APPPHYS225 - Tuesday 4 November 2008

Entropy of Entanglement

On several occasions we have remarked that joint pure states can be 'checked' for entanglement by forming the reduced density matrix $\tilde{\rho}$ of one subsystem and computing its *purity*, $\mathrm{Tr}[\tilde{\rho}^2]$. In fact, the purity of the reduced density matrix can be regarded as a quantitative measure of entanglement for a *pure* joint state of two systems. For example, if we consider

$$|\Psi_{AB}\rangle = \frac{1}{2}(|0_A0_B\rangle + |0_A1_B\rangle + |1_A0_B\rangle + e^{i\phi}|1_A1_B\rangle),$$

$$\begin{aligned}
|\Psi_{AB}\rangle\langle\Psi_{AB}| = \frac{1}{4}(&|0_A0_B\rangle\langle 0_A0_B| + |0_A1_B\rangle\langle 0_A0_B| + |1_A0_B\rangle\langle 0_A0_B| + e^{i\phi}|1_A1_B\rangle\langle 0_A0_B| \\
&+ |0_A0_B\rangle\langle 0_A1_B| + |0_A1_B\rangle\langle 0_A1_B| + |1_A0_B\rangle\langle 0_A1_B| + e^{i\phi}|1_A1_B\rangle\langle 0_A1_B| \\
&+ |0_A0_B\rangle\langle 1_A0_B| + |0_A1_B\rangle\langle 1_A0_B| + |1_A0_B\rangle\langle 1_A0_B| + e^{i\phi}|1_A1_B\rangle\langle 1_A0_B| \\
&+ e^{-i\phi}|0_A0_B\rangle\langle 1_A1_B| + e^{-i\phi}|0_A1_B\rangle\langle 1_A1_B| + e^{-i\phi}|1_A0_B\rangle\langle 1_A1_B| + |1_A1_B\rangle\langle 1_A1_B|),
\end{aligned}$$

we have

$$\begin{aligned}
\tilde{\rho}_A &= \mathrm{Tr}_B[|\Psi_{AB}\rangle\langle\Psi_{AB}|] \\
&= \langle 0_B||\Psi_{AB}\rangle\langle\Psi_{AB}||0_B\rangle + \langle 1_B||\Psi_{AB}\rangle\langle\Psi_{AB}||1_B\rangle \\
&= \frac{1}{4}(|0_A\rangle\langle 0_A| + |1_A\rangle\langle 0_A| + |0_A\rangle\langle 1_A| + |1_A\rangle\langle 1_A| \\
&\quad + |0_A\rangle\langle 0_A| + e^{i\phi}|1_A\rangle\langle 0_A| + e^{-i\phi}|0_A\rangle\langle 1_A| + |1_A\rangle\langle 1_A|) \\
&\leftrightarrow \begin{pmatrix} \frac{1}{2} & \frac{1}{4} + \frac{1}{4}e^{-i\phi} \\ \frac{1}{4} + \frac{1}{4}e^{i\phi} & \frac{1}{2} \end{pmatrix},
\end{aligned}$$

and thus

$$\tilde{\rho}_A^2 \leftrightarrow \begin{pmatrix} \frac{3}{8} + \frac{1}{8}\cos\phi & \frac{1}{4} + \frac{1}{4}e^{-i\phi} \\ \frac{1}{4} + \frac{1}{4}e^{i\phi} & \frac{3}{8} + \frac{1}{8}\cos\phi \end{pmatrix}, \quad \mathrm{Tr}[\tilde{\rho}_A^2] = \frac{3}{4} + \frac{1}{4}\cos\phi.$$

Hence the purity of $\tilde{\rho}_A^2$ varies from 1 ($\phi = 0,$ unentangled) down to $\frac{1}{2}$ ($\phi = \pi$).

It is actually more common to use the von Neumann entropy of the reduced density matrix rather than its purity, as a measure of pure-state entanglement, which is then called the *entropy of entanglement*. Note that it generally does not matter which system we trace over when performing a calculation of this kind. In our above example the joint pure state was completely symmetry between A and $B,$ but we can consider the following more illustrative example:

$$
\begin{aligned}
|\Psi_{AB}\rangle &= \frac{1}{\sqrt{3}}(|0_A0_B\rangle + |0_A1_B\rangle + |1_A2_B\rangle), \\
|\Psi_{AB}\rangle\langle\Psi_{AB}| &= \frac{1}{3}(|0_A0_B\rangle\langle 0_A0_B| + |0_A1_B\rangle\langle 0_A0_B| + |1_A2_B\rangle\langle 0_A0_B| \\
&\quad + |0_A0_B\rangle\langle 0_A1_B| + |0_A1_B\rangle\langle 0_A1_B| + |1_A2_B\rangle\langle 0_A1_B| \\
&\quad + |0_A0_B\rangle\langle 1_A2_B| + |0_A1_B\rangle\langle 1_A2_B| + |1_A2_B\rangle\langle 1_A2_B|),
\end{aligned}
$$

where subsystem A lives in a two-dimensional Hilbert space but B lives in a three-dimensional space. We can compute,

$$
\begin{aligned}
\tilde{\rho}_A &= \mathrm{Tr}_B[|\Psi_{AB}\rangle\langle\Psi_{AB}|] \\
&= \langle 0_B||\Psi_{AB}\rangle\langle\Psi_{AB}||0_B\rangle + \langle 1_B||\Psi_{AB}\rangle\langle\Psi_{AB}||1_B\rangle + \langle 2_B||\Psi_{AB}\rangle\langle\Psi_{AB}||2_B\rangle \\
&= \frac{1}{3}(|0_A\rangle\langle 0_A| + |0_A\rangle\langle 0_A| + |1_A\rangle\langle 1_A|) \\
&\leftrightarrow \begin{pmatrix} \frac{2}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix},
\end{aligned}
$$

