Diffusion of Behavior and Equilibrium Properties in Network Games

Matthew O. Jackson*† Leeat Yariv‡

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Abstract

We analyze games on social networks where agents select one of two actions (whether or not to adopt a new technology, withdraw money from the bank, become politically active, etc.). Agents' payoffs from each of the two actions depend on how many neighbors she has, the distribution of actions among her neighbors, and a possibly idiosyncratic cost for each of the actions. We analyze the diffusion of behavior when in each period agents choose a best response to last period's behavior. We characterize how the equilibrium points of such a process and their stability depend on the network architecture, the distribution of costs, and the payoff structure. We also illustrate how the dynamics of behavior depends on the number of neighbors that agents have. Our results have implications and applications to marketing, epidemiology, financial contagions, and technology adoption.

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†Department of Economics, Stanford University, Stanford CA 94305, USA. http://www.stanford.edu/∼jacksonm e-mail: jacksonm@stanford.edu

‡Division of the Humanities and Social Sciences, Caltech, Pasadena, CA 91125. http://www.hss.caltech.edu/˜lyariv/index.htm e-mail: lyariv@hss.caltech.edu

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1 Introduction

Situations in which agents' choices depend on choices of those in close proximity, be it social or geographic, are ubiquitous. Selecting a new computer platform, signing a political petition, or even catching the flu are examples in which social interactions have a significant role in one's ultimate choice or state. Some behaviors or states propagate and explode within the population (e.g., the Windows OS, the HIV virus, etc.[1]) while others do not (e.g., certain computer viruses[2]). Our goal in this paper is twofold. First, we provide a general dynamic model in which agents' choices depend on the underlying social network of connections. Second, we show the usefulness of the model in determining when a given behavior expands within a population or disappears as a function of the environment's fundamentals.

In more detail, we start with a framework in which agents each face a choice between two actions, 0 and 1 (e.g., whether or not to pursue a certain level of education, switch to the Linux OS, sign a petition, or go on strike). Agents are linked through a social network, modeled by a graph. An agent's payoffs from each action depends on her location within the network (specifically, the number of neighbors she has), her neighbors' choices, and a random cost determined at the outset. The diffusion process is defined so that at each period, the agent best responds to the actions taken by her neighbors in the previous period, assuming that her neighbors follow the population distribution of actions (a mean-field approximation). For instance, if actions are strategic complements, then over time as an agent's neighbors become more inclined to choose the action 1, she becomes more inclined to switch to the action 1 as well. Steady states correspond to equilibria of the static game. Under some simple conditions, equilibria take one of two forms. Some are stable, so that a slight perturbation to any such equilibrium would lead the diffusion process to converge back to that equilibrium point. Other equilibria are unstable, so that, for example, a slightly higher level of 1 adopters ultimately leads to an increase in the number of action 1 adopters (and eventually to a stable

[1]See Gladwell (2000) and Rogers (1995).

[2]See virus prevalence data at http://www.virusbtn.com/, and summarized statistics in Pastor-Satorras and Vespignani (2000).

steady state). Such equilibria are called *tipping points*.

There are four main insights that come out of our inquiry that relate the social network and the payoff structure to the set of equilibria. *First*, increasing the costs of action 1 (in terms of a First Order Stochastic Dominance shift of the cumulative distribution of costs) increases tipping points and decreases the stable equilibria levels of action 1 adopters. Increasing costs thus makes it harder for behavior to diffuse, and leads to lower overall levels of action 1 adoption in a stable equilibrium. If there are positive externalities, this means that average welfare decreases. *Second*, a shift of the degree distribution of the social network generates higher probabilities of encountering neighbors of higher degrees, which in the case of strategic complements yields lower tipping points and higher stable equilibria, so that behavior diffuses more easily and converges to higher equilibrium levels. *Third*, under some assumptions on the cost distribution, increasing the variance of degrees similarly lowers tipping points and raises the stable equilibria of the system. *Fourth*, we identify conditions which ensure that the level of diffusion in the society exhibits an "S shape," well-documented in the empirical literature on diffusion, where adoption initially speeds up until a threshold point in time at which it starts slowing down. In terms of location within the network, agents of different levels of connectedness exhibit different adoption paths as well as different ultimate levels of adoption. This hints at the importance of understanding underlying network parameters when empirically estimating fundamentals such as revenues to a new action or cost distributions.

There are numerous potential applications of this framework, ranging from marketing (e.g., understanding which consumers should optimally be targeted with advertising), to financial markets (gaining insights on when bank runs, not necessarily encompassing the entire population, are formed), to epidemiology (establishing the fundamental characteristics for the evolution of an epidemic or spread of a computer virus), to politics (identifying the underlying attributes conducive to political uprise, providing a first step toward understanding the structure of information transmission). Given that social networks differ substantially and systematically in structure across settings (e.g., ethnic groups, professions, etc.), understanding the implications of social structure on diffusion is an important undertaking.

In terms of related work, there is a previous literature on spread of diseases on social networks (e.g., see Pastor-Satorras and Vespignani (2000, 2001), Newman (2002), Lopez-Pintado (2004), Jackson and Rogers (2007)).[3] This enriches the analysis in terms of distinguishing between tipping points and stable equilibria, and in terms of moving from a mechanical spread of disease to a strategic interaction; and also provides general results on how social structure impacts diffusion.

