# Learning and the Wisdom of Crowds in Networks

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Calit2,  UCSD Nov 2, 2007

# Introduction

- Beliefs, opinions, behaviors are all influenced by social networks
- When do beliefs and behaviors converge over time?
- How does convergence/speed depend on the social structure?
- Who is influential?
- When do naïve groups become wise as a whole?

# Related Literature

- DeGroot (1974) – sufficient conditions for consensus
  - model variations: Berger (1981), Friedkin and Johnsen (1997), Krause (1997), Deffuant, Weisbuch et al (2000), DeMarzo, Vayanos, Zweibel (2003), Lorenz (2005)
- Observational Learning – Bala Goyal (1998)
  - see neighbors actions and payoffs, consensus reached in decisions
  - + optimism implies correct decision

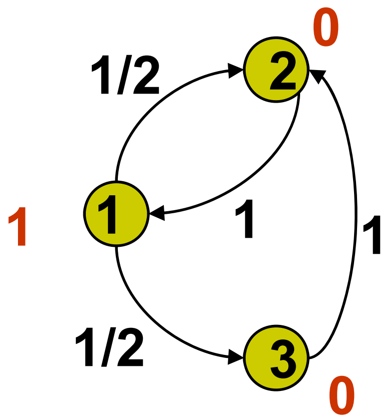
# Questions

- **Social structure missing from above answers**
- When do beliefs and behaviors converge over time (when no new observations)?
- How does convergence/speed depend on the social structure?
- Who is influential?
- When do naïve groups become wise as a whole?

# DeGroot Social Interaction Model

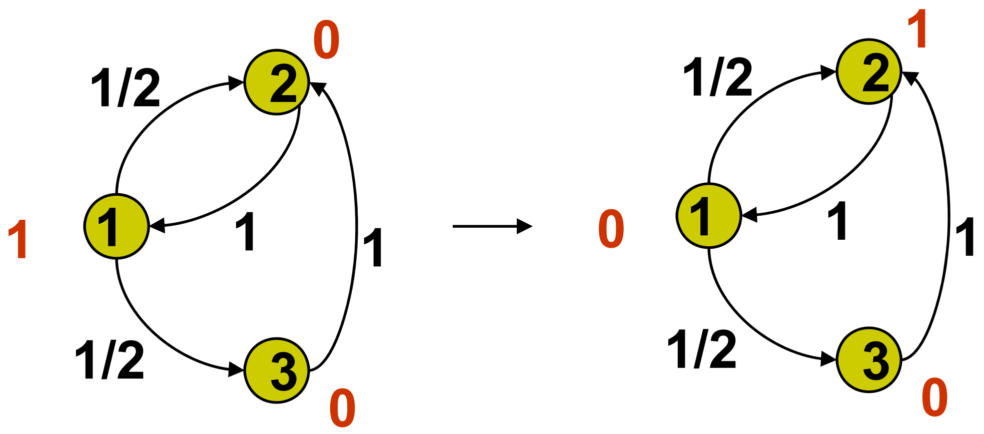
- Individuals {1, ... n}
- **T** weighted directed network, stochastic matrix
  - For talk – take to be strongly connected
- Start with beliefs, attitude, etc. $p_i(0)$ in [0,1]
  - can also have these be vectors…
- Updating: $p_i(t) = \sum_j T_{ij}\, p_j(t-1)$

# Example




$$T = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

# Example



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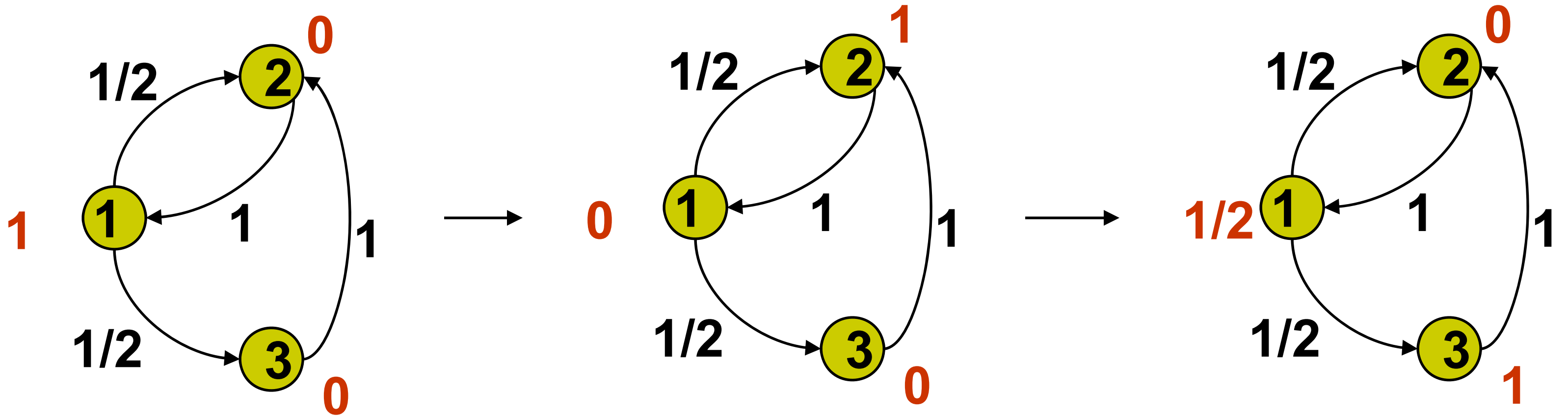