so that $\mathrm{Tr}[\tilde{\rho}_A^2] = \frac{5}{9}$, while

$$
\begin{aligned}
\tilde{\rho}_B &= \mathrm{Tr}_A[|\Psi_{AB}\rangle\langle\Psi_{AB}|] \\
&= \langle 0_A||\Psi_{AB}\rangle\langle\Psi_{AB}||0_A\rangle + \langle 1_A||\Psi_{AB}\rangle\langle\Psi_{AB}||1_A\rangle \\
&= \frac{1}{3}(|0_B\rangle\langle 0_B| + |1_B\rangle\langle 0_B| + |0_B\rangle\langle 1_B| + |1_B\rangle\langle 1_B| + |2_B\rangle\langle 2_B|) \\
&\leftrightarrow \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}, \quad \tilde{\rho}_B^2 \leftrightarrow \begin{pmatrix} \frac{2}{9} & \frac{2}{9} & 0 \\ \frac{2}{9} & \frac{2}{9} & 0 \\ 0 & 0 & \frac{1}{9} \end{pmatrix},
\end{aligned}
$$

so $\mathrm{Tr}[\tilde{\rho}_B^2] = \frac{5}{9}$ as well. It turns out that the eigenvalues of $\tilde{\rho}_B$ are $\{0, \frac{1}{3}, \frac{2}{3}\}$, so we have $-\mathrm{Tr}[\tilde{\rho}_A \ln \tilde{\rho}_A] = -\mathrm{Tr}[\tilde{\rho}_B \ln \tilde{\rho}_B]$ as well.

Schmidt decomposition

Consider a joint pure state of subsystems A and B, where $H^A = \mathrm{span}\{|0_A\rangle, |1_A\rangle, \ldots, |M_A\rangle\}$ and $H^B = \mathrm{span}\{|0_B\rangle, |1_B\rangle, \ldots, |N_B\rangle\}$. We can write, in general,

$$
|\Psi_{AB}\rangle = \sum_{i=1}^{M} \sum_{j=1}^{N} c_{ij} |i_A j_B\rangle.
$$

It turns out that one can always rewrite such a joint pure state in terms of a *Schmidt decomposition*,

$$
|\Psi_{AB}\rangle = \sum_{k=1}^{\min(M,N)} s_k |u_A^k\rangle \otimes |v_B^k\rangle,
$$

where for each value of k, $|u_A^k\rangle \in H^A$ and $|v_B^k\rangle \in H^B$ satisfying

$$\left\langle u_A^i | u_A^{i'} \right\rangle = \delta_{ii'}, \quad \left\langle v_B^i | v_B^{i'} \right\rangle = \delta_{ii'}.$$

The states $\{|u_A^k\rangle \otimes |v_B^k\rangle\}$ are sometimes called the Schmidt basis for $H^A \otimes H^B$ induced by the given joint pure state $|\Psi_{AB}\rangle$. If we denote by $K \leq \min(M,N)$ the number of non-zero Schmidt coefficients s_k, which is sometimes called the *Schmidt rank*, we can write

$$|\Psi_{AB}\rangle\langle\Psi_{AB}| = \sum_{k,k'=1}^{K} s_{k'}^* s_k |u_A^k\rangle\left\langle u_A^{k'}\right| \otimes |v_B^k\rangle\left\langle v_B^{k'}\right|,$$

and

$$\begin{aligned}\tilde{\rho}_A &= \sum_{j=1}^{K}\left\langle v_B^j\right| |\Psi_{AB}\rangle\langle\Psi_{AB}| \left| v_B^j\right\rangle \\ &= \sum_{j=1}^{K}\left\langle v_B^j\right|\left(\sum_{k,k'=1}^{K} s_{k'}^* s_k |u_A^k\rangle\left\langle u_A^{k'}\right| \otimes |v_B^k\rangle\left\langle v_B^{k'}\right|\right)\left| v_B^j\right\rangle \\ &= \sum_{j=1}^{K} |s_j|^2 \left| u_A^j\right\rangle\left\langle u_A^j\right|,\end{aligned}$$

and similarly

$$\begin{aligned}\tilde{\rho}_B &= \sum_{j=1}^{K}\left\langle u_A^j\right|\left(\sum_{k,k'=1}^{K} s_{k'}^* s_k |u_A^k\rangle\left\langle u_A^{k'}\right| \otimes |v_B^k\rangle\left\langle v_B^{k'}\right|\right)\left| u_A^j\right\rangle \\ &= \sum_{j=1}^{K} |s_j|^2 \left| v_B^j\right\rangle\left\langle v_B^j\right|.\end{aligned}$$

Note that because of the assumed orthonormality of the Schmidt basis states, these are actually spectral decompositions for the reduced density operators. It is thus clear from these expressions that $\tilde{\rho}_A$ and $\tilde{\rho}_B$ have the same eigenvalues and therefore the same entropies of entanglement. We can also see from these expressions that the Schmidt basis states can be computed by finding eigenvectors of the reduced density matrices.

As an example, consider again

$$|\Psi_{AB}\rangle = \frac{1}{\sqrt{3}}(|0_A 0_B\rangle + |0_A 1_B\rangle + |1_A 2_B\rangle),$$

which is easily seen to be equivalent to

$$|\Psi_{AB}\rangle = \sqrt{\frac{2}{3}}\,|0_A\rangle \otimes \left(\frac{1}{\sqrt{2}}|0_B\rangle + \frac{1}{\sqrt{2}}|1_B\rangle\right) + \frac{1}{\sqrt{3}}|1_A\rangle \otimes |2_B\rangle.$$

Hence

$$s_1 = \sqrt{\frac{2}{3}}, \quad |u_A^1\rangle = |0_A\rangle, \quad |v_B^1\rangle = \frac{1}{\sqrt{2}}(|0_B\rangle + |1_B\rangle),$$
$$s_2 = \frac{1}{\sqrt{3}}, \quad |u_A^2\rangle = |1_A\rangle, \quad |v_B^2\rangle = |2_B\rangle,$$

and $K = 2 \leq \min[\dim(H^A) = 2, \dim(H^B) = 3],$ while we can easily verify the orthogonality conditions on the Schmidt basis states.

