Existence of Equilibrium in Auctions and Discontinuous Bayesian Games: Endogenous and Incentive Compatible Sharing Rules.*

by

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August, 1999

This draft: November 22, 1999

Abstract

We consider discontinuous games with incomplete information. Auctions are a leading example. With standard tie breaking rules (or more generally, sharing rules), these games may not have equilibria. We consider sharing rules that depend on the private information of players. We show that there exists an equilibrium of an augmented game with an incentive compatible sharing rule in which players reveal their private information for the purpose of determining sharing. We also show that for a large class of private value auctions, ties never occur in the equilibrium of the augmented game. This establishes existence of equilibria in such auctions with standard tie breaking rules.

Keywords: Bayesian Games, Existence, Equilibrium, Endogenous Sharing, Tie Breaking, Auctions.

JEL Classification Codes: C62,C63,D44,D82

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*Independent work by Simon and Zame [24] establishes results similar to some of those in this paper. We became aware of the overlap in our work in October 1999. We thank Leo Simon and Bill Zame for helpful conversations concerning existence in auctions in October 1997. We also thank Kim Border, Martin Cripps, John Nachbar, Larry Samuelson, Mark Satterthwaite, and Tianxiang Ye for helpful comments and suggestions.

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1 Introduction

Many Bayesian games involve discontinuities in payoffs as strategies are varied. For example, in auctions, there are typically discontinuities in payoffs at points where bids are tied. Moreover, at such tied bids, a bidder may make inferences concerning the value of an object conditional on whether or not the tie is broken in her favor. These discontinuities make proofs of existence of equilibrium very difficult, and, in a number of examples lead to non-existence. This is important because in many settings, such as auctions, the continuum bidding spaces are a natural and very handy modeling tool, and so it is disturbing that there seems to be a real difference between games with finite action spaces (which have equilibria under mild conditions) and games with continuum type spaces.

We consider Bayesian games in which tie-breaking (or more generally, "sharing") is allowed to be "endogenous" in the sense that it depends on the private information of players. This establishes the correct limits of finite action games and eliminates the disturbing non-existence at the limit problems. To do this, we construct an augmented game in which an announcement of type is added to the original strategy space. The type announcement is irrelevant except at points of indeterminacy of the original game (e.g., at ties in auctions). We show that under mild conditions such games have an equilibrium in which the announcement is truthful. Thus, for example, auctions which fail to have a normal equilibrium do have equilibria if one allows an endogenous determination of how ties are broken. And, for the right tie-breaking rule it is incentive compatible for the players to reveal their true types.

An important corollary to our results concerns a large class of private value auctions. This class includes single and double discriminatory and uniform auctions and allows for multiple unit demands and supplies and for correlations of demands and supplies. Our results imply that these auctions have equilibria in the augmented game where tie-breaking is type dependent. However, we show that if distributions over the private valuations are atomless (in a sense to be made precise) then the tie-breaking rule is never relevant in equilibrium and so the bidding behavior in the equilibrium of the augmented game is in fact an equilibrium of the game with standard tie-breaking rules. We also prove that each auction in our class has an equilibrium where trade occurs with positive probability. So, our results are not vacuous in settings where there exist degenerate no-trade equilibria (as in a double auction in which all sellers ask for a very high price, and all buyers offer 0). As far as we know, our result about positive trade equilibria in e.g., double auction settings is completely new to the auctions literature.[1]

Both the method of proof and the spirit of our existence result for games with endogenous sharing owe a debt to Simon and Zame [23] (henceforth SZ). They establish existence of equilibrium for a class of games with complete information but with discontinuous payoffs. The key to their construction is to endogenize payoffs at combinations of strategies that lead to discontinuities in such a way as to recreate the necessary continuity. In location games, for example, this amounts to allowing a choice of market shares if the two firms locate in the same place. Our existence result extends this style of construction to Bayesian games, by allowing the sharing rule to depend not only on the parameters of the game, but also on the private information of players.

The construction used in establishing our results about auctions also owes a debt to Maskin and Riley [14]. The strategy of their proof is to consider auctions in which, in the event of a tie, a Vickrey auction takes place. For the auctions they consider, the Vickrey auction is enough to guarantee that payoffs are preserved in the limit as the finite grid of bids grows fine. They then argue that these ties could not have been occurring anyway, and so the equilibrium is in fact a standard one. So, exactly like them, our idea is to show existence in a game where one does something strange (but incentive compatible) in the event of a tie and then work backward to show that in many cases of interest this was irrelevant. Our results for private value auctions, cover a substantially broader set of cases than those of Maskin and Riley, essentially because our construction of endogenous sharing allows for much more general methods of allocating objects at discontinuities than does Maskin and Riley's use of the Vickrey auction. Hence, our results are stronger because our tie breaking method is potentially stranger. They show existence for single unit one-sided high bid auctions, and for the private value case need affiliation of types. Most importantly, we do not need affiliated types, and cover a variety of payment rules, multiple unit demands and supplies, and double auctions. On the other hand, their results hold for some non-private value auctions while Theorem 5 does not. It is an open question how far

[1] Such no-trade equilibria are notoriously difficult to overcome in some other settings. For instance, see the discussion of positive trade equilibria in market games in Dubey and Shubik [7] and Peck, Shell, and Spear [17].

techniques like those in this paper can extend beyond private value auctions.

This paper is also related to the other literature on existence in games with continuum type spaces, including, for example, Dasgupta and Maskin [6] and Simon [22]. Many of these papers have been generalized by Reny [19], and so we focus on the relationship of our paper to Reny's (see Reny for an excellent discussion of how his paper in turn relates to the previous existence literature). Reny's approach is based on a condition termed *better reply security.* Essentially, better reply security requires that when there is a profitable deviation from a strategy, then some player has a deviation which is profitable even if other players play a little distance from the base strategy. Under *better reply security* plus some other conditions, games have Nash equilibria. Reny shows that his condition applies in a multiple unit, private value, pay your bids auction, a case for which we also prove existence (see his Example 5.2). Our auction result is stated for a substantially broader class of auctions that Reny's, although one suspects that his method would apply to many of the cases we cover that he does not. More importantly, Reny's results only hold in situations in which one can argue a priori that there is a best response in increasing strategies whenever one's opponents use increasing strategies. This requires fairly tight restrictions on how values are related. In our work, these restrictions are unnecessary. See Example 3 below.

Reny, in his discussion of SZ, ([19], page 1050) states that an approach such as theirs is unlikely to be helpful in, for example, auction settings, because "One cannot leave some of the payoffs [at discontinuities] unspecified, to somehow be endogenously determined." In a sense, the point of this paper is that, "in fact you can, and in such a way that it is incentive compatible for players to reveal the information you need to do so."

The other major difference between our paper, and previous existence papers is that in games which do not have Nash equilibria, we are able to exhibit a "close by" game (the augmented game) that does.

Lebrun [12],[13] also shows existence in single unit first price auctions. His strategy is based on finding conditions such that a set of differential equations characterizing equilibrium has a solution. This approach allows for interesting comparative statics results. See also Bajari [3] on this approach. Using the differential equations approach, Lizzeri and Persico [11] show existence in first price auctions with a reserve price.

Athey [2] addresses existence in a different way. She considers conditions on games such that a monotone comparative statics result applies to the best bid of a player as his signal varies. Essentially, one imposes a condition under which, if all of i's opponents are using an increasing strategy, i has a best response in increasing strategies. A strength of Athey's result is that it does not rest on private values. It does however, require a single dimensional type space with something akin to MLRP (see again Example 3). Athey provides a number of examples of auctions in which her conditions are satisfied.

Finally, Ye [28] has some interesting work on existence in double auctions. Unfortunately, he is unable to rule out the no-trade equilibrium. Our belief is that our positive trade result along with his derivation constitute an alternative proof of existence in the class of double auctions he studies, one that does not rely on the machinery used here.

Section 2 presents an example with no sensible equilibrium under standard tie breaking. The example is not covered by the previous literature on existence of equilibria even though it has a very natural structure. We show how, in this example, the limit of behavior in games with finite approximations on the bid space implies type dependent tie breaking in the game with continuum strategy space. This leads to the general conjecture that games with endogenous sharing rules have equilibria. Section 3 contains the relevant notation and definitions. Section 4 presents our central theorems about existence with endogenous sharing rules. In Section 5, we show how this result can be used to demonstrate existence of standard equilibria in private value auctions. Section 6 concludes. Two appendices contain proofs.

2 An Example

Consider a two player auction with the following features.

- Utility for player i is a convex combination $at_i + (1-a)q$ of a private value t_i and a common value q.
- The private value t_i takes on value either 0 or 1 with equal probability, and independently of the other parameters of the auction.
- The common value takes on values 0 and v, again with equal probability.
- Bidders each observe a signal taking on three possible values (low, middle, high) about the common value component. Conditional on

> the low signal, a bidder knows that $q = 0$. Conditional on the middle signal, it is equally likely that $q = 0$ or $q = v$ and so this signal is uninformative. Conditional on the high signal the bidder knows that $q = v$. We will refer to bidders who receive either extreme signal as "sure," and bidders who receive the middle signal as "unsure." A bidder has a probability m of being unsure.

When $a = 0$ the setting is one of pure common values, and with $a = 1$ of pure private values. When $1 > a > 0$ there are both private and common components to an agent's valuation.[2] In the situation where $1 > a > 0$, so that both common and private components are important, higher bids by an opponent do not always translate into better news about a player's value. An opponent's bid could reflect information about private value, common value, or both. This invalidates the usual arguments that higher bids translate into better news that underlie constructions of equilibria in the previous auctions literature (e.g., Milgrom and Weber [16], Maskin and Riley [14], or Athey [2]).

Consider a sealed bid Vickrey auction. Each bidder i submits a bid b_i. The high bidder is awarded the object and pays the second highest bid. Ties are broken by a flip of a fair coin.[3]

As usual for second price auctions, there is the trivial asymmetric equilibrium in which every type of player 1 bids $b > v + 1$, while every type of player 2 bids 0. With a variety of weak domination or perfection notions, or an appeal to symmetry, all sure types must bid their value. So, sure types bid in $\{0, a, (1 - a)\, v, a + (1 - a)v\}$. Assume that a and v are such that $a < (1 - a)v$, so that these elements are ordered as indicated. In words, $q = v$ is more important than $t_i = 1$ in terms of payoffs. Assume further that $a < v/2$.

Consider the bids of unsure types for whom $t_i = 0$. A bid above $(1 - a)v$ can be ruled out by weak dominance, since $(1 - a)v$ is the maximum possible value of the object for a bidder with $t_i = 0$. A bid of $(1 - a)v$ sometimes wins

[2]Various extensions to the example are possible. See the working paper by Jackson [9]. See Jackson [8] and Pesendorfer and Swinkels [18] and for analyses of efficiency properties of auctions where such preferences are assumed.

[3]In a three person version of this game, the non-existence result here extends to any tie breaking rule that does not use private information. So, this example fails existence even with SZ style endogenous sharing. For further details of this and other aspects of this example, see [9].

when $q = 0$ (since opponent types who have $t_i = 1$ and know $q = 0$ bid a) and so such a bid results in a negative expected utility. A bid in $[a, (1-a)\,v)$ wins in only two circumstances: either the opponent is also unsure, in which case the object has expected utility $v/2$, or the opponent is sure that $q = 0$ (and has $t = 1$) in which case the utility is 0. For m small enough, the later event is much more likely than the former, and so these bids also earn negative expected utility. So, unsure types for whom $t_i = 0$ must bid in $[0, a)$. But over this range, anytime a bidder wins, either the other player is also unsure, and so the object has expected utility $v/2 > a$, or the other player is sure, in which case the price is 0. So, if one is winning with probability less than one in the event that the other player is also unsure, one has an incentive to raise ones bid closer to (but not equal to!) a. Since both players can't be winning with probability one conditional on their opponent being unsure and having $t_i = 0$, there is no equilibrium.

Essentially, we have an openness problem: there is no "largest bid less than a." And, if one considers a finite grid of bids, there is a perfectly sensible equilibrium. To simplify, consider modifying the example so that unsure types have $t_i = 0$ always. Then, there is an equilibrium of the auction with a finite grid of bids where sure types bid at the grid points immediately below a, $(1-a)v$, and $a + (1-a)v$ and unsure types bid b^*, where b^* is the second highest grid point *below* a.[4] Unsure types do not want to lower their bid, because at b^*, both bidders strictly prefer to win. They do not want to raise their bid because winning at the grid point below a involves a disastrous winner's curse.

This example illustrates why existence results based on finite approximations are problematic for discontinuous Bayesian games. While the above is an equilibrium for each finite grid, the limit of these behaviors is not an equilibrium of the auction with continuum bid space. In particular, in the limit, both unsure types (with $t_i = 0$) and sure types who know $q = 0$ and have $t_i = 1$ bid a, and hence the allocation changes to one where unsure types sometimes win against sure types.

Consider the following modification to the auction with continuous bidding spaces. In the event of a tie, the players simultaneously announce their signal and type. In this example, only the announced signal will turn out to matter. If both bidders announce that they are sure or both unsure, the ob-

[4] When one includes unsure types with $t_i = 1$, the equilibrium is as described for the other types, while with $t_i = 1$, unsure bidders follow a mixed strategy.

ject is randomly assigned. If one player announces she is sure and the other player announces that he is unsure, then the object goes to the player who said she was sure. In this limit game, it is an equilibrium for unsure types to bid a, and for all players to announce their types truthfully in the event of a tie. The only time (given equilibrium play) that the announcements matter is when there is a tie at a. In this event, the announcement game has the effect of recreating what was going on in the limit of the auction with finite bid spaces. The sure player is given the object and the unsure player avoids the selection problem. And, any allocation that a player can achieve by a false announcement in the continuum game is one he could have come arbitrarily close to achieving by deviating late in the sequence of finite games, and so players will not wish to make false announcements.

Generously interpreted, this example says that while (a non-weakly dominated) equilibrium does not exist in this auction with the standard tie breaking rule, such an equilibrium exists if players can "talk" in the event of a tie.

The heart of this paper is to show that this construction is general. Consider a Bayesian game satisfying some regularity conditions on beliefs, payoffs, and strategy spaces, but allowing for discontinuities, and more generally, ambiguity at some points in the strategy space about the allocation. Fix a sharing rule at points of ambiguity (so, in an auction, specify a tie breaking rule). For each finite grid on strategies, there is an equilibrium with this sharing rule. Now, consider the limit of this equilibrium as the grid becomes fine. We present an augmented game differing from the original game only in that players make announcements whose sole use is to affect the manner in which any given ambiguity in outcome is resolved. This augmented game always has an equilibrium where announcements are incentive compatible and which replicates what was happening in the limit of the games with increasingly fine finite strategy spaces. Moreover, we can find such an equilibrium that satisfies a perfection notion.

3 Model and Definitions

In this section, we introduce a class of Bayesian games that allows for discontinuities in payoffs. Obvious examples include auctions, wars of attrition (essentially all pay auctions), and various research and development models. We then show how to extend these games to include a type announcement and discuss the various concepts that will be useful in what follows.

3.1 A Class of Bayesian Games

Players

$N = \{1, \ldots, n\}$ is the set of *players*.

Types

Θ_i is the set of *types* for player i, a compact metric space. $\Theta = \Theta_1 \times \cdots \times \Theta_n$.

Uncertainty

P is a (Borel) probability measure on Θ that describes the uncertainty over types. The marginal of P on Θ_i is P_i. We assume that P is absolutely continuous with respect to $P_1 \times \cdots \times P_n$, with continuous Radon-Nikodym derivative $f(\theta)$.

This condition is satisfied automatically if Θ is finite or if types are independent, but is nonetheless restrictive. To see a case that is ruled out, let $\Theta_1 = \Theta_2 = [0, 1]$ and let P be uniform on (θ, θ). Thus, types are perfectly correlated, and so P is not absolutely continuous with respect to $P_1 \times P_2$, the uniform distribution on $[0, 1]^2$. On the other hand, types can be "almost perfectly correlated" in the sense that they could be distributed uniformly on an arbitrarily small neighborhood of (θ, θ).

Strategies

S_i is the set of *pure strategies* for player i, a compact metric space. $S = S_1 \times \cdots \times S_n$.

