Network Games*

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Abstract

In contexts ranging from public goods provision to information collection, a player's well-being depends on own action as well as on the actions taken by his or her neighbors. We provide a framework to analyze such strategic interactions when neighborhood structure, modeled in terms of an underlying network of connections, affects payoffs. In our framework, individuals are partially informed about the structure of the social network. The introduction of incomplete information allows us to provide general results characterizing how the network structure, an individual's position within the network, the nature of games (strategic substitutes versus complements and positive versus negative externalities), and the level of information, shape individual behavior and payoffs.

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1 Introduction

In a range of social and economic interactions – including public goods provision, job search, political alliances, trade, friendships, and information collection – an agent's well being depends on his or her own actions as well as on the actions taken by his or her neighbors. For example, the decision of an agent of whether or not to buy a new product, or to attend a meeting, is often influenced by the choices of his or her friends and acquaintances (be they social or professional). The empirical literature identifying the effects of agents' neighborhood patterns (i.e., their social network) on behavior and outcomes has grown over the past several decades.[1] The emerging empirical evidence motivates the theoretical study of network effects. We would like to understand how the pattern of social connections shape the choices that individuals make and the payoffs they can hope to earn. We would also like to understand how changes in the network matter as this tells us how individuals would like to shape the networks in which they are located.

Attempts at the study of these basic questions have been thwarted by a fundamental theoretical problem: even the simplest games played on networks have multiple equilibria, which display a bewildering range of possible outcomes. The literature on global games illustrates how the introduction of (a small amount of) incomplete information can sometimes resolve the problem of multiplicity as well as provide interesting and novel economic intuitions.[2] Recently, this approach has faced the critique that the equilibrium selection achieved depends on the specifics of the incomplete information that is assumed, a point made convincingly by Weinstein and Yildiz (2007). However, in the context of network games there is a natural way to introduce incomplete information that eliminates this ambiguity, which is having uncertainty about the identity of their future neighbors and the number of neighbors that they will have. There are many decisions that are made at times where a player has a good forecast of the number of her connections (her degree) but has incomplete information about the degrees of others.[3]

[1] The literature is much too vast to survey here; but influential works include Katz and Lazarsfeld (1954), Coleman (1966), Granovetter (1994), Foster and Rosenzweig (1995), Glaeser, Sacerdote, Scheinkman (1996), Topa (2001), and Conley and Udry (2005).

[2] Starting with the work of Carlsson and van Damme (1993), there is now an extensive literature on global games. For a survey of this work see Morris and Shin (2003).

[3] For discussion of the knowledge of individuals about the network see, e.g., Kumbasar, Romney, and Batchelder

Indeed, in many circumstances individuals are aware of their proclivity to interact with others, but do not know who these partners at the time of choosing actions. For instance, students who are planning a career of international diplomacy may anticipate how many individuals each of them will most likely interact with, but do not know who these individuals will be when deciding the number of foreign languages to study; or researchers choosing software based on compatibility may know the number of coauthors they expect to have in the future, but not necessarily who these people will be; or individuals deciding whether to get a medical vaccine may anticipate the volume of people they will interact with, but not specifically who these people will be. For these kind of environments, our model highlights the following two features: (i) agents have a good sense of the volume of agents each of them will interact with (their respective degree), and (ii) action choices are taken prior to the actual network of connections being realized (i.e. there is incomplete information regarding the identity of neighbors, neighbors's neighbors, etc.).

Motivated by these considerations we develop a model of games played on networks, in which players have private and incomplete information about the network. Their private knowledge about the network is interpreted as their type, and we study the Bayes-Nash equilibria of this game. We find that much of the equilibrium multiplicity that arises under complete information is no longer sustainable under incomplete information. Specifically, the key insight is that when players have limited information about the network they are unable to condition their behavior on its fine details and this leads to a significant simplification and sharpening of equilibrium predictions.

There are two other important aspects of our framework that we would like to stress here. One is that individuals are allowed to have beliefs about degrees of their neighbors that depend on their own degree. We capture correlations in the degrees of neighbors through a weakening of the notion of affiliation, which is a measure widely used in economics to capture joint correlations in types. As we explain below, many of the real-world contexts studied by the network literature display degree correlations (positive or negative) that fall into one of the scenarios considered here. This is also true for much of the theoretical work concerned with alternative models of network formation.

A second important feature of our approach is that we allow for alternative scenarios on how a

(1994).

player's payoffs are affected by the actions of others. This is motivated by our desire to develop an understanding of how the payoffs interact with the network structure. We focus, therefore, on two canonical types of interaction: strategic complements and strategic substitutes.[4] These two cases cover many of the game-theoretic applications studied by the economic literature.

We now provide an overview of our main results. Our first result shows existence of an equilibrium involving monotone (symmetric) strategies. In particular, in the case of strategic substitutes equilibrium actions are non-increasing in players' degrees, while under strategic complements equilibrium actions are non-decreasing in players' degrees. We also provide conditions under which all (symmetric) equilibria are monotone. In turn, the monotonicity property of equilibrium actions implies that social connections create personal advantages irrespective of whether the game exhibits strategic complements or substitutes: in games with positive externalities well connected players earn more than poorly connected players.[5] This provides a first illustration of the additional structure afforded by our assumption of incomplete information. Building upon it, our second objective is to understand how changes in the perceived social network affect equilibrium behavior and welfare within the different payoff scenarios. We start by considering the effects induced by increased connectivity, as embodied by shifts in the degree distribution that suitably extend the standard notion of First Order Stochastic Dominance (FOSD). This is proven to have unambiguous effects on equilibrium behavior under strategic substitutes for binary-action games, as well as for general games with strategic complements. For binary-action games, we also derive results that involve arbitrary changes in the degree distribution relative to the equilibrium actions. Finally, we explore the implications of endowing agents with deeper (but still local) information on the network. We find that this may lead to non-monotonic equilibrium behavior.

[4] For instance, strategic complements arise whenever the benefit that an individual obtains from buying a product or undertaking a given behavior is greater as more of his partners do the same. This might be due to direct effects of having similar or compatible products (such as in the case of computer operating systems), peer pressures (as in the case of drug use), and so forth. The strategic substitutes case encompasses many scenarios that allow for free riding or have a public-good structure of play, such as costly experimentation or information collection. Formal definitions of these games are given in Section 3.

[5] The idea that social connections create personal advantages is a fundamental premise of the influential work of Granovetter (1994) and it is central to the notion of structural holes developed by Burt (1994). A number of recent empirical studies document the role of connections in providing personal advantages – ranging from finding jobs, getting promotions, gaining competitive advantages in markets.

Our paper is a contribution to the growing literature that, in recent years, has undertaken the study of games played on networks (for an extensive overview of the networks literature, see Goyal (2007) and Jackson (2008)). For instance, decisions to undertake criminal activity (Ballester, Calvó-Armengol, and Zenou (2006)), public-good provision (Bramoullé and Kranton (2007)), the purchase of a product (Galeotti (2008)), and research collaboration among firms (Goyal and Moraga-Gonzalez (2001)) have been studied for specific network structures under complete information.[6] We would also like to mention Jackson and Yariv (2005), Galeotti and Vega-Redondo (2006), and Sundararajan (2006), who study games with incomplete network knowledge in specific contexts. The principal contribution of our paper is the development of a general framework for the study of games in such an incomplete-information setup. We accommodate a large class of games with strategic complements and strategic substitutes, including practically all the applications mentioned above as special cases. Our approach also allows naturally for general patterns of correlations across the degrees of neighbors, and this is important as empirical work suggests that real world networks display such features. To the best of our knowledge, our paper is the first attempt to incorporate general patterns of degree correlations in the study of network games.[7]

There is also a literature in computer science that examines games played on a network; see, e.g., the model of "graphical games" as introduced by Kearns, Littman, and Singh (2001), also analyzed by Kakade, Kearns, Langford, and Ortiz (2003), among others.[8] The graphical-games literature has focused on the complexity of and algorithms for computing equilibria in two-action complete information games played on networks. Here, we allow for more general games and examine different information structures. Importantly, our focus is on the structure of equilibria and its interaction with the underlying network, rather than with the computational complexity of equilibria.

The rest of the paper is organized as follows. In Section 2 we discuss some simple examples that convey many of the insights to be gathered from the general analysis. Our abstract theoretical

[6] In particular, regular networks (in which all players have the same degree) and core-periphery structures (the star network being a special case) have been extensively used in the literature.

[7] Jackson and Yariv (2007) follows up on the approach introduced in this paper and obtains complementary results. It examines the multiplicity of equilibria of games on networks with incomplete information, but with a binary action model and a different formulation of payoffs. See also Jackson and Yariv (2008) for a review of related results.

[8] There are also models of equilibria in social interactions where players care about the play of certain other groups of players. See Glaeser and Scheinkman (2003) for an overview.

framework for the study of games played on networks is then introduced in Section 3. Section 4 presents results on the existence and monotonicity of equilibria. Section 5 takes up the study of the effects of network changes on equilibrium behavior and payoffs. While the analysis in Sections 4 and 5 focuses on a setting in which players know their own degree and have some beliefs about the rest of the network, Section 6 takes up the issues that arise when players have deeper knowledge about the network. Section 7 concludes. All the proofs are gathered in the Appendix.