If we consider the much less obvious example

$$|\Psi_{AB}\rangle = \frac{1}{2}(|0_A0_B\rangle + |0_A1_B\rangle + |1_A0_B\rangle + i|1_A1_B\rangle)$$
$$= \frac{1}{2}|0_A\rangle \otimes (|0_B\rangle + |1_B\rangle) + \frac{1}{2}|1_A\rangle \otimes (|0_B\rangle + i|1_B\rangle),$$

where we note that we cannot just use the same simple ‘factoring’ trick since

$$\{|0_B\rangle + |1_B\rangle, |0_B\rangle + i|1_B\rangle\}$$

are not an orthogonal pair of states, we must work from the reduced density matrix

$$\tilde{\rho}_A = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} + \frac{1}{4}e^{-i\phi} \\ \frac{1}{4} + \frac{1}{4}e^{i\phi} & \frac{1}{2} \end{pmatrix},$$

which we computed above. With $\phi \to \pi/4$ we find that the eigenvectors and eigenvalues of $\tilde{\rho}_A$ are

$$\begin{pmatrix} \pm\frac{1}{2}(1-i) \\ \frac{1}{\sqrt{2}} \end{pmatrix} \leftrightarrow \frac{\sqrt{2} \pm 1}{2\sqrt{2}},$$

hence

$$|u_A^1\rangle = \frac{1-i}{2}|0_A\rangle + \frac{1}{\sqrt{2}}|1_A\rangle,$$
$$|u_A^2\rangle = \frac{i-1}{2}|0_A\rangle + \frac{1}{\sqrt{2}}|1_A\rangle,$$

and we check that

$$\langle u_A^1|u_A^2\rangle = \left(\frac{1+i}{2}\right)\left(\frac{i-1}{2}\right) + \frac{1}{2} = \frac{i-1-1-i}{4} + \frac{1}{2} = 0.$$

We see that

$$|0_A\rangle = \frac{1}{1-i}(|u_A^1\rangle - |u_A^2\rangle), \quad |1_A\rangle = \frac{1}{\sqrt{2}}(|u_A^1\rangle + |u_A^2\rangle),$$

hence

$$
\begin{aligned}
|\Psi_{AB}\rangle &= \frac{1}{2}|0_A\rangle \otimes (|0_B\rangle + |1_B\rangle) + \frac{1}{2}|1_A\rangle \otimes (|0_B\rangle + i|1_B\rangle) \\
&= \frac{1}{2-2i}(|u_A^1\rangle - |u_A^2\rangle) \otimes (|0_B\rangle + |1_B\rangle) + \frac{1}{2\sqrt{2}}(|u_A^1\rangle + |u_A^2\rangle) \otimes (|0_B\rangle + i|1_B\rangle) \\
&= |u_A^1\rangle \otimes \left\{ \frac{1}{2-2i}(|0_B\rangle + |1_B\rangle) + \frac{1}{2\sqrt{2}}(|0_B\rangle + i|1_B\rangle) \right\} \\
&\quad + |u_A^2\rangle \otimes \left\{ \frac{1}{2\sqrt{2}}(|0_B\rangle + i|1_B\rangle) - \frac{1}{2-2i}(|0_B\rangle + |1_B\rangle) \right\} \\
&= |u_A^1\rangle \otimes \frac{1}{2\sqrt{2}(1-i)} \left\{ \left(1+\sqrt{2}-i\right)|0_B\rangle + \left(1+\sqrt{2}+i\right)|1_B\rangle \right\} \\
&\quad + |u_A^2\rangle \otimes \frac{1}{2\sqrt{2}(1-i)} \left\{ \left(1-\sqrt{2}-i\right)|0_B\rangle + \left(1-\sqrt{2}+i\right)|1_B\rangle \right\}.
\end{aligned}
$$

Checking the normalizations,

$$
\begin{aligned}
\left(1+\sqrt{2}-i\right)\left(1+\sqrt{2}+i\right) &= 1+\sqrt{2}+i+\sqrt{2}+2+\sqrt{2}\,i-i-\sqrt{2}\,i+1 = 4+2\sqrt{2}, \\
\left(1-\sqrt{2}-i\right)\left(1-\sqrt{2}+i\right) &= 1-\sqrt{2}+i-\sqrt{2}+2-\sqrt{2}\,i-i+\sqrt{2}\,i+1 = 4-2\sqrt{2},
\end{aligned}
$$

hence

$$
\begin{aligned}
|v_B^1\rangle &= \frac{1+\sqrt{2}-i}{\sqrt{8+4\sqrt{2}}}|0_B\rangle + \frac{1+\sqrt{2}+i}{\sqrt{8+4\sqrt{2}}}|1_B\rangle, \\
|v_B^2\rangle &= \frac{1-\sqrt{2}-i}{\sqrt{8-4\sqrt{2}}}|0_B\rangle + \frac{1-\sqrt{2}+i}{\sqrt{8-4\sqrt{2}}}|1_B\rangle,
\end{aligned}
$$

and

$$
\begin{aligned}
|\Psi_{AB}\rangle &= \frac{\sqrt{8+4\sqrt{2}}}{2\sqrt{2}(1-i)}|u_A^1\rangle \otimes |v_B^1\rangle + \frac{\sqrt{8-4\sqrt{2}}}{2\sqrt{2}(1-i)}|u_A^2\rangle \otimes |v_B^2\rangle \\
&= \frac{\sqrt{1+\frac{1}{\sqrt{2}}}}{1-i}|u_A^1\rangle \otimes |v_B^1\rangle + \frac{\sqrt{1-\frac{1}{\sqrt{2}}}}{1-i}|u_A^2\rangle \otimes |v_B^2\rangle \\
&= \frac{1+i}{2}\sqrt{1+\frac{1}{\sqrt{2}}}\,|u_A^1\rangle \otimes |v_B^1\rangle + \frac{1+i}{2}\sqrt{1-\frac{1}{\sqrt{2}}}\,|u_A^2\rangle \otimes |v_B^2\rangle,
\end{aligned}
$$

and we check that

$$
\begin{aligned}
|s_1|^2 &= \frac{1}{2} + \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}+1}{2\sqrt{2}}, \\
|s_2|^2 &= \frac{1}{2} - \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}-1}{2\sqrt{2}},
\end{aligned}
$$

match the eigenvalues of $\tilde{\rho}_A$ computed above.