Outcomes

$O : S \rightarrow\rightarrow [0, 1]^K$ is the *outcome correspondence*. O is assumed to be upper hemi-continuous and non-empty, convex and compact valued.

Utility functions

$u_i : [0, 1]^K \times S \times \Theta \rightarrow [0, 1]$ is the utility function for player i. u_i is continuous for every i.

When we refer to *a game* in the sequel, we mean one with the above features.

There is an extra layer of definition here relative to standard definitions of games, as we define an outcome correspondence and have payoffs depend on outcomes and strategies rather than simply on strategies. This is consistent with our interest in existence of equilibrium in games such as auctions, where tie breaking involves decisions over allocations of objects and payments and only implicitly over utilities. The auctioneer often cannot observe the true preferences of the players and can only define sharing rules in terms of the allocations. In equilibrium allocations implicitly determine utilities.

Examples

The following are three examples of games meeting our conditions. Example 1 is provided simply to see the formalization in action for a familiar example. Example 2 is another example (along with the one of Section 2) in which existence without endogenous tie breaking fails and so existing existence theorems do not apply, but to which ours does. Example 3 is a game to which no existing existence theorem applies, but which falls into the class for which we show existence (with standard tie breaking) in Section 5. See that section for a number of examples for which our techniques allow one to prove the existence of standard equilibria in private value auctions.

Example 1 Consider a standard single unit private value auction. Player i has value θ_i for the object, and submits bid s_i. The various θ_i are *i.i.d.*

In this example, the outcome $O(s)$ can be interpreted as specifying the probability with which the object is given to each player as a function of bids. So, $o \in O(s)$ is an n-dimensional vector $o \in [0,1]^n$ such that $\sum o_i \leq 1$ (allowing for the possibility of a reserve price so that the object is not given away). $O(s)$ is singleton valued whenever there is a unique highest bid $s_i > s_j$ for each $j \neq i$, and then if s_i is at least the reserve price $o_i = 1$ and $o_j = 0$ for $j \neq i$. To make O upper hemi-continuous we allow that in the event of a tie at the highest bid, the allocation can be made with any probability among the tied bidders.

Note that O does *not* specify payments or values for the object, simply who gets it. Player i's utility function can incorporate the payment and valuation. So $u_i(o, s, \theta) = o_i (\theta_i - s_i)$. Note that u_i is continuous in o, θ and s. The only discontinuity is of o in s.

Example 2 Maskin and Riley [14] consider the following example of an auction with negative dependence on types. Two buyers have type either 0 (probability 2/3) or 1 (probability 1/3), drawn independently. Player i's utility for the object is $3 + s_i - 2s_j$. A first price sealed bid auction is held.

Maskin and Riley show that there is no equilibrium for this auction, even if ties are broken by holding a subsequent Vickrey auction among the tied bidders. As established in Theorem 1 below, there does exist an equilibrium of this auction if tie-breaking is endogenous.

Example 3 Consider a two player private value first price auction. Values are uniformly distributed over $\{(v_1, v_2)|v_1 \geq 0, v_2 \geq 0, v_1 + v_2 < 12\}$. This game has no Nash equilibrium in which bids are increasing in values. To see why not, assume such an equilibrium exists. Suppose $b_2(v) > v$ for some v. Then, since there is a positive probability conditional on $v_2 = v$ that $v_1 < v$, one of players 1 and 2 is with positive probability winning with a bid above value, and so has a profitable deviation. Hence, $b_2(v) \leq v$ for all v. Hence, when $v_1 = 12-\varepsilon$,and so $v_2 \leq \varepsilon$, $b_2 \leq \varepsilon$. So, $b_1(12-\varepsilon) \leq \varepsilon$. As this holds for all $\varepsilon > 0$, the only possible equilibrium in increasing strategies is $b_1(v) = b_2(v) = 0$ for all v, which is clearly absurd.

As we show in Section 5 this game has an equilibrium. However, because this equilibrium is not in non-decreasing strategies, neither the techniques of Reny nor of Athey can be made to apply here. And, while this example is stylized for expositional simplicity, the broader point remains valid. When values are anything but affiliated, existing methods of analysis are in trouble.

3.2 Bayesian Games With Endogenous Tie Breaking

Augmented Strategies

Let $Z_i = S_i \times \Theta_i$. Players choose a strategy in the Bayesian game and make a (possibly false) announcement of their own type.

Sharing Rules

A *sharing rule* is a function $o : Z \to [0, 1]^K$ such that $o(s, \theta) \in O(s)$ for each $(s, \theta) \in Z$.

While true types are unobservable, we allow the mechanism to ask for announcements of types and the sharing rule to depend on these announcements. Note that the announcement of types does not affect which outcomes are feasible, but may be used in the selection of an outcome.

Distributional Strategies

A *distributional strategy* for player i is a probability measure m_i on $Z_i \times \Theta_i$ with marginal P_i on Θ_i.

See Milgrom and Weber [16] for details.

Equilibrium

Following Milgrom and Weber [16], we can define the payoff to player i as a function of the profile of distributional strategies $(m_1, \ldots, m_n)$ and the sharing rule o as

$$\pi_i(o, m_1, \ldots, m_n) = \int u_i(o(z), s, \theta) dm_1(z_1|\theta_1) \ldots dm_n(z_n, |\theta_n) dP(\theta).$$

The pair m, o form an *equilibrium* if

$$\pi_i(o, m_1, \ldots, m_n) \geq \pi_i(o, m_{-i}, \widehat{m}_i)$$

for each i and distributional strategy $\widehat{m}_i$ for player i.

An equilibrium is thus a Bayesian equilibrium where the allocation rule has been endogenously determined.

We will prove the existence of equilibria where the announcement of types is truthful and thus where o is incentive compatible in the following sense.

Incentive Compatibility

Let $D = \{(s, \theta', \theta) \in Z \times \Theta \mid \theta' = \theta)\}$ and $D_i = \{(s_i, \theta'_i, \theta_i) \in Z_i \times \Theta_i \mid \theta'_i = \theta_i)\}$ be the diagonals of $Z \times \Theta$ and $Z_i \times \Theta_i$. Thus, these are the sets where announced types correspond to true types.

An equilibrium m, o has an *incentive compatible* sharing rule if $m_i(D_i) = 1$ for each i (or equivalently $m(D) = 1$).

The fact that incentive compatibility of o is defined relative to an equilibrium m is important. A given type is interested in inferring the types of other players (given that u_i can depend on θ_{-i} as well as θ_i). This inference, of course, depends on the distributional strategies being played.

For the actual operation of the game, the sharing rule can be run at a stage *after* s has been submitted. So players can be asked to announce their types only in situations where a realized s has led to the necessity of selecting an outcome or breaking a tie.

Tie-Breaking

For certain results about measurability, we refer to a class of games where O is derived from tie-breaking.

Let us write outcome o as a vector $(o_1, \ldots, o_n) \in [0, 1]^K$, where $u_i(o, s, \theta)$ depends only on o_i, s, θ. (Taken alone, this is without loss of generality, simply by taking the n-fold product of o and letting K be nK' where K' was the original dimension of o. However, it is restrictive in conjunction with what follows.)

An outcome correspondence O is *derived from tie breaking* if for every s, $o \in O(s)$, $o' \in O(s)$ and $o'_i \neq o_i$ imply that $s_i = s_j$ for some $j \neq i$.

Thus, an outcome correspondence is derived from tie-breaking if there is only discretion in i's outcome when i has submitted the same strategy as some other individual. For example, in an auction where each bidder can buy at most one object, *if i's bid has not tied with any other bid then i has either won an object or not. Only when i's bid is exactly tied with some other bid is there ambiguity about whether i should receive an object.

3.3 Weakly Perfect Equilibria[5]

As discussed in Section 2, it is important to exclude degenerate equilibria such as the one in a Vickrey auction where one bidder bids an amount greater than any possible value, and other bidders all bid zero. Such equilibria can be ruled out by domination arguments. However, looking for equilibrium in undominated strategies can be a problem with continuous action spaces. To see why, consider the following example.

Example 4 Consider a complete information, two person, first price auction in which both players have value $v = 1$ with probability 1. This auction has a unique Nash equilibrium (regardless of the tie breaking rule) where both players bid 1. These strategies always lead to a payoff of 0 and so are weakly dominated by any lower bid.

This equilibrium is quite natural (it is equivalent to a Bertrand equilibrium) and one that we wish to admit in an existence result. So, we cannot simply look for equilibria where no player ever plays a weakly dominated strategy. However, note that in the example, the strategies are the limit of ones nearby that are not weakly dominated.

In order to establish a general existence result, we need to allow for such situations and consider a notion of equilibrium that rules out "clearly" dominated behavior, but not behavior that might be the limit of reasonable play. The following extension of Selten's trembling hand perfect equilibrium concept [21] accomplishes this.

[5]Most of what follows can be absorbed without a detailed understanding of weak perfection. The reader interested in getting to the heart of the paper can thus skip this section, at least on a first reading.

Consider a finite grid S_i^* of strategies for each i. Given $\varepsilon \geq 0$, let

$$C_i^\varepsilon(S_i^*) = \{m_i \mid m_i(\{s_i\} \times E) \geq \varepsilon P_i(E)\ \forall s_i \in S_i^* \text{ and closed } E \subseteq \Theta_i\}.$$

Thus $C_i^\varepsilon(S_i^*)$ is the set of distributional strategies for i for which (almost) every type plays every strategy in S_i^* with probability at least ε.

We say that o, m is an *ε-constrained equilibrium* of the game with strategy space S^* if $m_i \in C_i^\varepsilon$ for each i and $\pi_i(o, m) \geq \pi_i(o, m_{-i}, \widehat{m}_i)$ for every $\widehat{m}_i \in C_i^\varepsilon$.

We say that o, m is *a weakly perfect equilibrium* (of the game with strategy space S) if it is an equilibrium, and there exists a sequence of finite grids $\{S^r\} \to S$ (in Hausdorff distance), a sequence $\varepsilon^r \to 0$, and a sequence $\{o^r, m^r\}$ of ε^r-constrained equilibria of the game with strategy space S^r, such that m^r weakly converges to m and $o^r m^r$ weakly converges to om.[6]

Thus, a weakly perfect equilibrium is an equilibrium that is the limit of a sequence of equilibria of approximating constrained games, where the constrained games have finite strategy spaces and interior mixed strategies.

If S_i is finite for each i and Θ_i is a singleton, then this is trembling hand perfection. The added complications come from handling the continuum strategy and type spaces.

In finite games, a weakly perfect equilibrium uses only strategies that are not weakly dominated. With a continuum of pure strategies, however, weak perfection does not rule out the use of weakly dominated strategies. This is easily checked with regard to Example 3 of a complete information auction where the unique equilibrium is for both players to bid 1. One might conjecture that the definition of weak perfection would allow for strategies that are weakly dominated provided they are the limit of undominated strategies, but would rule out strategies that are not the limit of undominated strategies. This conjecture (and in fact even weaker conjectures) are false. To see why, consider the following example.

Example 5 There are two players with strategy spaces $S_i = [0, 1]$. There is complete information, so Θ is a singleton. We suppress reference to θ for the remainder of the example. The outcome space is $[0, 1]$. Preferences are represented by $u_1(o, s) = o$ and $u_2(o, s) = s_2$. Thus, player 2 prefers to play higher strategies, regardless of what player 1 does or what the outcome is,

[6]Here, $o^r m^r$ denotes the measure defined by $\nu^r(E) = \int_E o^r(z) dm^r(z, \theta)$ where $E \subset Z \times \Theta$ is Borel.

while player 1 only cares about the outcome. The outcome correspondence is given by $O(s) = [0, 1]$ if $s_1 = s_2 = 0$ and $O(s) = \{0\}$ otherwise.

Consider the selection o where $o(s) = 1$ if $s_1 = s_2 = 0$, and the strategy m, where m has each player choose 1 with probability 1. Here, o, m is a weakly perfect equilibrium. To see this, set the grids $S_i^r = \{1/r, 2/r, ..., (r-1)/r\}$ and let $\varepsilon^r = 1/r^2$. Let $o^r = o$ for each r, and let m_i^r put ε^r weight on each strategy other than $(r-1)/r$ and the remaining weight on $(r-1)/r$. This is an equilibrium of the ε^r constrained game, and so o, m is weakly perfect. This is despite the fact that player 1's only undominated strategy is $s_1 = 0$.

The fact that $s_1 = 0$ is not identified is due to the fact that it is a single strategy (at a point of discontinuity) out of a continuum, that only performs better than other strategies against a single strategy of player 2. In order to push player 1 to play $s_1 = 0$, one would need a notion of perfection that was much stronger, involving, e.g., a continuum of approximating games.[7] However, such a strengthening would lead to non-existence in situations like Example 4.

Despite the fact that weak perfection does not rule out the use of certain dominated strategies in Examples 4 and 5, it turns out that many weakly dominated strategies are eliminated by weak perfection. In particular, assume that (i) each pure strategy in a closed neighborhood $E \subseteq S_i$ is weakly dominated for a given type θ (ii) a weakly dominating strategy exists for each sufficiently fine grid on strategies, and (iii) the set of strategies for the opponents on which strategies in E perform strictly worse than their dominator is open. Then, no strategy on the interior of E will be used by i in a weakly perfect equilibrium. Conditions (ii) and (iii) guarantee that for fine grids, payoffs under the perturbed games generated by weak perfection will reflect the weak dominance of elements in E. Condition (i) takes care of the fact that convergence is weak.

So, in the continuum, strategies that are "robustly" weakly dominated, in the sense of (i)-(iii) above, are ruled out by weak perfection. This eliminates the degenerate equilibrium in the Vickrey auction, without eliminating the natural equilibrium in Example 3. To see a proof that relies on these features, see the proof of Lemma 4.

[7] We could build a richer version of the example in which $O(s)$ blows up at an uncountable number of places (say whenever $s_1 = s_2$). No choice of the approximating grid would then take care of the issue.

4 Existence of Equilibrium

In this section, we establish our main result: augmented games have equilibria. To establish this for the case of infinite type spaces we use an extra condition.

Payoffs that are Affine in Outcomes

Payoffs are affine in outcomes if for each i there exist continuous functions $v_i : S \times \Theta \to [0,1]^K$ and $w_i : S \times \Theta \to [0,1]$ such that

$$u_i(o, s, \theta) = v_i(s, \theta) \cdot o + w_i(s, \theta).$$

In auctions, where outcomes may be thought of as probabilities that an agent is awarded an object or objects, payoffs are naturally affine in outcomes. So, for example, recall that in Example 1, $u_i(o, s, \theta) = o_i(\theta_i - s_i)$, so $v_i(s, \theta) = \theta_i - s_i$. Note that, because von Neumann-Morgenstern utility functions are linear in probabilities, affineness does not rule out risk aversion (or risk loving) on the part of bidders: we could equally well have taken $u_i(o, s, \theta) = o_i U(\theta_i - s_i)$ for any continuous U. Because the functions v_i and w_i may depend on s, many relevant applications have a similarly natural interpretation of the outcomes as probabilities over different configurations of transactions.

The condition that payoffs are affine is central to the proof. As we use finite approximations of the strategy sets to get existence in the continuum strategy game, we need to keep track of payoffs at the limit to make sure that we are selecting outcomes correctly at strategy combinations where $O(s)$ is multi-valued. Without the affine assumption, it is possible to start with some simple exogenous selection from O, $\overline{o}$, have a sequence of equilibria $\overline{o}, m^r$ of the finite approximations, so that $\overline{o}m^r$ weakly converge to some limit om, and yet have the corresponding utilities fail to converge to the limit utilities.[8] If utility functions are affine in outcomes (and continuous

[8] To see a simple example: let $S = [0,1]$, and let m^r place $1/2$ weight on $s = 1/2 - 1/r$ and $1/2$ weight on $s = 1/2 + 1/r$. So m^r weakly converges to m, where m puts probability 1 on $s = 1/2$. Let $\overline{o}(s) = 0$ if $s < 1/2$, $\overline{o}(s) = 1/2$ if $s = 1/2$, and $\overline{o}(s) = 1$ if $s > 1/2$. However, let $u(o) = \sqrt{o}$. Then $\int u(\overline{o}(s))dm^r = 1/2$ fails to converge to $\int u(o(s))dm = \sqrt{1/2}$. The role of affiness is seen by noting that it is the case that $\int v(s)\overline{o}(s)dm^r$ converges to $\int v(s)\overline{o}(s)dm$ for any continuous v.

in other variables) then weak convergence of $\bar{o}m^r$ implies the corresponding convergence of utilities.