2 Effects of Networks on Behavior and Payoffs: Examples

This section presents and analyzes two simple games played on networks – reflecting strategic substitutes and strategic complements, respectively – to illustrate the main insights of the paper.

We start with the setting studied by Bramoullé and Kranton (2007) – henceforth referred to as BK. It is a model of the local provision of information (or a local public good) and agents' actions are strategic substitutes. We compare the equilibrium predictions under the assumption of complete information and incomplete information.

Consider a society of n agents, each of them identified with a node in a social network. The links between agents reflect social interactions, each pair of connected agents is said to be "neighbors." It is posited that every individual must choose independently an action in $X = \{0, 1\}$, where action 1 may be interpreted as acquiring information, getting vaccinated, etc. and action 0 as not doing so. To define the payoffs, let $y_i \equiv x_i + \bar{x}_{N_i}$ where x_i is the action chosen by agent i, N_i is the set of i's neighbors, and $\bar{x}_{N_i} \equiv \sum_{j \in N_i} x_j$ is the aggregate action in N_i. The *gross* payoff to agent i is assumed equal to 1 if $y_i \geq 1$, and 0 otherwise. On the other hand, there is a cost c, where $0 < c < 1$ for choosing action 1, while action 0 bears no cost. Gross payoffs minus costs define the (net) payoffs of the game.[9] Thus, an agent would prefer that someone in his or her neighborhood take the action 1, and would rather not take the action himself/herself. The agent would, however, be willing to take the action 1 if nobody in the neighborhood does.

We start with the informational assumption made by BK: agents have complete information on the social network and thus the natural equilibrium concept to focus on is Nash equilibrium. It is

[9] The game is sometimes referred to as the best-shot game. For a more detailed presentation of it, see Section 3.

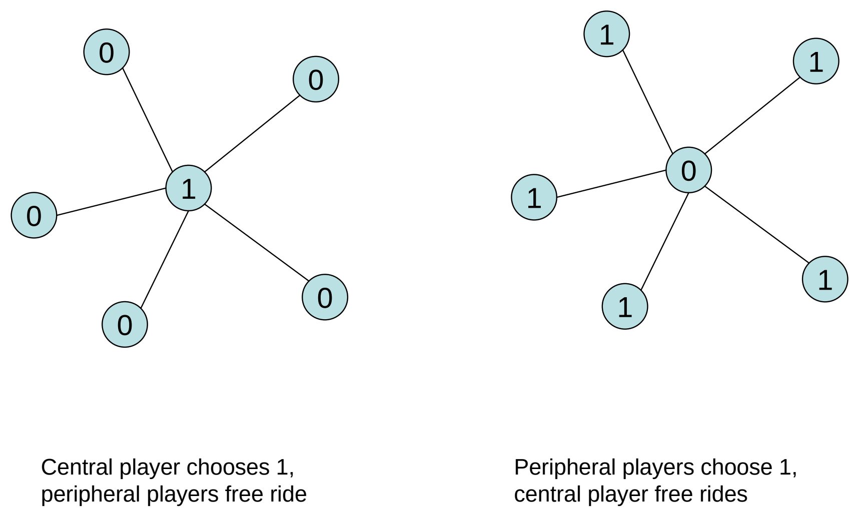

Figure 1: Strategic substitutes with complete information

immediate to observe that, since $c < 1$, $y_i \geq 1$ for every player $i \in N$ in any Nash equilibrium. Let us first turn to understanding the relation between network connections and actions. In general, such a complete-information context allows for a very rich set of Nash equilibria of the induced games. To see this, consider the simple case of a star network and note that there exist two equilibria. In one equilibrium, the center chooses 1 and the peripheral players choose 0, while in the second equilibrium the peripheral players choose 1 and the center chooses 0. In the former equilibrium, the center earns less than the peripheral players, while in the latter equilibrium it is the opposite. Figure 1 depicts these possibilities. Hence, even in the simplest networks there exist multiple equilibria and, most importantly, the relation between network connections, equilibrium actions and payoffs may exhibit very different patterns *even when all agents of the same degree choose the same actions.*

Still remaining under the assumption of complete information, note that the effects of adding links to a network on equilibrium actions and aggregate payoffs depend very much on the details of the network and where the links are added (a point made by BK). To see this, consider a network with two stars each of which contain 5 peripheral players. Fix a symmetric equilibrium in which the two centers choose action 1 while the peripheral players all choose 0. The aggregate payoff in this equilibrium is $12 - 2c$. Now suppose that a link is added between a center of one star and a peripheral player of some other star. In the new network the old action profile still constitutes an equilibrium. Next, consider adding a link between the centers of the two stars. In this case the old profile of actions is no longer an equilibrium. In fact, there is *no* equilibrium where both of the original centers choose 1. There is, however, an equilibrium in which the peripheral players of the stars choose 1 and the centers choose 0. In this equilibrium there is a clear change in profile of actions, and the aggregate payoffs are given by $12 - 10c$. There is another equilibrium where one of the two centers takes the action 1, and the other does not, and this leads to aggregate payoffs of $12 - 6c$. It follows that in any of the equilibria associated with the addition of the link aggregate payoffs are lower than in the starting equilibrium. Figure 2 illustrates these outcomes. Interestingly, if a link is added between the center node of one star and a peripheral player on the other star, as in the bottom of Figure 2, the original equilibrium actions remain part of an equilibrium.

Now let us relax the assumption of complete information on the social network and assume, instead, that players do not know the whole network but are informed only of their own degree. For example, agents' learning may occur prior to the network being realized (say, taking agricultural classes in college prior to opening a winery), or agents may decide to get an immunization (for the flu, hepatitis, and so on) before knowing the individuals they will interact with over the course of the year.

Moreover, assume that players' beliefs about the rest of the network are summarized by a probability distribution over the degrees of their neighbors. For expositional simplicity, suppose also that these beliefs are independent across neighbors as well as of own degree. Under these conditions, a player's (pure) strategy can be identified with a mapping σ specifying the action $\sigma(k) \in X$ chosen for each player of degree k. This game can be studied within the framework of Bayesian games of incomplete information by identifying player types with their corresponding degrees.

For concreteness, suppose that a link between any two of n agents is formed independently with probability $p \in (0, 1)$ (commonly referred to as an Erdös-Rényi network). Asymptotically, beliefs about neighbors' degrees then follow a binomial distribution. The probability that any randomly selected neighbor is of degree k is the probability the neighbor is connected to $k-1$ additional agents

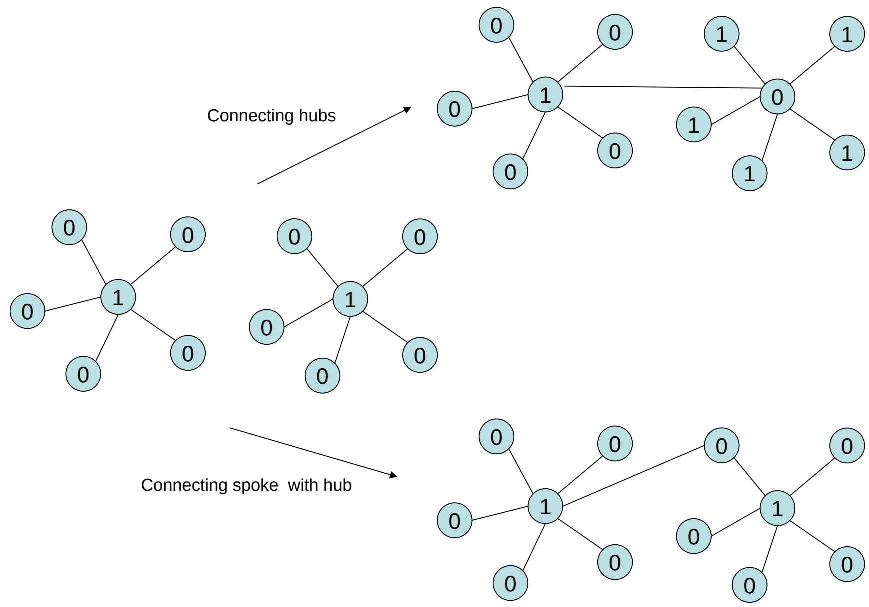

Figure 2: The effects of adding links

of the remaining $N-2$ agents, and is therefore given by:

$$Q(k;p) = \binom{N-2}{k-1} p^{k-1} (1-p)^{N-k-1}. \qquad (1)$$

If an agent of degree k chooses action 1 in equilibrium, it follows from degree independence (again, assuming for the sake of the example that n is infinitely large) that an agent of degree $k-1$ faces a lower likelihood of an arbitrary neighbor choosing the action 1, and would be best responding with action 1 as well. In particular, any equilibrium is characterized by a threshold.