There is also a recent literature examining games on networks (e.g., Glaeser and Scheinkman (2000), Kearns, Littman and Singh (2000), Morris (2000), Galeotti, Goyal, Jackson, Vega-Redondo, and Yariv (2005), henceforth GGJVY, Lopez-Pintado and Watts (2005), and Sundararajan (2006)). The analysis here is a natural complement to the analysis of static equilibria of network games (e.g., Glaeser and Scheinkman (2000) or GGJVY) in that it expands our understanding of the diffusion of behavior and of the stability properties of equilibria, as well as enriches the set of applications that are covered. The closest analog in approach to that taken here is Jackson and Yariv (2005). They also analyze the influence of network structure on diffusion of behavior in network games. Their results, however, apply more narrowly in terms of payoff structures and cost distributions. Furthermore, the results here provide for far more extensive characterizations of the stability of equilibria, the general methodology, and the diffusion over time.

[3]There is also a large literature on diffusion (see Rogers (1995) for a comprehensive survey), including some analysis of stability of equilibria (e.g., Granovetter (1978)). Again, our contribution is the examination of the impacts of the social structure on stability and diffusion. The same is true regarding the connection to various studies of the stability of equilibria in economic settings (e.g., see Fisher (1989) for an overview of the stability of general equilibria).

2 The Model

2.1 Social Networks and Payoffs

We consider a set of agents and capture the social structure by its underlying network.[4] We model the network through the distribution of the number of neighbors, or *degree*, that each agent has. Agent i's degree is denoted d_i. The fraction of agents in the population with d neighbors is described by the degree distribution $P(d)$ for $d = 0, 1, ..., D$ (with the possibility that $D = \infty$), where $\sum_{d=1}^{D} P(d) = 1$.

Let

$$\widetilde{P}(d) \equiv \frac{P(d)d}{\overline{d}},$$

where $\overline{d} = E_P[d] = \sum_d P(d)d$. This corresponds to the usual calculation of the probability of the degree of an agent conditional on that agent being at the end of a randomly chosen link in the network.

Agents each have a choice between taking an action 0 or an action 1. Without loss of generality, we consider the action 0 to be the default behavior (for example, the status-quo technology). Agent i has a cost of choosing 1, denoted c_i. Costs are randomly and independently distributed across the society, according to a distribution H^c. Let $u_{d_i}(a, x)$ denote the utility of agent i of degree d_i when she takes the action $a \in \{0, 1\}$ and expects each of her neighbors to independently choose the action 1 with probability x. Agent i's added payoff from adopting behavior 1 over sticking to the action 0 is then

$$v(d_i, x) - c_i,$$

where

$$v(d_i, x) = u_{d_i}(1, x) - u_{d_i}(0, x).$$

This captures how the number of neighbors that i has, as well as their propensity to choose

[4] The cardinality of the set of agents is not invoked in our analysis, as we study a Bayesian equilibrium structure where agents are unsure of the larger network in which they reside.

the action 1, affects the benefits from adopting 1. Thus, i prefers to choose the action 1 if

$$c_i \leqslant v(d_i, x). \tag{1}$$

For some applications it is convenient to normalize $u_d(0,x) = 0$ for all d and x. This normalization is with some loss of generality in terms of welfare conclusions (see below).

In general, it will be useful to distinguish between two classes of utilities. We say that the game exhibits *positive externalities* if for each d, and for all $x \geqslant x'$, $u_d(a,x) \geqslant u_d(a,x')$, $a = 0, 1$. Analogously, we say that the game exhibits *negative externalities* if for each d, and for all $x \geqslant x'$, $u_d(a,x) \leqslant u_d(a,x')$, $a = 0, 1$.

Let $H(d,x) \equiv H^c(v(d,x))$. In words, $H(d,x)$ is the probability that a random agent of degree d chooses the action 1 when anticipating that each neighbor will choose 1 with an independent probability x.

We illustrate the generality of the framework by noting a few special cases.

Examples

- $v(d,x) = u(dx)$, so that an agent's payoffs are a function of the expected number of neighbors adopting the action 1. This is a reasonable assumption in cases such as the learning of a new language, the adoption of a communication technology (skype, google talk, etc.), and so on.[5]
- $v(d,x) = u(x)$, so that agents care only about the average play of their neighbors. In this case, the dependence on degree is lost and network structure plays no role.
- $v(d,x)$ is a step function, for instance taking one value if x lies below a threshold (where the threshold can depend on d), and taking another value if x exceeds the threshold. A metaphor of a political revolt fits this structure, where an agent joins a revolt if either the *number* of neighbors (e.g., Chwe (2000)) or *fraction* of neighbors (e.g., Granovetter (1978)) surpasses a certain threshold.

[5]Formally, this corresponds to the framework analyzed in GGJVY.

2.2 Bayesian Equilibrium

We consider *symmetric Bayesian equilibria* of the network game as follows:

1. Each agent i knows only her own degree d_i and cost c_i, the distribution of degrees in the population, and assumes that degrees and cost parameters are independently allocated. Thus, the game is a Bayesian game in the Harsanyi sense where types are given by degrees and costs.

2. The play is symmetric in the sense that any agent perceives the distribution of play of each of her neighbors to be independent and to correspond to the distribution of play in the population.[6]

Existence of symmetric Bayesian equilibria follows standard arguments. In cases where v is non-decreasing in x for each d, it is a direct consequence of Tarski's Fixed Point Theorem. In fact, in this case, there exists an equilibrium in pure strategies. In other cases, provided v is continuous in x for each d, we can still find a fixed point by appealing to standard fixed point theorems (e.g., Kakutani) and admitting mixed strategies.[7]

A simple equation is sufficient to characterize equilibria. Let x be the probability that a randomly chosen neighbor chooses the action 1. Then $H(d, x)$ is the probability that a random (best responding) neighbor of degree d chooses the action 1. Thus, it must be that

$$x = \phi(x) \equiv \sum_{d} \widetilde{P}(d) H(d, x). \tag{2}$$

Equation 2 characterizes equilibria in the sense that any symmetric equilibrium results in an x which satisfies the equation, and any x that satisfies the equation corresponds to an equilibrium where type (d_i, c_i) chooses 1 if and only if inequality (1) holds.[8]

[6] This is an extension of the concept from GGJVY, as in that model agents have identical costs. The equilibrium is symmetric in that it depends only on an agent's type (d_i, c_i), and not her label i.

