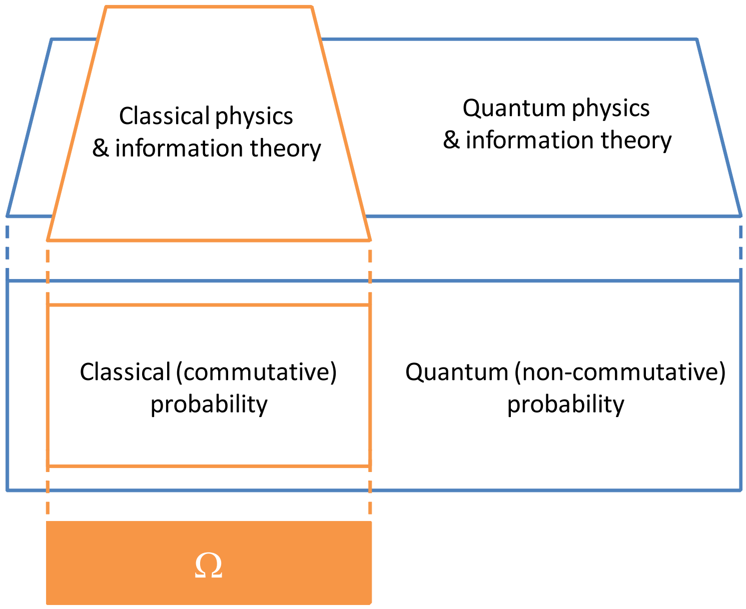
APPPHYS225 - Tuesday 18 November 2008

At the beginning of the term we briefly discussed the following diagram:



Hopefully the structure of the diagram makes more sense now that we have had time to discuss classical versus quantum probability, and classical versus quantum information. In particular, we have seen that quantum probability represents a non-commutative generalization of familiar classical probability, which thereby loses its connection to the notion of an underlying sample space $\Omega = \{\omega_1, \ldots, \omega_N\}$. We have looked at some consequences of this generalization for the information theory that is built on top of probabilistic foundations. But we have not really directly addressed the physicists' question of why we should be driven to consider non-commutative probability outside the realm of abstract mathematics. Unfortunately there is no good answer to this question, as of yet, other than the rather glib answer that quantum mechanics is a physical theory based on non-commutative probability and it has broader applicability than Liouville mechanics, which is the closest equivalent classical theory based on commutative probability theory.

Historically, many prominent scientists have extolled the central importance of symmetry considerations in the foundations of physical theory. Our aim for today will be to look at some ways in which symmetry considerations can be seen as encourage us to consider non-commutative probability models. In the last week of class we will survey some alternative approaches to understanding—in physicists' terms—what non-commutative probability could all be about.

Symmetries and probability models

Early in the term we introduced the idea that a probability model is defined by an operator algebra and a state (consistent assignment of an expectation value to every observable in the algebra). We noted that for the familiar type of classical probability model based on a sample space Ω and probability density function $m : \Omega \to [0,1]$, the observables are random variables (which we can view as functions from Ω to R) forming an algebra under pointwise addition and multiplication. The state on this algebra is then given by expectation value with respect to $m(\cdot)$. We also noted that this structure could be embedded in the linear algebra of $N \times N$ diagonal real matrices, where N is the number of elementary configurations in Ω, which results in the expectation map

$$\langle \mathbf{A} \rangle = \mathrm{Tr}[\boldsymbol{\rho}\mathbf{A}],$$

where $\mathbf{A}$ is the matrix representing an observable and $\boldsymbol{\rho}$ is the matrix with the values of $m(\cdot)$ on its diagonal.

The most familiar sorts of (finite-dimensional) quantum probability models are then constructed by taking the operator algebra to be the full matrix algebra $\boldsymbol{GL}(N,\boldsymbol{C})$ ($N \times N$ complex matrices), with observables corresponding to the Hermitian elements within this algebra. The most general state on such a set of observables is then a trace-1 positive operator, known as the density matrix. In physical terms we can (for example) think of such a model as representing the state and observables of a spin-J particle, with $N = 2J + 1$, or of an N-level atom.