Let t be the smallest integer for which

$$1 - \left[1 - \sum_{k=1}^{t} Q(k;p))\right]^{t} \geq 1 - c. \qquad (2)$$

It is easy to check that an equilibrium σ must satisfy $\sigma(k) = 1$ for all $k < t$, $\sigma(k) = 0$ for all $k > t$, and $\sigma(t) \in [0,1]$. In particular, $\sigma(k)$ is non-increasing.

Observe that social connections create personal advantages: players with degree greater than t obtain higher expected payoffs as compared to the less connected players of degree less than t. In general, the existence and uniqueness of such a symmetric threshold equilibrium follows from simple arguments for binary-action games, both for the present case of strategic substitutes and for the case of strategic complements.[10] For general games, we establish a similar conclusion that every symmetric equilibrium strategy is monotone decreasing.

We now look at how equilibrium play is affected by changes in the network. Consider, in particular, a change in the probability distribution over the degrees of players' neighbors that reflects an unambiguous increase in connectivity, as given by the standard criterion of First Order Stochastic Dominance (FOSD). Specifically, suppose we move from p to p' where $p' > p$, so that $Q(k;p')$ FOSD $Q(k;p)$. From (2), it follows that the (unique) threshold t' corresponding to p' must be higher than t. This has a two-fold implication. First, contingent on any given type, the extent of information acquisition (or public-good contribution) does not fall – it remains unchanged for agents with degrees lower than t or greater than t', and increases for all other agents. Second, the probability that any *randomly selected* neighbor of an agent exerts a positive contribution falls – for consistency, it must be that $\sum_{k=1}^{t'} Q(k;p') \leq \sum_{k=1}^{t} Q(k;p)$.

This example illustrates the existence of a unique non-increasing symmetric equilibrium, and the two effects of an increase in connectivity: generating a (unique) equilibrium with greater contribution, though reducing the probability that any random neighbor contributes. Our results generalize these insights to a wide array of games exhibiting strategic substitutability, allowing for more general action spaces, payoff structures, and neighbor degree correlations.

We next study a simple game where actions are strategic complements. Again, consider a context were $X = \{0,1\}$ is the action space, but now let the payoffs of any particular agent i be given by $(\alpha \bar{x}_{N_i} - c)x_i$. Assuming that $c > \alpha > 0$, these payoffs define a coordination game where, depending on the underlying network and the information conditions, there can generally be multiple equilibria (even under incomplete information).

As before, we start our discussion with the case of complete information, i.e., with the assumption that the prevailing network is common knowledge. Clearly, the induced game always allows for an equilibrium where $x_i = 0$ for all i. There are generally other equilibria and we illustrate this for a

[10] Naturally, if actions are strategic complements, playing action 1 is prescribed by the equilibrium strategy if the type is *no lower* than the corresponding threshold. On this issue, see our second example in this section.

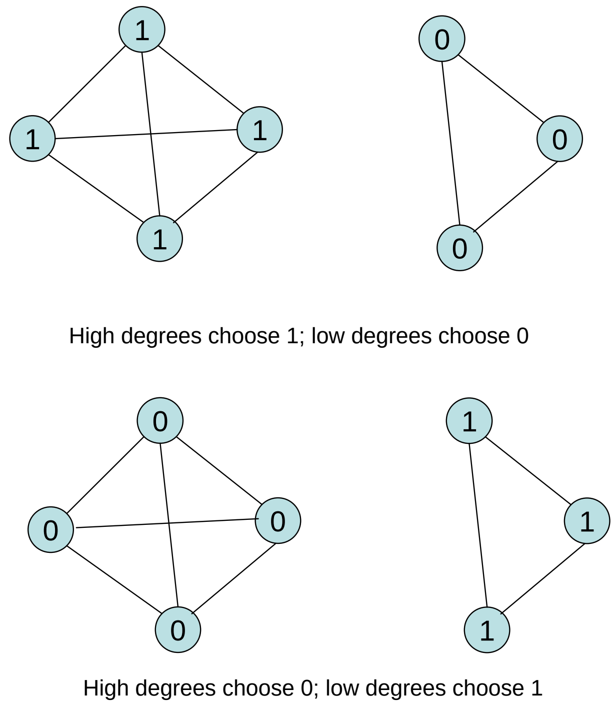

Figure 3: Strategic complements with complete information

simple network with 7 players, split into two complete components with 3 and 4 players, respectively. It is easy to see that there is an equilibrium in which all players in the larger component choose 1, while all players in the smaller component choose 0. However, the reverse pattern, in which all players in the large component choose 0, while all players in the small component choose 1 is also an equilibrium. These are depicted in Figure 3.

By contrast, if we make the assumption that each player is only informed of her own degree (and has independent beliefs on the degrees of neighbors), we find much more definite predictions with regard to equilibrium behavior. Take, for example, the Erdös-Renyi model and the resulting binomial beliefs considered above. Note that independence of neighbor degrees implies that the probability a random neighbor chooses the action 1 cannot depend on one's own degree. In particular, the expectation of the sum of actions $\bar{x}_{N_i}$ of any agent i with $|N_i| = k$ neighbors is increasing in k. The structure of payoffs then assures that if a degree k agent is choosing the action 1 in equilibrium, any agent of degree greater than k must be best responding with the action 1 as well, and so every equilibrium is determined by a threshold and is non-decreasing. Certainly, everyone choosing the action 0 is a symmetric (threshold) equilibrium. For sufficiently large p there exists $t < N-1$, an integer, for which

$$\alpha\,(t-1)\sum_{k=t}^{N-1} Q(k;p) < c \;\text{ and }\; \alpha t\sum_{k=t}^{N-1} Q(k;p) \geq c.$$

Such a t corresponds to an equilibrium that satisfies $\sigma(k) = 0$ for all $k < t$, $\sigma(t) \in [0,1]$, and $\sigma(k) = 1$ for all $k > t$.

Furthermore, increasing connectedness, as before, by shifting p to p', where $p' > p$, thereby inducing an FOSD shift in neighbors' degree distribution, implies that $\sum_{k=t}^{N-1} Q(k;p') > \sum_{k=t}^{N-1} Q(k;p)$. Hence, there exists an equilibrium threshold t' corresponding to p' and satisfying $t' \leq t$. Intuitively, the shift to p' increases perceived connectivity and therefore, ceteris paribus, the probability each random neighbor chooses the action 1. Thus, the value of the action 1 increases, and if all agents use the threshold t, the best response of any particular agent would be to use a threshold lower than t. Continuing such a process iteratively, we generate t'. Note also that $t' \leq t$ implies that the probability that a random neighbor chooses the action 1 in the t'-equilibrium under p', given as $\sum_{k=t'}^{N-1} Q(k;p')$, is greater than the probability that a random neighbor chooses the action 1 in the original equilibrium (with threshold t) under p.

Our results in Section 5 extend these observations to a general class of games with complements, allowing for a wide scope of action spaces, payoff structures, and neighbor degree correlations.

To summarize, under complete information there is no systematic relation between social networks and individual behavior and payoffs (even if we restrict attention to equilibrium in which players with the same degree choose the same action). By contrast, under incomplete network information, both in games of strategic substitutes as well as in games of strategic complements, we obtain a clear cut relation between networks and individual behavior and payoffs. Moreover, our discussion clarifies how networks have systematically different effects in games with substitutes and in games with complements.

3 The Model

This section presents our theoretical framework. We start by describing the modeling of a network game, comprised of the degree distribution and each agent's payoffs. We then discuss our equilibrium concept, symmetric Bayesian equilibrium.

3.1 Networks and Payoffs

There is a finite set of agents, $N = \{1, 2, ..., n\}$. The connections between them is described by a network that is represented by a matrix $g \in \{0, 1\}^{n \times n}$, with $g_{ij} = 1$ implying that i's payoff is affected by j's behavior. We follow the convention of setting $g_{ii} = 0$ for all $i \in N$.

Let $N_i(g) = \{j | g_{ij} = 1\}$ represent the set of neighbors of i. The *degree* of player i, $k_i(g)$, is the number of i's connections:

$$k_i(g) = |N_i(g)|.$$

Each player i takes an action x_i in X, where X is a compact subset of $[0, 1]$. Without loss of generality, we assume throughout that $0, 1 \in X$. We consider both discrete and connected action sets X. The payoff of player i when the profile of actions is $x = (x_1, ..., x_n)$ is given by:

$$v_{k_i(g)}(x_i, x_{N_i(g)})$$

where $x_{N_i(g)}$ is the vector of actions taken by the neighbors of i. Thus, the payoff of a player depends on her own action and the actions that her neighbors take.

Note that the payoff function depends on the player's degree k_i but not on her identity i. Therefore, any two players i and j who have the same degree ($k_i = k_j$) have the same payoff function. We also assume that v_k depends on the vector $x_{N_i(g)}$ in an anonymous way, so that if x' is a permutation of x (both k-dimensional vectors) then $v_k(x_i, x) = v_k(x_i, x')$ for any x_i. If X is not a discrete set then we assume that it is connected, in which case v_k is taken to be continuous in all its arguments and concave in own action.