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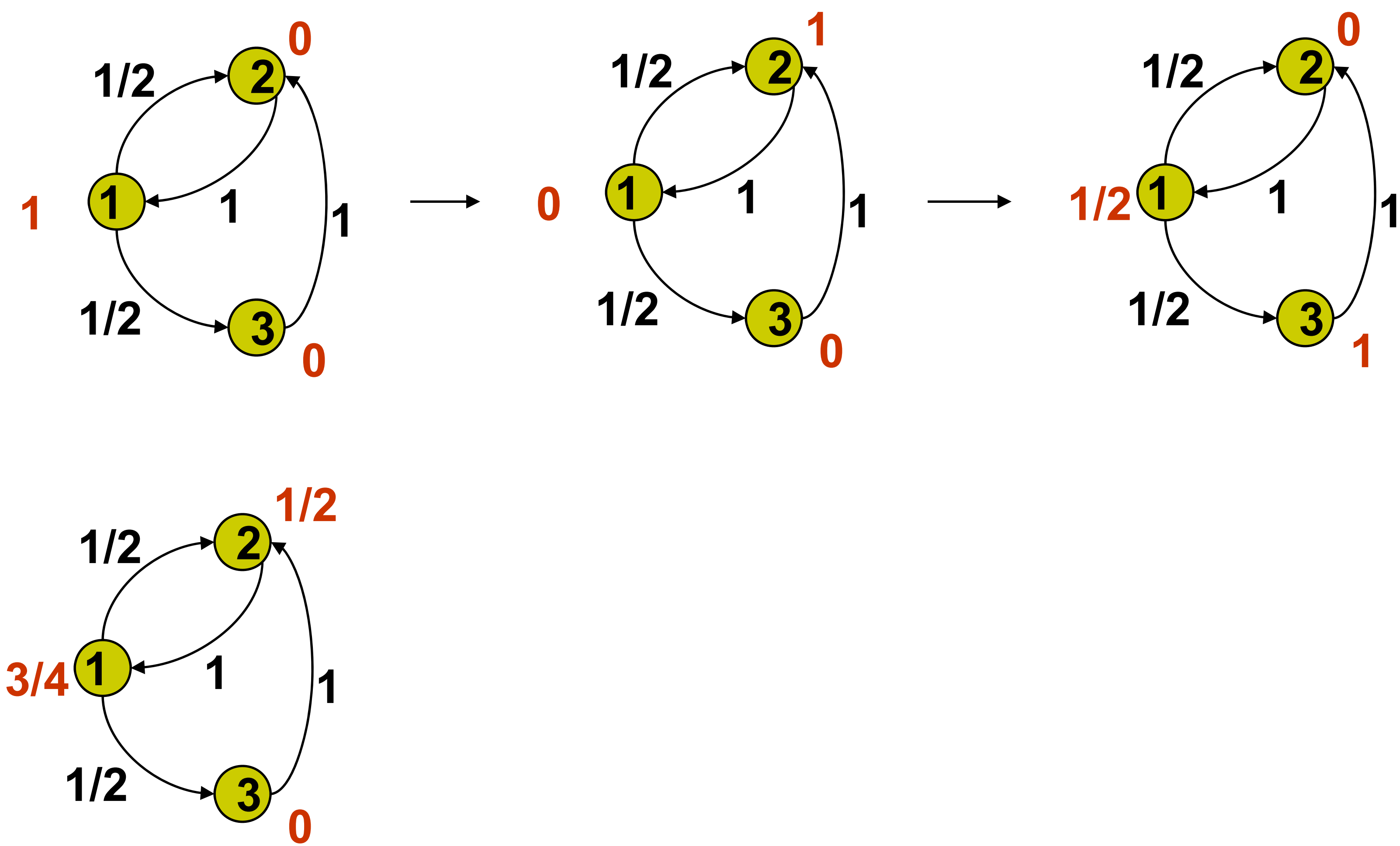

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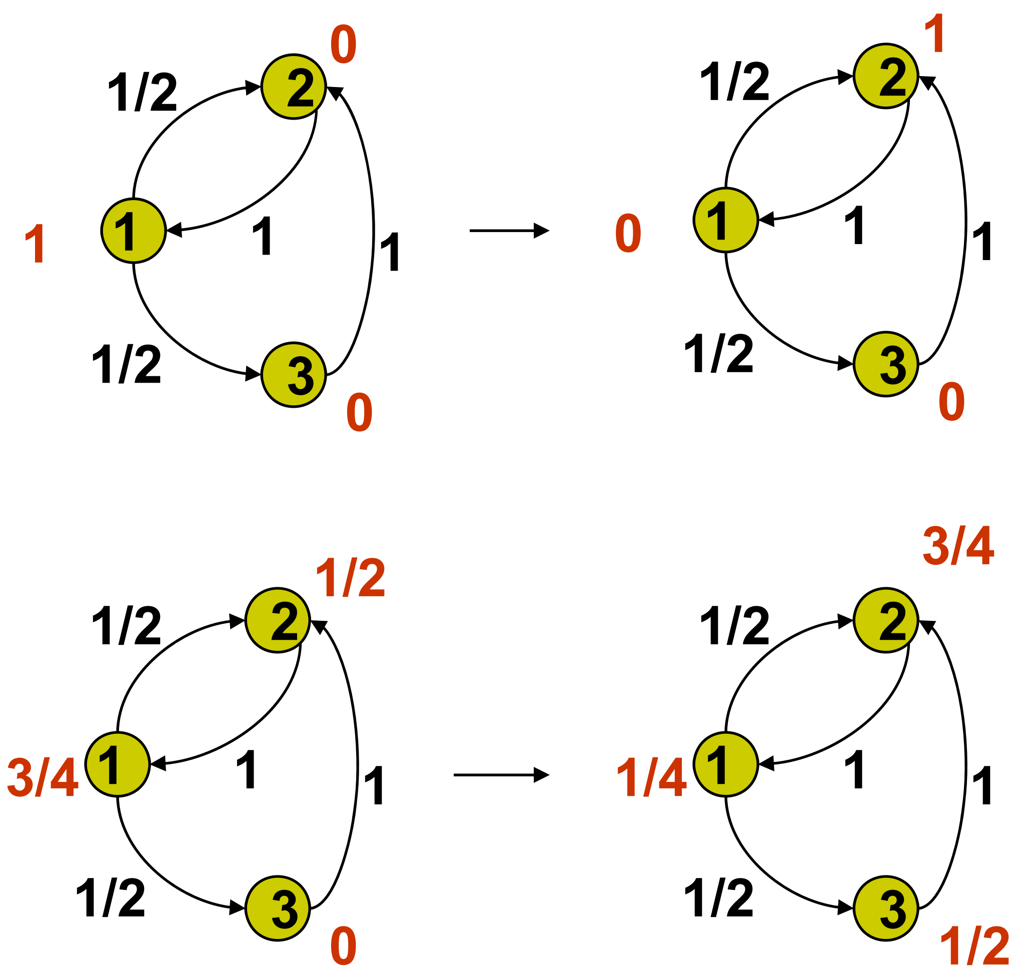
# Example