[7] In such a case, the best response correspondence (allowing mixed strategies) for any (d_i, c_i) as dependent on x is upper hemi-continuous and convex-valued. Taking expectations with respect to d_i and c_i, we also have a set of population best responses as dependent on x that is upper hemi-continuous and convex valued.

[8] The above statement corresponds to situations where there are no atoms in H^c at $v(d, x)$ for any d. In cases where there is an atom in H^c at $v(d, x)$ for some d, then some equilibria might involve mixed strategies

Given that equilibria can be described by their corresponding x, we often refer to some value of x as being an "equilibrium."

2.3 A Diffusion Process

Consider a diffusion process governed by best responses in discrete time. At time $t = 0$, a fraction x^0 of the population is exogenously and randomly assigned the action 1, and the rest of the population is assigned the action 0. At each time $t > 0$, each agent, *including the agents assigned to action* 1 *at the outset*, best responds to the distribution of agents choosing the action 1 in period $t-1$, presuming that their neighbors will be a random draw from the population.

Thus, the dynamics are governed by myopic best responses, where the best responses are relative to the Bayesian structure of the game. The analysis provides a two-fold contribution. First, it provides a characterization of the structure of the Bayesian equilibria and their stability or instability. Second, it provides a heuristic investigation of diffusion. To the extent that the network is large and types are uncorrelated, it offers an approximation of diffusion.[9]

Let x_d^t denote the fraction of those agents with degree d who have adopted behavior 1 at time t, and let x^t denote the link-weighted fraction of agents who have adopted the behavior at time t. That is,

$$x^t = \sum_d \widetilde{P}(d) x_d^t.$$

Then at each date t,

$$x_d^t = H(d, x^{t-1})$$

among the positive measure of types who have degree d and cost exactly equal to $v(d,x)$. In such a case, equation (2) holds relative to a selection of $H(d,x)$ corresponding to the percent of agents with a degree of d expected to take action 1, rather than $H^c(v(d,x))$.

[9] Allowing for a richer information structure where agents react to more comprehensive information, potentially including the actions of the realized types of their neighbors, may be more realistic in some contexts, particularly for small group interactions. However, this would introduce numerous technical difficulties (see examples in GGJVY). Since in most big groups agents cannot track the full map of social connections, we view the analysis presented here as a natural first step.

and therefore

$$x^t = \sum_d \widetilde{P}(d) H(d, x^{t-1}).$$

Any rest point of the system corresponds to a static Bayesian equilibrium of the system.

If payoffs exhibit complementarities, then convergence of behavior from any starting point is monotone, either upwards or downwards. So, once an agent (voluntarily) switches behaviors, the agent will not want to switch back at a later date.[10] Thus, although these best responses are myopic, any eventual changes in behavior are equivalently forward-looking.

3 Equilibrium Structure

In the remainder of the analysis, unless otherwise stated, let H^c be atomless, so that there is no need for selections of H and all equilibria are characterized by (2).

The following observation regarding equilibrium structure is helpful in illustrating some of the driving forces behind our results.

Observation 1 *If v is non-decreasing (non-increasing) in d for each x, then any symmetric equilibrium entails agents with higher degrees choosing action 1 with weakly higher (lower) probability. Furthermore, if $u_d(0, x) = 0$ for all d, then agents of higher degree have higher (lower) expected payoffs.*

Indeed, consider any symmetric equilibrium generating a probability of x for a random neighbor to choose action 1. If v is non-decreasing in d, then the expected payoff of a degree $d+1$ agent is $v(d+1, x) \geqslant v(d, x)$ and so $H^c(v(d+1, x)) \geqslant H^c(v(d, x))$ and agents with higher degrees choose 1 with weakly higher probabilities. Since an agent of degree $d+1$ can imitate the decisions of an agent of degree d and gain at least as high a payoff, the observation follows.[11]

[10] If actions are strategic substitutes, convergence may not be guaranteed for all starting points. However, our results will still be useful in characterizing the potential rest points, or equilibria, of such systems.

[11] The observation provides an analog to Proposition 2 from GGJVY, on a simpler action space but applying to a much wider set of payoff structures.

3.1 Multiplicity

The multiplicity of equilibria is determined by the properties of ϕ, which, in turn, correspond to properties of $\widetilde{P}$ and H. For instance,

- if $H(d, 0) > 0$ for some d in the support of P, and H is concave in x for each d, then there exists at most one fixed point, and
- if $H(d, 0) = 0$ for all d and H is strictly concave or strictly convex in x for each d, then there are at most two equilibria - one at 0, and possibly an additional one, depending on the slope of $\phi(x)$ at $x = 0$.[12]

In general, as long as the graph of $\phi(x)$ crosses the 45 degree line only once, there is a unique equilibrium (see Figure 1 below). We note that there is a conceptual connection between our analysis and the recent literature on global games identifying a variety of forms of heterogeneity guaranteeing uniqueness when the underlying game with complementarities admits multiple equilibria (see Morris and Shin (2002, 2003) and references therein). The heterogeneity determining uniqueness in our general setup is introduced through the differential costs agents experience, as well as through the different degrees agents have within the social network. In a sense, the current analysis is more general in that it allows us to study the impact of changes in a variety of fundamentals on the set of stable and unstable equilibria, regardless of multiplicity, in a rather rich environment. Moreover, we can tie the equilibrium structure to the network of underlying social interactions.[13]

3.2 Stability

We are interested in equilibria that are robust to small perturbations, and are therefore stable, and equilibria that are not robust, from which small perturbations lead to significant changes in the distribution of play in the population. Formally,

[12] as is standard, the slope needs to be greater than 1 for there to be an additional equilibrium in the case of strict concavity and less than 1 in the case of strict convexity.