Entanglement concentration

Here we will briefly introduce an entanglement concentration scheme presented by C. H. Bennett *et al.*, “Concentrating partial entanglement by local operations,” Phys. Rev. A **53**, 2046 (1996). Please refer to the paper and your upcoming homework for further details.

Consider a situation in which Alice and Bob share a large number of pairs of partially entangled two-level systems. Let the state of each pair be

$$|\Psi_i\rangle = \cos\theta|0_{Ai}1_{Bi}\rangle + \sin\theta|1_{Ai}0_{Bi}\rangle.$$

Overall, the state of N such particle pairs will look like

$$|\Psi_{1\ldots N}\rangle = (\cos\theta|0_{A1}1_{B1}\rangle + \sin\theta|1_{A1}0_{B1}\rangle) \otimes (\cos\theta|0_{A2}1_{B2}\rangle + \sin\theta|1_{A2}0_{B2}\rangle)\cdots(\cos\theta|0_{AN}1_{BN}\rangle + \sin\theta|1_{AN}0_{BN}\rangle$$

We recognize that when the product is distributed, we’ll end up with a range of terms with coefficients of the form $\{\cos^N\theta, \cos^{N-1}\theta\sin\theta, \cos^{N-2}\theta\sin^2\theta, \ldots\cos\theta\sin^{N-1}\theta, \sin^N\theta\}$. If we group together all the terms that have coefficient $\cos^{N-k}\theta\sin^k\theta,$ these will span a subspace of the total $2N$-particle Hilbert space of dimension

$$D_k = \binom{N}{k},$$

that is, ‘N choose k’ dimensions. We note that each subspace basis state identified in this manner is a product state. For example, the subspace with $k = 1$ has N basis states of the form

$$|k=1,i\rangle \equiv |1_{Ai}0_{Bi}\rangle \bigotimes_{j\neq i} |0_{Aj}1_{Bj}\rangle = |0_{A1}0_{A2}\ldots 1_{Ai}\ldots 0_{AN}\rangle \otimes |1_{B1}1_{B2}\ldots 0_{Bi}\ldots 1_{BN}\rangle, \quad i = 1\ldots N.$$

It thus follows that either Alice or Bob can perform a measurement composed of rank-D_k partial projectors into the k subspaces constructed in this way. The probability of obtaining result k' in such a measurement is

$$\Pr(k') = \binom{N}{k'}\cos^{2N-2k'}\theta\sin^{2k'}\theta.$$

After such a measurement is performed, with result k', the joint state of all the particles is projected into the subspace $\mathrm{span}\{|k',i\rangle\}$. There will be $D_{k'}$ terms in the conditional state, each of which has equal magnitude. Hence, except in the relatively rare cases $k' = 0$ or $k' = N,$ we end up with a maximally entangled state in some subspace of the total $2N$-particle Hilbert space.

Entropy of Formation

So far we have discussed only pure states of a joint system. What about mixed states? This turns out to be quite a complicated matter...

We have previously discussed the fact that a given density matrix can be ‘decomposed’ into ensembles of pure states in countless ways. Consider for example

$$\rho_{AB} = \frac{1}{2}|0_A0_B\rangle\langle 0_A0_B| + \frac{1}{2}|1_A1_B\rangle\langle 1_A1_B|,$$

which represents the ensemble density matrix for both

$$p_0 = \frac{1}{2}, \quad |\psi_0\rangle = |0_A0_B\rangle,$$

$$p_1 = \frac{1}{2}, \quad |\psi_1\rangle = |1_A1_B\rangle,$$

and

$$p_0 = \frac{1}{2}, \quad |\phi_0\rangle = \frac{1}{\sqrt{2}}(|0_A 0_B\rangle + |1_A 1_B\rangle),$$
$$p_1 = \frac{1}{2}, \quad |\phi_1\rangle = \frac{1}{\sqrt{2}}(|0_A 0_B\rangle - |1_A 1_B\rangle).$$

The first correspondence should be obvious, and to see the second just note that all the 'cross-terms' will have opposite signs and therefore disappear when the ensemble density matrix is formed. While $\{|\phi_0\rangle, |\phi_1\rangle\}$ ensemble, which contains only highly entangled states, might suggest that ρ_{AB} could be considered to be an entangled mixed state, the existence of the alternative ensemble $\{|\psi_0\rangle, |\psi_1\rangle\}$ presumably proves that it should not be so considered.

The *entanglement of formation* of a mixed state ρ_{AB} is defined to be the minimum average entropy of entanglement among all consistent ensembles of pure states. That is,

$$E_F(\rho_{AB}) = \min_{\{p_i, |\Psi_i\rangle\}} \sum_i p_i E(|\Psi_i\rangle), \quad \rho_{AB} = \sum_i p_i |\Psi_i\rangle\langle\Psi_i|,$$

where $E(|\Psi_i\rangle)$ denotes the entropy of entanglement of the pure state $|\Psi_i\rangle$. Efficient methods for calculating E_F for subsystems of arbitrary dimension are not known (in fact, even just deciding whether it is nonzero seems to be quite hard), but for the special case of two two-dimensional systems (qubits) there is an efficient method due to Wootters [W. K. Wootters, "Entanglement of Formation of an Arbitrary State of Two Qubits," Phys. Rev. Lett. **80**, 2245 (1998)].

