\sigma_x \otimes i\sigma_y, \quad (\sigma_z \otimes \sigma_z)(\sigma_z \otimes 1) = 1 \otimes \sigma_z,\\
&(\sigma_x \otimes \sigma_x)(\sigma_z \otimes \sigma_z) = -i\sigma_y \otimes -i\sigma_y, \quad (\sigma_x \otimes \sigma_x)(\sigma_x \otimes \sigma_x) = 1 \otimes 1,\\
&(\sigma_x \otimes \sigma_x)(\sigma_y \otimes \sigma_x) = i\sigma_z \otimes 1, \quad (\sigma_x \otimes \sigma_x)(\sigma_z \otimes 1) = -i\sigma_y \otimes \sigma_x,\\
&(\sigma_y \otimes \sigma_x)(\sigma_z \otimes \sigma_z) = i\sigma_x \otimes -i\sigma_y, \quad (\sigma_y \otimes \sigma_x)(\sigma_x \otimes \sigma_x) = -i\sigma_z \otimes 1,\\
&(\sigma_y \otimes \sigma_x)(\sigma_y \otimes \sigma_x) = 1 \otimes 1, \quad (\sigma_y \otimes \sigma_x)(\sigma_z \otimes 1) = i\sigma_x \otimes \sigma_x,\\
&(\sigma_z \otimes 1)(\sigma_z \otimes \sigma_z) = 1 \otimes \sigma_z, \quad (\sigma_z \otimes 1)(\sigma_x \otimes \sigma_x) = i\sigma_y \otimes \sigma_x,\\
&(\sigma_z \otimes 1)(\sigma_y \otimes \sigma_x) = -i\sigma_x \otimes \sigma_x, \quad (\sigma_z \otimes 1)(\sigma_z \otimes 1) = 1 \otimes 1.
\end{aligned}
$$

Hence the only new elements generated are $\sigma_y \otimes \sigma_y$, $\sigma_x \otimes \sigma_y$, and $1 \otimes \sigma_z$. One can easily see that nothing further gets generated. Hence, our subalgebra consists of elements of the form

$$
\Sigma = a1 \otimes 1 + b1 \otimes \sigma_z + c\sigma_z \otimes 1 + d\sigma_x \otimes \sigma_x + e\sigma_y \otimes \sigma_y + f\sigma_z \otimes \sigma_z + g\sigma_x \otimes \sigma_y + h\sigma_y \otimes \sigma_x,
$$

and

$$
\begin{aligned}
\langle \Sigma \rangle &= \mathrm{Tr}\left[\begin{array}{c}
a\rho \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\rho \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + c\rho\sigma_z \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\
+d\rho\sigma_x \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + e\rho\sigma_y \otimes \begin{pmatrix} 0 & -i \\ 0 & 0 \end{pmatrix} + f\rho\sigma_z \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\
+g\rho\sigma_x \otimes \begin{pmatrix} 0 & -i \\ 0 & 0 \end{pmatrix} + h\rho\sigma_y \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}
\end{array} \right] \\
&= \mathrm{Tr}\left[\begin{pmatrix} (a+b+c+f)\rho_{11} & \square & (a+b-c-f)\rho_{12} & \square \\ \square & 0 & \square & \square \\ (a+b+c+f)\rho_{21} & \square & (a+b-c-f)\rho_{22} & \square \\ \square & \square & \square & 0 \end{pmatrix} \right] \\
&= (a+b+c+f)\rho_{11} + (a+b-c-f)\rho_{22}.
\end{aligned}
$$

Hence we can map

$$
\Sigma \mapsto \sigma : \Omega \to R, \quad \sigma(\omega_1) = a+b+c+f, \quad \sigma(\omega_3) = a+b-c-f,
$$

and then just use the state $p(\omega)$ we derived above to assign an expectation value to every observable. Alternatively we may write

$$
\begin{aligned}
&\Sigma \mapsto s(\omega) \equiv \mathrm{Tr}\left[\begin{pmatrix} a+b+c+f & 0 \\ 0 & a+b-c-f \end{pmatrix} \tilde{\rho}(\omega) \right],\\
&\langle \Sigma \rangle = \sum p(\omega) s(\omega).
\end{aligned}
$$

Now we are asked to define $S = e^{i\pi\sigma_y/4} \otimes 1$ and $U' = S^{-1}US = S^*US$.

g. What happens in *c.* – *e.* if we use U' instead of U?

Let's just have a look at the matrices:

$$e^{-i\pi\sigma_y/4} = \exp\begin{pmatrix} 0 & -\frac{\pi}{4} \\ \frac{\pi}{4} & 0 \end{pmatrix} = \exp\left\{\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\begin{pmatrix} -\frac{1}{4}i\pi & 0 \\ 0 & \frac{1}{4}i\pi \end{pmatrix}\begin{pmatrix} \frac{1}{2} & -\frac{1}{2}i \\ \frac{1}{2} & \frac{1}{2}i \end{pmatrix}\right\}$$

$$= \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\begin{pmatrix} e^{-i\pi/4} & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}\begin{pmatrix} \frac{1}{2} & -\frac{1}{2}i \\ \frac{1}{2} & \frac{1}{2}i \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

$$S = e^{i\pi\sigma_y/4} \otimes 1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \quad S^{-1} = S^* = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix},$$

$$U' = S^*US = \frac{1}{2}\begin{pmatrix} 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \end{pmatrix}, \quad (U')^{-1} = (U')^* = U.$$

It's clear from the form of S that this represents a modified controlled-not gate, which applies σ_x to the probe spin if the system spin is in the $|1_x\rangle$ eigenstate. Note that we can write,

$$U' = e^{i\pi\sigma_y/4} \otimes 1(|0\rangle\langle 0| \otimes 1 + |1\rangle\langle 1| \otimes \sigma_x)e^{-i\pi\sigma_y/4} \otimes 1$$

$$= e^{i\pi\sigma_y/4}|0\rangle\langle 0|e^{-i\pi\sigma_y/4} \otimes 1 + e^{i\pi\sigma_y/4}|1\rangle\langle 1|e^{-i\pi\sigma_y/4} \otimes \sigma_x$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \otimes 1 + \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \otimes \sigma_x$$

$$= \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \otimes 1 + \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \otimes \sigma_x$$

$$= |0_x\rangle\langle 0_x| \otimes 1 + |1_x\rangle\langle 1_x| \otimes \sigma_x$$

We thus expect that the overall procedure will implement an indirect measurement of σ_x rather than σ_z for the system.

Contingency of least-squares in quantum measurement theory

M. R. James, "Risk-sensitive optimal control of quantum systems," Phys. Rev. A **69**, 032108 (2004).