[13] A special case of our analysis is one where the network is complete, which is equivalent to one where all agents interact with one another.

Definition [Stability and Tipping]. *An equilibrium x is* stable *if there exists $\varepsilon' > 0$ such that $\phi(x-\varepsilon) > x-\varepsilon$ and $\phi(x+\varepsilon) < x+\varepsilon$ for all $\varepsilon' > \varepsilon > 0$. An equilibrium x is* unstable or a tipping point *if there exists $\varepsilon' > 0$ such that $\phi(x-\varepsilon) < x-\varepsilon$ and $\phi(x+\varepsilon) > x+\varepsilon$ for all $\varepsilon' > \varepsilon > 0$.*

The point $x = 0$ has a special status in several of the germane applications of the model (contagion, fashions, etc.). Depending on the setting it can turn out to be either an equilibrium or not, and when it is an equilibrium it can be stable or unstable. More specifically, since $H(d,x) \geqslant 0$ for all d and x, 0 is an equilibrium if and only if $H(d,0) = 0$ for all d in the support of P. This occurs if $\min_d v(d,0)$ is no greater than the minimal cost of adopting the action 1. Stability then depends on the behavior of ϕ in the vicinity of $x = 0$. Supposing that 0 is an equilibrium, if $H(d,x)$ is differentiable in x at $x = 0$, then 0 is stable if $\phi'(0) < 1$ and unstable if $\phi'(0) > 1$.

For example, if $v(d,x) = dx$, so that returns depend on the *number* of agents adopting the new technology, or being contaminated, instability at 0 corresponds to the condition $\frac{E[d^2]}{E[d]} > \frac{1}{H^{c\prime}(0)}$.[14]

In order to make comparisons across environments, we need to be able to keep track of the changes in equilibria. As there are multiple equilibria, they may be changing in a variety of ways. The kinds of changes we are interested in are ones that either make diffusion easier or more difficult, in a well-defined sense.

Definition [Greater Diffusion] *One environment, with corresponding mapping $\widetilde{\phi}(x)$,* generates greater diffusion *than another, with corresponding mapping $\phi(x)$, if for any stable equilibrium of the latter there exists a (weakly) higher stable equilibrium of the former, and for any unstable equilibrium of the latter there is either a (weakly) lower unstable equilibrium of the former or else $\widetilde{\phi}(0) > 0$.*

[14]When specializing H^c to be uniform, this coincides with the condition ensuring the stability of 0 in the epidemiology literature (e.g., see Pastor-Satorras and Vespignani (2000, 2001).

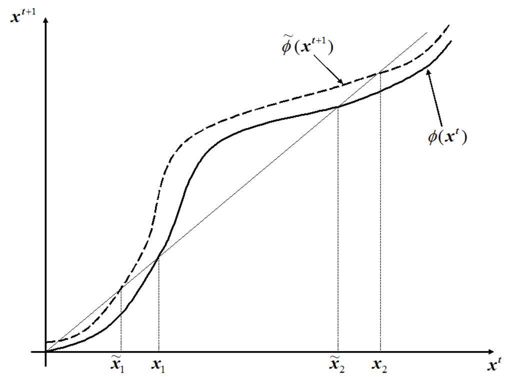

Figure 1: The Effects of Shifting $\phi(x)$ Pointwise

One environment thus has greater diffusion than another if its tipping points or unstable equilibria are lower, thus making it easier to get diffusion started, and its stable equilibria are higher, and so the eventual resting points are higher.

Before turning to our results, we need one technical definition.

Definition [Regular Environment]. *An environment is* regular *if all fixed points are either stable or unstable and H is continuous.*

Note that in a regular environment, if x is an unstable fixed point, and x' is the next largest fixed point, then x' must be stable.

The following Proposition illustrates how pointwise increases in ϕ affect the tipping points and stable points of the system. It will be key in the analysis of the effects of changes in fundamentals.

Proposition 1 *Consider $\overline{\phi}$ and ϕ corresponding through (2) to two regular environments. If $\overline{\phi}(x) \geqslant \phi(x)$ for each x, then $\overline{\phi}$ has greater diffusion than ϕ.*

The proposition is straightforward and so its proof is omitted.

Proposition 1 implies that a small upward shift in a (continuous) ϕ leads to locally lower tipping points and higher stable equilibria, as illustrated in Figure 1.[15]

With respect to welfare, if externalities are positive, then higher equilibrium points correspond to (weakly) higher expected utilities to agents of *each* degree regardless of their action, as the payoff from either of their actions has increased and so their best response must lead to a higher payoff.

4 Comparative Statics

We now identify how equilibrium structure depends on three fundamentals in our environment: the cost distribution H^c, the return function v, and the network structure as described by P and $\widetilde{P}$.