Theorem 1 *Every game with payoffs that are affine in outcomes has a weakly perfect equilibrium with an incentive compatible sharing rule. If O is derived from tie-breaking, then the sharing rule can be chosen to be Borel measurable.*

We establish Borel measurability only in the special setting of tie-breaking where the discontinuities in the outcome correspondence come from ties in the strategies. While this is a restrictive assumption, it allows for an easy proof of measurability.[9]

Let us first outline how Theorem 1 is proved and where the problem arises. We then explain why the tie-breaking condition solves it.

The outcome selection o in the equilibrium of the augmented game is determined from the limit of a sequence of equilibria of games with a finite grid of strategies. The difficulty is that this limit is only determined by weak convergence. So, outcomes are not completely tied down at the limit. And, if the outcome for even a single strategy z is misspecified, then players may wish to deviate from the proposed equilibrium. Thus, one has to identify such problem z's and correct o at those points. We correct the outcome selection o at each problem point z by reconstructing the limit of what the finite game specifies at z. However, even though the set of problem points is Borel, there is no guarantee that this pointwise correction does not create a selection that is not Borel measurable.

Under the condition that O is derived from tie breaking, problem points arise only at tied strategies. The belief of a given player regarding other player's strategies can have at most a countable set of atoms, and only at those atoms can there be a problem. Thus, we end up patching together a countable set of measurable functions

While Borel measurability is desirable, even when o is not Borel measurable it is still integrable with respect to the equilibrium strategies and can be evaluated with respect to pure strategy deviations. Also, there is no difficulty implementing the allocation rule, since this can be done after the type announcements have been made. However, (and we thank Simon and Zame

[9]Simon and Zame [24] establish Borel measurability of the allocation rule without requiring that O is derived from tie-breaking. We understand that their result is based on a new theorem on measurable selections from correspondences.

for pointing this out), there is a question whether it is enough to check pure strategy deviations in a setting where the payoffs to some mixed strategy deviations are potentially undefined.

To this, we note the following. While the integral over payoffs following a mixed strategy deviation may be undefined, payoffs are, point by point, less than or equal to the payoffs to the equilibrium strategy. With nonmeasurable sets, there exist pairs of lotteries over which it is unclear that an agent should have a well defined preference ordering. But, it seems clear that an agent should weakly prefer a lottery which gives him 1 always over one which gives him at most 1 and sometimes less, even if the set of events on which it gives less than 1 is not measurable. Essentially, the second lottery is weakly dominated in its realizations.

In Section 5 we show that in a broad class of private value auctions there is no need to worry about the selection of the outcome function in any case.

Games with Finite Type Spaces

Consider a game where Θ is finite and let $T = \#\Theta$. Let $U : S \rightarrow\rightarrow [0,1]^{NT}$ be defined by

$$U(s) = \{u \in [0,1]^{NT} \mid \exists o \in O(s) \text{ s.t. } u = (u_i(o,s,\theta))_{i \in N, \theta \in \Theta}\}.$$

$U(s)$ is the set of vectors of utility realizations (across i and θ) corresponding to the outcomes $o \in O(s)$. Given the continuity of u_i, it follows from the properties of O that U is a compact valued and upper hemi-continuous correspondence.

Theorem 2 *Every game with a finite Θ and convex-valued U has a weakly perfect equilibrium with an incentive compatible sharing rule. If O is derived from tie-breaking, then the sharing rule can be chosen to be Borel measurable.*

The strengthening of Theorem 2 relative to Theorem 1 is that it does not require payoffs to be affine in outcomes, but only that U is convex-valued.

Symmetric Games

A game is symmetric if $\Theta_i = \Theta_j$, $S_i = S_j$, $u_i = u_j$, for each i and j, P is symmetric, and O is symmetric.[10]

[10] Given a permutation (bijection) $\sigma : N \rightarrow N$, for any n-dimensional vector x let x^σ be such that $x^\sigma_{\sigma(i)} = x_i$; and for $E \subset \Theta$, let $E^\sigma = \{\theta^\sigma | \theta \in E\}$. P is symmetric if for each E and permutation σ, $P(E) = P(E^\sigma)$. O is symmetric if for each permutation σ, $o \in O(s)$ implies $o^\sigma \in O(s^\sigma)$. When we say $u_i = u_j$ we mean that $u_i(o,s,\theta) = u_j(o^{\sigma_{ij}}, s^{\sigma_{ij}}, \theta^{\sigma_{ij}})$ for all o, s, θ, where the permutation σ_{ij} switches only i and j.

An equilibrium is *symmetric* if $m_i = m_j$ for each i and j and o is symmetric.

Theorem 3 *Every symmetric game with payoffs that are affine in outcomes, or a finite type space and U that is convex-valued, has a symmetric weakly perfect equilibrium with an incentive compatible sharing rule. If O is derived from tie-breaking, then the sharing rule can be selected to be Borel measurable.*

The results in the sequel can be extended to symmetric equilibrium in the same manner, when the underlying auctions are symmetric.

5 Existence in Private Value Auctions

In this section, we show that our results showing the existence of equilibrium with endogenous sharing rules imply that for a large class of private value auction there exist equilibria with any standard tie breaking rule (such as an equally weighted randomization across tied bids).

5.1 A General Auction Model

We begin by presenting a general model of private value auctions. The model includes auctions in which both demands and supplies may be for multiple units, values may be correlated (but not perfectly) and in which the supply and number of active bidder is stochastic.[11] In addition, the model is broad enough to include double auctions, all pay auctions, auctions in which players are treated asymmetrically, and combinations thereof.

Players and Endowments

There are $n+1$ players. There exists $m < \infty$ such that each player $i \in \{0, 1, \ldots, n\}$ has an endowment of $e_i \in \{0, 1, \ldots, m\}$ indivisible objects. Objects are identical. Let $e = (e_0, e_1, \ldots, e_n)$.

Valuations

Player 0 is non-strategic (which allows for an exogenous supply of objects to be auctioned, as for example in a one sided auction) and has no value for

[11] So the model subsumes the auctions for which Swinkels [25], [26] establishes asymptotic efficiency results, except that it does not allow for atoms in values.

the objects. Each player $i \geq 1$ desires at most m objects, and has valuations which are represented by $v_i = (v_{i1}, \ldots, v_{im})$. The interpretation is that i has marginal value v_{ih} for an h^{th} object, and so receives value $\sum_{h=1}^{H} v_{ih}$ from having H objects. Players have non-increasing marginal valuations so that $v_{ih} \geq v_{i,h+1}$ for each $h < m$. Let $v = (v_1, \ldots, v_n)$.

For $h \leq e_i$, say that v_{ih} is a *sell value.* For $h > e_i$, say that v_{ih} is a *buy value.* We assume that there is $\bar{v} < \infty$ such $v_{ih} \in [-1, \bar{v}]$ for buy values, and $v_{ih} \in [0, \bar{v}+1]$ for sell values. The different supports of buy and sell values allow a player to be uninterested in selling or buying beyond any given number of units.[12]

Types and Uncertainty

A player's type θ_i is e_i, v_i. The vector (e, v) lies in $\Theta \equiv \{0, 1, \ldots, m\}^{n+1} \times [-1, \bar{v}+1]^{mn}$, with the restriction that buy values are in $[-1, \bar{v}]$, sell values are in $[0, \bar{v}+1]$, and marginal values are non-increasing. The vector (e, v) is drawn according to a probability measure P on Θ. The marginal of P onto (e_i, v_i) is P_i. P_0 is the marginal of P onto e_0. As before, it is assumed that P is absolutely continuous with respect to $\prod_{i=1}^{n} P_i$, with continuous Radon-Nikodym derivative $f(\theta)$. This puts no restriction on how i's endowment and various values are related, but does impose that the various player's endowments and values are "not too dependent." In particular, note well that it does not impose any sort of affiliation among different player's values. So, for example, Example 3 is within our setting.

Values are *atomless* in the sense that $P_i(\{v_{ih} = a\}) = 0$ for all i, h and a. This rules out that there is a positive probability that player i values an object at some particular a. This is stronger than assuming that P_i is atomless as we are assuming that no component h has an atom. However, the assumption does not put any other restrictions on P_i. So for, example, P_i need not satisfy any sort of full support condition and could have arbitrary correlation structures among the various v_{ih}, and between values and endowments.

Strategies

Player 0 announces a reserve price vector $b_0 = (b_{01}, \ldots, b_{0m}) \in [-1, \bar{v}+1]$ (arranged in non-increasing order), where $\bar{v}+1 \geq b_{0h} \geq 0$ for $h \in \{1, \ldots, e_i\}$ and where $b_{0h} = -1$ for $h > e_i$ (which will imply that 0 cannot be a net buyer) before the other players bid. Essentially, 0 will not part with his

[12]So, for example, m can be chosen very large to allow player 0 to have a large supply available for sale, but this in no way imposes that any given player is interested in buying more that say 2 objects. Values v_{ih} for $h > 2$ can be packed in $[-1, 0)$.

h^{th} object unless he receives at least b_{0h} for it. Each player $i \in \{1, \ldots, n\}$ observes (e_i, v_i) and the reserve price vector.[13] Player i then submits $b_i = (b_{i1}, \ldots, b_{im}) \in [-1, \bar{v}+1]$. Player i's bids are arranged in descending order, $b_{ih} \geq b_{i,h+1}$, $h = 1, 2, \ldots, m-1$. Let $\mathcal{B}$ be the set of such bids. Players other than 0 bid simultaneously.

Transformed Bids and the Allocation of Objects

To allow for auctions that treat players asymmetrically, we work with transformed bids instead of the bids themselves in determining the winners. This allows for auctions such as the PCS auctions (see, e.g., Cramton [5]) which subsidized bids by minority owned firms.

For each $i \in \{0, 1, \ldots, n\}$, let $g_i(b_i) \equiv (g_{i1}(b_{i1}), \ldots, g_{im}(b_{im}))$, where g_i maps $\mathcal{B}$ into $\mathcal{B}$. Also, let each g_{ih} be a continuous and strictly increasing function, mapping $[0, \bar{v}]$ onto $[0, \bar{v}]$.[14] For auctions which treat all players identically, g_{ih} can be taken to be the identity for all i and h.

An allocation is a specification of the probabilities that each bidder receives various numbers of objects. Before giving a formal definition, let us verbally describe how objects may be allocated. We consider any allocation rule for the base auction that does not rely on private information and such that (a) if $g_{ih}(b_{ih}) > g_{jh'}(b_{jh'})$, then i gets an h^{th} object before j gets an h'^{th} and (b) the right number of objects are given away.[15] The inclusion of b_0 in this construction allows for reserve prices. So, in auctions that treat players symmetrically, one object is awarded to each player (in the case of a seller, being "awarded" an object means that she does not sell that object) for each bid she made that is unambiguously among the $\sum_{i=0}^{n} e_i$ highest. In the event that for some b, there are less than k bids above b but more than k bids of b, some form of randomization takes place. There are several interpretations of what "fair" randomization might mean. For instance, as in Swinkels [25] one could allocate an object with equal probability to each player with an unfilled bid at b and repeat until the objects are gone. Another sensible procedure is

[13] Secret reserve prices are handled by having player 1 have the only positive endowment.

[14] To handle situations where bids are subsidized by a given proportion or addition, we can work with a subset of the strategy space having such subsidies and set the strategy space large enough relative to the support of values so that we can still have $f_{ih}(0) = 0$ and $f_{ih}(\bar{v}) = \bar{v}$.

[15] In many real world situations, such as the PCS auctions, the actual implementation is that if one bids x, one pays based on some percentage of x. We instead have payments based on x, but whether one wins based on a scaling up of x. This simplifies the development that follows, and comes to the same thing.

to allocate objects with likelihood in proportion to the number of bids at b submitted (as a single bidder may have more than one bid tied at b). For our results, any tie breaking rule which does not depend on private information will work, e.g., giving the objects to the people whose names come earliest in the alphabet.

This is formalized as follows. An allocation is a vector $o = (o_{eih}) \in [0,1]^{(m+1)^{n+1}(n+1)m}$, with the interpretation that o_{eih} is the probability that i receives exactly h objects when the endowment vector is e. The outcome correspondence depends on the (non-augmented) strategy vector b, and is represented as $O(b)$ which is the set of all o such that for every e:

- $o_{eih} > 0 \Rightarrow \sum_{w=h'}^{m} o_{ejw} = 1$ for all j, h' such that $g_{jh'}(b_{jh'}) > g_{ih}(b_{ih})$;

and,

- $\sum_{i=0}^{n} \sum_{h=1}^{m} h o_{eih} = \sum_{i=0}^{n} e_i$.

The first condition states that if j's h'^{th} bid exceed i's h^{th}, and i ever gets an h^{th} object under o, then j always gets at least h' objects. The second is the condition that the right number of objects are allocated (in expectation). In Appendix B, we verify that the outcome correspondence O is upper hemi-continuous, nonempty, and convex and compact valued.

Ties

Fix a selection o from O. Say that i's h^{th} bid is involved in a tie given bid vector b and endowment e if $\sum_{h'=h}^{m} o_{eih'}(b_{-i}, b'_i) = 1$ for any b'_i with $b'_{ih} > b_{ih}$, while $\sum_{h'=h}^{m} o_{eih}(b_{-i}, b'_i) = 0$ for any b'_i with $b'_{ih} < b_{ih}$.

This implies, for example, that there is a tie if there is one object, two bidders, and $g_1(b_1) = g_2(b_2)$. However, there is *not* a tie if there is one object left at $\widehat{b}$, $b_{ih} = b_{ih+1} = \widehat{b}$, but no other player bids at $\widehat{b}$. Nor is there a tie if two players make the same bid, but all the objects are allocated to strictly higher bids or there are enough objects that both players receive an object.

Payments and Utility Functions

The payment a player makes (or receives) depends continuously on the vector of submitted bids and the number of objects won (or sold). The payments are modeled by the continuous and weakly increasing function $C_i = (C_{i0}, C_{i1}, \ldots, C_{im}) : \{0, \ldots, m\}^{n+1} \times \mathcal{B}^{n+1} \to [-\bar{v}, \bar{v}]^{m+1}$, such that i's payment if the bid profile is b, the endowment vector is $e \in \{0, \ldots, m\}^{n+1}$ and he receives h objects is $C_{ih}(e, b)$. Typically, $C_{ih}(e, b)$ will have the sign of $h - e_i$.

Player i has a continuous and strictly increasing von Neumann-Morgenstern utility function U_i over her net payoff.[16] We assume that U_i has a first derivative, and that this derivative is bounded from 0 and ∞, so that in particular, there is $\Lambda < \infty$ such that $U'(x)/U'(y) < \Lambda$ for all x and y. Thus, i's expected utility given an outcome function o, a bid profile b, valuation vector v_i, and endowment profile e is

$$u_i(o, b, v_i, e) = \sum_{h=1}^{m} o_{eih}(b) U_i \left(\sum_{h'=1}^{h} v_{ih'} - C_{ih}(e, b) \right).$$

Given $s = b$ and $\theta_i = (e_i, v_i)$, it is clear that $u_i(o, s, \theta) = u_i(o, b, v_i, e)$ is affine in outcomes.

Payments at Ties

Any time that i's h^{th} bid is involved in a tie at b for endowment e, we require that $C_{ih}(e, b) - C_{i,h-1}(e, b) \leq b_{ih}$ if b_{ih} is a buy bid. That is, if i's h^{th} bid is on the threshold between winning and losing, then i pays at most b_{ih} for an h^{th} object. If b_{ih} is a sell bid involved in a tie, then $C_{ih}(e, b) - C_{i,h-1}(e, b) \geq b_{ih}$. Rewriting this as $(-C_{i,h-1}(e, b)) - (-C_{ih}(e, b)) \geq b_{ih}$, this is the statement that the revenue from selling an h^{th} object is at least b_{ih}.