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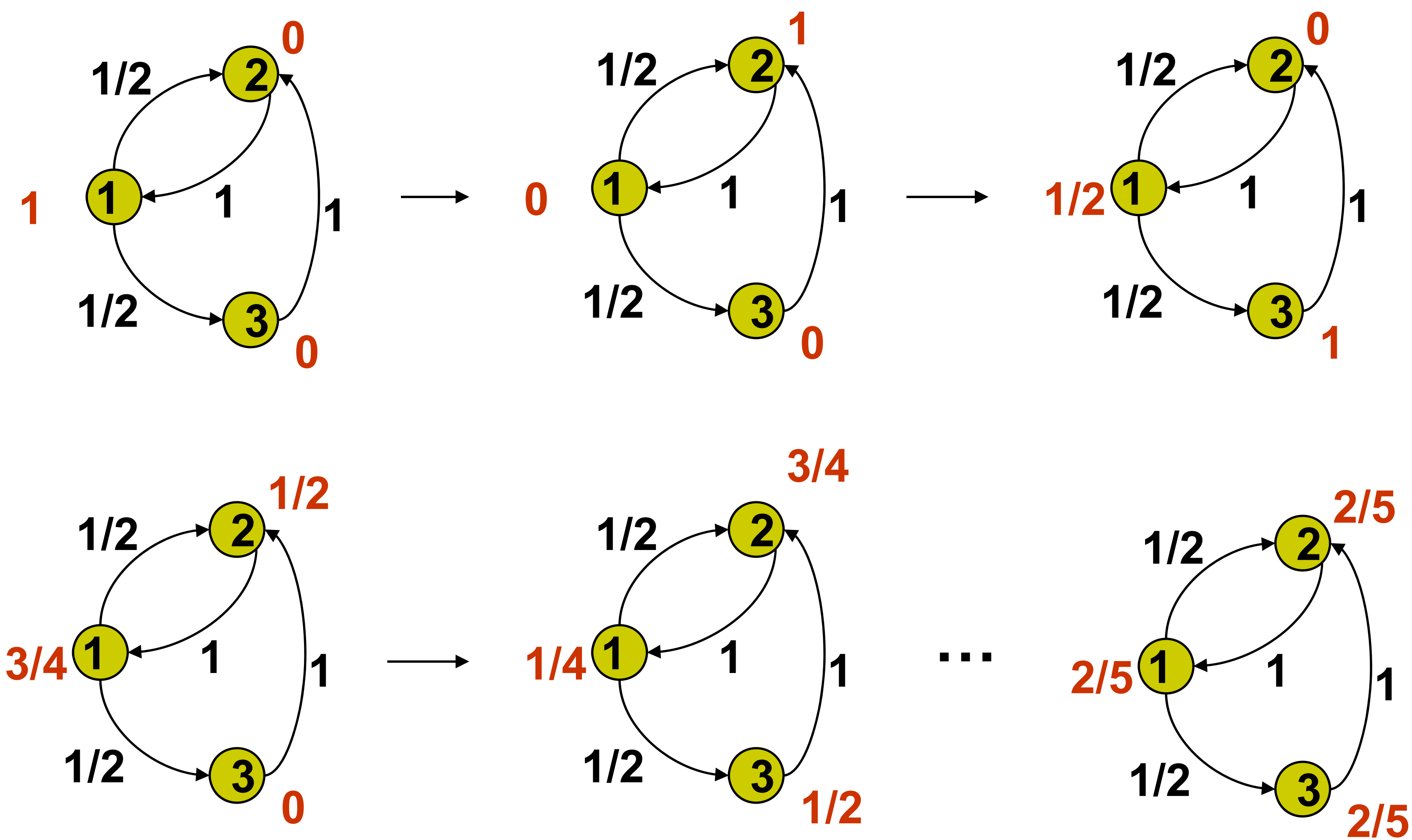
# Example



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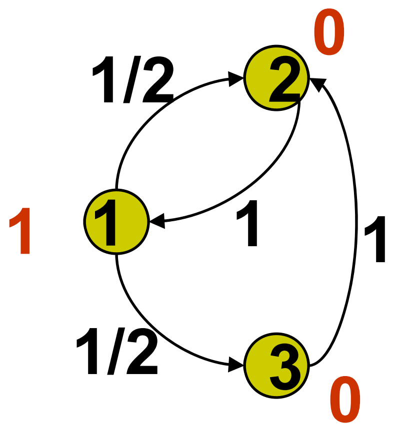
# Example - Convergence:




$$T=\begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$p(0)=\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

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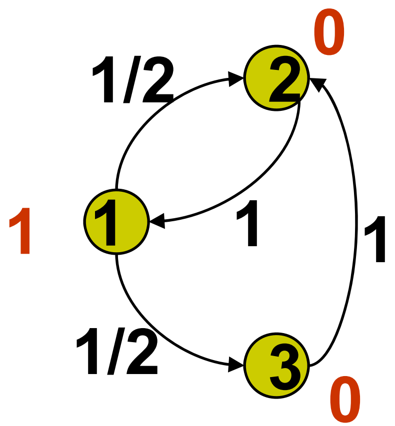
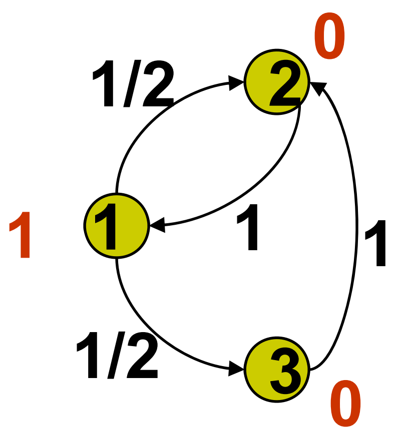



$$T = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$p(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad p(1) = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