Given that

$$\phi(x) = \sum_d \widetilde{P}(d) H^c(v(d,x)),$$

and Proposition 1, we can deduce much about changes in the structure of equilibria by considering changes in costs, returns, and network structure that shift $\phi(x)$ upwards for all x or downwards for all x.

4.1 Changes in Cost Distribution

We consider increases in costs in terms of first order stochastic dominance shifts of H^c.

Proposition 2 (Increasing Costs) *If $\overline{H}^c$ FOSD H^c, then the corresponding $\overline{\phi}(x), \phi(x)$ satisfy $\overline{\phi}(x) \leqslant \phi(x)$ for each x. Thus, if $\overline{H}^c$ and H^c (given v) correspond to regular environments, then H^c generates greater diffusion than $\overline{H}^c$.*

[15] In fact, for sufficiently small shifts of ϕ as in the proposition, there are stronger conclusions one can make in terms of the changes in equilibria. Indeed, there is a one-to-one mapping between equilibria, and each equilibrium point has a unique counterpart that has shifted as claimed.

Proof of Proposition 2: By first order stochastic dominance,

$$\overline{\phi}(x) = \sum_d \widetilde{P}(d)\overline{H}^c(v(d,x)) \leqslant \sum_d \widetilde{P}(d)H^c(v(d,x)) = \phi(x).$$

The implications regarding tipping points and stable equilibria then follow directly from Proposition 1. ▮

As costs increase, agents are generally less prone to take action 1, and so tipping points are shifting up and stable equilibria are shifting down.

Note that if the game exhibits positive externalities, then clearly if x^* is a stable equilibrium under H^c and $\overline{x}^* \leqslant x^*$ is a stable equilibrium under $\overline{H}^c$, then expected utility of *all* agents goes up and expected welfare under H^c when x^* is played, is higher than under $\overline{H}^c$ when $\overline{x}^*$ is played.

We remark that the conclusions of Proposition 2 pertaining to shifts of the equilibria set hold regardless of whether v is monotonic in either of its arguments.

4.2 Changes in Network Structure

We consider two types of changes to the network architecture. The first pertains to the number of expected neighbors each agent has. The second relates to the heterogeneity of connectedness within the population.

Proposition 3 (FOSD shifts) *If $\widetilde{P}$ FOSD $\widetilde{P}'$ and $H(d,x)$ is non-decreasing (non-increasing) in d for all x, then $\phi(x) \geqslant \phi'(x)$ ($\phi(x) \leqslant \phi'(x)$) for each x. Thus, if the environments corresponding to P and P' are regular, then P generates greater diffusion than P'.*

Proof of Proposition 3: If $H(d,x)$ is non-decreasing in d, then by the definition of FOSD,

$$\phi(x) = \sum_d \widetilde{P}(d)H(d,x) \geqslant \sum_d \widetilde{P}'(d)H(d,x) = \phi'(x),$$

and so the result follows directly from Proposition 1 (and analogously for $H(d,x)$ non-increasing). ▮

Proposition 3 tells us that if H is non-decreasing, tipping points are lower and stable equilibria are higher under $\widetilde{P}$, and that the opposite holds when H is non-increasing. A related result appears in GGJVY (Propositions 4 and 5), but with several differences. The result does not distinguish between stable and unstable equilibria and only applies to a special class of payoff functions. On the other hand, the result there applies to more general action spaces (in the case where H is non-decreasing).

To gain intuition for Proposition 3, consider a case in which $v(d, x)$ is non-decreasing in d. The observation in Section 3 tells us that any symmetric equilibrium entails higher degree agents choosing action 1 with higher probability. Start then with any such equilibrium under P' and consider a shift from P for which $\widetilde{P}$ FOSD $\widetilde{P}'$. Without any change in strategies, each agent would perceive her neighbors to be more likely to have higher degrees. Thus, a best response would entail a greater propensity to choose the action 1. We can now iterate best responses and converge to an equilibrium involving a (weakly) higher rate of agents of each type choosing the action 1. In particular, it is easier to get the action 1 adopted and tipping points are lower.

As for welfare, consider for the sake of illustration the case of positive externalities and $u_d(0, x) = 0$ for all d and x. A FOSD change in the degree distribution of neighbors affects players in the same direction in terms of their expected payoffs (that is, expected payoffs go up for all d). In particular, expected payoffs corresponding to stable equilibria increase for agents of any given type, and if the underlying degree distribution itself is shifted in the sense of FOSD (that is, P FOSD P' in the proposition), then more weight is shifted to higher expected payoff agents (recall that H^c is atomless) and overall welfare increases. This condition is naturally satisfied when, e.g., P is a simple translation of the distribution P'. Unfortunately, more general forms of FOSD shifts in the distribution of neighbors' degrees, $\widetilde{P}$, do not always correspond to FOSD shifts in the original degree distribution P and so welfare implications are, in general, ambiguous.[16]

[16]For more on this, see GGJVY. As an example, consider a network comprised of agents of degrees $1, 10, 100$ and 1000, and consider P and P' such that $P(1) = P(100) = P(1000) = \frac{1}{3}$ and $P'(1) = P'(10) = \frac{1}{4}, P'(100) =$

We now examine changes in the heterogeneity of connectedness in the form of mean-preserving spreads of the degree distribution.