Existence of Equilibria in Augmented Auctions

Let Φ^A be the set of auctions satisfying the conditions given above. By Theorem 1 each $A \in \Phi^A$ has a weakly perfect equilibrium with an endogenous sharing rule.[17] Given that O is single-valued except at ties, announced types affect the outcome only at ties.

Example 6 The following are some examples of auctions in Φ^A. Unless otherwise stated, $g_{ih}(b_{ih}) = b_{ih}$ for all i and h, tie breaking is standard, and player 0 submits e_0 bids of 0 and $m - e_0$ bids of -1.

(1) A standard first price single unit auction.

[16] To fit with the first part of the paper, note that we could have let U_i map into $[0, 1]$. In particular, this would have required us to rescale and add a constant, which does not affect equilibrium behavior.

[17] As player 0 is non-strategic and announces b_0 before the other players, simply fix 0's strategy space to be a singleton.

$m = 1$. $\Pr(\{e = (1, 0, \ldots, 0)\}) = 1$, $C_{i1}(e, b) = b_{i1}$, $C_{i0}(e_i, b) = 0$, $b_{01} = 0$, $o_i(b) = 0$, for $i \notin W$, where $W \equiv \{i \in \{0, \ldots, n\} | b_{i1} \text{ is maximal}\}$ is the set of bidders who submitted the highest bid. $o_i(s) = 1/\#W$ if $i \in W$.

(2) A standard first price single unit auction with a known reserve price.

As in (1), except $b_{01} = r$, where b_{01} is announced before bids are chosen

(3) A single unit Vickrey auction.

As in (1), except that $C_{i1}(e, b) = b^2$, where b^2 is the second highest bid submitted.

(4) An all pay single unit auction (and thus, various implementations of the war of attrition).

$\Pr(\{e = 1, 0, \ldots, 0\}) = 1$, $C_{i0}(e, b) = C_{i1}(e, b) = b_i$. o_i as above.

(5) An all pay auction in which player 1 has an innate advantage.

$\Pr(\{e = (1, 0, \ldots, 0)\}) = 1$, $C_{i0}(e, b) = b_{i1}$, $C_{i1}(e, b) = b_{i1} + d$, $g_i(b_i) = b_i$, $i \neq 1$, $g_1(b_1) = b_1 + 5$.

This might model, for example, a research and development situation in which firms decide how much to spend, b_i, during the research phase, the winner gets the patent, the winner needs to spend a development cost d, and firm 1 has a lead in the technology under consideration.

(6) A standard double auction.

Players 1 through n_b are potential buyers having $e_i = 0$. Players $n_b + 1$ through $n = n_b + n_s$ are potential sellers having $e_i = 1$. Let $p = \left(b' + b''\right)/2$, where b' is the n_s^{th} highest bid, and b'' the $n_s + 1^{th}$. Then, $C_{i0}(e, b) = -pe_i$, while $C_{i1}(e, b) = p(1 - e_i)$.

(7) A generalized double auction.

Players 0 through n draw a realization of (e_i, v_i), and submit bid vectors. Objects are allocated to the $\sum_{i=0}^{n} e_i$ highest bidders, with ties between buy bids and sell bids broken in favor of buyers (so that trade occurs), and ties among buyers or among sellers broken randomly. Given the allocation, determine who is in the position of a net seller or net buyer. For each net seller, let $\widehat{b}_i$ be formed by replacing the last $m - e_i$ elements of b_i by -1. For each net buyer, let $\widehat{b}_i$ be formed by replacing the first e_i elements of b_i by $\bar{v} + 1$. Let $p = g\left(b', b''\right)$, for some weakly increasing and continuous function such that $b' \geq g(b, b) \geq b$, where b' is the $(\sum_{i=0}^{n} e_i)^{th}$ highest bid given $\widehat{b}$, and b'' the $(\sum_{i=0}^{n} e_i) + 1^{th}$. $C_{ih}(e, b) = p\,(h - e_i)$.

The point of the construction of $\widehat{b}$ from b is to clear the order books of sell bids by net buyers and buy bids by net sellers before setting the price. The problem if a player can act on both sides of the market at once is that a player who is fairly sure he will end up as a net seller may wish to distort his buy bids upward in an attempt to affect the market price. This introduces complexities to the analysis that we have not been able to overcome.

(8) A double discriminatory auction.

Players 0 through n draw a realization of (e_i, v_i), and submit b_i. If a player ends up as a net buyer with h objects, he pays $\sum_{h'=e_i+1}^{h} b_{ih'}$. If i ends up as a net seller, he receives $\sum_{h'=h+1}^{e_i} b_{ih'}$. The auctioneer pockets the difference.

Weak Perfection

We work with weakly perfect equilibria. For each of the examples above, weak perfection implies that (with probability 1) i bids at least v_{ih} for $h \leq e_i$, and at most v_{ih} for $h > e_i$. The precise argument for this turns out to differ depending on the specifics of the auction. In the discriminatory and uniform price auctions, strategies which put weight on bids above value are weakly dominated by a strategy which moves this weight to the value instead. In the all pay auction, and the research and development auction, strategies which put weight above v need *not* be weakly dominated by those which but weight on v instead. So for example, in an all pay auction with $v_{i1} = 11$ and an opponent who always bids 12, 13 is a better bid than 11. In these auctions, bidding 13 *is* weakly dominated by bidding 0. As discussed in Example 6.7, a uniform price double auction in which a player can act on both sides of the market simultaneously may not meet this condition.

So, we will simply assume:

No Dumb Bids (NDB) C is such that it in any weakly perfect equilibrium, with probability 1, i bids at least v_{ih} for $h \leq e_i$, and at most v_{ih} for $h > e_i$.

Under NDB and weak perfection, buy bids are weakly below value and sell bids weakly above value. This implies that $b_{ih} \leq \bar{v}$ for $h > e_i$ and $b_{ih} \geq 0$ for $h \leq e_i$. Hence, a buy bid below 0 by a buyer or a sell bid above $\bar{v}$ means "not interested." In particular, given the assumption that $b_{0h} \geq 0$ for $h \leq e_i$, it follows that are always at least $\sum_{i=0}^{n} e_i$ bids at or above 0. Similarly, there are never more that $\sum_{i=0}^{n} e_i$ bids above $\bar{v}$. So, the market always "clears" somewhere in $[0, \bar{v}]$.

Since NDB is not an easily interpretable condition on primitives, we offer the following lemma which establishes two sets of conditions under which the NDB holds. The first condition is the generalization of what drives the weak domination of bids above value in discriminatory and uniform auctions. Hence, it covers all examples except 6.4 and 6.5. The second, which applies only to one sided auctions in which each player wishes to purchase a single unit, applies to examples 6.4 and 6.5.[18]

Lemma 4 *Suppose that C is such that if i is a net buyer (seller), then his sell bids (buy bids) do not affect his payment. If C satisfies one of*

(a) If b_{ih} is involved in a tie then, $C_{ih}(e, b) - C_{i,h-1}(e, b) = b_{ih}$

(b) $\Pr(e_i = 0,\ i \neq 0,\ v_{ih} < 0, h > 1) = 1$. $C_{i0}(e, b) \geq 0$. $C_{i1}(e, b) \geq b_{i1}$. $C_{i0}(e, b) = C_{i1}(e, b) = 0$ *whenever* $b_{i1} = 0$.

Then NDB is satisfied.

Theorem 5 below states that equilibria will not involve ties between buyers or between sellers (but may involve ties between a buyer and a seller). To see why a condition like NDB matters, consider the following example.

Example 7 Consider a one sided first price auction in which 1's value is uniform on $[2, 3]$, and 2's value is uniform on $[4, 5]$. Then, it is an equilibrium of the augmented game for each player to bid 4 with probability 1, with the object always allocated to player 2. In this equilibrium, ties are probability 1, and it is clear that there is no nearby equilibrium with a standard tie breaking rule. Note further that an equilibrium such as is described here could have

[18] Condition (b) is restrictive because we are not sure what multiple unit all pay auction should look like.

arisen even from the limit of discrete games with standard tie breaking. In particular, let b^r and c^r be the two consecutive bids in approximation r which are just below 4. Then, in the discrete game with standard tie breaking, it is an equilibrium (in weakly dominated strategies) for player 1 to bid b^r and player 2 c^r. This yields as its limit a situation in which both players bid 4, but the tie is always resolved in favor of player 2. So, it is key to our result that we work with some version of weak dominance or perfection.

Although there is no equilibrium near this one with standard tie breaking, there is one if the auctioneer can recognize and favor player 2. While this takes more information than the standard auction (and requires the auctioneer to understand a great deal about the value structure of the players he is facing), it does not require any sort of announcement game such as was needed in our example of a two dimensional auction. So, this example satisfies existence under a game form strictly more complicated than is needed for standard auctions, but less complicated than for our general theorem.

5.2 Analysis of the Auction Model

We are now ready to state our main theorem of this section.

Theorem 5 *Let $A \in \Phi^A$ be a (non-augmented) auction satisfying NDB and having a tie breaking rule o which does not depend on private information and under which ties between buyers and sellers result in trade. Then, A has a weakly perfect equilibrium with the tie breaking rule o.*

The proof of Theorem 5 appears in Appendix B. The idea is as follows. Begin with an auction with a standard exogenous tie-breaking rule. Theorem 1 establishes that there is a weakly perfect equilibrium with an endogenous incentive compatible sharing rule. First, we show that in this equilibrium ties where the object is transferred depending on a type announcement are a zero probability event. This follows in two parts. First, at a tied bid, there is only a single (atomless) valuation for which a player is indifferent between obtaining or not obtaining the object. If there were a positive probability of a tie where a player did not get the object with probability either 0 or 1 for almost every valuation, then the player could improve by increasing or decreasing the bid slightly depending on his value. Second, under NDB players offer at or above their sell values and bid at or below their buy values.

So, if a player is happy to have probability 0 of obtaining an object at a tie then the player must be selling the object, and if the player is happy to have probability 1 of obtaining an object at a tie then the player must be buying the object. Therefore, if there are tied bids that occur with positive probability, they must be between a buyer and a seller with the transfer taking place for certain (which is then not in fact a "tie" under our definition). So, consider changing the allocation rule at ties in the augmented game to the standard tie breaking rule. Given that ties occur with zero probability, it follows that payoffs are unchanged relative to equilibrium strategies. Consider any deviating strategy. If it only uses bids outside of $\widehat{B}_{-i}$ (the set of atoms in the induced distribution over bids by i's opponents) then its payoffs are as before. If it uses bids in $\widehat{B}_{-i}$, then the strategy may now have a different payoff. However, because of the assumption of private values, it turns out that if the strategy is a profitable deviation, then (approximately) the strategy in which buy offers in $\widehat{B}_{-i}$ are moved to slightly higher bids and sell offers to slightly lower offers must also be profitable. But, this altered strategy can be chosen to involve ties with probability zero, and so would also have been a profitable deviation in the original augmented game, a contradiction.

5.3 No Trade Equilibria

There is a major weakness in the existence result as currently stated. In particular, while we have shown existence of an equilibrium, we have not shown that this equilibrium involves trade.[19] Consider, for example, the double auction of Example 6.6. It is a weakly perfect equilibrium in this auction for sellers to always bid $\bar{v}$, and buyers 0. Buyers know that sellers are never making serious bids and sellers know that buyers are never making serious bids, and thus their own non-serious bids are best responses. Weak perfection is satisfied since a seller could imagine a buyer bidding at $\bar{v}$ and a buyer could imagine a seller offering at 0. So, it could be that Theorem 5 has only proven that this degenerate equilibrium exists. We now show that in fact a non-degenerate equilibrium exists.

Consider first the case that the non-strategic player 0 in our auction model is *active* in the sense that $\Pr(\{e_0 > 0\}) > 0$ and that when $e_0 > 0$, 0 sets a reserve price for at least one of his units so that there is a positive probability

[19] We are grateful to Mark Satterthwaite for pointing this out.

that some buyer prefers paying the reserve price to autarchy. Then, there is clearly trade in equilibrium (otherwise the buyer should deviate).

So, let us turn to establishing existence of equilibria involving trade when player 0 is not active. Our next theorem exhibits a class of auctions each of which has at least one weakly perfect equilibrium (with endogenous tie-breaking) involving a positive probability of trade. This implies the existence of a positive trade equilibrium in the associated standard auctions. We use the following assumptions.

Conditions for positive trade

The first assumption we use is that in addition to P being absolutely continuous with respect to $\prod_i P_i$ (with continuous Radon-Nikodym derivative), the converse holds as well.

Somewhat Independent Priors (SIP). $f(\theta) > 0$ for all $\theta \in \Theta$, where f is the Radon-Nikodym derivative of P with respect to $\prod_i P_i$.

Given that f is a continuous function and Θ is compact, $f(\theta) > 0$ implies that there exists $\infty > M > M' > 0$ such that $M > f(\theta) > M'$ for all $\theta \in \Theta$. The new content in SIP, is of course, the existence of M'.

SIP rules out that the support of i's beliefs about his opponents varies in his type, although his beliefs within that support can vary greatly. In the Appendix (see Lemma 9) we show that the fact that types are not perfectly correlated translates into the same feature for bids.

The following condition ensures that there is some possibility of serious competition.

Competition for Gains from Trade (CGT). There is a positive probability of a realization of (e, v) such that either (1) there are $m+1$ buy values above a sell value or (2) there are $m+1$ sell values below a buy value.

In a standard double auction, this simply means that sometimes there are two sellers both of whom have value lower than a single buyer, or vice versa. Hence, we are ruling out the case of a double auction with one buyer and one seller.

Say that an equilibrium is *non-degenerate* if there is a positive probability of trade. That is, there is a positive probability that some player ends up with a number of objects different from his endowment.

Theorem 6 *Each auction in Φ^A that satisfies SIP, CGT, and NDB has a non-degenerate weakly perfect equilibrium.*

The proof is based on a fairly simple idea. Consider a sequence of modified auctions in which we add an $n+1^{th}$ player who $1/t$ of the time has $e_{n+1} = 1$ and makes a sell offer which is uniform on $[0, \bar{v}]$, $1/t$ of the time has $e_{n+1} = 0$ and makes a buy offer which is uniform on $[0, \bar{v}]$ and the remainder of the time has $e_{n+1} = 0$ and makes no bids, where all of this is independent of (e, v). Of course, this rules out no trade as part of an equilibrium since somebody should behave in a way to trade with $n+1$ when he is active. More importantly however, is that no matter how large is t, once some buyers are acting to take account of the possibility of player $n+1$, sellers will change away from the no-trade equilibrium to take account of these buyers. Buyers in turn now have even more of an incentive to make serious offers. The key to the proof is to show that this implies a positive level of trade which is independent of t. But, as $t \to \infty$, this generates a positive trade equilibrium of the original game.

By using the equilibrium established in Theorem 6 as the starting equilibrium in the proof of Theorem 5, and recalling our discussion of auctions in which player 0 is active, we obtain the following corollary.

Corollary 7 *Let $A \in \Phi^A$ be a (non-augmented) auction satisfying NDB and having a tie breaking rule o which does not depend on private information and under which ties between buyers and sellers result in trade. Assume either that player 0 is active (as will be true in any one sided auction), or that SIP and CGT are satisfied. Then, A has a non-degenerate weakly perfect equilibrium with the tie breaking rule o.*

6 Conclusion

An important question for further work is to establish the limits of the type of construction we used in the private values case. That is, how generally does our existence result in augmented games also imply existence in standard games? The example in Section 2 shows that for certain atomic uncertainty structures there are auctions that only have equilibria with endogenous tie-breaking and not with a standard tie-breaking rule. As settings with both private and common components to valuations are quite natural, and as results on limit efficiency of equilibria in such settings (see Pesendorfer and Swinkels [18]) depend on the existence of equilibria with exogenous tie-breaking, it is important to establish such existence. The example in Section 2 has a rather degenerate uncertainty structure. One might then conjecture that this type of auction "typically" has an equilibrium, and that such a result could be proven by the same sort of "ties turn out not to matter" argument used here for the private values case.[20]

In cases where there are no equilibria with standard sharing rules, we are stuck with using incentive compatible endogenous ones. The sharing rule generated in our construction can potentially be very complicated, requiring the game designer to have a deep knowledge of the underlying structure of the game. It would be worth understanding for which classes of games various simple "all purpose" sharing rules suffice. For example, one might, in an auction setting, consider games in which players do not announce types, which may lie in a very complicated space, but rather some information about where they would like to be ranked conditional on being in a tie.