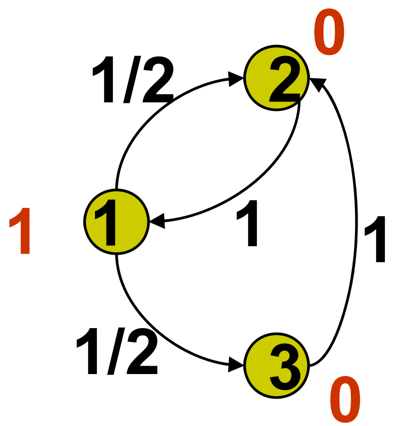
# Example - Convergence:




$$T=\begin{pmatrix} 0 & ½ & ½ \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$p(1)=\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad p(2)=\begin{pmatrix} 0 & ½ & ½ \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 0 \\ 1 \end{pmatrix}$$

# Example - Convergence:




$$T = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\text{Limit} \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2/5 \\ 2/5 \\ 2/5 \end{pmatrix}$$

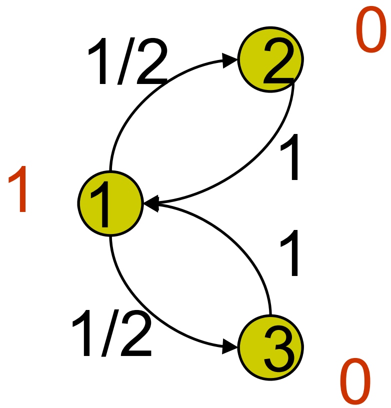
# Applications:

- Myopic Best reply when want to match weighted average of neighbor’s behavior
- Social influence (Friedkin and Johnsen…)
- Centrality measures – (Katz, Bonacich), etc.
- Google page-rank...,power depends on power of neighbors
- Recursive utility calculations (Rogers)
- ...

# Outline / Questions

- How does convergence depend on social structure, **T**?
- How much influence does each agent have?
- When do beliefs become accurate?
- How quickly do beliefs converge?

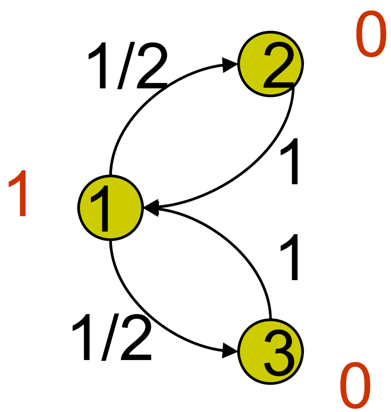
# Example – No Convergence:




$$T=\begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$p(0)=\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

# Example – No Convergence:

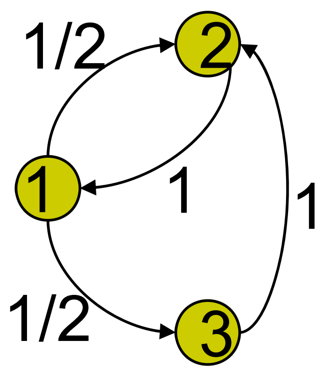



$$T=\begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

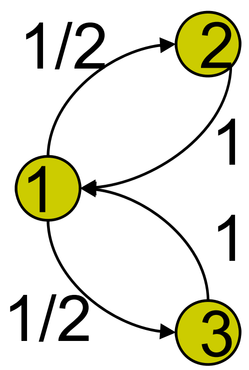
$$p(0)=\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad p(1)=\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cdots \rightarrow$$

# Convergence and Consensus

- T *converges* if $\lim T^t p$ exists for all p
- T is *aperiodic* if the greatest common divisor of its simple directed cycle lengths is one




aperiodic




periodic

# Preliminaries: Theorems on Convergence and Consensus

**T is convergent if and only if it is aperiodic.**

**Agents reaches a consensus if and only T is aperiodic.**

``If'' is standard (Perron-Frobenius+Stochastic Matrix Thm)

``Only if'' by algorithm constructing nonconvergent p

Consensus: look at min or max belief

# Influence:

$$\text{Limit} \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2/5 \\ 2/5 \\ 2/5 \end{pmatrix}$$

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# Example

$$T = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad T^2 = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \end{pmatrix}$$

$$T^3 = \begin{pmatrix} 1/2 & 1/4 & 1/4 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{pmatrix} \qquad T^4 = \begin{pmatrix} 1/4 & 1/2 & 1/4 \\ 1/2 & 1/4 & 1/4 \\ 1/2 & 1/2 & 0 \end{pmatrix}$$

$$T^5 = \begin{pmatrix} 1/2 & 3/8 & 1/8 \\ 1/4 & 1/2 & 1/4 \\ 1/2 & 1/4 & 1/4 \end{pmatrix} \qquad T^\infty = \begin{pmatrix} 2/5 & 2/5 & 1/5 \\ 2/5 & 2/5 & 1/5 \\ 2/5 & 2/5 & 1/5 \end{pmatrix}$$

# Influence Measure

- Look for a row vector **s** indicating the *relative influence* each agent has – limit belief is **s p**