The *distillable entanglement* E_D of a mixed state ρ_{AB} is defined to be the asymptotic yield of arbitrarily pure singlets that can be prepared locally by *entanglement purification protocols* analogous to the pure-state entanglement concentration procedure sketched above. Surprisingly, there exist mixed states that cannot be prepared without some input of entanglement yet have $E_D = 0$ provably [D. Yang *et al.*, "Irreversibility for All Bound Entangled States," Phys. Rev. Lett. **95**, 190501 (2005)].

Entanglement of Assistance

Consider now the three-part state

$$|\Psi_{ABC}\rangle = \frac{1}{\sqrt{2}}(|0_A 1_B \alpha_C\rangle + |1_A 0_B \beta_C\rangle),$$
$$|\Psi_{ABC}\rangle\langle\Psi_{ABC}| = \frac{1}{2}(|0_A 1_B \alpha_C\rangle\langle 0_A 1_B \alpha_C| + |1_A 0_B \beta_C\rangle\langle 0_A 1_B \alpha_C|$$
$$+ |0_A 1_B \alpha_C\rangle\langle 1_A 0_B \beta_C| + |1_A 0_B \beta_C\rangle\langle 1_A 0_B \beta_C|),$$

where we are working in a joint Hilbert space $H^A \otimes H^B \otimes H^C$ with

$$H^A = \text{span}\{|0_A\rangle, |1_A\rangle\}, \quad H^B = \text{span}\{|0_B\rangle, |1_B\rangle\}, \quad H^C = \text{span}\{|\alpha_C\rangle, |\beta_C\rangle\}.$$

Assume that Alice and Bob have posession of the particles whose states live in H^A and H^B, respectively, and that Charlie has the particle from spaace H^C.

If Alice and Bob get together but Charlie runs off on his own, what can Alice and

Bob say about their marginal state? As we have done with computing reduced density matrices for one subsystem out of two, we can here just take a partial trace over H^C to obtain a reduced joint density matrix:

$$\begin{aligned}\tilde{\rho}_{AB} &= \mathrm{Tr}_C[|\Psi_{ABC}\rangle\langle\Psi_{ABC}|] \\ &= \langle\alpha_C||\Psi_{ABC}\rangle\langle\Psi_{ABC}||\alpha_C\rangle + \langle\beta_C||\Psi_{ABC}\rangle\langle\Psi_{ABC}||\beta_C\rangle \\ &= \frac{1}{2}(|0_A 1_B\rangle\langle 0_A 1_B| + |1_A 0_B\rangle\langle 1_A 0_B|).\end{aligned}$$

This is clearly a mixed state with zero entanglement of formation. As we have previously discussed, we can think of the partial trace here as corresponding to a scenario in which Charlie measures his state in the $\{|\alpha_C\rangle, |\beta_C\rangle\}$-basis but does not inform Alice and Bob of the result. But note that it would not help Alice and Bob to know the result of Charlie's measurement if what they want is to end up with some entanglement.

We know that it does not matter what basis (for H^C) we use for the partial trace, but what if Charlie actually does perform a measurement on his particle and then does inform Alice and Bob of the result? They would then end up with a state derived by selective evolution, which we will compute by applying a partial projection before factoring out the state of particle C. An illustrative example is obtained if we have Charlie perform his measurement in the basis $\{|\pi_C\rangle, |\nu_C\rangle\}$, where

$$\begin{aligned}|\pi_C\rangle &= \frac{1}{\sqrt{2}}(|\alpha_C\rangle + |\beta_C\rangle), \quad |\nu_C\rangle = \frac{1}{\sqrt{2}}(|\alpha_C\rangle - |\beta_C\rangle), \\ |\alpha_C\rangle &= \frac{1}{\sqrt{2}}(|\pi_C\rangle + |\nu_C\rangle), \quad |\beta_C\rangle = \frac{1}{\sqrt{2}}(|\pi_C\rangle - |\nu_C\rangle).\end{aligned}$$

Rewriting our initial joint state with this basis for H^C in mind, we have

$$\begin{aligned}|\Psi_{ABC}\rangle &= \frac{1}{\sqrt{2}}(|0_A 1_B \alpha_C\rangle + |1_A 0_B \beta_C\rangle) \\ &= \frac{1}{2}(|0_A 1_B \pi_C\rangle + |0_A 1_B \nu_C\rangle + |1_A 0_B \pi_C\rangle - |1_A 0_B \nu_C\rangle) \\ &= \frac{1}{2}(|0_A 1_B\rangle + |1_A 0_B\rangle) \otimes |\pi_C\rangle + \frac{1}{2}(|0_A 1_B\rangle - |1_A 0_B\rangle) \otimes |\nu_C\rangle.\end{aligned}$$

Hence we see that if Charlies makes his measurement and obtains the π result, Alice and Bob are left with the *entangled* pure joint state

$$|\Psi_{AB}\rangle \mapsto_\pi |\Psi^+_{AB}\rangle = \frac{1}{\sqrt{2}}(|0_A 1_B\rangle + |1_A 0_B\rangle),$$

whereas if he obtains the ν result,

$$|\Psi_{AB}\rangle \mapsto_\nu |\Psi^-_{AB}\rangle = \frac{1}{\sqrt{2}}(|0_A 1_B\rangle - |1_A 0_B\rangle).$$

Note of course that the ensemble density operator corresponding to non-selective evolution with this scenario is still

$$\tilde{\rho}_{AB} = \frac{1}{2}|\Psi^+_{AB}\rangle\langle\Psi^+_{AB}| + \frac{1}{2}|\Psi^-_{AB}\rangle\langle\Psi^-_{AB}| = \frac{1}{2}(|0_A 1_B\rangle\langle 0_A 1_B| + |1_A 0_B\rangle\langle 1_A 0_B|),$$

where again the last equivalence can be seen by noting that all the 'cross-terms' cancel. It thus appears that Charlie's choice of measurement basis, while it cannot change Alice and Bob's non-selective reduced density matrix, does pick out a specific ensemble decomposition of $\tilde{\rho}_{AB}$. In particular if Charlie picks a measurement basis that generates an ensemble containing highly entangled states, then Alice and Bob can end up with a highly entangled state if Charlie informs them of the measurement result. On the other hand Charlie may be able to pick a measurement basis that leads to conditional states for Alice and Bob that have lower entanglement, or even no entanglement if $E_F(\tilde{\rho}_{AB}) = 0$.