We saw that the formalism is capable of describing probability models that are non-classical but not equivalent to a full matrix algebra. For instance, the set of matrices of the form

$$\mathbf{M} = \begin{pmatrix} a & c \\ d & b \end{pmatrix}, \quad a, b, c, d \in \boldsymbol{R},$$

forms a closed matrix algebra $\boldsymbol{GL}(2,\boldsymbol{R})$ (clearly a sub-algebra of $\boldsymbol{GL}(2,\boldsymbol{C})$) containing observables of the form

$$\mathbf{O} = \begin{pmatrix} a & c \\ c & b \end{pmatrix}, \quad a, b, c \in \boldsymbol{R}.$$

These observables do not all commute, since

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix}\begin{pmatrix} d & f \\ f & g \end{pmatrix} - \begin{pmatrix} d & f \\ f & g \end{pmatrix}\begin{pmatrix} a & c \\ c & b \end{pmatrix} = \begin{pmatrix} 0 & af - cd - bf + cg \\ cd - af + bf - cg & 0 \end{pmatrix},$$

and therefore this model is not equivalent to a purely classical model. While familiar spin-1/2 observables such as

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

are clearly included in the algebra,

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

clearly is not.

What sorts of practical reasons could we have for considering a particular operator algebra? In the classical case we are basically building matrix algebras starting from the sample space. In the usual quantum setting with full matrix algebras (with complex entries), we could say that we are simply taking the most ‘unrestricted’ matrix algebras of a given dimension. However, the connection with spin-J particles points to another possible answer, which is that our non-commutative quantum models are associated with irreps of the rotation group.

Symmetry groups give rise to matrix algebras in many different ways. One simple way is that we ask for the smallest matrix algebra that contains all the matrices in a faithful (one-to-one) linear or projective representation of some given symmetry group. We have seen that $\boldsymbol{GL}(2,\boldsymbol{C})$ contains all the matrices in the following irreducible linear representation of the $\boldsymbol{R}^3$ rotation group:

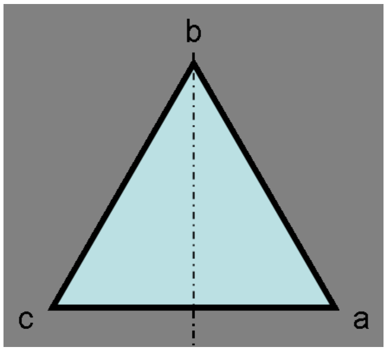
$$(\hat{n},\varphi) \mapsto \begin{pmatrix} \cos\frac{\varphi}{2} - in_z\sin\frac{\varphi}{2} & (-in_x - n_y)\sin\frac{\varphi}{2} \\ (-in_x + n_y)\sin\frac{\varphi}{2} & \cos\frac{\varphi}{2} + in_z\sin\frac{\varphi}{2} \end{pmatrix}.$$

The matrices of this representation include

$$(\hat{x},\pi) \leftrightarrow \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad (\hat{y},\pi) \leftrightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad (\hat{z},\pi) \leftrightarrow \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \quad (\hat{n},0) \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and $\boldsymbol{GL}(2,\boldsymbol{C})$ is in fact the smallest algebra that contains the linear span of these.

Along similar lines, let us consider the dihedral group $\mathbf{D}_3$, which corresponds to the symmetries of an equilateral triangle.