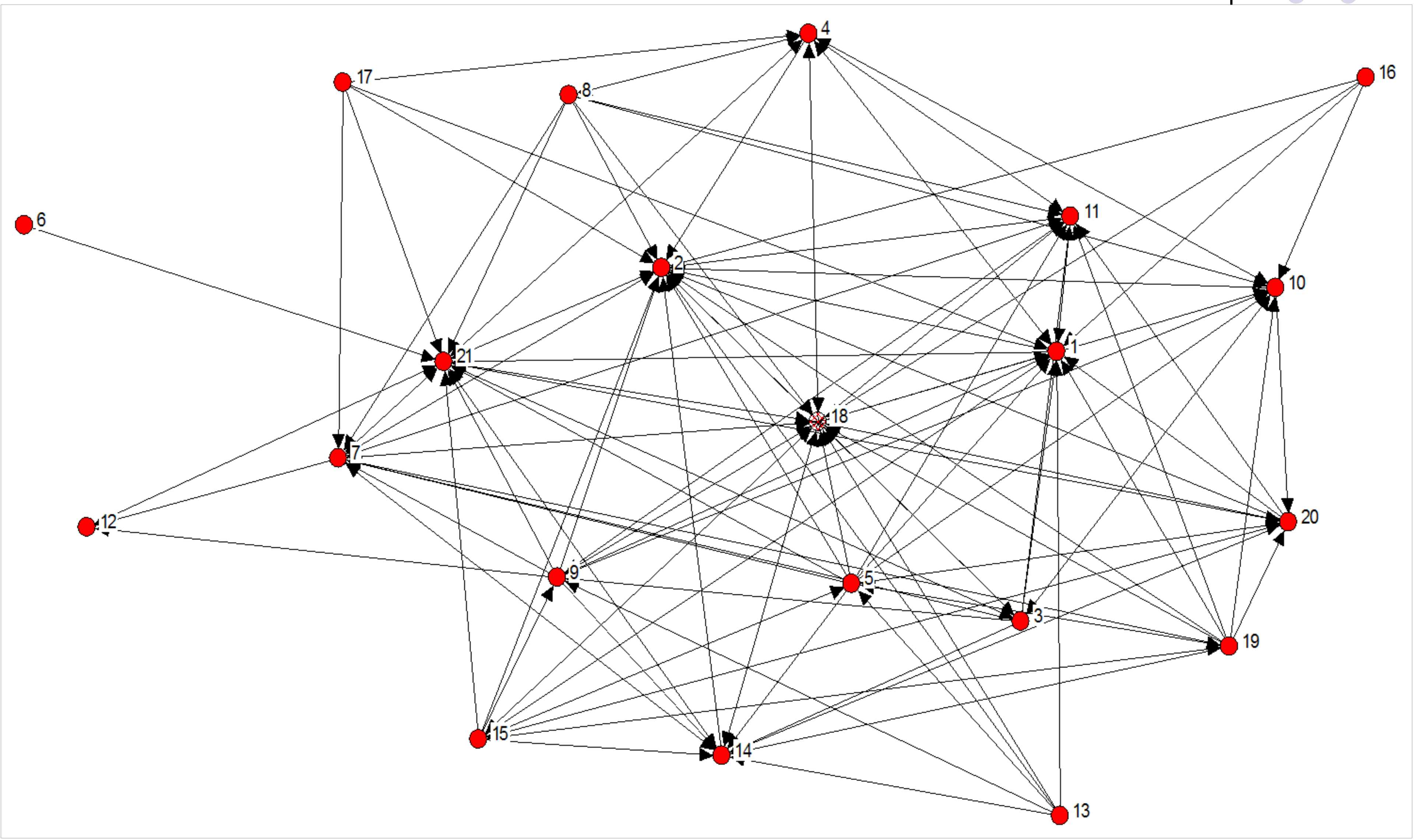
- Note that **s p** = **s T p**

- So, **s** = **s T** : **s** is the left unit eigenvector

# Who has influence?

- $s_i = \sum_j T_{ji} s_j$
- High influence from being paid attention to by people with high influence...
- Again, page ranks…

# Krackard's advice network:



| label | s | level | dept. | age | tenure |
|---|---|---|---|---|---|
| 1 | 0.043 | 3 | 4 | 33 | 9.3 |
| 2 | 0.132 | 2 | 4 | 42 | 19.6 |
| 3 | 0.039 | 3 | 2 | 40 | 12.8 |
| 4 | 0.052 | 3 | 4 | 33 | 7.5 |
| 5 | 0.002 | 3 | 2 | 32 | 3.3 |
| 6 | 0.000 | 3 | 1 | 59 | 28 |
| 7 | 0.143 | 1 | 0 | 55 | 30 |
| 8 | 0.007 | 3 | 1 | 34 | 11.3 |
| 9 | 0.015 | 3 | 2 | 62 | 5.4 |
| 10 | 0.024 | 3 | 3 | 37 | 9.3 |
| 11 | 0.053 | 3 | 3 | 46 | 27 |
| 12 | 0.051 | 3 | 1 | 34 | 8.9 |
| 13 | 0.000 | 3 | 2 | 48 | 0.3 |
| 14 | 0.071 | 2 | 2 | 43 | 10.4 |
| 15 | 0.015 | 3 | 2 | 40 | 8.4 |
| 16 | 0.000 | 3 | 4 | 27 | 4.7 |
| 17 | 0.000 | 3 | 1 | 30 | 12.4 |
| 18 | 0.106 | 2 | 3 | 33 | 9.1 |
| 19 | 0.002 | 3 | 2 | 32 | 4.8 |
| 20 | 0.041 | 3 | 2 | 38 | 11.7 |
| 21 | 0.201 | 2 | 1 | 36 | 12.5 |

# Wise Crowds:

- Suppose true state is $\mu$
- Agent i sees $p_i(0) = \mu + \varepsilon_i$
- $\varepsilon_i$ has 0 mean and finite variance, bounded below and above,
- signal distributions may differ across agents, but are independent conditional on $\mu$

# Wise Crowds

- Let the society grow

- If they pooled their information, they would have an accurate estimate of μ

- When does Prob $[\ |\sum_i s_i^n p_i(0) - \mu| > \delta\ ] \to 0$ for all δ?

# A WLLN Lemma:

$$\text{plim} \sum s_i^n p_i(0) - \mu \quad = \text{plim} \sum s_i^n \varepsilon_i \rightarrow 0$$

*if and only if* $\max_i s_i^n \rightarrow 0$

Proof: Chebyshev

wise crowds if and only if max influence vanishes

# Proposition

Suppose that T is column stochastic (so each agent receives weight one). Then s=(1/n,...1/n), and so T is wise.

So, **reciprocal trust** implies wisdom.