This type of consideration motivates the definition of a quantity called *entanglement of assistance*,

$$E_A(\rho_{AB}) = \max_{\{p_i,|\Psi_i\rangle\}} \sum_i p_i E(|\Psi_i\rangle), \quad \rho_{AB} = \sum_i p_i |\Psi_i\rangle\langle\Psi_i|,$$

the definition of which is quite like the definition of E_F except that the minimum is replaced by a maximum. The general idea is that if the impurity of a given bipartite mixed state ρ_{AB} is assumed to result entirely from entanglement with a third system, but nothing is known about the structure of that entanglement, E_F and E_A bound the minimum and maximum expected entropy of entanglement (respectively) between A and B that could result from partial projection of the third system.

The 'quantum eraser' scenario

The simple calculations we have performed above are quite similar to those involved in the analysis of the so-called 'quantum eraser' scenario.

Suppose for example that we have two optical cavities, each prepared initially in a one-photon Fock state. We can model these as having quantum states that live in two-dimensional Hilbert spaces,

$$H^A = \text{span}\{|0_A\rangle, |1_A\rangle\}, \quad H^B = \text{span}\{|0_B\rangle, |1_B\rangle\},$$

where the labels of the given basis states indicate photon number. Our initial state is thus

$$|\Psi_{AB}\rangle = |1_A 1_B\rangle.$$

We need to enlarge the system under consideration in order to include a very simplistic description of energy decay, via leakage of the photons stored in the cavities into outgoing modes of the electromagnetic field. We will assume that each of these outgoing modes is monitored by perfect photon-counting detectors with high time resolution Δt, assumed to be much smaller than the cavity decay time constant.

If we add two more Hilbert spaces,

$$H^\alpha = \text{span}\{|0_\alpha\rangle, |1_\alpha\rangle\}, \quad H^\beta = \text{span}\{|0_\beta\rangle, |1_\beta\rangle\},$$

we can model the leakage of photons from cavity A into the α mode, and of photons from cavity B into the β mode, via the interaction Hamiltonian

$$\mathbf{H}_{\text{int}} = i\kappa\hbar\left(\mathbf{a}\boldsymbol{\alpha}^\dagger - \mathbf{a}^\dagger\boldsymbol{\alpha}\right) + i\kappa\hbar\left(\mathbf{b}\boldsymbol{\beta}^\dagger - \mathbf{b}^\dagger\boldsymbol{\beta}\right).$$

(Note that a proper quantum-optical decay model would be far more complicated, involving multimode electromagnetic fields outside the cavity and quantum Markov approximations—see below—but this toy model will suffice for our purposes.) Here the annihilation operators are defined by

$$\mathbf{a} = |0_A\rangle\langle 1_A|, \quad \mathbf{b} = |0_B\rangle\langle 1_B|, \quad \boldsymbol{\alpha} = |0_\alpha\rangle\langle 1_\alpha|, \quad \boldsymbol{\beta} = |0_\beta\rangle\langle 1_\beta|,$$

and κ is a parameter related to the rate of photon leakage. We assume that the initial state of the four-part system is now

$$|\Psi_{AB\alpha\beta}\rangle = |1_A 1_B 0_\alpha 0_\beta\rangle.$$

We proceed by evolving this state with coarse-grained timestep $\Delta t,$ interleaving periods of unitary (Schrödinger) evolution with measurements of the photon number in modes α and $\beta.$ If the result of the measurement on the α output channel is $m_\alpha \in \{0,1\}$ and the result of the measurement on the β output channel is $n_\beta \in \{0,1\},$ we have the selective evolution

$$\begin{aligned} |\Psi(t)\rangle &\mapsto |\Psi(t+\Delta t)\rangle = \exp(-i\Delta t\mathbf{H}_{\text{int}}/\hbar)|\Psi(t)\rangle \\ &\mapsto \frac{\mathbf{A}_{m_\alpha n_\beta}\exp(-i\Delta t\mathbf{H}_{\text{int}}/\hbar)|\Psi(t)\rangle}{\sqrt{\langle\Psi(t)|\exp(+i\Delta t\mathbf{H}_{\text{int}}/\hbar)\mathbf{A}^\dagger_{m_\alpha n_\beta}\mathbf{A}_{m_\alpha n_\beta}\exp(-i\Delta t\mathbf{H}_{\text{int}}/\hbar)|\Psi(t)\rangle}}, \end{aligned}$$

where

$$\begin{aligned} \mathbf{A}_{0_\alpha 0_\beta} &= \mathbf{1}^A \otimes \mathbf{1}^B \otimes |0_\alpha\rangle\langle 0_\alpha| \otimes |0_\beta\rangle\langle 0_\beta|, \quad \mathbf{A}_{0_\alpha 1_\beta} = \mathbf{1}^A \otimes \mathbf{1}^B \otimes |0_\alpha\rangle\langle 0_\alpha| \otimes |0_\beta\rangle\langle 1_\beta|, \\ \mathbf{A}_{1_\alpha 0_\beta} &= \mathbf{1}^A \otimes \mathbf{1}^B \otimes |0_\alpha\rangle\langle 1_\alpha| \otimes |0_\beta\rangle\langle 0_\beta|, \quad \mathbf{A}_{1_\alpha 1_\beta} = \mathbf{1}^A \otimes \mathbf{1}^B \otimes |0_\alpha\rangle\langle 1_\alpha| \otimes |0_\beta\rangle\langle 1_\beta|, \\ \mathbf{A}^\dagger_{m_\alpha n_\beta}\mathbf{A}_{m_\alpha n_\beta} &\equiv \mathbf{1}^A \otimes \mathbf{1}^B \otimes |m_\alpha\rangle\langle m_\alpha| \otimes |n_\beta\rangle\langle n_\beta|. \end{aligned}$$