Finally, we turn to the relation between players' strategies and their payoffs. We say that a payoff function exhibits *strategic complements* if it has increasing differences: for all k, $x_i > x_i'$, and $x \geq x'$:

$$v_k(x_i, x) - v_k(x_i', x) \geq v_k(x_i, x') - v_k(x_i', x').$$

Analogously, we say that a payoff function exhibits *strategic substitutes* if it has decreasing differences: for all k, $x_i > x_i'$, and $x \geq x'$:

$$v_k(x_i, x) - v_k(x_i', x) \leq v_k(x_i, x') - v_k(x_i', x').$$

These notions are said to apply strictly if the payoff inequalities are strict whenever $x \neq x'$.

We also keep track of the effects of others' strategies on a player's payoffs. We say that a payoff function exhibits *positive externalities* if for each k, and for all $x \geq x'$, $v_k(x_i, x) \geq v_k(x_i, x')$. Analogously, we say that a payoff function exhibits *negative externalities* if for each k, and for all $x \geq x'$, $v_k(x_i, x) \leq v_k(x_i, x')$. Correspondingly, the payoff function exhibits *strict* externalities (positive or negative) if the above payoff inequalities are strict whenever $x \neq x'$.

We now present some economic examples to illustrate the scope of our framework in terms of the payoff structures it allows for (that will be layered upon the social network configurations we describe below).

Example 1 *Payoffs Depend on the Sum of Actions*
Player i's payoff function when she chooses x_i and her k neighbors choose the profile $(x_1, ..., x_k)$ is:

$$v_k(x_i, x_1, ..., x_k) = f\left(x_i + \lambda \sum_{j=1}^{k} x_j\right) - c(x_i), \tag{3}$$

where $f(\cdot)$ is non-decreasing and $c(\cdot)$ is a "cost" function associated with own effort. The parameter $\lambda \in \mathbb{R}$ determines the nature of the externality across players' actions. This example exhibits (strict) strategic substitutes (complements) if, assuming differentiability, $\lambda f''$ is negative (positive).

The case where f is concave, $\lambda = 1$, and $c(\cdot)$ is increasing and linear corresponds to the case of information sharing as a local public good studied by Bramoullé and Kranton (2007), where actions are strategic substitutes. In contrast, if $\lambda = 1$, but f is convex (with $c'' > f'' > 0$), we obtain a model with strategic complements, which nests a model studied by Goyal and Moraga-Gonzalez (2001) regarding collaboration among firms. In fact, the formulation in (3) is general enough to accommodate a good number of further examples in the literature such as human capital investment (Calvó-Armengol and Jackson (2008)), crime networks (Ballester, Calvó-Armengol, and Zenou (2006)), some coordination problems (Ellison (1993)), and the onset of social unrest (Chwe (2000)). ■

An interesting special case of example 1 is the Best-Shot game described in the opening example of Section 2.

Example 2 *"Best-Shot" Public Goods Games*
The Best-Shot game is a good metaphor for many situations in which there are significant spillovers between players' actions. $X = \{0, 1\}$ and the action 1 can be interpreted as acquiring information (or providing any local and discrete public good). We suppose that $f(0) = 0$, $f(x) = 1$ for all $x \geq 1$, so that acquiring one piece of information suffices. Costs, on the other hand, satisfy $0 = c(0) < c(1) < 1$ so that no individual finds it optimal to dispense with the information but prefers one of her neighbors to gather it. This is a game of strategic substitutes and positive externalities.[11] ■

In the above examples, a player's payoffs depend on the sum of neighbors strategies and all of them satisfy the following general property.

Property A $v_{k+1}(x_i, (x, 0)) = v_k(x_i, x)$ for any $(x_i, x) \in X^{k+1}$.

Under Property A, adding a link to a neighbor who chooses action 0 is payoff equivalent to not having an additional neighbor. The above discussion clarifies that many economic examples studied so far satisfy Property A. There is however a prominent case where the payoffs violate Property A: this arises when payoffs depend on the average of the neighbors' actions. Our framework allows for a consideration of such games as well.

Example 3 *Payoffs Depend on the Average of Neighbors' Actions*

[11] For instance, consumers learn from relatives and friends (Feick and Price, 1987), innovations often get transmitted between firms, and experimentation is often shared amongst farmers (Foster and Rosenzweig, 1995, Conley and Udry, 2005). For a discussion of best-shot games, see Hirshleifer (1983).

Let $X = \{0, 1\}$. Player i's payoff function when she chooses x_i and her k neighbors choose the profile $(x_1, ..., x_k)$ is:

$$v_k(x_i, x_1, ..., x_k) = x_i f\left(\frac{\sum_{j=1}^{k} x_j}{k}\right) - c(x_i), \tag{4}$$

where $f(.)$ is an increasing function. This is a game of strategic complements and positive externalities. ■

3.2 Information

We study an environment in which individuals are aware of their proclivity to interact with others, but do not know who these others will be when taking actions. For instance, a researcher choosing an operating system may know the number of coauthors they tend to work with at any given time, but not necessarily who these people will be during the upcoming year. These considerations motivate the informational assumptions in our model: individuals know the number of their contacts and have information on the distribution of connections in the population at large.

Formally, let the degrees of the neighbors of a player i of degree k_i be denoted by $\mathbf{k}_{N(i)}$, which is a vector of dimension k_i. The information a player i of degree k_i has regarding the degrees of her neighbors is captured by a distribution $P(\mathbf{k}_{N(i)} \mid k_i)$. Throughout, we model players' beliefs with a common prior and ex-ante symmetry. Players may end up with different positions in a network and conditional beliefs, but their beliefs are only updated based on their realized position and not on their names. This means that the information structure is given by a family of anonymous conditional distributions $\mathbf{P} \equiv \left\{[P(\mathbf{k} \mid k)]_{\mathbf{k} \in \mathbb{N}^k}\right\}_{k \in \mathbb{N}}$. In some of our results, we also need to refer to the underlying unconditional degree distribution, which is denoted by $P(\cdot)$.

We would like to emphasize that our framework allows for correlation between neighbors' degrees. This means that the conditional distributions concerning neighbors' degrees can in principle vary with a player's degree. This is particularly important in face of the empirical evidence illustrating that social networks generally display such internode correlations. Newman (2003), for example, summarizes empirical results in this respect across different contexts. He reports, specifically, that some networks such as those of scientific collaboration (reflecting joint authorship of papers) or actor collaboration (film co-starring) display significant positive degree correlation while others, such as the internet (physical connections among routers) or the world wide web (hyperlinks between webpages), have a negative one. Since these correlations, positive or negative, may well have some bearing (in interplay with game payoffs) on the strategic problem faced by agents, they should be accommodated by the model.

To deal with this issue, we generalize (i.e., weaken) a standard definition of affiliation that has been amply used in the economic literature to capture statistical correlations between collections of random variables (e.g., individual valuations in auctions, as in Milgrom and Weber (1982)).[12] In order to introduce the notion formally, denote by $\mathbf{k}_{N(i)} = (k_1, k_2, \ldots k_{k_i})$ the degrees of the neighbors of a typical player i with degree k_i. Then, given any function $f : \{0, 1, \ldots, n-1\}^m \to \mathbb{R}$ where $m \leq k_i$, let

$$E_{P(\cdot|k_i)}[f] = \sum_{\mathbf{k}_{N(i)}} P(\mathbf{k}_{N(i)} \mid k_i) f(k_1, \ldots k_m). \tag{5}$$

The above expression simply fixes some subset $m \leq k_i$ of i's neighbors, and then takes the expectation of f operating on their degrees. We say that $\mathbf{P}$ exhibits *positive neighbor affiliation* if, for all $k' > k$, and any non-decreasing $f : \{0, 1, \ldots, n-1\}^k \to \mathbb{R}$.

$$E_{P(\cdot|k')}[f] \geq E_{P(\cdot|k)}[f]. \tag{6}$$

Analogously, $\mathbf{P}$ exhibits *negative neighbor affiliation* if the reverse inequality holds for each $k' > k$ and non-decreasing f.

As indicated, our notion of neighbor affiliation is weaker than what affiliation (positive or negative) among the whole vector of random variables $(k_i, \mathbf{k}_{N(i)})$ would entail.[13] It simply embodies the idea that higher degrees for a given player are correlated with higher or lower degree (depending on whether it is positive or negative, respectively) of *all* her neighbors. Obviously, it is satisfied in

[12] Affiliation, in turn, can be viewed as a strengthening of the notion of association that is common in the statistical literature – see, e.g., Esary, Proschan, and Walkup (1967) for a useful reference. In this paper, we are interested in both the notion of positive affiliation (which is the usual case postulated in the literature) as well as a negative one – the conditions and implications, however, are obviously fully symmetric in each case.