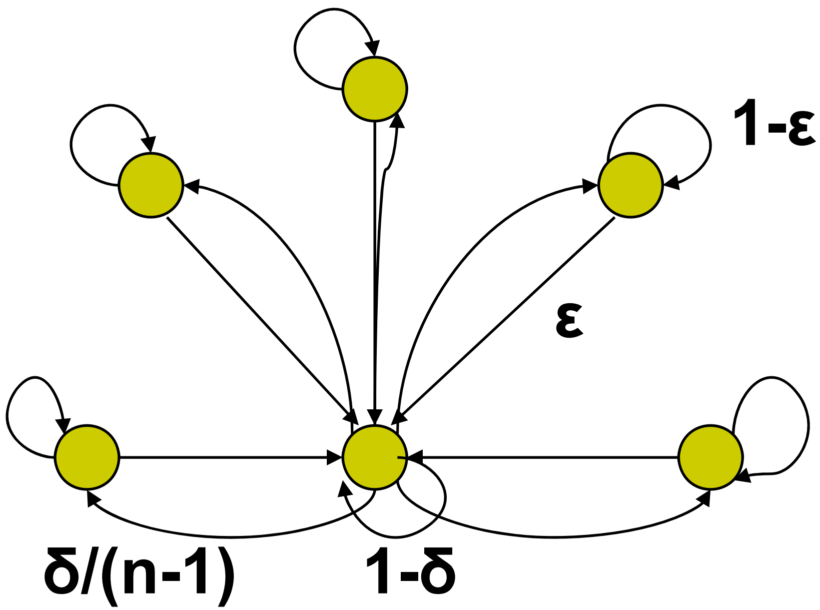
But that is a very strong condition…

- $s_i = \sum_j T_{ji} s_j$

- If there is some i with $T_{ji} > a > 0$ for all j, then $s_i > a$

- So clearly cannot have too strong an ``opinion leader''

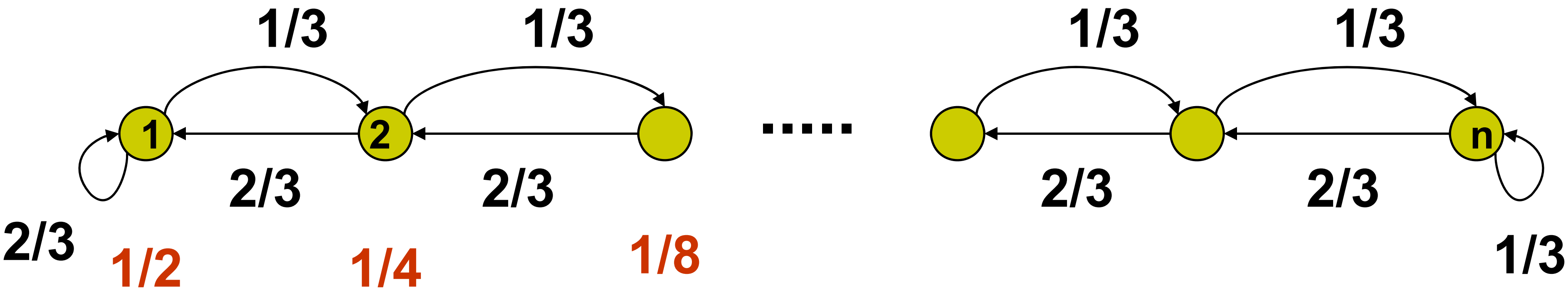
# Even small weights add up




$$s_{hub} = \varepsilon / (\varepsilon + \delta)$$

- Hub agent has too much influence – relative weights matter

# What if bound relative weight to any agent?




Problem: *relative* indirect weight matters

# Sufficient Conditions for Wise Crowds:

## Balance:

$|B_n| < n/2$ implies $\sup T^n_{B_n^c B_n} / T^n_{B_n B_n^c} < \infty$

**No group of less than half society can be getting infinitely more weight in than it puts out**

(rules out unboundedly large hubs, groups)

# Minimal Out Dispersion

Exists ε, x such that |A|>x, |B|>n/2 implies $T^n_{AB} > \varepsilon$

**Any large enough group must give some minimal weight to a group of more than half the society**

(rules out the uneven ``line'' we saw)

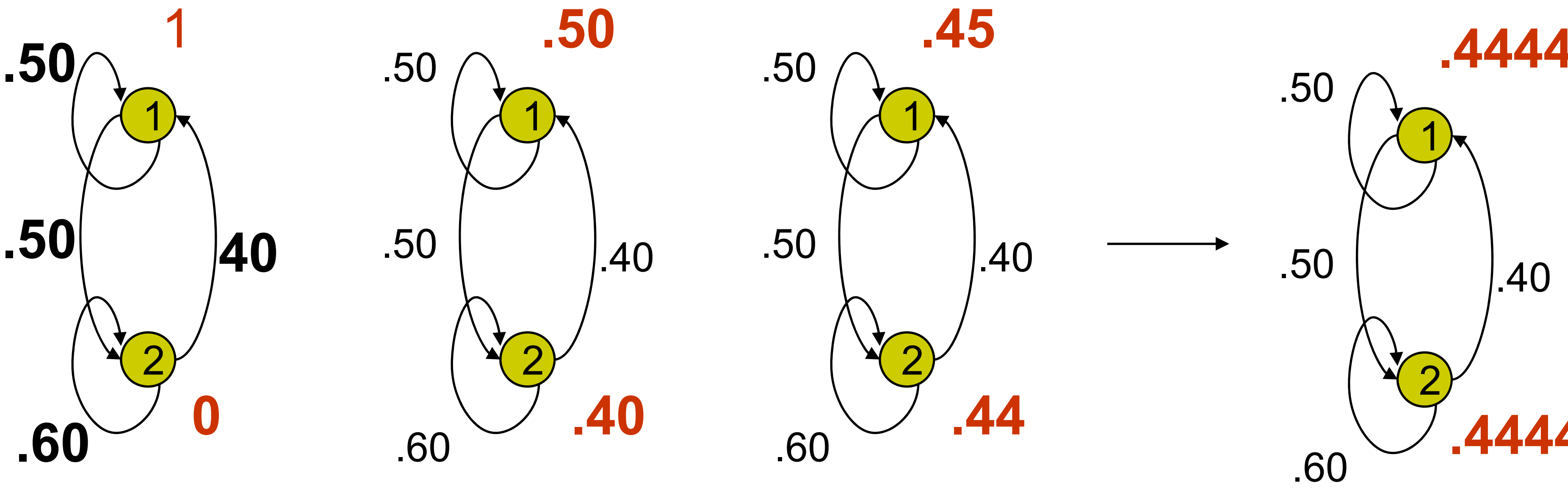
# Theorem – Sufficient conditions for wise crowds

Balance and minimal out dispersion are sufficient for wise crowds.