Under the assumption of small Δt we have

$$\begin{aligned} \exp(-i\Delta t\mathbf{H}_{\text{int}}/\hbar) &\approx \mathbf{1} - i\Delta t\mathbf{H}_{\text{int}}/\hbar \\ &= \mathbf{1} + \sqrt{\varepsilon}\left(\mathbf{a}\boldsymbol{\alpha}^\dagger + \mathbf{a}^\dagger\boldsymbol{\alpha}\right) + \sqrt{\varepsilon}\left(\mathbf{b}\boldsymbol{\beta}^\dagger + \mathbf{b}^\dagger\boldsymbol{\beta}\right), \end{aligned}$$

where $\sqrt{\varepsilon} \equiv \kappa\Delta t$, and so in the very first time-step

$$\begin{aligned} |\Psi(\Delta t)\rangle &\approx \left\{\mathbf{1} + \sqrt{\varepsilon}\left(\mathbf{a}\boldsymbol{\alpha}^\dagger + \mathbf{a}^\dagger\boldsymbol{\alpha}\right) + \sqrt{\varepsilon}\left(\mathbf{b}\boldsymbol{\beta}^\dagger + \mathbf{b}^\dagger\boldsymbol{\beta}\right)\right\}|1_A 1_B 0_\alpha 0_\beta\rangle \\ &= |1_A 1_B 0_\alpha 0_\beta\rangle + \sqrt{\varepsilon}\,|0_A 1_B 1_\alpha 0_\beta\rangle + \sqrt{\varepsilon}\,|1_A 0_B 0_\alpha 1_\beta\rangle. \end{aligned}$$

We then have measurement probabilities, to first order in ε,

$$\Pr(m_\alpha = 0) \approx 1, \quad \Pr(m_\alpha = 1) \approx \varepsilon, \quad \Pr(n_\beta = 0) \approx 1, \quad \Pr(n_\beta = 1) \approx \varepsilon.$$

It thus follows that the most likely ‘effect’ of each evolution-measurement sequence will be for $|\Psi(t)\rangle$ to be projected back to the $|1_A 1_B 0_\alpha 0_\beta\rangle$. In each time step, however, there is a small but finite probability ε for a click in either the α or β detector (we neglect $\Pr(m_\alpha = 1, n_\beta = 1)$ as vanishingly small). In general we thus expect to have a state ‘trajectory’ that looks either like

$$|1_A 1_B 0_\alpha 0_\beta\rangle \to |0_A 1_B 0_\alpha 0_\beta\rangle \to |0_A 0_B 0_\alpha 0_\beta\rangle,$$

or

$$|1_A 1_B 0_\alpha 0_\beta\rangle \to |1_A 0_B 0_\alpha 0_\beta\rangle \to |0_A 0_B 0_\alpha 0_\beta\rangle,$$

depending on which detector clicks first. We expect the complete decay to $|0_A 0_B 0_\alpha 0_\beta\rangle$ to occur over a time interval of order $\varepsilon^{-1}.$ Note that no entanglement between cavities A and B is ever generated.

A different picture results if we add a 50/50 beamsplitter to the setup. Suppose we place it to mix the outputs of cavities A and B before the photon counting detectors.

The net effect of this change in setup is to modify the measurement operators. In particular, the annihilation operators for the new, ‘mixed’ modes are

$$\boldsymbol{\gamma} = \frac{1}{\sqrt{2}}(\boldsymbol{\alpha}+\boldsymbol{\beta}) = \frac{1}{\sqrt{2}}(|0_\alpha\rangle\langle 1_\alpha| + |0_\beta\rangle\langle 1_\beta|),$$
$$\boldsymbol{\delta} = \frac{1}{\sqrt{2}}(\boldsymbol{\alpha}-\boldsymbol{\beta}) = \frac{1}{\sqrt{2}}(|0_\alpha\rangle\langle 1_\alpha| - |0_\beta\rangle\langle 1_\beta|),$$

according to fundamental quantum-optical relations for a 50/50 beamsplitter, and we will now want to consider states such as

$$|0_\gamma 0_\delta\rangle = |0_\alpha 0_\beta\rangle,$$
$$|1_\gamma 0_\delta\rangle = \boldsymbol{\gamma}^\dagger|0_\alpha 0_\beta\rangle = \frac{1}{\sqrt{2}}(|1_\alpha 0_\beta\rangle + |0_\alpha 1_\beta\rangle),$$
$$|0_\gamma 1_\delta\rangle = \boldsymbol{\delta}^\dagger|0_\alpha 0_\beta\rangle = \frac{1}{\sqrt{2}}(|1_\alpha 0_\beta\rangle - |0_\alpha 1_\beta\rangle),$$

where we note that

$$|1_\gamma 0_\delta\rangle\langle 1_\gamma 0_\delta| = \frac{1}{2}(|1_\alpha 0_\beta\rangle + |0_\alpha 1_\beta\rangle)(\langle 1_\alpha 0_\beta| + \langle 0_\alpha 1_\beta|)$$
$$= \frac{1}{2}(|1_\alpha 0_\beta\rangle\langle 1_\alpha 0_\beta| + |0_\alpha 1_\beta\rangle\langle 1_\alpha 0_\beta| + |1_\alpha 0_\beta\rangle\langle 0_\alpha 1_\beta| + |0_\alpha 1_\beta\rangle\langle 0_\alpha 1_\beta|),$$
$$|1_\gamma 0_\delta\rangle\langle 1_\gamma 0_\delta| = \frac{1}{2}(|1_\alpha 0_\beta\rangle - |0_\alpha 1_\beta\rangle)(\langle 1_\alpha 0_\beta| - \langle 0_\alpha 1_\beta|)$$
$$= \frac{1}{2}(|1_\alpha 0_\beta\rangle\langle 1_\alpha 0_\beta| - |0_\alpha 1_\beta\rangle\langle 1_\alpha 0_\beta| - |1_\alpha 0_\beta\rangle\langle 0_\alpha 1_\beta| + |0_\alpha 1_\beta\rangle\langle 0_\alpha 1_\beta|),$$