[13] To see this, refer to Theorem 5 in Milgrom and Weber (1982), which establishes that affiliation implies that the counterpart of (6) must hold when we condition on any subset of the random variables in $(k_i, \mathbf{k}_{N(i)})$ and compute the expected value for any nondecreasing function of those random variables. More precisely, our notion of neighbor affiliation is identical to the concept of *positive regression dependence* with respect to k_i, as formulated by Lehman (1966). Esary, Proschan, and Walkup (1967) show that this concept is weaker than the standard one of association, except for bivariate random variables

the case where neighbors' degrees are all stochastically independent. This is, for example, a condition that holds asymptotically in many models of random networks, including the classical model of Erdös-Rényi or the more recent configuration model (see, e.g., Newman (2003), Vega-Redondo (2007), and Jackson (2008) for discussions). Positive neighbor affiliation, on the other hand, is a feature commonly found in other models of network formation that have a dynamic dimension – cf. the model of Barabási and Albert (1999) based on preferential attachment, or the models by Vazquez (2003) and Jackson and Rogers (2007) reflecting network-based search.[14] In addition, an important motivation for internode degree correlations is empirical. For, as mentioned, many of the studies on real social networks undertaken in recent years find strong evidence for either positive or negative correlations, depending on the kind of network examined and the main driving forces at work. Neighbor affiliation, while entailing some restrictions, provides a workable tool for capturing these observations.

Finally, we also need a way of comparing situations where the network (and thus the corresponding beliefs) undergo changes in connectivity. Again, we focus on changes that reflect unambiguous increases or decreases in the distribution of agents' degrees. So we use a suitable extension of the standard notion of First Order Stochastic Dominance (FOSD) to embody changes in the degree distributions that capture the idea of link addition. Specifically, we say that $\mathbf{P}'$ *dominates* $\mathbf{P}$ if for all k, and any non-decreasing $f : \{0, 1, \ldots, n-1\}^k \to \mathbb{R}$

$$E_{P'(\cdot|k)}[f] \geq E_{P(\cdot|k)}[f].$$

This concept of dominance is a generalization of stochastic dominance relationships adapted to vectors and families of distributions.

To conclude, a network game is fully described and is henceforth denoted by a quadruple $(N, X, \{v_k\}_k, \mathbf{P})$. In certain cases, concentrating on degree distributions that exhibit independence between neighbors' degrees allows us to derive further insights. In these cases, the entire set of conditionals is captured by the underlying distribution P and so we denote the corresponding network

[14] See the working paper version Galeotti, Goyal, Jackson, Vega-Redondo, and Yariv (2006) for a formal description of neighbor affiliation attributes of commonly observed and studied network formation procedures.

game by $(N, X, \{v_k\}_k, P)$.

3.3 The Bayesian Game

A *strategy* for player i is a mapping $\sigma_i : \{0, 1, ..., n-1\} \to \Delta(X)$, where $\Delta(X)$ is the set of probability distributions on X. So, $\sigma_i(k)$ is the mixed strategy played by a player of degree k. We analyze (symmetric) Bayesian equilibria of this game and they can be represented simply as a (mixed) strategy, $\sigma(.)$.[15]

More formally, given a player i of degree k_i let $\psi(x_{N_i(g)}, \sigma, k_i)$ be the probability distribution over $x_{N_i(g)} \in X^{k_i}$ induced by the beliefs $P(\cdot \mid k_i)$ over the degrees of i's neighbors when composed with the strategy σ. Thus, the expected payoff to a player i with degree k_i when other players use strategy σ and i chooses action x_i is

$$U(x_i, \sigma, k_i) \equiv \int_{x_{N_i(g)} \in X^{k_i}} v_{k_i}(x_i, x_{N_i(g)}) \mathrm{d}\psi(x_{N_i(g)}, \sigma, k_i). \tag{7}$$

A strategy σ comprises a symmetric *Bayesian equilibrium* (or just an equilibrium, for short) if $\sigma(k_i)$ is a best response, for each degree k_i to the strategy σ being played by other players. That is, σ is an equilibrium if for every degree k_i displayed by any typical agent i, the following holds:

$$U(x_i, \sigma, k_i) \geq U(x_i', \sigma, k_i), \forall x_i' \in X, x_i \in \mathbf{supp}(\sigma(k_i)). \tag{8}$$

Our interest is in understanding the effects of networks on behavior and welfare. To bring out these effects clearly, we focus on symmetric Bayes-Nash equilibria, i.e. configurations where all players with the same network characteristic (which, under our informational assumptions, is their degree) choose the same strategy. This is further motivated by the observation that, in fact, *all* equilibria of the game must be symmetric when the following two conditions apply:

(a) the underlying network-formation mechanism is anonymous and the population very large;

[15] Static equilibrium refinements are not so useful in our case, as our equilibria are typically strict; e.g., in our applications (as, say, in best-shot games of the sort discussed in Section 2) the equilibria are typically strict, both in the complete- and incomplete-information scenarios. Usual equilibrium refinements, therefore, have no bite. Finally, it is worth noting, refinements that require dynamic stability in terms of an adjustment process can encounter nonexistence problems. As an illustration, consider the notion of stable equilibrium used by Bramoullé and Kranton (2007) for their analysis of local public goods in networks. As they show, these equilibria exist only for networks whose maximal independent set has two nodes in every non-provider's neighborhood, which rules out many networks.

(b) the payoff function is strictly concave in own action.

For, in this case, all agents of any given degree face the same decision problem (from (a)) and the optimal choice in it is unique (by (b)). This leads to symmetric behavior.[16]

It is worth emphasizing as well that the contrast between complete and incomplete information that is the heart of our analysis remains in force when we restrict attention to symmetric equilibria in both cases. To illustrate this, recall the star networks which were considered in section 2 (see especially Figure 1). There, the restriction of symmetry under complete information requires that all peripheral players choose the same action. But, as we saw, this allows for two polar and very different Nash equilibria. Instead, symmetry under incomplete information singles out a unique equilibrium outcome in which the center does not contribute. Our discussion in section 2 suggests that analogous observations hold for games with strategic complements (see Figure 3).

In order to relate network structure and the primitives of the payoffs to features of equilibrium, we need to relate strategies to degrees. Some basic definitions of monotonicity are thus useful in stating our results.

A strategy σ is *non-decreasing* if $\sigma(k')$ first-order stochastically dominates $\sigma(k)$ for each $k' > k$. Similarly, σ is *non-decreasing* if the domination relationship is reversed.

Expected payoffs exhibit *degree complementarity* if

$$U(x_i, \sigma, k_i) - U(x_i', \sigma, k_i) \geq U(x_i, \sigma, k_i') - U(x_i', \sigma, k_i'),$$

whenever $x_i > x_i'$, $k_i > k_i'$, and σ is non-decreasing. Analogously, payoffs exhibit *degree substitution* if the inequality above is reversed in the case where σ is non-increasing.

Degree complementarity captures the idea that if a high strategy is more attractive than a low strategy for a player of some degree, then the same is true for a player of a higher degree when the

[16] Formally, the statement here is in effect of an asymptotic nature, pertaining to the limit equilibrium behavior as the population size grows infinite. To be precise, consider the relatively simple case where the underlying network-formation mechanism is random, the degree distribution has a uniformly bounded support, and every two networks differing only in some arbitrary permutation of player indices have an identical ex-ante probability. Under these conditions, the probability that any two agents be connected becomes insignificant for large populations and, therefore, if they have the same degree, they must also face a probability distribution over neighbor's actions that is essentially the same. Then, by strict concavity and continuity of payoffs, the claim follows.

strategy being played by other players is non-decreasing. Degree complementarity arises in many contexts that are covered by our framework. We illustrate this by considering two cases of interest.

Recall that Property A says that $v_{k+1}(x_i,(x,0)) = v_k(x_i,x)$ for any $(x_i,x) \in X^{k+1}$. We note that Property A, strategic complements of $v_k(.,.)$ and positive neighbor affiliation of $\mathbf{P}$ ensure degree complementarity. To see why this is true consider a strategy σ which is non-decreasing and suppose that $k' = k+1$. Now observe that

$$\begin{array}{ll} & U(x_i,\sigma,k) - U(x_i',\sigma,k) \\ = & \int_{x\in X^k}[v_k(x_i,x) - v_k(x_i',x)]\mathrm{d}\psi(x,\sigma,k) \\ = & \int_{x\in X^k}[v_{k'}(x_i,(x,0)) - v_{k'}(x_i',(x,0))]\mathrm{d}\psi(x,\sigma,k) \\ \leq & \int_{x\in X^k}[v_{k'}(x_i,(x,0)) - v_{k'}(x_i',(x,0))]\mathrm{d}\psi((x,0),\sigma,k') \\ \leq & \int_{(x,x_{k+1})\in X^{k'}}[v_{k'}(x_i,(x,x_{k+1})) - v_{k'}(x_i',(x,x_{k+1}))]\mathrm{d}\psi((x,x_{k+1}),\sigma,k') \\ = & U(x_i,\sigma,k') - U(x_i',\sigma,k'), \end{array}$$

where the second equality follows from Property A, the first inequality follows from positive neighbor affiliation, σ being non-decreasing and strategic complements, while the second inequality follows from strategic complements. Analogous considerations establish that Property A, strategic substitutes of $v_k(.,.)$, and negative neighbor affiliation of $\mathbf{P}$ ensure degree substitution.