This group has six elements that can be interpreted as transformations on the set of labeled corner points:

$$g_1 : \{a,b,c\} \mapsto \{a,b,c\}, \quad g_2 : \{a,b,c\} \mapsto \{b,c,a\}, \quad g_3 : \{a,b,c\} \mapsto \{c,a,b\},$$
$$g_4 : \{a,b,c\} \mapsto \{a,c,b\}, \quad g_5 : \{a,b,c\} \mapsto \{c,b,a\}, \quad g_6 : \{a,b,c\} \mapsto \{b,a,c\}.$$

These satisfy a convenient decomposition,

$$g_1 = S^0R^0, \quad g_2 = S^0R^1, \quad g_3 = S^0R^2, \quad g_4 = S^1R^1, \quad g_5 = S^1R^0, \quad g_6 = S^1R^2,$$

where S represents reflection through a ‘bisecting’ symmetry axis (in the plane of the triangle) and R represents rotation by $2\pi/3$ (about a central axis perpendicular to the plane of the triangle). The group $\boldsymbol{D}_3$ has only three distinct irreps, two of which (including the trivial irrep) are one-dimensional while the third is two-dimensional:

$$D(g_1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad D(g_2) = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}, \quad D(g_3) = \frac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix},$$

$$D(g_4) = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}, \quad D(g_5) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad D(g_6) = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}.$$

We note that the linear span of these will include

$$D(g_2) - D(g_3) = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix} = \begin{pmatrix} 0 & -\sqrt{3} \\ \sqrt{3} & 0 \end{pmatrix},$$

$$D(g_4) - D(g_6) = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix} = \begin{pmatrix} 0 & \sqrt{3} \\ \sqrt{3} & 0 \end{pmatrix},$$

which together with $D(g_1)$ and $D(g_5)$ provide a basis for all of $\boldsymbol{GL}(2,\boldsymbol{R})$, which is the non-classical yet not-fully-quantum probability model mentioned above.

Symmetry groups can also generate matrix algebras via their regular representations. In order to do this, we first construct a vector space with an orthonormal basis vector v_g for each group element g. We then construct a linear representation of the group by associating each group element g with the permutation matrix $D(g)$ that correctly implements the multiplication table

$$D(g_1)v_{g_2} = v_{g_1 g_2}.$$

To provide a very simple example, we can take $G = \{e,a\}$ with $a^2 = e$, which is the group commonly known as $\boldsymbol{C}_2$. We form the vector space

$$v_e = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad v_a = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

and find the regular representation

$$D(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad D(a) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

which we have considered before. Clearly the span of these two matrices is all matrices of the form

$$\mathbf{M} = \begin{pmatrix} a & b \\ b & a \end{pmatrix},$$

which is closed under matrix addition and multiplication:

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix}\begin{pmatrix} c & d \\ d & c \end{pmatrix} = \begin{pmatrix} ac+bd & ad+bc \\ ad+bc & ac+bd \end{pmatrix}.$$

The observables in this algebra are then all matrices with $a,b \in \boldsymbol{R}$, and if we consider the general form of an expectation value

$$
\begin{aligned}
\langle \mathbf{O} \rangle &= \mathrm{Tr}\left[\begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{12}^* & 1-\rho_{11} \end{pmatrix}\begin{pmatrix} a & b \\ b & a \end{pmatrix}\right] \\
&= \mathrm{Tr}\left[\begin{pmatrix} a\rho_{11}+b\rho_{12} & a\rho_{12}+b\rho_{11} \\ a\rho_{12}^* - b(\rho_{11}-1) & b\rho_{12}^* - a(\rho_{11}-1) \end{pmatrix}\right] \\
&= a\rho_{11}+b\rho_{12}+b\rho_{12}^* - a(\rho_{11}-1) \\
&= a + 2b\,\mathrm{Re}[\rho_{12}],
\end{aligned}
$$

we see that a state on the algebra is actually completely specified by just a single real number corresponding to $\mathrm{Re}[\rho_{12}]$. If we note that

$$
\langle \sigma_x \rangle = \mathrm{Tr}\left[\begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{12}^* & 1-\rho_{11} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] = 2\,\mathrm{Re}[\rho_{12}],
$$

we see that we can think of the state as being fully specified by $-1 \le \langle \sigma_x \rangle \le 1$.