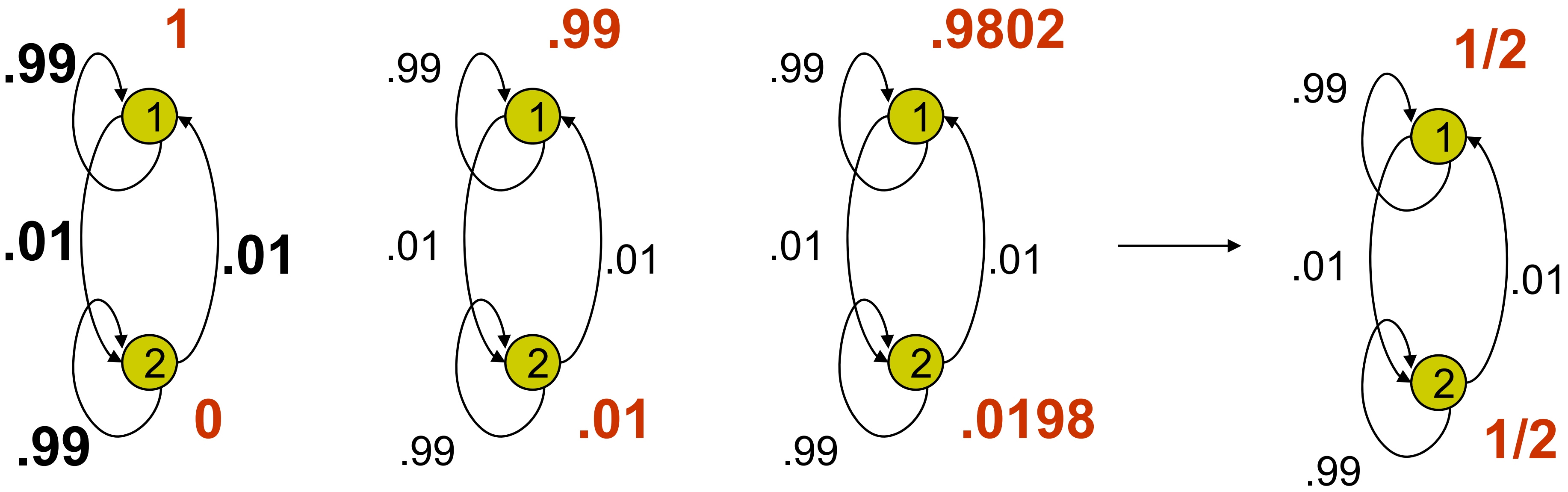
So, need influence to die out for all agents:

- Small group cannot get too much more weight in than it sends out
- Large enough groups must give some minimal weight to very large groups

# How Fast is Convergence?



# Slow convergence



# Theorem (adapted from Markov chain results (e.g., Seneta 1973))

Generically, if T is strongly connected and aperiodic, with second largest eigenvalue $\lambda_2$ then exist C,c, :