7 Appendix A: Existence Proofs

The following lemma is useful in the proof of Theorem 1.

Lemma 8 *Let T be a compact measure space, and μ^r a sequence of positive measures on T that weakly converge to μ. If E is a closed subset of T, then*

$$\int_E g \ d\mu \geq \text{lim sup}_\text{r} \int_E g \ d\mu^r$$

for any bounded and continuous $g : T \to \mathbb{R}_+$.

Proof of Lemma 8

$$\text{lim sup}_\text{r} \int_E g \ d\mu^r = \text{lim sup}_\text{r} \int (1 - I_{E^C}) g \ d\mu^r,$$

where I_{E^C} is the indicator function of the complement of E, E^C. Since E is closed, it follows that $I_{E^C} g$ is lower semi-continuous. By Lemma 1 in SZ

$$\text{lim inf}_\text{r} \int I_{E^C} g \ d\mu^r \geq \int I_{E^C} g d\mu.$$

[20]See Goeree and Offerman [10] for an interesting example of an auction in this class that does have a well behaved equilibrium.

Thus,

$$\int -I_{E^C} g d\mu \geq \lim \sup\nolimits_{\mathrm{r}} \int -I_{E^C} g \; d\mu^r.$$

Since by weak convergence $\int g \; d\mu^r \to \int g \; d\mu$, it follows that

$$\int (1 - I_{E^C}) g d\mu \geq \lim \sup\nolimits_{\mathrm{r}} \int (1 - I_{E^C}) g \; d\mu^r,$$

which establishes the lemma. ■

Proof of Theorem 1

The proof follows the same general outline as the proof of the main theorem in SZ. The specifics differ significantly as the Bayesian aspect of the game introduces significant complications. We first present the proof of existence, and then discuss the modification to weak perfection.

Let $f = \frac{dP}{d\prod_i P_i}$, which by assumption exists and is continuous. We can replace $dm_1(z_1|\theta_1) \ldots dm_n(z_n, |\theta_n) dP(\theta)$ in the definition of π_i so that

$$\begin{aligned} \pi_i(o, m_1, \ldots, m_n) &= \int u(o(z), s, \theta) dm_1(z_1|\theta_1) \ldots dm_n(z_n, |\theta_n) f(\theta) dP_1(\theta_1) \ldots dP_n(\theta_n) \\ &= \int u(o(z), s, \theta) f(\theta) dm_1(z_1, \theta_1) \ldots dm_n(z_n, \theta_n). \end{aligned}$$

Step 1: In this step, we approximate the game by a finite grid of strategies and find an equilibrium for each finite grid.

Fix a Borel measurable selection $\overline{o}$ from O. For each $r \in \{1, 2, \ldots\}$ pick a finite subset $S_i^r \subset S_i$ such that the Hausdorff distance between S_i^r and S_i is less than $\frac{1}{r}$. Consider the game where the strategy space is S_i^r and the outcome is determined according to $\overline{o}$. By Theorem 1 in Milgrom and Weber [16], this game has an equilibrium $(\overline{m}_1^r, \ldots, \overline{m}_n^r)$ in distributional strategies.

For any $z \in S \times \Theta$, let s_z and θ_z be the strategy and type announcement profile in $S \times \Theta$ specified by z (so $z = (s_z, \theta_z)$). Extend each game described above to the strategy space $Z_i^r = S_i^r \times \Theta_i$, by setting $o(z) = \overline{o}(s_z)$ for any $z \in Z$. Extend $\overline{m}^r$ defined on $S \times \Theta$ to a corresponding measure m^r on $Z \times \Theta$ to have players truthfully announce their types in the strategy z (i.e., $m_i^r(D_i) = 1$ for each i).[21] It follows that m^r is an equilibrium of the

[21] To do this, set $m_i^r(E) = \overline{m}_i^r((E \cap D_i)_{Z_i})$ where $(E \cap D_i)_{Z_i}$ is the projection of $E \cap D_i$ onto Z_i.

game with strategy space Z^r and outcome function o, since the extended o is independent of the announcement of θ_z.

Step 2: In this step we define the limit of the outcomes corresponding to the equilibrium strategies m^r.

Select a subsequence so that m_i^r converges weakly to a distributional strategy m_i on $Z_i \times \Theta_i$ for each i.[22]

Note that $m_i^r(D_i) = 1$ for all i implies that $m_i(D_i) = 1$ for all i (since D_i is closed apply Theorem 2.1 (iii) in Billingsley [4]). Incentive compatibility will be satisfied if we can find $\widehat{o}$ for which $m, \widehat{o}$ is an equilibrium.

For each Borel set $E \subseteq Z \times \Theta$, define

$$\mu^r(E) = \int_E o(z) dm^r(z, \theta).$$

Taking a further subsequence (noting that μ^r is a K dimensional vector-valued measure), μ^r converges weakly to some μ.

Step 3: In this step we find $\widetilde{o}$ so that om^r (i.e., μ^r) weakly converges to $\widetilde{o}m$ (so this implies that $d\mu = \widetilde{o}dm$).

By Lemma 2 in SZ,[23] there is a Borel measurable selection o' as a function of z, θ from O such that for any Borel set $E \subseteq Z \times \Theta$,

$$\mu(E) = \int_E o'(z, \theta) dm(z, \theta).$$

Define $\widetilde{o}$ as a function of z by $\widetilde{o}(z) = o'(s_z, \theta_z, \theta_z)$. Given that $m(D) = 1$, it follows that

$$\mu(E) = \int_E \widetilde{o}(z) dm(z, \theta). \tag{1}$$

Step 4: One would like $\widetilde{o}, m$ to be an equilibrium, as it is defined from limits of equilibria o, m^r. However, since $\widetilde{o}$ was defined by means of weak convergence, it is possible for $\widetilde{o}$ to vary on sets of $m-$measure 0. This is a potential problem, as even having an incorrect outcome for a single strategy

[22] To see that m_i is a distributional strategy, note that since the marginal of m_i^r on Θ_i is P_i, it follows that $\int f dm_i^r = \int f dP_i$ for any bounded continuous $f : \Theta_i \to \mathbb{R}$ and any r. So by weak convergence $\int f dm_i = \int f dP_i$ for any bounded continuous $f : \Theta_i \to \mathbb{R}$. Theorem 1.3 in Billingsley [4] then implies that the marginal of m_i on Θ_i is P_i.

[23] To be careful, write O to be a correspondence O' on $Z \times \Theta$ where $O'(z, \theta) = O(s_z)$ for any (z, θ), and then apply Lemma 2 in SZ.

can attract players to deviate to that strategy. So, we need to identify the z's where $\widetilde{o}$ is misspecified and correct it at those points. H_i is the set of these points, and is constructed as the set of z_i such that a neighborhood of types of player i prefer to deviate to play z_i rather than play according to m_i. (Given the continuity of beliefs and the continuity of u_i in θ, if there are no deviations of this form, then there are no attractive deviations; so these are the only problem points we need to worry about.) The main point of Step 4 is to show that $m_i(H_i) = 0$. In Step 5 this will allow us to correct $\widetilde{o}$ at these points without disturbing the expected utility to the m (which will turn out to be the equilibrium strategies).

As this step is perhaps the most complicated of the proof, let us spend some time explaining it more carefully. On one level, the fact that the set of points where $\widetilde{o}$ is misspecified is of zero measure seems obvious. If there are an m_i-positive measure of strategies in the limit that are improving deviations for i for a positive measure of his types, then, since $\widetilde{o}$ should be tied down on positive measure sets by weak convergence, i should have deviated to an attractive strategy far enough along the sequence. The difficulty is that weak convergence of om^r to $\widetilde{o}m$ does not tell us that $om^r_{-i}\widehat{m}^r_i$ weakly converges to $\widetilde{o}m_{-i}\widehat{m}_i$ for deviating strategies $\widehat{m}^r_i$ weakly converging to a deviation $\widehat{m}^r$. So we need to construct deviations as continuous variations of the original m^r_i and m_i. We show that if H_i is of positive measure, then we can construct improving deviations $\widetilde{m}^r_i$ that converge to a deviation $\widetilde{m}_i$, and such that these deviations can be rewritten as continuous deformations of the corresponding m^r_i's and m_i. Since expectations of continuous functions are well behaved under weak continuity, this then reaches a contradiction.

Let $\overline{N}^{\varepsilon}(\overline{\theta}_i)$ denote the closed ε neighborhood of $\overline{\theta}_i$. Let

$$H^k_i(\overline{\theta}_i) = \left\{ \overline{z}_i \mid \int_{Z_{-i}\times\overline{N}^{1/k}(\overline{\theta}_i)\times\Theta_{-i}} u_i(\widetilde{o}(\overline{z}_i, z_{-i}), s_{\overline{z}_i}, s_{z_{-i}}, \theta) f(\theta) dm_{-i}(z_{-i}, \theta_{-i}) dP_i(\theta_i) \right.$$

$$\left. > \int_{Z\times\overline{N}^{1/k}(\overline{\theta}_i)\times\Theta_{-i}} u_i(\widetilde{o}(z), s_z, \theta) f(\theta) dm \right\}.$$

So, $H^k_i(\overline{\theta}_i)$ is the set of strategies to which i would find it profitable to deviate conditional on θ_i being in the neighborhood $\overline{N}^{1/k}(\overline{\theta}_i)$. Note that $H^k_i(\overline{\theta}_i)$ is a Borel set for each $\overline{\theta}_i$ and k. Let H_i be the union of $H^k_i(\overline{\theta}_i)$ over k and $\overline{\theta}_i \in \Theta^D_i$ where Θ^D_i is a countable dense subset of Θ_i (recall that Θ_i is a compact metric space). So H_i is the set of problem points.

Let us show that $m_i(H_i \times \Theta_i) = 0$ for each i. Suppose to the contrary that $m_i(H_i \times \Theta_i) > 0$ for some i. It follows that for some k and $\overline{\theta}_i \in \Theta_i^D$, $m_i(H_i^k(\overline{\theta}_i) \times \Theta_i) > 0$.

Let m_{Z_i} be the marginal of m_i on Z_i. It follows from the definition of $H_i^k(\overline{\theta}_i)$ that

$$\frac{1}{\int_{H_i^k(\overline{\theta}_i)} dm_{Z_i}} \int_{H_i^k(\overline{\theta}_i)} \left[\int_{Z_{-i} \times \overline{N}^{1/k}(\overline{\theta}_i) \times \Theta_{-i}} u_i(\widetilde{o}(z), s_z, \theta) f(\theta) dm_{-i} dP_i \right] dm_{Z_i}$$

$$> \int_{Z \times \overline{N}^{1/k}(\overline{\theta}_i) \times \Theta_{-i}} u_i(\widetilde{o}(z), s_z, \theta) f(\theta) dm. \tag{2}$$

Changing the order of integration (appealing here and in what follows to Fubini's theorem) (2) becomes

$$\frac{1}{\int_{H_i^k(\overline{\theta}_i)} dm_{Z_i}} \int_{\overline{N}^{1/k}(\overline{\theta}_i)} \left[\int_{H_i^k(\overline{\theta}_i) \times Z_{-i} \times \Theta_{-i}} u_i(\widetilde{o}(z), s_z, \theta) f(\theta) dm_{-i} dm_{Z_i} \right] dP_i$$

$$> \int_{Z \times \overline{N}^{1/k}(\overline{\theta}_i) \times \Theta_{-i}} u_i(\widetilde{o}(z), s_z, \theta) f(\theta) dm. \tag{3}$$

By the definition of the marginal m_{Z_i} we rewrite (3)

$$\frac{1}{\int_{H_i^k(\overline{\theta}_i) \times \Theta_i} dm_i} \int_{\overline{N}^{1/k}(\overline{\theta}_i)} \left[\int_{H_i^k(\overline{\theta}_i) \times Z_{-i} \times \Theta} u_i(\widetilde{o}(z), s_z, \theta'_i, \theta_{-i}) f(\theta'_i, \theta_{-i}) dm(z, \theta) \right] dP_i(\theta'_i)$$

$$> \int_{Z \times \overline{N}^{1/k}(\overline{\theta}_i) \times \Theta_{-i}} u_i(\widetilde{o}(z), s_z, \theta) f(\theta) dm. \tag{4}$$

Given the regularity of the measure m_i (e.g., see Billingsley [4] Theorem 1.1.) for any $\varepsilon > 0$ we can find a closed set Q_i^ε and an open set F_i^ε, where $Q_i^\varepsilon \subseteq H_i^k(\overline{\theta}_i) \subseteq F_i^\varepsilon$ such that $m_i(F_i^\varepsilon - Q_i^\varepsilon) < \varepsilon$. By Urysohn's Lemma there exists a continuous $h^\varepsilon : Z_i \to [0,1]$ such that $h^\varepsilon(z_i) = 1$ if $z_i \in Q_i^\varepsilon$ and $h^\varepsilon(z_i) = 0$ if $z_i \notin F_i^\varepsilon$. ¿From (4) it follows that for small enough ε,

$$\frac{1}{\int_{Z_i \times \Theta_i} h^\varepsilon \, dm_i} \int_{\overline{N}^{1/k}(\overline{\theta}_i)} \left[\int_{Z \times \Theta} h^\varepsilon(z_i) u_i(\widetilde{o}(z), s_z, \theta'_i, \theta_{-i}) f(\theta'_i, \theta_{-i}) dm(z, \theta) \right] dP_i(\theta'_i)$$

$$> \int_{Z \times \overline{N}^{1/k}(\overline{\theta}_i) \times \Theta_{-i}} u_i(\widetilde{o}(z), s_z, \theta) f(\theta) dm. \tag{5}$$

Given the weak convergence of $\bar{o}m^r$ to $\widetilde{o}m$ from Step 3, and the affinity of u_i in o, and the continuity (on compact spaces) of u_i, f, and h^ε, it follows that

$$\int_{Z\times\Theta} h^\varepsilon(z_i)u_i(\bar{o}(z), s_z, \theta_i', \theta_{-i})f(\theta_i', \theta_{-i})dm^r(z,\theta)$$

$$= \int_{Z\times\Theta} h^\varepsilon(z_i)\left(v_i(s_z, \theta_i', \theta_{-i})\bar{o}(z) + w_i(s_z, \theta_i', \theta_{-i})\right) f(\theta_i', \theta_{-i})dm^r(z,\theta)$$

$$\to \int_{Z\times\Theta} h^\varepsilon(z_i)\left(v_i(s_z, \theta_i', \theta_{-i})\widetilde{o}(z) + w_i(s_z, \theta_i', \theta_{-i})\right) f(\theta_i', \theta_{-i})dm(z,\theta).$$

$$= \int_{Z\times\Theta} h^\varepsilon(z_i)u_i(\widetilde{o}(z), s_z, \theta_i', \theta_{-i})f(\theta_i', \theta_{-i})dm(z,\theta). \tag{6}$$

Viewing the inside integral on the left hand side of (5) as a function of θ_i', it follows from (6) and the Dominated Convergence Theorem that for large enough r we can rewrite (5) as

$$\frac{1}{\int_{Z_i\times\Theta_i} h^\varepsilon\ dm_i^r}\int_{\overline{N}^{1/k}(\overline{\theta}_i)}\left[\int_{Z\times\Theta} h^\varepsilon(z_i)u_i(\bar{o}(z), s_z, \theta_i', \theta_{-i})f(\theta_i', \theta_{-i})dm^r(z,\theta)\right]dP_i(\theta_i')$$

$$> \int_{Z\times\overline{N}^{1/k}(\overline{\theta}_i)\times\Theta_{-i}} u_i(\widetilde{o}(z), s_z, \theta)f(\theta)dm. \tag{7}$$

Let $\widetilde{m}_i^r$ be the distributional strategy for i defined by

$$\widetilde{m}_i^r(E) = \frac{\int_{(z_i,\theta_i',\theta_i)\in E\times\Theta_i} h^\varepsilon(z_i)dm_i^r(z_i,\theta_i)dP_i(\theta_i')}{\int_{Z_i\times\Theta_i} h^\varepsilon\ dm_i^r}.$$

Then, (7) can be rewritten as

$$\int_{\overline{N}^{1/k}(\overline{\theta}_i)}\int_{Z\times\Theta_{-i}} u_i(\bar{o}(z), s_z, \theta_i', \theta_{-i})f(\theta_i', \theta_{-i})d\widetilde{m}_i^r(z_i,\theta_i')dm_{-i}^r(z_{-i},\theta_{-i})$$

$$> \int_{Z\times\overline{N}^{1/k}(\overline{\theta}_i)\times\Theta_{-i}} u_i(\widetilde{o}(z), s_z, \theta)f(\theta)dm. \tag{8}$$

That is, far enough along the sequence, $\widetilde{m}_i^r$ does strictly better than i's equilibrium payoff conditional on $\theta_i \in \overline{N}^{1/k}(\overline{\theta}_i)$. It remains to show that this implies a profitable deviation late in the sequence. But, since $\overline{N}^{1/k}(\overline{\theta}_i)$ is closed, v_i, w_i and f are bounded and continuous, and v_i and f are nonnegative[24], it follows from Lemma 8 that

$$\int_{Z\times\overline{N}^{1/k}(\overline{\theta}_i)\times\Theta_{-i}} u_i(\widetilde{o}(z), s_z, \theta)f(\theta)dm$$
$$= \int_{Z\times\overline{N}^{1/k}(\overline{\theta}_i)\times\Theta_{-i}} (v_i(s_z,\theta)\widetilde{o}(z) + w_i(s_z,\theta))f(\theta)dm$$
$$\geq \limsup\nolimits_{\mathrm{r}} \int_{Z\times\overline{N}^{1/k}(\overline{\theta}_i)\times\Theta_{-i}} (v_i(s_z,\theta)\overline{o}(z) + w_i(s_z,\theta))f(\theta)dm^r. \quad (9)$$

For r sufficiently far along the sequence, combining (8) and (9) thus leads to a contradiction of the fact that m^r is an equilibrium. Hence, our supposition was incorrect and $m_i(H_i \times \Theta_i) = 0$.