While Property A (taken along with the corresponding properties on $\mathbf{P}$ and $v_k(.,.)$) is sufficient to establish degree complementarity and substitution, it is not necessary. The following discussion, which builds on example 3, illustrates this point.

Example 4 *Degree complements and substitutes without Property A.*

Suppose that payoffs are as in example 3. In addition, let $\mathbf{P}$ be such that neighbors' degrees are stochastically independent (for example, as in an the asymptotic Erdös-Rényi random network discussed in Section 2). When neighbors' degrees are independent, $\frac{kP(k)}{\langle k\rangle}$ captures the probability that a random neighbor is of degree k (see, e.g., Jackson (2008)). Let Y_m be a random variable that has a binomial distribution with m draws each with probability $\sum_k \frac{kP(k)}{\langle k\rangle}\sigma(k)$, the expected action of any neighbor. Then, the expected payoffs to a player i are given by:

$$U(x_i,\sigma,k_i) = E\left[x_i f\left(\frac{Y_{k_i}}{k_i}\right)\right] - c(x_i),$$

and thus

$$U(1,\sigma,k_i) - U(0,\sigma,k_i) = E\left[f\left(\frac{Y_{k_i}}{k_i}\right)\right] - c(1) + c(0).$$

Note that $\frac{Y_{k'}}{k'}$ is a mean-preserving spread of $\frac{Y_k}{k}$ when $k' < k$. Thus, if f is concave, we have degree complementarity, while if f is convex then degree substitution obtains. ■

4 Equilibrium: Existence and Monotonicity

We start by showing existence of an equilibrium involving monotone strategies. We then provide conditions under which all equilibria are monotone. Finally, we close the section by exploring the relationship between network degree and equilibrium payoffs. The latter analysis, in particular, identifies conditions under which payoffs increase/decrease with network degree, thereby clarifying the contexts in which network connections are advantageous and disadvantageous, respectively.

Recall that a strategy σ is *non-decreasing* if $\sigma(k')$ first-order stochastically dominates $\sigma(k)$ for each $k' > k$. Similarly, σ is *non-increasing* if the domination relationship is reversed. Based on these notions, the following existence result easily follows.

Proposition 1 *There exists a symmetric equilibrium, and if the game has degree complements, then there exists a symmetric equilibrium in pure strategies. If there is degree complementarity (substitution) then there is a symmetric equilibrium that is non-decreasing (non-increasing).*

To show the validity of this result, we start by addressing the existence of a symmetric equilibrium. It has been assumed that players have identical action sets X, the payoff functions are also the same, and player's beliefs concerning network are ex-ante symmetric. The game, therefore, is a symmetric one of incomplete information. Given that the action set is compact, the payoff function is continuous in all arguments (when the action set is non-discrete) and concave in own action, it is then straightforward to adapt the usual fixed-point argument to show that there exists a symmetric equilibrium, possibly in mixed strategies. Moreover, the fact that this symmetric equilibrium can be chosen in pure strategies under degree complements follows from standard strategic complements arguments (e.g., see Milgrom and Shannon (1994)).

On the other hand, concerning monotonicity, one can readily exploit the degree complements/substitutes property to show that for a player faced with a monotone strategy played by the rest of the population, there always exists a monotone best-reply. Then, since the set of monotone strategies is convex and compact, the existence of a monotone equilibrium derives from standard arguments (see, e.g., Milgrom and Shannon (1994) or van Zandt and Vives (2007)).

Next, we elaborate on two aspects of Proposition 1. *First*, we discuss whether *every* symmetric equilibrium is monotone. Consider a game with action set $X = \{0,1\}$ and payoffs $v_k\left(x_i, x_{N_i(g)}\right) = x_i \sum_{j \in N_i(g)} x_j - cx_i$, where $0 < c < 1$ (a special case of the second example in Section 2). This example satisfies Property A and the underlying game displays strategic complements. Now suppose that there is perfect degree correlation so that players are connected to others of the same degree. It is then clear that *any* symmetric pure strategy profile defines an equilibrium.[17] This example suggests that the possibility of non-monotone equilibria is related to the correlation in degrees. This point is highlighted by the following result.

Proposition 2 *Suppose that payoffs satisfy Property A and that the degrees of neighboring nodes are independent. Then, under strict strategic complements (substitutes) every symmetric equilibrium is non-decreasing (non-increasing).*

The key point to note here is that, under independence, degree k and degree $k' = k+1$ players have the same beliefs about the degree of each of their neighbors. If the $k+1^{\text{th}}$ neighbor is choosing 0 then under Property A the degree k' player will choose the same best response as the degree k player; if the $k+1^{\text{th}}$ neighbor chooses a positive action then strict complementarities imply that the degree k' player best responds with a higher action.[18]

Going back to some of our motivating examples, Proposition 2 has very clear implications. Consider the student aiming at a career of diplomacy and contemplating learning a new language. Since the value of knowing a language is increasing with the number of connected individuals who speak that language (it is a game of complements), we would expect that student to be more likely to take on the study of the new language than a student who aims at a less interactive career.

A *second* issue is whether the nature of degree correlation – positive neighbor affiliation under strategic complements, or negative neighbor affiliation under strategic substitutes – is essential for

[17] In fact, the best response of a degree k player is to choose 0 (1) if all other degree k players also choose 0 (1).

[18] The strictness is important for the result. For instance, if players were completely indifferent between all actions, then non-monotone equilibria are clearly possible.

existence of monotone equilibria. Consider a special case of Example 1 in which $X = [0, 1]$, $f(y) = \gamma y + \alpha y^2$, $y = x_i + \sum_{j \in N_i(g)} x_j$, and $c(x_i) = \beta x_i^2$ for some $\gamma, \alpha, \beta > 0$. This game exhibits strategic complements. Next suppose that the unconditional degree distribution satisfies $P(1) = P(2) = \varepsilon$ and $P(\bar{k}) = 1 - 2\varepsilon$ for some small ε and a given large $\bar{k}$. Further suppose that $P(\bar{k} \mid 1) = P(2 \mid 2) = 1$, i.e., all agents with degree 1 are connected to those of degree $\bar{k}$ and all those of degree 2 are connected among themselves. Note that this pattern of connections violates positive neighbor affiliation. It is now possible to verify that if $\beta > \alpha$ then *every* equilibrium is interior; moreover if $\bar{k}$ is large enough and ε sufficiently small then σ satisfies $\sigma_2 < \sigma_1 < \sigma_{\bar{k}}$ and is not monotone.

A recurring theme in the study of social structure in economics is the idea that social connections create personal advantages. In our framework the relation between degrees and payoffs is the natural way to study network advantages. Let us consider games with positive externalities and positive neighbor affiliation, and look at a player with degree $k+1$. Suppose that all of her neighbors follow the monotone increasing equilibrium strategy, but her $k+1^{\text{th}}$ neighbor chooses the minimal 0 action. Property A implies that our $(k+1)$ degree player can assure herself an expected payoff which is at least as high as that of any k degree player by simply using the strategy of the degree k player. These considerations lead us to state the following result.

Proposition 3 *Suppose that payoffs satisfy Property A. If* $\mathbf{P}$ *exhibits positive neighbor affiliation and the game displays positive externalities (negative externalities), then in every non-decreasing symmetric equilibrium the expected payoffs are non-decreasing (non-increasing) in degree. If* $\mathbf{P}$ *exhibits negative neighbor affiliation and the game displays positive externalities (negative externalities), then in every non-increasing symmetric equilibrium the expected payoffs are non-decreasing (non-increasing) in degree.*

We emphasize that under positive externalities, players with more neighbors earn higher payoffs irrespective of whether the game exhibits strategic complements or substitutes (under the appropriate monotone equilibrium). These network advantages are especially striking in games with strategic substitutes (such as local public goods games) and negative neighbor affiliation: here higher degree players exert lower efforts but earn a higher payoff as compared to their less connected peers.

5 The Effects of Changing Networks

We now investigate how changes in a network – such as the addition/deletion of links or the re-distribution of links away from a regular network to highly unequal distributions that characterize empirically observed networks – affect the behavior and welfare of players. We start with games of strategic substitutes and then take up games of strategic complements.

5.1 Games with Strategic Substitutes

We refer to games where payoffs are of strict strategic substitutes and satisfy Property A and where **P** exhibits negative neighbor affiliation as *binary network games of substitutes*, and we focus on such games in the following analysis. An attractive feature of binary action network games with substitutes is that there is a unique symmetric equilibrium strategy σ, and it involves a threshold.

Proposition 4 *Consider a binary network game of substitutes. There exists some threshold* $t \in \{0, 1, 2, ...\}$ *such that the probability* $\sigma(1|\cdot)$ *of choosing action* 1 *in the unique non-increasing symmetric equilibrium strategy* σ *satisfies* $\sigma(1|k_i) = 1$ *for* $k_i < t$, $\sigma(1|k_i) = 0$ *for all* $k_i > t$, *and* $\sigma(1|t) \in (0, 1]$ *for* $k_i = t$.