- $|p_i(\infty)-p_i(t)| \leq C \, |\lambda_2|^t$

There exist p(0)'s and i's such that

- $|p_i(\infty)-p_i(t)| \geq c \, |\lambda_2|^t$

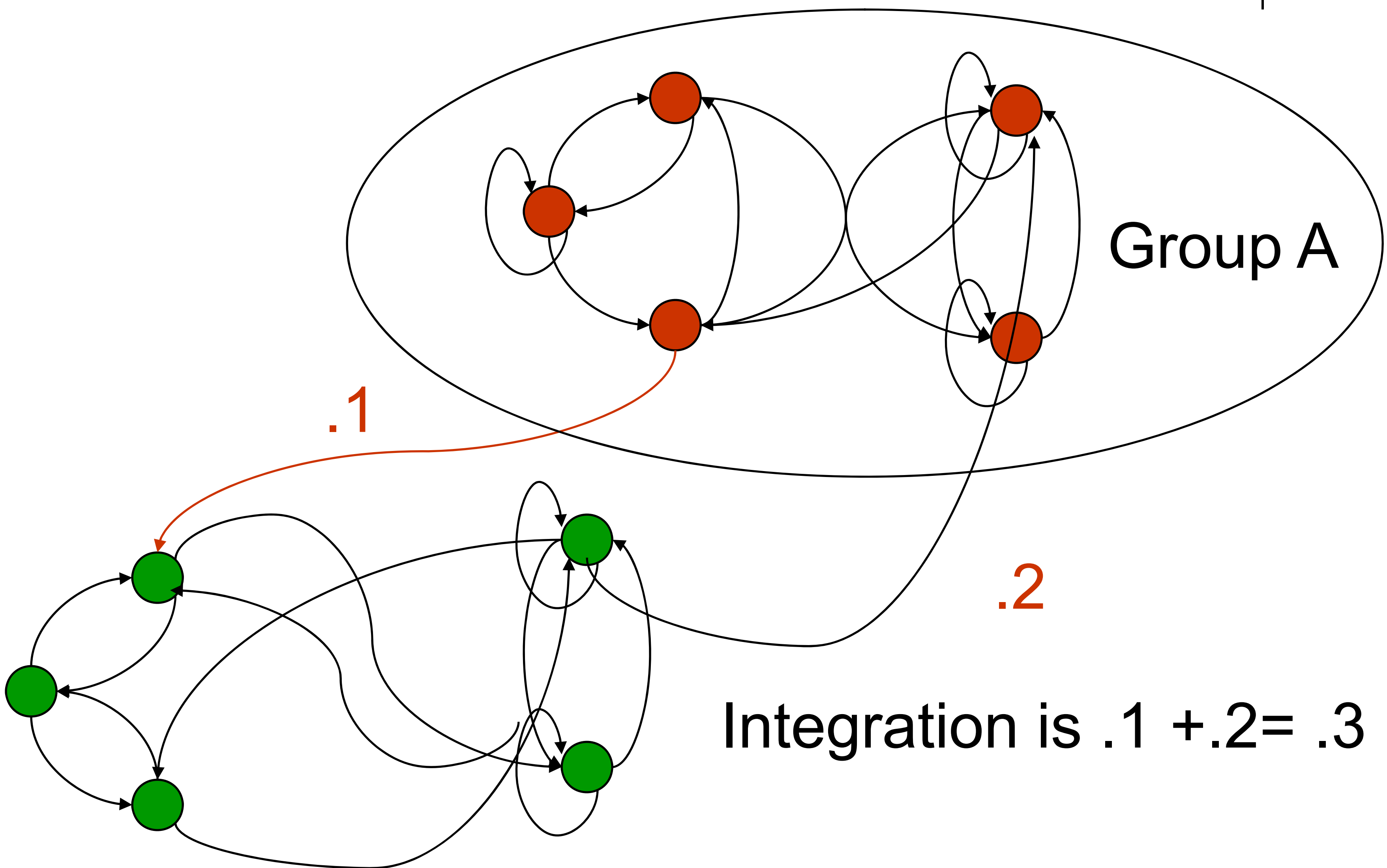
# Krackhardt's network

- second eigenvalue is .48
- upper bound on C is n-1
- so distance is at most 20 $(.48)^t$
- less than .02 by t=10, .00002 by t=20

- So ``fast'' when second eigenvalue is low
- ``slow'' (at least for initial periods) when second eigenvalue is high
- What does second eigenvalue capture??

# A measure of integration:

$$\mathrm{Int} = \min_{A,\, B \text{ disjoint}} \left(T_{A,N-A} + T_{B,N-B}\right)$$




Integration is .1 +.2= .3

# Theorem

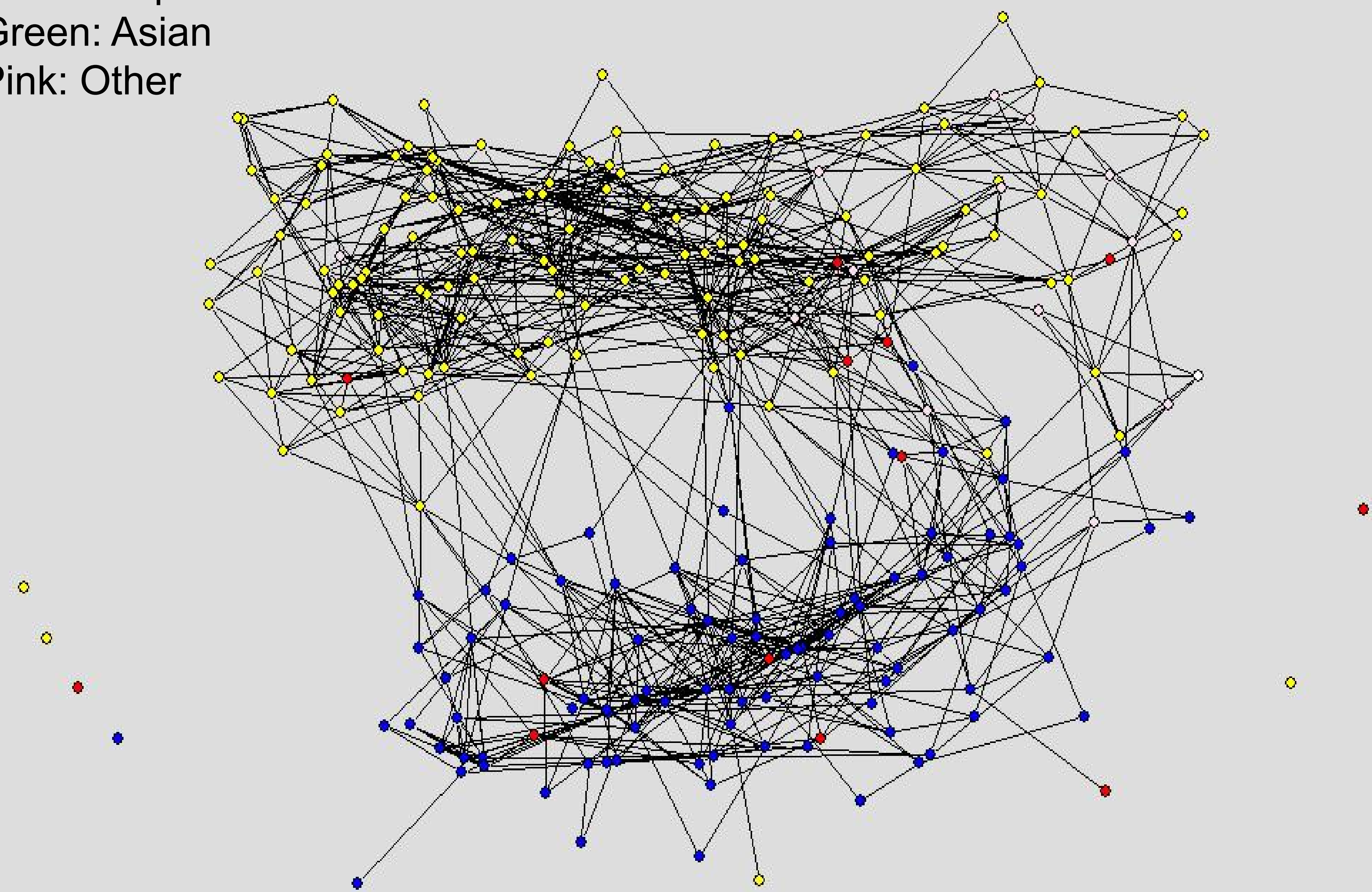
Second eigenvalue is close to 1 if and only if integration is close to 0.

Convergence will be ``slow'' if and only if integration is close to 0.

[Note: integration = 1-second eigenvalue when n=2, bounds it more generally (Use a theorem on stochastic matrices and eigenvalues by Hartfiel and Meyer (1998)).]

Yellow: Whites
Blue: Blacks
Reds: Hispanics
Green: Asian
Pink: Other

# Int<.03



# Summary

- **Convergence** if and only if **aperiodicity**
- **Limiting influence** related to eigenvectors and weights from **influential neighbors**
- **Wise crowds**:  nobody retains too much influence
  - **balance** between groups
  - **dispersion** out, or limit on relative weight in
- **Convergence is ``slow''** if and only if there are at least two **mutually distrustful groups**