Step 5: Next, let us correct $\widetilde{o}$ at the problem points.

For any i and $z_i = (s_i, \theta_i) \in H_i$, select a sequence of $s_i^r \in S_i^r$ such that s_i^r converges to s_i, on a subsequence where m_{-i}^r weakly converges to m_{-i}. Let $\delta^{s_i^r}$ denote the Dirac measure on s_i^r. For any Borel set $E \subseteq S \times \Theta_{-i} \times \Theta_{-i}$, define

$$\mu_{s_i}^r(E) = \int_E \overline{o}(s)dm_{-i}^r(s_{-i}, \theta'_{-i}, \theta_{-i})d\delta^{s_i^r}.$$

Taking a further subsequence, $\mu_{s_i}^r$ converges weakly to some μ_{s_i}. By Lemma 2 in SZ, there is a Borel measurable selection[25] o_{s_i} from O which is a function of $(s, \theta'_{-i}, \theta_{-i})$ such that for any Borel set $E \subseteq S \times \Theta_{-i} \times \Theta_{-i}$,

$$\mu_{s_i}(E) = \int_E o_{s_i}(s, \theta'_{-i}, \theta_{-i})dm_{-i}(s_{-i}, \theta'_{-i}, \theta_{-i}).$$

Given that $m_j(D_j) = 1$ for each j, following an analogous argument to that in Step 3, we can redefine o_{s_i} as a function of s_i, z_{-i} rather than $(s, \theta'_{-i}, \theta_{-i})$.

We are now prepared to correct $\widetilde{o}$ at the problem points in each H_i and define $\widehat{o}$ such that $m, \widehat{o}$ is an equilibrium. Note that we cannot simply fix $\widetilde{o}$ by repairing it whenever $z_i \in H_i$ for some i as we might encounter z such that $z_i \in H_i$ and $z_j \in H_j$ for $i \neq j$. However, since each H_j is a measure 0 set under m_j, from any agent i's perspective even conditional on a particular

[24] f is bounded since Θ is a compact space.

[25] To apply the lemma, extend O to be defined on $S \times \Theta_{-i} \times \Theta_{-i}$ by setting $O(s, \theta'_{-i}, \theta_{-i}) = O(s)$.

strategy z_i, the set H_j is a measure 0 event. So we need only correct things at the points where we fall in a single H_i at a time and can ignore the other points. Let

$$A_i = \{z \mid z_i \in H_i, z_j \notin H_j \; \forall j \neq i\}.$$

As each H_i and H_j is a Borel set, so is A_i. Define the selection $\widehat{o}$ of O as follows. Let $\widehat{o}(z) = o_{s_i}(s_i, z_{-i})$ if $z = (s, \theta) \in A_i$, and let $\widehat{o}(z) = \widetilde{o}(z)$ if $z \in \cap_i A_i^C$.[26]

Step 6: We verify that $m, \widehat{o}$ is an equilibrium.

Suppose the contrary. Given the continuous type distribution (i.e., the continuity of f), there exists i, $\overline{z}_i$, k and $\overline{\theta}_i \in \Theta_i^D$ such that $m_i|_{\overline{N}^{1/k}(\overline{\theta}_i)}\overline{z}_i$ is a better response than m_i to m_{-i} and $P_i(\overline{N}^{1/k}(\overline{\theta}_i)) > 0$, where $m_i|_{\overline{N}^{1/k}(\overline{\theta}_i)}\overline{z}_i$ is the distributional strategy that plays $\overline{z}_i$ if $\theta_i \in \overline{N}^{1/k}(\overline{\theta}_i)$ and according to m_i otherwise.

Case 1: $\overline{z}_i \notin H_i$.

In this case, it follows from the definition of $\widehat{o}$ that $\widehat{o}(z) = \widetilde{o}(z)$ for $m_{-i}, m_i|_{\overline{N}^{1/k}(\overline{\theta}_i)}\overline{z}_i$ a.e. (z, θ). The same is true for m almost every z, θ, since $m_j(H_j \times \Theta_j) = 0$ for each j by Step 4. Thus, since $\overline{z}_i \notin H_i$ it follows from the definition of H_i that $m_i|_{\overline{N}^{1/k}(\overline{\theta}_i)}\overline{z}_i$ cannot be a better response than m_i to m_{-i}.

Case 2: $\overline{z}_i \in H_i$.

Let $\overline{s}_i$ be such that $\overline{z}_i = (\overline{s}_i, \theta_i')$. By supposition,

$$\int_{Z_{-i}\times\Theta_{-i}\times\overline{N}^{1/k}(\overline{\theta}_i)} u_i(\widehat{o}(\overline{z}_i, z_{-i}), \overline{s}_i, s_{-i}, \theta) f(\theta) dm_{-i} dP_i$$
$$> \int_{Z\times\Theta_{-i}\times\overline{N}^{1/k}(\overline{\theta}_i)} u_i(o(z), s, \theta) f(\theta) dm.$$

By the condition that payoffs are affine, this implies that

$$\int_{\overline{N}^{1/k}(\overline{\theta}_i)} \left[\int_{Z_{-i}\times\Theta_{-i}} \left(v_i(\overline{s}_i, s_{-i}, \theta)\widehat{o}(\overline{z}_i, z_{-i}) + w_i(\overline{s}_i, s_{-i}, \theta)\right) f(\theta) dm_{-i}\right] dP_i$$
$$> \int_{Z\times\Theta_{-i}\times\overline{N}^{1/k}(\overline{\theta}_i)} \left(v_i(s, \theta)\widehat{o}(z) + w_i(s, \theta)\right) f(\theta) dm. \qquad (10)$$

[26] As we do this pointwise, it is here that the measurability issue arises.

By the definition of $\widehat{o}$ in Step 5, we know that

$$\int_{\overline{N}^{1/k}(\overline{\theta}_i)} \left[\int_{Z_{-i}\times\Theta_{-i}} \left(v_i(\overline{s}_i, s_{-i}, \theta)\widehat{o}(\overline{z}_i, z_{-i}) + w_i(\overline{s}_i, s_{-i}, \theta)\right) f(\theta) dm_{-i} \right] dP_i =$$

$$\int_{\overline{N}^{1/k}(\overline{\theta}_i)} \left[\int_{Z_{-i}\times\Theta_{-i}} \left(v_i(\overline{s}_i, s_{-i}, \theta) o_{\overline{s}_i}(\overline{s}_i, z_{-i}) + w_i(\overline{s}_i, s_{-i}, \theta)\right) f(\theta) dm_{-i} \right] dP_i.$$

So given the continuity of v_i, f, and w_i, for large enough r it follows from the weak convergence of $\mu_{s_i^r}$ to μ_{s_i} (in Step 5) that we can rewrite (10) as

$$\int_{\overline{N}^{1/k}(\overline{\theta}_i)} \left[\int_{Z_{-i}\times\Theta_{-i}} \left(v_i(\overline{s}_i, s_{-i}, \theta)\overline{o}(\overline{s}_i, z_{-i}) + w_i(\overline{s}_i, s_{-i}, \theta)\right) f(\theta) dm^r_{-i} \right] dP_i.$$

$$> \int_{Z\times\Theta_{-i}\times\overline{N}^{1/k}(\overline{\theta}_i)} \left(v_i(s,\theta)\widehat{o}(z) + w_i(s,\theta)\right) f(\theta) dm. \tag{11}$$

Since $P_i(\overline{N}^{1/k}(\overline{\theta}_i)) > 0$ and $m(A_i) = 0$ it follows that

$$\int_{Z\times\Theta_{-i}\times\overline{N}^{1/k}(\overline{\theta}_i)} \left(v_i(s,\theta)\widehat{o}(z) + w_i(s,\theta)\right) f(\theta) dm$$

$$= \int_{Z\times\Theta_{-i}\times\overline{N}^{1/k}(\overline{\theta}_i)} \left(v_i(s,\theta)\widetilde{o}(z) + w_i(s,\theta)\right) f(\theta) dm. \tag{12}$$

Since $\overline{N}^{\frac{1}{k}}$ is closed, it follows from Steps 1 and 3 and Lemma 8 that for large enough r

$$\int_{Z\times\Theta_{-i}\times\overline{N}^{1/k}(\overline{\theta}_i)} \left(v_i(s,\theta)\widetilde{o}(z) + w_i(s,\theta)\right) f(\theta) dm.$$

$$\geq \int_{Z\times\Theta_{-i}\times\overline{N}^{1/k}(\overline{\theta}_i)} \left(v_i(s,\theta)\overline{o}(s) + w_i(s,\theta)\right) f(\theta) dm^r. \tag{13}$$

Combining (11), (12), and (13) we find that

$$\int_{\overline{N}^{1/k}(\overline{\theta}_i)\times Z\times\Theta_{-i}} u_i(\overline{o}(s), s, \theta) f(\theta) dm^r_{-i} d\delta^{s_i^r} dP_i$$

$$> \int_{Z\times\Theta_{-i}\times\overline{N}^{1/k}(\overline{\theta}_i)} u_i(\overline{o}(s), s, \theta) f(\theta) dm^r.$$

This contradicts the fact that m^r is an equilibrium as defined from Step 1.

Step 7: We verify that if O is derived from tie breaking, then the construction of $\widehat{o}$ in Step 5 is made in a Borel measurable way.

For any s_i, let

$$T_i(s_i) = \{(s_{-i}, \theta_{-i}) \in Z_{-i} \mid s_j = s_i \text{ for some } j \neq i\}.$$

This is the set of strategies of that can conceivably tie when i uses plays s_i. Fix any i. Note that $\{s_j | m_j(\{s_j\} \times \Theta_j \times \Theta_j) > 0\}$ is a countable set for each j. Thus, $X_i = \{s_i | m_{-i}(T_i(s_i) \times \Theta_{-i}) > 0\}$ is a countable set.

We claim that if $z_i = (s_i, \theta_i) \in H_i$, then $s_i \in X_i$. Suppose the contrary. Then $z_i \in H_i^k(\overline{\theta}_i)$ for some k and $\overline{\theta}_i$ and $s_i \notin X_i$. Thus, $m_{-i}(T_i(s_i) \times \Theta_{-i}) = 0$ and so from the definition of O being derived from tie breaking, it follows that $\widetilde{o}_i(z) = \overline{o}_i(s)$ for m_{-i}-a.e. $z_{-i} = (s_{-i}, \theta_{-i})$. Then, an argument similar to that of case 2 in Step 6 reaches a contradiction.

Thus, by the definition of $\widehat{o}$ in Step 5 for any measurable set of outcomes E, we can write

$$\widehat{o}^{-1}(E) = \big(\cup_i [\cup_{s_i \in X_i} (o_{s_i}^{-1}(E) \cap A_i)]\big) \cup [\widetilde{o}^{-1}(E) \cap (\cap_i A_i^C)].$$

Given that X_i is countable, A_i is Borel, o_{s_i} and $\widetilde{o}$ are Borel measurable for each i and s_i, it follows that $\widehat{o}^{-1}$ is a Borel set.

Step 8: To complete the proof, we discuss the modification to prove existence of a weakly perfect equilibrium.

In Step 1, select ε^r so that $(\max_i |S_i^r|)\varepsilon^r \to 0$, and then let $\overline{m}_i^r$ be a ε^r-constrained equilibrium.[27] In Step 4, $\widetilde{m}^r$ should be modified to place ε^r weight on each strategy, and in Steps 5 and 6, $\delta^{s_i^r}$ should be taken to be the strategy placing all available weight on s_i^r subject to being in $C_i^{\varepsilon^r}$. The rest of the proof is unchanged. ■

Proof of Theorem 2 Let $u_i(o, s)$ denote the $\#\Theta$-dimensional vector $(u_i(o, s, \theta))_\theta$. Let

$$\widehat{O}(s) = \{(u_1(o, s), \ldots, u_n(o, s)) \mid o \in O(s)\}.$$

[27]To establish existence of a constrained equilibrium one can extend Milgrom and Weber's theorem to constrained distributional strategies (on a finite strategy space). To do this, note that by Lemma 8 the set C_i^ε is closed. Thus, C_i^ε is compact and convex (following the arguments of Milgrom and Weber), and the remainder of their proof proceeds as stated. Alternatively, following standard perfection style arguments instead of constraining the strategies, one can alter the payoffs so that payoffs are those that would occur if each opponent's strategy trembles. One could then apply their theorem to the game with altered payoffs and then convert the equilibrium strategies to lie in C_i^ε.

For s and θ, let $\widehat{v}_i(s,\theta)$ be the vector such that $v_i(s,\theta)_\theta = 1$ and $v_i(s,\theta)_{\theta'} = 0$ for $\theta' \neq \theta$. Thus,

$$u_i(o,s,\theta) = v_i(s,\theta) \cdot u_i(o,s).$$

As, $\widehat{O}(s) = U(s)$ is upper-hemicontinuous, compact and convex valued, we can apply Theorem 1 to establish the existence of an equilibrium $m, \widehat{o}$, where $\widehat{o}$ is incentive compatible and a selection from $\widehat{O}$. By the definition of $\widehat{O}$, one can find a selection o from O such that $\widehat{o}(s) = (u_i(o(s),s,\theta))_{i,\theta}$. It follows that m, o is an equilibrium and o is incentive compatible.

Let us show that if $\widehat{o}$ is Borel measurable, then we can choose o to be Borel measurable. Let

$$F(s) = \{o \mid \max_{i,\theta} |\widehat{o}(s)_{i\theta} - u_i(o,s,\theta)| = 0\}.$$

Given the continuity of u_i this correspondence is closed-valued. This correspondence is also measurable given the continuity of u_i in o and the measurability of $\widehat{o}$ and u_i in s (see Corollary 14.82 in Aliprantis and Border [1]). Thus by the Kuratowski-Ryll-Nardzewski theorem, (see [1]) there exists a Borel measurable selection from F.[28] ■

Proof of Theorem 3 We describe the modifications to the proof of Theorem 1 to verify that a symmetric equilibrium exists.

In Step 1, pick $\bar{o}$ to be symmetric and measurable. To see that this can be done, pick an arbitrary measurable selection o from O. For each permutation σ over players we have a corresponding o^σ. Let $\bar{o}(s) = \sum_\sigma \frac{o^\sigma(s)}{k}$ for each s, where k is the number of permutations σ. By the convexity of $O(s)$, $\bar{o}(s) \in O(s)$. By construction $\bar{o}$ is symmetric and measurable.