Proposition 4 (MPS in P) *If $H(d,x)$ is non-decreasing and convex (non-increasing and concave) in d, then P is a MPS of P' implies that $\phi(x) \geqslant \phi'(x)$ ($\phi(x) \leqslant \phi'(x)$) for all x, and so P generates greater (lesser) diffusion than P'.*

Furthermore, if $u_d(0,x) = 0$ for all d and x, the game exhibits positive externalities, and $v(d,x)$ is convex (concave) in d, then if x^* is a stable equilibrium under P and $\overline{x}^* \leqslant x^*$ ($\overline{x}^* \geqslant x^*$) is a stable equilibrium under P', the expected welfare under P when x^* is played, is higher (lower) than under P' when $\overline{x}^*$ is played.

Proof of Proposition 4: When $H(d,x)$ is non-decreasing and convex, $H(d,x)d$ is a convex and increasing function of d. Since $\phi(x) = \frac{1}{\overline{d}} \sum_d H(d,x)P(d)d$, the result then follows directly from the definition of MPS. Regarding welfare, assume indeed that $v(d,x)$ is convex and let $\overline{x}^*, x^*$ be given as stable points satisfying the statement of the Proposition. The welfare level under P when x^* is played is given by $\sum_d v(d,x^*)P(d)$. Now,

$$\sum_d v(d,x^*)P(d) \geqslant \sum_d v(d,\overline{x}^*)P(d) \geqslant \sum_d v(d,\overline{x}^*)P'(d),$$

where the first inequality follows from the fact that v is non-decreasing in x and the second from the definition of MPS. Since the right hand side is precisely the welfare level under P' when $\overline{x}^*$ is played, the proposition's claim follows. The case in which $H(d,x)$ is non-increasing and concave follows analogously. ∎

The convexity and monotonicity conditions in Proposition 4 are clearly satisfied when $v(d,x)$ is multiplicatively separable, $v(d,x) = \widetilde{v}(d)f(x)$, $f(x) > 0$ increasing, $\widetilde{v}(d)$ is non-decreasing and convex, and H^c is convex.[17] In that case, MPS shifts of the degree distribution

$\frac{3}{8}, P'(1000) = \frac{1}{8}$. In this case, $\widetilde{P}$ FOSD $\widetilde{P}'$, but P does not FOSD P'. Consider $v(1,x) \equiv 0, v(10,x) = 100x, v(100,x) = 101x, v(1000,x) = 102x$. Note that $v(d,x)$ is non-decreasing in d for all x, but that for any fixed x, welfare under P is given by $67\frac{2}{3}x$ and is lower than under P', which is given by $75\frac{5}{8}x$. Thus, for appropriate choice of cost distribution, welfare may be impacted negatively by a shift from P' to P.

[17]For instance, consider $\widetilde{v}(d) = d$ and H uniform on $\left[0, \max_{\mathrm{supp} P(d)} d\right]$.

increase the number of low and high degree nodes. From the observation in Section 3, higher degree agents adopt the action 1 with greater propensity and so given the complementarities ($\widetilde{v}(d)f(x)$ is increasing in x), the likelihood of any random agent choosing the action 1 goes up. The convexity of $H(d,x)$ guarantees that the effect of the increase in the higher degree agents at least offsets the effect of an increase in the number of low degree agents, thereby generating higher stable equilibria (and lower tipping points).

Note that this result implies that if $H(d,x)$ is non-decreasing and convex, then power, Poisson, and regular degree distributions with identical means generate corresponding values of $\phi^{power}, \phi^{Poisson}$, and $\phi^{regular}$ such that

$$\phi^{power}(x) \geqslant \phi^{Poisson}(x) \geqslant \phi^{regular}(x)$$

for all x.[18]

4.3 Changes in Returns to Adoption

We now contemplate changes in the returns to the action 1 and their effects on the eventual adoption rate. This inquiry is interesting for a variety of applications. For example, it is germane to directed advertising in marketing, which may effectively increase the propensity to adopt a product by particular individuals; in epidemiology, the question of whom to immunize first is very much related to the effects of changes in specific values of $v(d,x)$; and so on and so forth.

We concentrate on the special case in which $v(d,x)$ is multiplicatively separable, $v(d,x) = \widetilde{v}(d)f(x)$, where $\widetilde{v}(d), f(x) > 0$ for all d and x.

Consider starting with a given $\widetilde{v}(d)$ and then reordering it to become $\widetilde{v}'(d)$. Formally, $\widetilde{v}$ and $\widetilde{v}'$ are reorderings of one another if there is a permutation π of $1, 2, \ldots$ such that $\widetilde{v}(\pi(d)) = \widetilde{v}'(d)$ for each d.

[18] This is consistent with the simulation-based observations regarding to tipping points in the epidemiology literature (see Pastor-Satorras and Vespignani (2000, 2001)).

Let us say that a reordering $\widetilde{v}'$ of $\widetilde{v}$ is *weight increasing* if the following condition holds: For any d and d' such that $\widetilde{v}'(d) \neq \widetilde{v}(d)$ and $\widetilde{v}'(d') \neq \widetilde{v}(d')$, if $\widetilde{v}'(d') > \widetilde{v}'(d)$ then $P(d')d' \geqslant P(d)d$.

The condition states that any values of $\widetilde{v}$ that have been reordered should be reordered so that higher values are assigned to degrees that have higher conditional weight.

Proposition 5 (Weight Increasing Reorderings) *If $\widetilde{v}'$ is a weight increasing reordering of $\widetilde{v}$, then the corresponding $\phi'(x) \geqslant \phi(x)$ for all x, and so $\widetilde{v}'$ generates greater diffusion than $\widetilde{v}$.*

Proof of Proposition 5: It is enough to show that $\sum_d H^c(\widetilde{v}'(d)f(x))P(d)d \geqslant \sum_d H^c(\widetilde{v}(d)f(x))P(d)d$.