Now we ask: what is the effect of adding links on equilibrium behavior? We first observe that the best response of a player depends on the actions and hence the expectations concerning the degrees of her neighbors. Thus, the effects of link addition must be studied in terms of the change in the degree distribution of the neighbors.[19] We therefore approach the addition of links in terms of an increase in the degrees of a neighbor. In our context of non-increasing strategies, this means a fall in her action (on average), which, from strategic substitutes, suggests that the best response of the player in question should increase. However, this increase in action of every degree may come into conflict

[19] Indeed, it is important to note that the relationship between two underlying (unconditional) degree distributions does not imply a similar relation for the conditional distribution over neighbors' degrees, even under independence. As an illustration consider a case where degrees of neighbors are independent. Consider two degree distributions P and P', where P' assigns one half probability to degrees 2 and 10 each, while distribution P assigns one half probability to degrees 8 or 10 each. Clearly P FOSD P'. As mentioned above, when neighboring degrees are independent, the probability of having a link with a node is (at least roughly, depending on the process) proportional to the degree of that node, so that for all k, $P(k'|k) = k'P(k')/\sum P(l)l$. Let $\tilde{P}(k')$ be the neighbor's degree distribution. Under $\tilde{P}'$, the probability that a neighbor has degree 10 is 5/6, while under $\tilde{P}$, the same probability is 5/9. Thus, $\tilde{P}$ does not FOSD $\tilde{P}'$.

with the expectation that neighbors must be choosing a lower action, on average. The following result clarifies how this tension is resolved. Denote by t the threshold in the game $(N, X, \{v_k\}_k\}, \mathbf{P})$ and by t' the threshold in game $(N, X, \{v_k\}_k, \mathbf{P}')$.

Proposition 5 *Let $(N, X, \{v_k\}_k, \mathbf{P})$ and $(N, X, \{v_k\}_k, \mathbf{P}')$ be binary network games of substitutes. If $\mathbf{P}$ dominates $\mathbf{P}'$, then $t \geq t'$. However, for the threshold degree type t the probability that a neighbor chooses 1 is lower under $\mathbf{P}$.*

This result clarifies that an increase in threshold for choosing 1 is consistent with equilibrium behavior because each of the neighbors is more connected and chooses 1 with a lower probability (in spite of an increase in the threshold). The best shot game helps to illustrate the effects of dominance shifts in degrees which are derived in the above result.

Example 5 *Effects of increasing degrees in a best-shot game.*

Consider the best shot game discussed in the introduction and described in example 2. Set $c = 25/64$. Suppose that degrees take on values $1, 2$ and 3 and that the degrees of neighbors are independent. Note that, in view of Proposition 2 and Proposition 4, the assumption that the degrees of neighboring nodes are stochastically independent implies that there exists a unique symmetric equilibrium which is non-decreasing and it is fully characterized by a threshold.

Let us start with initial beliefs $\mathbf{P}'$ that assign probability one-half to neighboring players having degree 1 and 2. In the unique symmetric equilibrium, degree 1 players choose 1 with probability 1, while degree 2 players choose 1 with probability 0. Hence, at equilibrium, the probability that a neighbor of a degree 2 player chooses action 1 is 1/2.

Consider now a dominance shift to $\mathbf{P}$, so that neighboring players are believed to have degrees 2 and 3 with probability one-half each. It can be checked that the unique equilibrium involves degree 2 players choosing action 1 with probability 3/4 while degree 3 players choose 1 with probability 0. Consequently, the probability that a neighbor of a degree 2 player chooses action 1 is 3/8.

Overall, the dominance shift in the beliefs from $\mathbf{P}'$ to $\mathbf{P}$ leads to an increase in the threshold from 1 to 2. However, the threshold degree 2 player has lower expectation of action 1 under $\mathbf{P}$ as compared to $\mathbf{P}'$. ■

We now turn to the effects on welfare. The expected welfare is assessed by the expected payoff of a randomly chosen player (according to the prevailing degree distribution). Observe that dominance shifts in the interaction structure lower the expected probability that a randomly selected neighbor of a t-degree player (the threshold player under $\mathbf{P}$) chooses 1. If the degrees of neighbors are independent, then the average effort of a randomly selected neighbor of a player i does not depend on i's degree, and therefore all players expect lower action from each of their neighbors. However, in the presence of negative neighbor affiliation, matters are more complicated, and it is possible that the overall effect of a dominance shift in the distribution of connections can be positive for some degrees and negative for others.

Proposition 5 compares behavior across networks when there is an increase in the density of links in the sense of domination. However, there are many cases where we might be interested in comparing networks when there is not a clear cut domination relation. We now develop a result on the effect of *arbitrary* changes in the degree distribution.

For simplicity, we focus on the case where degrees of neighbors are independent. Let $\mathbf{P}$ and $\mathbf{P}'$ be two different sets of beliefs and suppose that $\tilde{F}$ and $\tilde{F}'$ are the corresponding induced cumulative distribution functions of the degree distributions, respectively. Let t and t' stand for the threshold types defining the (unique) threshold equilibria under $\mathbf{P}$ and $\mathbf{P}'$, respectively. Formally,

Proposition 6 *Let $(N, X, \{v_k\}_k, P)$ and $(N, X, \{v_k\}_k, P')$ be binary network games of substitutes with independent neighbor degrees. Let t and t' denote the unique equilibrium thresholds for these games. If $\tilde{F}(t') \leq \tilde{F}'(t'-1)$ then $t \geq t'$. Moreover, in these equilibria, the probability that any given neighbor chooses 1 in $(N, X, \{v_k\}_k, P)$ is lower than in $(N, X, \{v_k\}_k, P')$.*

The key issue here is the change in the probability mass relative to the threshold. If the probability of degrees equal or below the threshold goes down then the probability of action 1 decreases and from strategic substitutes, the best response of threshold type t must still be 1. In other words, the threshold rises weakly.

The contribution of Proposition 6 is that it allows us to examine the effect of *any change of the degree distribution*. A natural and important example of such changes is increasing the polarization of the degree distribution by shifting weights to the ends of the support of the degree distribution, as is done under a mean preserving spread (MPS) of the degree distribution. In particular, the above results can be directly applied to the case of strong MPS shifts in the degree distributions. Focusing on the unconditional beliefs (taken to coincide with the unconditional degree distributions because of independence), we say that $P(\cdot)$ is a *strong MPS* of $P'(\cdot)$ if they have the same mean and there exists L and H such that $P(k) \geq P'(k)$ if $k < L$ or $k > H$, and $P(k) \leq P'(k)$ otherwise. Proposition 6 implies that, in the context of binary-action games, the equilibrium effects of any such change can be inferred from the relative values of the threshold t, L, and H.

5.2 Games with Strategic Complements

This section studies the effects of changes in the network on equilibrium behavior and payoffs in games with strategic complements. From Proposition 1 we know that equilibria are increasing in degree in games with degree complementarities. As we shift weight to higher degree neighbors each player's highest best response to the original equilibrium profile would be at least as high as the supremum of her original strategy's support. We can now iterate this best response procedure. Since the action set is compact, this process converges and it is easy to see that the limit is a (symmetric) non-decreasing equilibrium that dominates the original one. The following result summarizes this argument.

We refer to a network game in which payoffs satisfy strict strategic complements and Property A and $\mathbf{P}$ exhibits positive neighbor affiliation as a *network game of complements.*

Proposition 7 *Let* $(N, X, \{v_k\}_k, \mathbf{P})$ *and* $(N, X, \{v_k\}_k, \mathbf{P}')$ *be network games of complements. If* $\mathbf{P}$ *dominates* $\mathbf{P}'$*, then for every non-decreasing equilibrium* σ' *of* $(N, X, \{v_k\}_k, \mathbf{P}')$ *there exists a non-decreasing equilibrium* σ *of* $(N, X, \{v_k\}_k, \mathbf{P})$ *that dominates it.*

The proof is straightforward and omitted.[20] Consider next the effect of a dominance shift in the social network on welfare. Recall that the expected welfare is assessed by the expected payoff

[20] Note that if $(N, X, \{v_k\}_k, \mathbf{P}')$ is a network game of complements, and $\mathbf{P}$ dominates $\mathbf{P}'$, it is not necessarily the case that $\mathbf{P}$ exhibits positive neighbor affiliation. In that case, we can still use similar arguments to construct a new symmetric equilibrium (under $\mathbf{P}$) that dominates σ, though it need not be non-decreasing.

of a randomly chosen player. Naturally, it must depend on whether the externalities are positive or negative. Suppose, for concreteness, that they are positive and let $\mathbf{P}$ dominate $\mathbf{P}'$. Then, from Proposition 7, we know that for every non-decreasing equilibrium σ' under $\mathbf{P}'$ there exists a non-decreasing equilibrium σ under $\mathbf{P}$ in which players' actions are all at least as high. Hence, the expected payoff of each player is higher under $\mathbf{P}$. However, since expected payoffs are non-decreasing in the degree of a player, to assess welfare it is also important to consider the relation between the corresponding unconditional degree distributions $P(\cdot)$ and $P'(\cdot)$. If, for example, $P(\cdot)$ FOSD $P'(\cdot)$, then, the above considerations imply that the *ex-ante* expected payoff of a randomly chosen player must rise when one moves from $\mathbf{P}'$ to $\mathbf{P}$. We summarize this argument in the following result. For a non-decreasing strategy profile σ under $\mathbf{P}$, define $W_{\mathbf{P}}(\sigma)$ to be the expected payoff of a node picked at random (under $P(\cdot)$).