Next, in Step 1 we need to show that we can find $\bar{m}^r$ that is a symmetric equilibrium. This follows from a simple extension of Theorem 1 in Milgrom and Weber [16]. To be precise, write $\pi(\bar{o}, \bar{m}_i, \bar{m}_j)$ to be the payoff to any player when that player plays $\bar{m}_i$ and the other players each play $\bar{m}_j$. Fixing $\bar{o}$, by the arguments in Milgrom and Weber π is a continuous (linear) function from a compact convex metric space. Let $M(\bar{m}_j) = argmax_{\bar{m}_i} \pi(\bar{o}, \bar{m}_i, \bar{m}_j)$. By Berge's theorem, this is upper-hemicontinuous and compact valued. Given the linearity of π in $\bar{m}_i$ it also follows that M is

[28] Note in the case where $\widehat{o}$ is not measurable, we can still apply this reasoning to Step 3 of the proof to ensure that we find a corresponding Borel measurable $\widetilde{o}$, and so it we will have an outcome function that is measurable up to the patching in Step 5.

convex valued. So, by Glicksberg's Theorem there exists a fixed-point of M. Such a fixed point is a symmetric equilibrium.

In Step 2, we can take $\bar{m}$ to be symmetric simply by choosing the same limit point for each i, given that the sequences of $\bar{m}_i^r$ are identical across i.

In Step 3 we select o to be symmetric. To do this, start with the o as defined in Step 3. Let $\widehat{o}(s) = \sum_\sigma \frac{o^\sigma(s)}{k}$. As argued above, $\widehat{o}$ is symmetric and measurable. Also, $\bar{o}m^r \to o^\sigma m$ for each σ implies that $\bar{o}m^r \to \widehat{o}m$.

In Step 5, H_i and A_i are the same for each player. So, define o_{s_i} for some i and then choose the same for each j. Then $\widetilde{o}$ will be symmetric.

The rest of the proof proceeds unaltered. ■

8 Appendix B: Auction Proofs

Properties of O(b) To see that $O(b)$ is nonempty, simply give the objects away in the order indicated by $g_{.,.}(.)$, and at ties give objects away in the order of player's labels. That O has a closed graph (which implies both compact values and upper hemi-continuity given the compact bid space) follows from the fact that along any convergent sequence of bid vectors and allocations $b^k, o^k \to b, o$: if $o_{eih} > 0$, then $o^k_{eih} > 0$ for all k sufficiently far along the sequence, and similarly, if $g_{ih}(b_{ih}) > g_{i'h'}(b_{i'h'})$, then $g_{ih}(b^k_{ih}) > g_{i'h'}(b^k_{i'h'})$ for all k sufficiently far along the sequence. Therefore, any restriction implied in the limit is also implied everywhere late in the sequence. To see that $O(b)$ is convex valued, let $o \in O(b)$ and $o' \in O(b)$. As the second condition is clearly satisfied by $\alpha o + (1-\alpha)o'$, let us check the first condition. Suppose $\alpha o_{eih} + (1-\alpha)o'_{eih} > 0$, and, without loss of generality, let $o_{eih} > 0$. If $o'_{eih} > 0$ also, then $\sum_{w=h'}^m (\alpha o_{ejw} + (1-\alpha)\, o_{ejw}) = 1$ for all j, h' such that $g_{jh'}\,(b_{jh'}) > g_{ih}\,(b_{ih})$ and so the first condition is satisfied. Consider the case where $o'_{eih} = 0$. Then for some i', h' (where only one of i' or h' needs to differ from i and h) $o'_{ei'h'} > o_{ei'h'} \geq 0$. Since $o'_{eih} = 0$, it must thus be that $g_{ih}\,(b_{ih}) \leq g_{i'h'}(b_{i'h'})$. Since $o_{ei'h'} < 1$, it must be that $g_{ih}(b_{ih}) \geq g_{i'h'}(b_{i'h'})$. So, $g_{ih}(b_{i'h'}) = g_{ih}(b_{ih})$. But then, each time that $o_{eih} > 0$ implies that a certain player must be winning for sure under $o_{e,.,.}$, the same implication for $o'_{e,.,.}$ follows from the fact that $o'_{ei'h'} > 0$. So, $\alpha o_{eih} + (1-\alpha)o'_{eih}$ again satisfies the first condition.

Proof of Lemma 4, Condition (a) Consider a sequence of grids and full support trembles, indexed by k. Assume there is $\varepsilon > 0$ and a subsequence along which

$$\Pr_k(b_{ih} - v_{ih} > \varepsilon | i \text{ does not tremble, } b_{ih} \text{ is a buy bid}) > \varepsilon$$

for some i. Consider any grid fine enough that there are always at least two available bids in any interval of length ε. Consider the deviation $d_i(b_i)$ that when the original strategy specifies a buy bid $b_{ih} > v_{ih} + \varepsilon$, i lowers b_{ih} (and any later bids that need to be) to b'_{ih}, defined as the first grid point above v_{ih}. Consider any outcome (under the equilibrium strategies) in which i would have won the same number of objects by $d_i(b_i)$ as under b_i. Then, since C is weakly increasing, and since in the event that i is a seller, his payments are unaffected by his buy bids, i would have been weakly better off by bidding $d_i(b_i)$. Consider any outcome in which i wins h'' objects with b_i and would have won $h' < h''$ objects by bidding $d_i(b_i)$. Think about moving from b_i to $d_i(b_i)$ in a sequence of changes, first lowering b_{im} to $d_{im}(b_i)$ then $b_{i,m-1}$ to $d_{i,m-1}(b_i)$ up until $e_i + 1$. For $h \notin \{h', \ldots, h''\}$, this does not change the number of objects won, but does weakly lower the price. Consider any step $h \in \{h', \ldots, h''\}$. Since b_i won an h^{th} object, but $d_i(b_i)$ does not, it must be that b_i was actually lowered. Thus, it must be that $b_{ih} > v_{ih}$, and, since $d_{ih}(b_i)$ does not win, it must be that there is $\widehat{b}_{ih} \in [d_{ih}(b_i), b_{ih}]$ such that if instead i lowered bid b_{ih} to $\widehat{b}_{ih}$, he would be tied and so would pay $\widehat{b}_{ih} \geq b_{ih} > v_{ih}$ for object h. Since C is weakly increasing, under b_i he is paying more than v_{ih} for object h. Hence, this step is strictly profitable. So, this deviation is weakly profitable, and strictly profitable any time that it actually results in winning fewer objects. But, given the tremble structure, there is a positive probability that all players place all bids on any given grid point. So, if there is a positive probability that i is making buy bids strictly above value, d must win fewer objects in expectation, and hence is a strictly profitable deviation. It follows that i places minimum weight on such strategies, and hence in the limit submits buy bids above value with probability 0. A symmetric argument holds on the sell side.

Condition (b) As before, assume $\Pr_k(b_{ih} - v_{ih} > \varepsilon | i \text{ does not tremble}) > \varepsilon$ for some i. If b_{i1} wins, it earns $v_{i1} - C_{i1}(b) < v_{i1} - b_{i1} < -\varepsilon$. If it does not win, it earns $-C_{i1}(b) \leq 0$. Given the tremble technology, b_{i1} does sometimes win. Hence, conditional on $b_{ih} - v_{ih} > \varepsilon$, m_i earns strictly negative profits. Consider the deviation that whenever i's equilibrium strategy specifies $b_{i1} - v_{i1} > \varepsilon$, i bids some grid point below 0 instead. Since $b_{0h} > 0$ for $h \leq e_0$, such a bid never wins. Hence, it earns 0, a strictly profitable improvement. So, in the limit, i submits such bids with probability 0. ■

Fix a profile of strategies $(m_1, \ldots, m_n)$ (equilibrium or otherwise). Given these strategies, let B be the induced measure over $Z \times \Theta$, where in this case, a typical element of $z_i \in Z_i$ has the form (b_i, θ_i') consisting of the submitted bid vector and announced value vector (remember that we are assuming that realizations of e_i are observable), and a typical element θ of Θ has the form (e_i, v_i). Let B_i be the marginal of B onto $Z_i \times \Theta_i$ (except that B_0 is simply P_0, the marginal onto e_0). Our next Lemma shows that an implication of the absolute continuity of P with respect to $\prod_{i=0}^n P_i$ is that B is absolutely continuous with respect to $\prod_{i=0}^n B_i$. So, events involving the set of submitted bids and realized values which are zero probability assuming players draw values and bid independently are also zero probability under the actual distribution over realized values and submitted bids. If in addition, $\prod_{i=0}^n P_i$ is absolutely continuous with respect to P (a condition which we use for Theorem 6) then the reverse implication will be true as well, so that positive probability events under $\prod_{i=0}^n P_i$ are positive probability under P.

Lemma 9 *(1) For each $A \in \Gamma^A$, or $A \in \Phi^A$ and for each strategy profile m in A, B is absolutely continuous with respect to $\prod_{i=0}^n B_i$.*

(2) If $\prod_i P_i$ is absolutely continuous with respect to P_i, then $\prod_{i=0}^n B_i$ is absolutely continuous with respect to B_i.

Proof of Lemma 9.

We provide the proof for augmented auctions. The proof for $A \in \Phi^A$ is obtained by replacing Z by S and z by s.

In what follows, think of player 0 as having a singleton strategy space, so that $dm_0(z_0|\theta_0) \equiv 1$, and $dm_0(z_0, \theta_0) = dP_0(\theta_0)$. Consider any Borel $E \subset Z \times \Theta$

$$B(E) = \int_E \prod_{i=0}^n dm_i(z_i|\theta_i) dP(\theta)$$

$$= \int_E f(\theta) \prod_{i=0}^n dm_i(z_i|\theta_i) dP_i(\theta_i)$$

$$= \int_E f(\theta) \prod_{i=0}^n dm_i(z_i, \theta_i)$$

$$= \int_E f(\theta) dm.$$

Given that $f(\theta) < M$, it follows that $B(E) \leq Mm(E)$. Since $m_i = B_i$,[29] we have established $B(E) \leq M \prod_i B_i(E)$ and hence Part 1. It also follows that $0 < M' < f(\theta)$ so that $B(E) \geq M'm(E)$. This implies that $B(E) \geq M' \prod_i B_i(E)$ establishing Part 2. ■

Say that $b \in [-1, \bar{v}+1]$, is a *bid-atom* for B_i if $B_i(\{b_{ih} = b \text{ for some } h\}) > 0$. For each i, let $\widehat{B}_i$ be the set of bid-atoms of B_i.

Let $g_i(\widehat{B}_i) \equiv \{g_{ih}(b) | h \in \{1, \ldots, m\}, b \in \widehat{B}_i\}$. So, $g_i(\widehat{B}_i)$ is the image of $\widehat{B}_i$ as transformed through the various g_{ih}'s. Since $\widehat{B}_i$ is countable, $g_i(\widehat{B}_i)$ is countable.

Recall that our definition of a tie was constructed to rule out irrelevant cases in which a small change in bid does not affect the allocation. Say that b_{ih} and $b_{jh'}$ are in a *pre-tie* if $g_{ih}(b_{ih}) = g_{jh'}(b_{jh'})$. Let $Y_i \equiv g_i^{-1}\left(\cup_{j \neq i} g_j\left(\widehat{B}_j\right)\right)$. Since the various g_{ih} are strictly increasing and continuous, Y_i is well defined and countable. By avoiding Y_i, i avoids pre-ties with any given $j \neq i$ using $b_{jh'} \in \widehat{B}_i$. It turns out that by doing so, i reduces the probability that he is involved in a pre-tie (and hence a tie) to 0.

Lemma 10 $\Pr\left(\{g_{ih}(b_{ih}) = g_{jh'}(b_{jh'})\} \cap \{b_{ih} \notin Y_i\}\right) = 0$. *That is, there is zero probability of a pre-tie involving i when i does not use bids in Y_i.*

Proof of Lemma 10 This is obvious if the B_i are independent. The result follows by absolute continuity of B with respect to $\prod_{i=1}^n B_i$. ■

Proof of Theorem 5 Let A be a non-augmented auction with a tie breaking rule o which does not depend on private information and in which ties between buyers and sellers always result in trade. Let A' be the augmented counterpart to A. By Theorem 1, A' has a weakly perfect equilibrium m', o'. Define m from m' by removing the type announcements. We show that o, m is an equilibrium, and that with probability one, outcomes under o, m are the same as under o', m'. (This establishes weak perfection as the same sequence weakly converging to o', m' can be used to converge to o, m.) The proof is in two steps. First, we show that ties occur in equilibrium in A' with zero

[29] B is defined by $dB(z, \theta) = \prod_i dm_i(z_i|\theta_i)dP(\theta)$. The careful reader can verify that the marginal of B on i, B_i, has the same form as as one would arrive at by directly taking the marginal of P onto P_i which leads to $dm_i(z_i|\theta_i)dP_i(\theta_i)$ (which for a distributional strategy is the same as $dm_i(z_i, \theta_i)$).

probability. Hence, players receive the same payoffs by playing m in A as they did playing m' in A'. Second, we argue that if a player has a profitable deviation d_i from m in A, then he has a profitable deviation $\widehat{d_i}$ which involves ties with probability 0. But then, $\widehat{d_i'}$, defined as $\widehat{d_i}$ along with truthful announcements, receives the same payment in A' as $\widehat{d_i}$ does in A, and so is a profitable deviation in A' as well.

Fix an arbitrary player i. Let $\widehat{o}_{eih}(b) \equiv \sum_{h'=h}^{m} o_{eih'}(b)$ be the probability that i ends up with h or more objects given bid vector b. Let T_i be the event that i is involved in a tie given equilibrium play. Let T_{ihLb} be the subset of T_i such that b_{ih} is a buy offer involved in a tie and $\widehat{o}_{eih}(b) < 1$. So, T_{ihLb} is the set of events where b_{ih} is involved in a tie and i does not get an h^{th} object for sure.

Assume that $\Pr(T_{ihLb}) > 0$. By Lemma 10, $\Pr(b_{ih} \in Y_i | T_{ihLb}) = 1$. By NDB, $\Pr(v_{ih} < b_{ih} | T_{ihLb}) = 0$. By atomlessness, $\Pr(v_{ih} \in Y_i | T_{ihLb}) = 0$. It follows that $\Pr(v_{ih} > b_{ih} | T_{ihLb}) = 1$.

Similarly, let T_{ihLs} be the subset of T such that b_{ih} is a sell offer involved in a tie and $\widehat{o}_{eih} > 0$. Recalling that in our set up, a sale occurs when ones bid is *not* accepted, this is the set of events where b_{ih} is involved in a tie, and i does not sell object h for sure. By an argument analogous to that establishing that $\Pr(v_{ih} > b_{ih} | T_{ihLb}) = 1$, it follows that $\Pr(v_{ih} < b_{ih} | T_{ihLs}) = 1$.

Define

$$\begin{aligned} \omega \equiv & \sum_{\{h | \Pr(T_{ihLs}) > 0\}} \Pr(T_{ihLs}) E(\widehat{o}_{eih}(b) \, (b_{ih} - v_{ih}) \, | T_{ihLs}) \\ & + \sum_{\{h | \Pr(T_{ihLb}) > 0\}} \Pr(T_{ihLb}) E((1 - \widehat{o}_{eih}(b)) \, (v_{ih} - b_{ih}) \, | T_{ihLb}). \end{aligned}$$

So, the first term sums the probabilities of not making a sale on unit h, multiplied by the expected profit on that sale, and the second sum does the same thing for purchases. If $\Pr(T_{ihLs})$ or $\Pr(T_{ihLb})$ is positive for any i, h, then ω is positive.