In passing we should briefly note a fact that we will invoke below. While the regular representation that we have just introduced may remind you of the permutation representation we considered last week for $\boldsymbol{D}_2$ (the symmetries of a rectangle), it is not quite the same. Recall that the $\boldsymbol{D}_2$ transformations induce permutations among high symmetry points of the rectangle according to

$$
\begin{aligned}
&e: 1 \mapsto 1,\ 2 \mapsto 2,\ 3 \mapsto 3,\ 4 \mapsto 4, \\
&a: 1 \mapsto 1,\ 2 \mapsto 4,\ 3 \mapsto 3,\ 4 \mapsto 2, \\
&b: 1 \mapsto 3,\ 2 \mapsto 2,\ 3 \mapsto 1,\ 4 \mapsto 4, \\
&c: 1 \mapsto 3,\ 2 \mapsto 4,\ 3 \mapsto 1,\ 4 \mapsto 2,
\end{aligned}
$$

which led us to the permutation representation

$$
D_p(a) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad D_p(b) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad D_p(c) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.
$$

On the other hand the multiplication table of the group is

$$
\begin{aligned}
&e: e \mapsto e,\ a \mapsto a,\ b \mapsto b,\ c \mapsto c, \\
&a: e \mapsto a,\ a \mapsto e,\ b \mapsto c,\ c \mapsto b, \\
&b: e \mapsto b,\ a \mapsto c,\ b \mapsto e,\ c \mapsto a, \\
&c: e \mapsto c,\ a \mapsto b,\ b \mapsto a,\ c \mapsto e,
\end{aligned}
$$

which would lead to the regular representation

$$D_r(a) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad D_r(b) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad D_r(c) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

where we verify

$$D_r(a)D_r(b) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = D_r(c),$$

$$D_r(a)D_r(c) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = D_r(b),$$

$$D_r(b)D_r(c) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = D_r(a),$$

and likewise for $D_r(b)D_r(a)$, $D_r(c)D_r(a)$ and $D_r(c)D_r(b)$. We see that the $D_r(\bullet)$ matrices are distinct from the $D_p(\bullet)$ matrices, although since the group multiplication table is commutative it follows that the regular representation is (like the permutation representation) reducible to the direct sum of one-dimensional irreps. Explicitly, with

$$S = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 \end{pmatrix}, \quad S^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix},$$

we find

$$D_r(a) \mapsto \frac{1}{4}\begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$D_r(b) \mapsto \frac{1}{4}\begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$D_r(c) \mapsto \frac{1}{4}\begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

But this is apparently not equivalent to the permutation representation since the matrices have different 'sign structures' in their diagonal forms (only one copy of the trivial irrep for $D_r(\bullet)$ while $D_p(\bullet)$ has two). Note that for $\boldsymbol{D}_2$ the permutation and regular representations have the same dimension, but for $\mathbf{D}_3$ they would not (three corners of the triangle, six elements of the group).

Covariant measurements

To see yet another way that a symmetry group can motivate the choice of a non-commutative probability model let us remind ourselves of the notion of a *covariant measurement*, which we introduced somewhat quickly last week.

To talk about covariance we should first pick a group G whose elements can be interpreted as transformations of a set of 'points' (perhaps rather abstractly). In a somewhat suggestive fashion, we'll use $\Omega = \{\omega_1, \omega_2, \ldots, \omega_N\}$ to denote the set of points, and a group element g acts on these via

$$\boxed{g : \omega_i \mapsto \omega_{g^{-1}(i)}.}$$

Above we considered the dihedral group $\boldsymbol{D}_2$ acting on a set of points $\{\omega_1, \omega_2, \omega_3, \omega_4\}$ corresponding to high symmetry points on the perimeter of a rectangle. Since the group element b (reflection through the horizontal axis) induces the permutation of points

$$\boxed{b : \ 1 \mapsto 3,\ 2 \mapsto 2,\ 3 \mapsto 1,\ 4 \mapsto 4,}$$

we have in our new notation $b(1) = 3,\ b(2) = 2,\ b(3) = 1,$ and $b(4) = 4$.