Let D be the set of d's on which any reordering has been made (i.e., d's such that $\widetilde{v}'(d) \neq \widetilde{v}(d)$). The definition of weight increasing reordering then implies that $\widetilde{v}'$ maximizes $\sum_{d\in D} w(d)P(d)d$ over all $w(d)$ that are reorderings of $\widetilde{v}$ on D.

Since H^c is a non-decreasing function, this implies that $\widetilde{v}'$ maximizes $\sum_{d\in D} H^c(w(d)f(x))P(d)d$ over all $w(d)$ that are reorderings of $\widetilde{v}$ on D. The result then follows. ∎

In the case of a uniform cost function, in order to increase ϕ pointwise, we do not need the reordering to be weight increasing, but rather just to increase weight on average. This is stated as follows.

Proposition 6 (Reorderings with Uniform Costs) *Suppose that H^c is uniform on $[0, M]$, where $M \geqslant \max_d \widetilde{v}(d)$. If $\sum_d P(d)\widetilde{v}'(d)d \geqslant \sum_d P(d)\widetilde{v}(d)d$, then $\phi'(x) \geqslant \phi(x)$ for all x, and so $\widetilde{v}'$ has greater diffusion than $\widetilde{v}$.*[19]

The implications of the Propositions are that in order to lower the set of tipping points and increase the set of stable equilibria, the appropriate choice of $\widetilde{v}(d)$ requires matching the ordering of $\widetilde{v}(d)$ with that of $P(d)d$.[20]

[19]Note that the proposition holds for any $\widetilde{v}$ and $\widetilde{v}'$, even if they are not reorderings of one another.

[20]Since the exercise in this subsection is to study the effects of changing returns of particular individuals, a comparison of welfare after such a shift seems somewhat unnatural. Of course, a shift that generates a higher stable equilibrium would also imply higher welfare under the original return functions.

This has clear implications for how effective diffusion will be. The simple intuition is that in order to maximize diffusion, one wants the types that are most prone to adopt a behavior to be those who are most prevalent in the society in terms of being most likely to be neighbors.

Interestingly, this leads to conclusions that are counter to what one sometimes concludes from thinking about trying to target specific types of nodes. If one can only target a specific number of nodes, then one would like to target those with the highest degree as they will have the greatest number of neighbors. This is the usual sort of "hub" idea. However, the exercise here is different. It is to ask which types are most influential, *when accounting for the population size and thus their likelihood to be neighbors.*

Examples Under a power distribution (e.g., the Pareto distribution) $P(d)d$ is *decreasing* in d. So, in order to maximize adoption rates we would want the lowest degree nodes to have the highest propensity to adopt. More generally, a decreasing $\widetilde{v}(d)$ would lead to a higher $\phi(x)$, than an increasing $\widetilde{v}(d)$. For a uniform degree distribution the reverse holds. For a Poisson distribution, note that $P(d)d$ is increasing up to the mean and then decreasing thereafter and the ideal ordering of $\widetilde{v}(d)$ would match this shape, and thus not be monotonic.

So, under a power distribution, it is the lowest degree nodes that are most influential in terms of those most likely to be any given node's neighbor. Those lowest degree nodes are thus the ones whose propensities to adopt a behavior has the greatest impact on the overall diffusion.

4.4 Optimal Networks

We can ask a related question, which is if we fix payoffs and costs, but consider different network structures with the same average degree $\overline{d}$ (and subject to some maximal degree $\overline{D}$), which degree distributions would maximize diffusion?

That is, we are asking the question of which P with a given average $\overline{d}$ and support in $1, \ldots, \overline{D}$ maximizes $\phi(x) = \sum_d H(d,x)\frac{P(d)d}{\overline{d}}$. If we have a distribution P that maximizes $\phi(x)$

pointwise, then we know that it leads to higher diffusion than any other degree distribution.

Here, we can make use of Proposition 4, which implies the following.

Corollary 7 *If $H(d,x)$ is non-decreasing and convex in d, then the P which maximizes diffusion (under our greater diffusion partial ordering) is one which has weight only on degree 1 and $\overline{D}$ (in proportions that yield average degree $\overline{d}$).*

If $H(d,x)$ is non-increasing and concave in d, then the P which maximizes diffusion (under our greater diffusion partial ordering) is a regular network with full weight on degree $\overline{d}$.

5 Convergence Patterns

We close with an analysis of convergence patterns. Here we study the structure of the best response dynamics as well as the paths and convergence points experienced by different degree agents.

5.1 S-Shaped Rates of Adoption

If we examine best response dynamics over time, starting from some x^0 and progressing to some x^1, x^2,..., we can get an idea of the "speed of convergence" of the system at different points by examining the difference $x^{t+1} - x^t$, or $\phi(x) - x$.

The following proposition summarizes the structure of the convergence paths.