If use m_γ and n_δ to denote the numbers of photons detected at the two post-beamsplitter detectors, we thus have

$$\mathbf{A}_{0_\gamma 1_\delta} = \mathbf{1}^A \otimes \mathbf{1}^B \otimes |0_\gamma 0_\delta\rangle\langle 0_\gamma 1_\delta|, \quad \mathbf{A}_{1_\gamma 0_\delta} = \mathbf{1}^A \otimes \mathbf{1}^B \otimes |0_\gamma 0_\delta\rangle\langle 1_\gamma 0_\delta|.$$

Note that

$$\mathbf{A}^\dagger_{1_\gamma 0_\delta}\mathbf{A}_{1_\gamma 0_\delta} + \mathbf{A}^\dagger_{0_\gamma 1_\delta}\mathbf{A}_{0_\gamma 1_\delta} = \mathbf{1}^A \otimes \mathbf{1}^B \otimes (|1_\gamma 0_\delta\rangle\langle 1_\gamma 0_\delta| + |1_\gamma 0_\delta\rangle\langle 1_\gamma 0_\delta|)$$
$$= \mathbf{1}^A \otimes \mathbf{1}^B \otimes (|1_\alpha 0_\beta\rangle\langle 1_\alpha 0_\beta| + |0_\alpha 1_\beta\rangle\langle 0_\alpha 1_\beta|)$$
$$= \mathbf{A}^\dagger_{1_\alpha 0_\beta}\mathbf{A}_{1_\alpha 0_\beta} + \mathbf{A}^\dagger_{0_\alpha 1_\beta}\mathbf{A}_{0_\alpha 1_\beta},$$

which means that

$$\Pr(m_\gamma = 1, n_\delta = 0) + \Pr(m_\gamma = 0, n_\delta = 1) = \Pr(m_\alpha = 1, n_\beta = 0) + \Pr(m_\alpha = 0, n_\beta = 1).$$

In other words, the probability per time step of obtaining a click is unchanged. On the other hand,

$$\mathbf{A}_{1_\gamma 0_\delta}|\Psi(\Delta t)\rangle = \frac{1}{\sqrt{2}}|0_\alpha 0_\beta\rangle(\langle 1_\alpha 0_\beta| + \langle 0_\alpha 1_\beta|)(|1_A 1_B 0_\alpha 0_\beta\rangle + \sqrt{\varepsilon}\,|0_A 1_B 1_\alpha 0_\beta\rangle + \sqrt{\varepsilon}\,|1_A 0_B 0_\alpha 1_\beta\rangle)$$
$$\propto |0_A 1_B 0_\alpha 0_\beta\rangle + |1_A 0_B 0_\alpha 0_\beta\rangle,$$
$$\mathbf{A}_{0_\gamma 1_\delta}|\Psi(\Delta t)\rangle = \frac{1}{\sqrt{2}}|0_\alpha 0_\beta\rangle(\langle 1_\alpha 0_\beta| - \langle 0_\alpha 1_\beta|)(|1_A 1_B 0_\alpha 0_\beta\rangle + \sqrt{\varepsilon}\,|0_A 1_B 1_\alpha 0_\beta\rangle + \sqrt{\varepsilon}\,|1_A 0_B 0_\alpha 1_\beta\rangle)$$
$$\propto |0_A 1_B 0_\alpha 0_\beta\rangle - |1_A 0_B 0_\alpha 0_\beta\rangle.$$

We see that $|0_\alpha 0_\beta\rangle$ factors out in either case, and Alice and Bob are left conditionally with entangled pure states $|\Psi^{\pm}_{AB}\rangle$.

The usual interpretation of the difference between these two scenarios is that while the unitarily evolved joint state

$$|\Psi(\Delta t)\rangle = |1_A 1_B 0_\alpha 0_\beta\rangle + \sqrt{\varepsilon}\,|0_A 1_B 1_\alpha 0_\beta\rangle + \sqrt{\varepsilon}\,|1_A 0_B 0_\alpha 1_\beta\rangle$$

is in fact very slightly entangled, by virtue of the fact that the 'decayed' kets in the superposition differ in terms of which cavity contributed the leaked photon, direct measurement of the output channels α and β breaks this interesting superposition by determining which cavity the detected photon came from. The role of the 50/50 beamspliter is to 'erase' this information - a click in counter γ or δ could have been caused by leakage of a photon from either cavity. Various analogues of this scheme are widely considered in modern quantum optics.

Again, we should be careful to recognize that the way 'decay' is treated in our toy model leaves much to be desired. For example, even though (with all parameters fixed) we have a 'coarse-grained rate' of photodetection $\sim \varepsilon/\Delta t$, we have the strange property that the probability per time step $\varepsilon \propto \Delta t^2$. Also, if we simply let the cavity fields and output fields evolve unitarily under the joint Hamiltonian $\mathbf{H}_{\rm int}$, without ever performing any measurements, photons would actually oscillate coherently between the cavity and output field modes. In order to fix these problems and obtain a more physically sensible model of irreversible exponential decay, we would need to do something similar to the Wigner-Weisskopf treatment of atomic spontaneous emission. See elementary texts in quantum optics for more details.