We next consider a measurement whose various *outcomes* can be identified with elements in Ω. For example, we might have a POVM $\{\mathbf{E}_1, \mathbf{E}_2, \ldots, \mathbf{E}_N\}$, where the outcome corresponding to $\mathbf{E}_i$ is associated with the point ω_i. Or we could be thinking about an observable $\mathbf{A}$, in which the outcome corresponding to an eigenprojector $\mathbf{\Pi}_i$ is associated with ω_i. Such a measurement is *covariant* with respect to a given representation $g \mapsto \mathbf{U}_g$ of the group G if

$$\boxed{\mathbf{U}_g^\dagger \mathbf{E}_i \mathbf{U}_g = \mathbf{E}_{g^{-1}(i)}, \quad \forall g \in G.}$$

For the permutation representation of $\mathbf{D}_2$ we can easily see that any observable of the following form is covariant:

$$\mathbf{A} = \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a_3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_4 \end{pmatrix} \equiv \sum_{i=1}^{4} a_i \mathbf{\Pi}_i,$$

where $a_1, a_2, a_3, a_4 \in \boldsymbol{R}$. Checking the covariance requirement for the group element b, for example,

$$D_p^\dagger(b)\mathbf{\Pi}_3 D_p(b) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \mathbf{\Pi}_1,$$

which is correct since $b = b^{-1}$ and $b(3) = 1$. We actually checked all the other cases already last week when we first introduced the permutation representation.

Last week we also considered a more subtle form of covariance, which actually involved the regular representation of $\boldsymbol{D}_2$. We defined something that we called a 'covariant POVM' of the form

$$\mathbf{E}_g = D(g)\mathbf{\Xi}D^\dagger(g), \quad \forall g \in G,$$

with $\mathbf{\Xi}$ a positive operator such that

$$\sum_{g \in G} \mathbf{E}_g = \mathbf{1}.$$

We then have that

$$\begin{aligned} D^\dagger(g_2)\mathbf{E}_{g_1}D(g_2) &= D^\dagger(g_2)D(g_1)\mathbf{\Xi}D^\dagger(g_1)D(g_2) = D(g_2^{-1})D(g_1)\mathbf{\Xi}D^\dagger(g_1)D^\dagger(g_2^{-1}) \\ &= D(g_2^{-1}g_1)\mathbf{\Xi}D^\dagger(g_2^{-1}g_1) = \mathbf{E}_{g_2^{-1}g_1}, \end{aligned}$$

which is covariant if we define 'points' in our above discussions via $\omega_i \leftrightarrow g_i$ and consider the group action

$$g_j : \omega_i \mapsto \omega_k, \quad g_j g_i = g_k.$$

This alternate formulation is motivated by the estimation problem we considered - rather than associating measurement outcomes with points on the rectangle, we wanted to associate them with transformations in the $\boldsymbol{D}_2$ symmetry group.

We can now propose that a given symmetry group of interest can motivate the choice of a particular operator algebra if we ask that the algebra be 'big enough' to support measurements that are covariant with respect to a faithful (one-to-one) representation of our symmetry group. This kind of criterion is meant to remind you, at least distantly, of the relativistic notion that fundamental physical equations and quantitites should be expressible in Lorentz-covariant form, where for example a Lorentz boost along the x-axis has matrix representation