Proposition 8 *Let $H(d,x)$ be twice continuously differentiable and increasing in x for all d.*

- *If $H(d,x)$ is strictly concave in x for all d, then there exists $x^* \in [0,1]$ such that $\phi(x) - x$ is increasing up to x^* and then decreasing past x^* (whenever $\phi(x)$ is neither 0 nor 1).*
- *If $H(d,x)$ is strictly convex in x for all d, then there exists $x^* \in [0,1]$ such that $\phi(x) - x$ is decreasing up to x^* and then increasing beyond x^* (whenever $\phi(x)$ is neither 0 nor 1).*

Proof of Proposition 8: Note that

$$(\phi(x) - x)' = \sum_d \widetilde{P}(d)\frac{\partial H(d,x)}{\partial x} - 1.$$

Let x^* be such that $\sum_d \widetilde{P}(d)\frac{\partial H(d,x)}{\partial x}|_{x^*} = 1$, if such a point exists. In the case where H is strictly concave in x, if $\sum_d \widetilde{P}(d)\frac{\partial H(d,x)}{\partial x} > 1$ for all x then set $x^* = 1$, and if $\sum_d \widetilde{P}(d)\frac{\partial H(d,x)}{\partial x} < 1$ for all x then set $x^* = 0$. In the case where H is strictly convex in x, if $\sum_d \widetilde{P}(d)\frac{\partial H(d,x)}{\partial x} > 1$ for all x then set $x^* = 0$, and if $\sum_d \widetilde{P}(d)\frac{\partial H(d,x)}{\partial x} < 1$ for all x then set $x^* = 1$.

Next, note that

$$(\phi(x) - x)'' = \sum_d \widetilde{P}(d)\frac{\partial^2 H(d,x)}{\partial x^2}.$$

Thus, if H is strictly concave in x, then the second derivative of $\phi(x) - x$ is negative. Therefore, the derivative of $\phi(x) - x$ is positive up to x^* and negative beyond it. The reverse holds for H being strictly convex, and the result then follows. ▮

Proposition 8 helps tell us more about how the diffusion process will work over time. Consider the case where H is strictly concave. From the concavity of H we know that there are three possible configurations of equilibria: 0 is a stable equilibrium and the only equilibrium, 0 is an unstable equilibrium and there is a unique stable equilibrium above 0, or 0 is not an equilibrium and there is a unique stable equilibrium above 0. In the first case, it must be that $\phi'(0) \leqslant 1$ and the dynamic process would converge to 0 regardless of the starting point. In the other cases, if $\phi'(0) > 1$, then x^* lies above 0 and Proposition 8 tells us that the adoption over time will exhibit an S-shape. From small initial levels of x the change in x will gain speed up to the level of x^*, and then it will start to slow down until eventually coming to rest at the adjacent stable equilibrium.

$S - shaped$ adoption curves are prevalent in empirical data on diffusion (see Mahajan and Peterson (1985) and references therein). The heuristic idea being that initially, a few agents adopt till the diffusion picks up, at which point there is a rapid increase in adopters that ultimately levels off (and so the speed of increase of adoption decreases). The intuition for this effect in our model is that the initial gain in speed is due to the complementarities in adoption among agents, while the eventual slowing is due to the concavity of the cost distribution. If instead the distribution of costs is convex, then the adoption will continue to gain speed rather than lose speed.[21]

5.2 Adoption Patterns by Degrees

The dynamic process corresponding to each degree d is given by:

$$x_d^t = H(d, x^{t-1})$$

where x_d^t is the fraction of agents of degree d who adopt at time t.

In particular, whenever $v(d,x) = \widetilde{v}(d)x$, then it is clear x_d^t is characterized by the same curvature properties that are discussed above for x itself. Moreover, the curves corresponding to the different x_d^t are ordered according to $\widetilde{v}(d)$. In particular, for any stable point x^*, the corresponding distribution according to degrees is given by $x_d^* = \widetilde{v}(d)x^*$ and the curvature of x_d^* follows that of $\widetilde{v}(d)$.

The distinction between different adoption paths corresponding to different degree players is important from an econometric point of view. Indeed, it provides additional restrictions on fundamentals arising from cross-sectioning data according to social degree.

As an illustration, consider the case in which $v(d,x) = dx$ and H^c is uniform on $[c, C]$, so that $H(d,x) = min\left[max\left(0, dx - c\right), C - c\right]/(C - c)$.

Figure 2 illustrates the resulting adoption dynamics corresponding to different degree agents in the case in which $c = 1, C = 5$, and the initial seed is 0.3.

In this example, higher degrees start adopting the action 1 earlier and have steeper slopes

[21] Bass (1969) and follow-ups was one of the first to provide a (network-free) contagion model explaining such general adoption paths. Recently, Young (2006) provides a different explanation based on learning. He studies an environment in which agent heterogeneity manifests itself through diverse priors regarding the value of an innovation. Initially, only agents with favorable priors or very favorable information adopt the innovation. This initial adoption shifts everyone's posteriors in favor of the innovation, which yields more adoption, etc, and an $S-shaped$ adoption curve is created. Young matches his model to Griliches (1957)'s data on hybrid corn adoption.

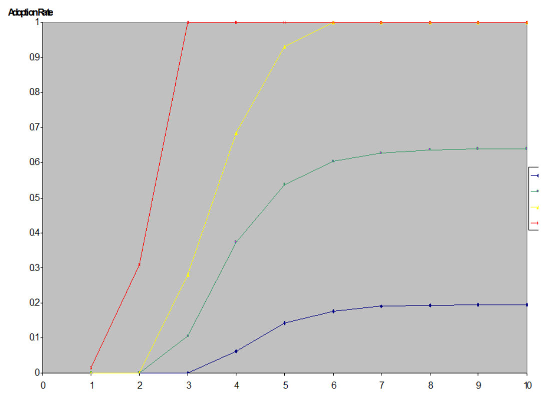

Figure 2: Fraction Adopting over Time by Degree

early in the process.[22]

Note that Figure 2 is consistent with the results in Proposition 8 in that the adoption curves are initially convex and then eventually concave and exhibit an $S - shaped$ pattern of adoption.

[22]This sits well with, e.g., the empirical observations on drug adoption by doctors in Coleman, Katz, and Menzel (1966), where the ultimate frequencies of adoptions were increasing in degrees.

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