Assume $\omega > 0$. Consider modifying m_i so that buy bids in Y_i are raised a little bit, and sell bids in Y_i are lowered a bit, in such a way that they no longer lie in Y_i. Formally, alter m_i as follows. Let $\{x_1, x_2, x_3, \ldots\}$ be an enumeration of Y_i. Fix a sequence $\{\varepsilon_k\}, \varepsilon_k \to 0$, and for each $x \in Y_i$, and k, define $y_k^b(x)$ as some bid in $[x, x + \varepsilon_k)$, but not in Y_i, and $y_k^s(x)$ as some bid in $(x - \varepsilon_k, x]$, but not in Y_i. Define a map from $\mathcal{B}$ to $\mathcal{B}$ that iteratively transforms buy bids as follows. If $b_{i,e_i+1} \in Y_i$, and $b_{ie_i} \neq b_{i,e_i+1}$, then $b'_{i,e_i+1} = y_k^b(b_{i,e_i+1})$, where ε_k is the first element of $\{\varepsilon_k\}$ such that $\varepsilon_k < \varepsilon$, $\varepsilon_k < (b_{ie_i} - b_{i,e_i+1})/3$. Otherwise, $b'_{i,e_i+1} = b_{i,e_i+1}$. Now, for $h = e_i + 2, \ldots, m$, if $b_{ih} \in Y_i$ and $b_{ih} < b'_{i,h-1}$, then $b'_{ih} = y^b_k(b_{ih})$ where ε_k is the first element of $\{\varepsilon_k\}$ such that $\varepsilon_k < \varepsilon$, $\varepsilon_k < (b'_{i,h-1} - b_{ih})$. Otherwise, $b'_{ih} = b_{ih}$. This results in raising all of i's buy bids except in the case $b_{ie_i} = b_{i,e_i+1}$. From NDB, $b_{ie_i} = b_{i,e_i+1}$ can only occur when $b_{ie_i} = v_{ie_i} = v_{i,e_{i+1}} = b_{i,e_i+1}$. Furthermore, when $b_{ie_i} = b_{i,e_i+1} = 0$, any bid not equal to v_{ie_i} is raised. So, the only time that this does not result in raising any buy bid in Y_i to a nearby bid not in Y_i is in the zero probability event that $v_{ie_i} \in Y_i$.

Similarly, iteratively define new sell bids as follows. If $b_{ie_i} \in Y_i$, and $b_{ie_i} \neq b_{i,e_i+1}$, then $b_{ie_i} = y^s_k(b_{i,e_i+1})$, where ε_k is the first element of $\{\varepsilon_k\}$ such that $\varepsilon_k < \varepsilon$, $\varepsilon_k < (b_{ie_i} - b_{i,e_i+1})/3$. Otherwise, $b'_{ie_i} = b_{ie_i}$. Now, for $h = e_i - 1, \ldots, 1$ if $b_{ih} \in Y_i$ and $b_{ih} > b'_{i,h+1}$ then $b'_{ih} = y^s_k(b_{ih})$ where ε_k is the first element of $\{\varepsilon_k\}$ such that $\varepsilon_k < \varepsilon$, $\varepsilon_k < \left(b'_{i,h+1} - b_{ih}\right)$. As before, the only time a bid at Y_i is left alone is in the zero probability event that $v_{ih} \in Y_i$.

This is a measurable mapping from $\mathcal{B}$ to $\mathcal{B}$. Hence, m_i^ε defined as the composition of m_i with this mapping is also measurable, and so is a feasible distributional strategy. Further, since ties involving i are zero probability given m_i^ε, any issues involving the potential non-Borel measurability of o at discontinuities are moot.

This change in strategy costs an amount which vanishes in ε, in terms of paying more for objects won, or receiving less for objects that would otherwise have been sold. And, it gains ω by transforming o_{eih} to 0 when a sell bid b_{ih} would have been involved in a tie, and o_{eih} to 1 when a buy bid b_{ih} would have been involved in a tie. Any extra objects bought beyond those at the original ties are worth almost as much as was paid for them (because the original strategy satisfied NDB), and similarly, any extra objects sold are worth not much more than the sales price. So, this deviation results in an increase in expected utility for ε sufficiently small. This is a contradiction. Hence, $\Pr(T_{ihLs}) = \Pr(T_{ihLb}) = 0$ for each i and h.

So, each buyer either is in a tie with probability 0, or wins ties with probability 1. So, in particular, it cannot be that he is with positive probability in a tie in which he wins and another buyer does not, since the other buyer also wins ties all the time. Similarly, it cannot be that a seller is with positive probability in a tie with another seller, and that he wins but the other seller does not. It is, however, possible that a seller and a buyer with positive probability make the same bid, but that the mechanism always decides in favor of trade in this event.[30] So, if one replaces the tie breaking rule of the augmented game by the standard one (subject to the proviso that when a buyer and a seller are tied, the object is actually transferred), then payoffs under the original equilibrium strategies are unaffected.

To complete the proof, we show that there are no improving deviations from o, m. Suppose to the contrary that some player i has a profitable deviating strategy μ_i. Without loss of generality, take μ_i to satisfy NDB..[31] Now, for given $\varepsilon > 0$, define μ_i^ε from μ_i in terms of the map defined above. Arguing as above, for small ε, μ_i^ε performs arbitrarily close to as well as μ_i, and so for small enough ε, is also a strictly profitable deviation. But, $\Pr(b_{ih} \in Y_i | \mu_i^\varepsilon) = 0$ for each h, and hence by Lemma 10, i is with probability 0 involved in a tie given μ_i^ε. Hence, μ_i^ε is also strictly profitable in the augmented game, a contradiction. ■

Proof of Theorem 6

Fix $A \in \Phi^A$ satisfying SIP and CGT. As in the proof of Theorem 1, we generate equilibria for a sequence of auctions with finite strategy spaces, so that the payoffs in the equilibrium constructed of the limit game is the limit of those in the sequence of finite games. Choose a sequence A^r of finite approximations to A, where in A^r, each coordinate of $\mathcal{B}$ is approximated within Hausdorff $1/r$ by a finite grid. For each such game, fix the tie breaking rule as symmetric randomization among tied bids, as described above. It suffices to show that there is $\rho > 0$ and $\widehat{r} < \infty$ such that for each $r > \widehat{r}$, there is an equilibrium of A^r that has probability of trade at least ρ.

Now, for any finite approximation A^r, and for each $t = \{1, 2, \ldots\}$, consider the auction A^{rt} modified from A^r as follows. With probability $1/t$ a non-strategic player $n+1$ has $e_{n+1} = 1$ and submits a sell offer which is uniform over the available bids in $[0, \bar{v}]$.[32] With probability $1/t$, $e_{n+1} = 0$ and $n+1$ submits a buy offer which is uniform on the available bids in $[0, \bar{v}]$. With residual probability, player $n+1$ is not involved. These probabilities are

[30] For example imagine that there are two buyers with demand in $[2, 3]$, and two sellers with a single unit for sale and value in $[0, 1]$. Then, it is an equilibrium for all players to bid 1.5, and for trade to always occur. This does not require the auctioneer to be able to observe anything about players except who is a seller and who a buyer. In this sense, this example is quite different from the example at the end of Section 5.1.

[31] For instance, transform bids to $\min(b_{ih}, v_{ih})$ if $h > e_i$ and $\max(b_{ih}, v_{ih})$ if $h \leq e_i$.

[32] This player is not the same as the non-strategic player 0 who represents the possibility of an exogenous supply (as in the case of a one sided auction). Player $n+1$ is just a construction for the proof who disappears from the limit auction A.

independent of P. Each A^{rt} has an equilibrium m^{rt} by Milgrom and Weber [16], Theorem 1. For each s, let $\rho_s = \inf_{r>s,t>s} \Pr\{\text{trade under } m^{rt}\}$. Assume that $\liminf \rho_s = \rho'$. Then, there is $s < \infty$ such that the probability of trade in any m^{rt} for $r > s, t > s$ is at least $\rho'/2$ for all t. By Milgrom and Weber [16], Theorem 2, the limit as $t \to \infty$ of any convergent subsequence of m^{rt} is an equilibrium m^r of A^r which has probability of trade at least $\rho'/2$. To prove the theorem, we need to show that $\rho' > 0$. Suppose the contrary so that there exists a sequence r_s, t_s such that $r_s \to \infty$, $t_s \to \infty$, and such that the probability of trade under the corresponding m^s goes to 0 along the sequence.

Consider the case where the second condition in CGT holds, so that there are sometimes $m+1$ sell values below a buy value. (The argument if the first condition holds is symmetric.) For notational convenience, let $\Pr_s(E)$ be the probability of event E happening in auction s. When the probability of an event does not depend on s, we write simply $\Pr(E)$

Without loss of generality consider players $1, 2, \ldots, j, j+1$ such that sometimes there are $m+1$ sell values among $J \equiv \{1, \ldots, j\}$ below $j+1$'s highest buy value, and such that J is minimal. That is,

$$\Pr\left\{e_{j+1} < m, \#\{(i,h) | i \in J, h \leq e_i, v_{ih} < v_{j+1,e_j+1}\} > m+1\right\} > 0,$$

but

$$\Pr\left\{e_{j+1} < m, \#\{(i,h) | i \in I, h \leq e_i, v_{ih} < v_{j+1,e_{j+1}+1}\} \geq m+1\right\} = 0,$$

for all $I \subset J$, $I \neq J$.

Let

$$v' = \min\{x | \Pr\{e_i < m, v_{i,e_{i+1}} > x\} = 0 \ \forall i \in N \backslash J\}$$

be the upper bound on buy values among players outside of J. Let $i^* \in N \backslash J$ be a player for whom this is binding. Let $\omega > 0$ be chosen so that

$$\Pr\{X\} > \omega$$

where

$$X \equiv \{\#\{(i,h)\,| i \in J, h \leq e_i, v_{ih} < v' - \omega\} \geq m+1\}$$

is the event that there are at least $m+1$ sell values below $v' - \omega$ in J. By the minimality of J, each $i \in J$ has at least one sell value below $v' - \omega$ in this event.

Consider an arbitrary $k \in \{1, 2, \ldots\}$. There is $\omega/k > \delta > 0$ and $\delta' > 0$ such that

$$\Pr\left(\{e_{i^*} < m, v_{i^*,e_i+1} > v' - \delta\} \cap \{\text{no other player has buy value} \geq v' - 2\delta\}\right) > \delta'.$$

That is, δ can be chosen such that there is a positive probability that i^* has a buy value above $v' - \delta$, but no other player has a buy value above $v' - 2\delta$. To see this, note that for any $\delta < \omega/2$, when each $i \in J$ has a sell value below $v' - \omega$, she has (by diminishing marginal utility) no buy values $\geq v' - 2\delta \geq v' - \omega$. And, by definition of v', and given that values are atomless, there is $\delta > 0$ small enough such that for each $i \in J \cup i^*$ there is a positive probability that i has no buy values $\geq v - 2\delta$. Since $\prod_i B_i$ is absolutely continuous with respect to B, there is some probability $\delta' > 0$ of all these things happening at once. By NDB, this implies

$$\Pr_s\{\text{no buy bids other than by } i^* \geq v - 2\delta\} > \delta'$$

for large enough s as well.[33]

For any given ω and δ satisfying the above, let Q_B be the number of buy bids greater than $v' - 2\delta$, and Q_S the number of sell bids at or below $v' - 2\delta$ by anybody other than i^*.

Now, $\Pr_s(Q_S > 0) \to 0$ must hold. To see this suppose the contrary. Note that by bidding at $v' - 3\delta/2$ whenever i^* has a buy value above $v' - \delta$, i^* wins whenever $Q_S > 0$, and all other buy bids are below $v' - 2\delta$. Since $\prod_i B_i$ is absolutely continuous with respect to B, this is a positive probability event, and so this strategy earns positive surplus independent of s, which contradicts the fact that the probability of trade is going to zero.

$\Pr_s(Q_B > 0) \to 0$ must also hold. To see this, note that if $\Pr_s(Q_B > 0) > \gamma$, then at least one of J, say 1, assesses probability at least $\gamma/2$ that one of his opponents makes a buy bid above $v' - 2\delta$, and hence probability at least $M'\gamma/2$ on such a bid conditional on having a sell value below $v' - \omega$. So, 1 gets a surplus of at least $(\omega - 2\delta)$ with probability at least $M'\gamma/2$ which contradicts the fact that the probability of trade is going to zero.

Let $\mu_s \equiv \max_{i \in \{0,\ldots,n\}} \Pr_s\{e_i < m, b_{i,e_i+1} \geq v' - 2\delta\}$ be the maximum probability that any player makes a buy bid above $v' - 2\delta$. By the above, μ_s must go to 0 as $\Pr_s\{Q_B > 0\} \geq \mu_s$.

[33] NDB applies to the limit auction, but then far enough along the sequence the probability of dumb bids is going to zero.

The probability that q players bid above $v' - 2\delta$ is less than $\binom{n}{q}(\mu^s)^q$. It follows that

$$\Pr_s\{Q_B > m\} < \kappa\left((\mu_s)^2 + (\mu_s)^3 + ...\right)$$

where $\kappa < \infty$ takes care of the combinatorial terms. Since $\Pr_s\{Q_B > 0\} \geq \mu_s$, and since $\mu_s \to 0$,

$$\Pr_s(Q_B \leq m | Q_B > 0) \to 1. \tag{14}$$

Now, since $\Pr_s(Q_S > 0) \to 0$, each $i \in J$ is in the limit only submitting sell offers above $v' - 2\delta$. Consider the deviation d_i for any given $i \in J$ that whenever i has a sell value v_{ih} below $v' - \omega$, i submits $b_{ih} = v' - 2\delta$ instead of his equilibrium bid. Let n_i be the random variable giving the number of such sell offers i submits with d_i. Now, since $\Pr_s(Q_S > 0) \to 0$, d_i in the limit sells $\min\{n_i, Q_B\}$ objects with probability 1, at a price at least $v' - 2\delta$.

For each Q_B, i only sells as many objects as he has bids among the Q_B lowest. So, by deviating, he sells an extra object (or objects) whenever his bid under the equilibrium strategy was among the Q_B lowest. Let i_L be the event that i has a sell bid not among the m lowest. Then, in particular, d_i sells at least one extra object whenever $\{X, i_L, 0 < Q_B \leq m\}$. Let y_i be this event. Thus, the deviation gains at least $\omega - 2\delta$ with probability at least $\Pr_s(y_i)$.

On the other hand, d_i only results in a worse outcome for i than the equilibrium strategy when i would already have sold objects, and in this event, the cost is at most $m2\delta$ and occurs with probability bounded by $\Pr(Q_B > 0)$. Since the utility function has bounded derivative, there exists $\Lambda < \infty$ such that $U'(x)/U'(y) < \Lambda$ for all x and y. So, for the deviation not the be profitable, it must be that

$$\Pr_s(y_i)(\omega - 2\delta) < \Pr_s(Q_b > 0) 2m\delta\Lambda$$

and so

$$(\omega/\delta - 2)\Pr_s(y_i) / \Pr_s(Q_b > 0) < 2m\Lambda, \tag{15}$$

where the division is well defined for s large, because for each s large, player $n+1$ sometimes makes a buy bid above $v' - 2\delta$.

Since, $\Pr(X) > \omega$, $\Pr_s(X|0 < Q_b \leq m) > M'\omega$. Hence,

$$\begin{aligned}
\Pr_s(y_i)/\Pr_s(Q_b &> 0) = \Pr_s(X, i_L,\ 0 < Q_b \leq m)/\Pr_s(Q_b > 0) \\
&= \Pr_s(X, i_L,\ 0 < Q_b \leq m | Q_b > 0) \\
&= \Pr_s(i_L|X, 0 < Q_b \leq m)\Pr_s(X|0 < Q_b \leq m)\Pr_s(0 < Q_b \leq m|Q_b > 0) \\
&\cong \Pr_s(i_L|X, 0 < Q_b \leq m)\Pr_s(X|0 < Q_b \leq m) \\
&\geq \Pr_s(i_L|X, 0 < Q_b \leq m)M'\omega.
\end{aligned}$$

where, from (14), the approximation is arbitrarily good when s is large. Substituting and summing (15) over $i \in J$, we have

$$(\omega/\delta - 2)\sum_{i\in J}\Pr_s(i_L|X, 0 < Q_b \leq m)M'\omega < 2jm\Lambda \tag{16}$$

for large s.

But, in event X there are at least $m+1$ sell values below $v' - \omega$ in J, and hence when $Q_b \leq m$, with probability 1 someone in J goes away with an unsold unit. Hence, (16)implies

$$(\omega/\delta - 2)\, M'\omega < 2j\Lambda m.$$

But, $\omega/\delta \geq k$, and so, since k was arbitrary, we have a contradiction. ■

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