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix},$$

where $\beta = v/c$ and $\gamma = 1/\sqrt{1-v^2/c^2}$. In the case of the irreducible representation of $\mathbf{D}_3$ (symmetries of the equilateral triangle) that we introduced above, one finds that the three normalized planar vectors

$$v_a = \frac{1}{2}\begin{pmatrix} \sqrt{3} \\ -1 \end{pmatrix}, \quad v_b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad v_c = \frac{1}{2}\begin{pmatrix} -\sqrt{3} \\ -1 \end{pmatrix},$$

give rise to projectors

$$\mathbf{\Pi}_a = \frac{1}{4}\begin{pmatrix} 3 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}, \quad \mathbf{\Pi}_b = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{\Pi}_c = \frac{1}{4}\begin{pmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix},$$

$$\mathbf{\Pi}_a + \mathbf{\Pi}_b + \mathbf{\Pi}_c = \frac{1}{4}\begin{pmatrix} 3 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & \frac{3}{2} \end{pmatrix}$$

that can be used to define a POVM

$$\left\{ \mathbf{E}_a = \frac{2}{3}\mathbf{\Pi}_a,\ \mathbf{E}_b = \frac{2}{3}\mathbf{\Pi}_b,\ \mathbf{E}_c = \frac{2}{3}\mathbf{\Pi}_c \right\},$$

that is covariant with respect to the irrep. Recalling

$$g_1 : \{a,b,c\} \mapsto \{a,b,c\}, \quad g_2 : \{a,b,c\} \mapsto \{b,c,a\}, \quad g_3 : \{a,b,c\} \mapsto \{c,a,b\},$$
$$g_4 : \{a,b,c\} \mapsto \{a,c,b\}, \quad g_5 : \{a,b,c\} \mapsto \{c,b,a\}, \quad g_6 : \{a,b,c\} \mapsto \{b,a,c\},$$

and

$$D(g_1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad D(g_2) = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}, \quad D(g_3) = \frac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix},$$
$$D(g_4) = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}, \quad D(g_5) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad D(g_6) = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix},$$

we have for example

$$D^\dagger(g_3)\mathbf{E}_b D(g_3) = \frac{1}{6}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{6}\sqrt{3} \\ \frac{1}{6}\sqrt{3} & \frac{1}{6} \end{pmatrix} = \mathbf{E}_c,$$

which is as desired since $g_3^{-1} = g_2$ and g_2 maps b to c. Note that

$$\Pi_b\Pi_c - \Pi_c\Pi_b \propto \begin{pmatrix} 3 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}\begin{pmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} - \begin{pmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}\begin{pmatrix} 3 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 4\sqrt{3} \\ -4\sqrt{3} & 0 \end{pmatrix},$$

which means that we can only construct this sort of POVM in a non-commutative matrix algebra.

If we want to have a commutative probability model in which there exist measurements covariant with respect to a faithful representation of $\mathbf{D}_3$, we actually need to go to 3×3 matrices. If we do that we can use the permutation representation, and find that the three orthogonal projectors

$$\Pi_a = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Pi_b = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Pi_c = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

do the trick. However the permutation representation defined by

$$g_1 = S^0R^0, \quad g_2 = S^0R^1, \quad g_3 = S^0R^2, \quad g_4 = S^1R^1, \quad g_5 = S^1R^0, \quad g_6 = S^1R^2,$$

$$S = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad R = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

is actually reducible. The easiest way to see this is to draw the vectors onto which $\Pi_{a,b,c}$ project, and to note that action of the 3×3 matrix representation is identical to that of the 2×2 irrep on a tilted plane that contains the points $\{(1,0,0),(0,1,0),(0,0,1)\}$. The axis orthogonal to this plane corresponds to a trivial irrep. Hence we see that if we are for some reason we are interested in probability models that can acommodate $\mathbf{D}_3$-covariant measurements, (note that $\mathbf{D}_3$ is the smallest non-commutative group), a non-commutative model is in some sense more natural than a classical one because the covariance can utilize an *irreducible* representation of the group.

We wrap up today with a quick look at some material from Section 1.2.3 of A. S. Holevo, *Statistical Structure of Quantum Theory* (Springer-Verlag, Berlin, 2001).