Proposition 8 *Let $(N, X, \{v_k\}_k, \mathbf{P})$ and $(N, X, \{v_k\}_k, \mathbf{P}')$ be network games of complements, in which payoffs satisfy positive externalities. Suppose that $\mathbf{P}$ dominates $\mathbf{P}'$ and $P(\cdot)$ FOSD $P'(\cdot)$. For any non-decreasing equilibrium σ' of $(N, X, \{v_k\}_k, \mathbf{P}')$, there exists a non-decreasing equilibrium σ of $(N, X, \{v_k\}_k, \mathbf{P})$ such that $W_{\mathbf{P}}(\sigma) \geq W_{\mathbf{P}'}(\sigma')$.*

The proof follows from the arguments above and is omitted. Propositions 7 and 8 pertain to dominance shifts in the conditional degree distributions.

As in the case of games of substitutes, in binary games with independent degree distributions, we can identify the effects of arbitrary changes in the degree distribution. Indeed, in those games, an analogue of Proposition 4 can be readily established and symmetric equilibria take the form of threshold equilibria: $\sigma(1|k_i) = 0$ for $k_i < t$, $\sigma(1|k_i) = 1$ for all $k_i > t$, and $\sigma(1|t) \in (0, 1]$ for $k_i = t$. Recalling that for any two distributions P and P', $\tilde{F}$ and $\tilde{F}'$ denote their respective cumulative distributions, we have:

Proposition 9 *Let $(N, X, \{v_k\}_k, P)$ and $(N, X, \{v_k\}_k, P')$ be binary network games of complements and independent neighbor degrees. Let t' be an equilibrium threshold of $(N, X, \{v_k\}_k, P')$. If $\tilde{F}(t') \leq \tilde{F}'(t'-1)$ then there is an equilibrium of $(N, X, \{v_k\}, P)$ with corresponding threshold type $t \leq t'$. Moreover, the probability that any given neighbor chooses 1 rises.*

The proof for this result follows along the lines of the proof of Proposition 6 and is omitted.

We conclude by observing that the strategic structure of payoffs has an important effect: recall from Subsection 5.1 that in the case of strategic substitutes, the probability that any neighbor chooses 1 falls when network connectivity grows. By contrast, in games of strategic complements the addition of links leads to an increase in the probability that a neighbor chooses action 1.

6 Deeper Network Information

So far we have focused on the case where players only know their own degree and best respond to the anticipated actions of their neighbors based on the (conditional) degree distributions. We now investigate the implication of increasing the information that players possess about their local networks. As a natural first step along these lines, we examine situations where a player knows not only how many neighbors she has, but also how many neighbors each of her neighbors has (e.g., a researcher deciding on an operating system may know the number and identity of their current coauthors, but not necessarily the full set of current coauthors of their coauthors). The arguments we develop in this section extend in a natural way to general radii of local knowledge. Indeed, in the limit, as this radius of knowledge grows, we arrive at complete knowledge of the arrangement of degrees in the network.[21]

Formally, the common type space $\mathcal{T}$ of every player i consists of elements of the form $(k; \ell_1, \ell_2, ..., \ell_k)$ where $k \in \{0, 1, 2, ..., n-1\}$ is the degree of the player and ℓ_j is the degree of neighbor j ($j = 1, 2, ..., k$), where (in an anonymous setup where the identity of neighbors is ignored) we may assume without loss of generality that neighbors are indexed according to decreasing degree (i.e., $\ell_j \geq \ell_{j+1}$). Given the multi-dimensionality of types in this case, the question arises as to how one should define monotonicity. In particular, the issue is what should be the order relationship $\succeq$ on the type space underlying the requirement of monotonicity. For the case of strategic complements, it is natural to say that two different types, $t = (k; \ell_1, \ell_2, ..., \ell_k)$ and $t' = (k'; \ell'_1, \ell'_2, ..., \ell'_{k'})$, satisfy $t \succeq t'$ if and only if $k \geq k'$ and $\ell_u \geq \ell'_u$ for all $u = 1, 2, ..., k'$. On the other hand, for the case of strategic substitutes, we write $t \succeq t'$

[21] For results on this limit case, see the earlier version of this paper, Galeotti, Goyal, Jackson, Vega-Redondo, and Yariv (2006).

if and only if $k \geq k'$ and $\ell_u \leq \ell'_u$ for all $u = 1, 2, 3, ...k'$. Given any such (partial) order on $\mathcal{T}$, we say that a strategy σ is monotone increasing if for all t_i, $t'_i \in \mathcal{T}$, $t_i \succeq t'_i \Rightarrow \sigma(t_i)$ FOSD $\sigma(t'_i)$. The notion of a monotonically decreasing strategy is defined analogously.

We first illustrate the impact of richer knowledge on the nature of equilibria. It is easier to see the effects of deeper network information in the simpler setting where the degrees of the neighbors are independent and so, for expositional simplicity, we assume independence of neighbors' degrees in this section.[22] Recall from Proposition 2 that under degree independence all symmetric equilibria are monotone increasing (decreasing) in the case of strategic complements (substitutes) when agents are only informed of own degree. The following example shows that greater network knowledge introduces non-monotone equilibrium even if the degrees of neighboring nodes are stochastically independent.

Example 6 *Non-monotone Equilibria with Knowledge of Neighbors' Degrees*

Consider a setting where nodes have either degree 1 or degree 2, as given by the corresponding probabilities $P(1)$ and $P(2)$. Suppose that the game is binary-action with $X = \{0, 1\}$ and displays strategic complements. Specifically, suppose that the payoff of a player only depends on his own action x_i and the sum $\bar{x}$ of his neighbors' actions as given by a function $v(x_i, \bar{x})$ as follows: $v(0, 0) = 0$, $v(0, 1) = 1/2$, $v(0, 2) = 3/4$, $v(1, 0) = -1$, $v(1, 1) = 1$, $v(1, 2) = 3$.

It is readily seen that, for any P with support on degrees 1 and 2, the following strategy σ defines a symmetric equilibrium: $\sigma(1|1; 1) = 1$; $\sigma(1|1; 2) = 0$; $\sigma(1|2; \ell_1, \ell_2) = 0$ for any $\ell_1, \ell_2 \in \{1, 2\}$. Here, two players that are only linked to each other both play 1, while all other players choose 0. ■

Similar non-monotonic equilibrium examples can be constructed for games with strategic substitutes. These observations leave open the issue of whether there exist *any* suitably increasing or decreasing monotone equilibria. The following result shows that a monotone equilibrium always exists if players have deeper network information.

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[22] We note that the assumption of stochastic independence of the degrees of neighboring nodes implies independence of degrees of neighbors of neighbors.

Proposition 10 *Suppose that neighbors' degrees are independent, players know their own degree and the degrees of their neighbors and payoffs satisfy Property A. Under strategic complements (strategic substitutes) there exists a symmetric equilibrium that is monotone increasing (decreasing).*

The proof of the proposition, which appears in the Appendix, extends naturally the ideas mentioned for the proof of Proposition 2, i.e., the best-reply to a monotone strategy can be chosen monotone and the set of all monotone strategies is compact and convex. A direct implication of the result is that there is always an equilibrium that, on average across the types $(k; \ell_1, \ell_2, ..., \ell_k)$ consistent with each degree k, prescribes an (average) action that is monotone in degree. Equipped with the above monotonicity result, it is also possible to recover most of the insights obtained earlier under the assumption that players only know their own degree.

7 Concluding Remarks

Empirical work suggests that the patterns of social interaction have an important influence on economic outcomes. These interaction effects have however been resistant to systematic theoretical study: even in the simplest examples games on networks have multiple equilibria that possess very different properties. The principal innovation of our paper is the introduction of the idea that players have incomplete network knowledge. In particular, we focus on an easily measurable aspect of networks, the number of personal connections/degree, and suppose that players know their own degree but have incomplete information concerning the degree of others in the network. This formulation allows us to develop a general framework for the study of games played on networks. On the one hand, it allows us to accommodate a large class of games with strategic complements and strategic substitutes. On the other hand, it allows us to capture features displayed by real world networks such as general patterns of correlations across the degrees of neighbors.

The analysis of this framework yields a number of powerful and intuitively appealing insights with regard to the effects of location within a network as well as with regard to changes in networks on equilibrium actions and payoffs. These results also clarify how the basic strategic features of the game – as manifest in the substitutes and complements property – combine with different patterns of degree correlations to shape behavior and payoffs.

In this paper we have focused on the degree distribution in a network. The research on social networks has identified a number of other important aspects of networks, such as clustering, centrality and proximity, and in future work it would be interesting to bring them into the model.

8 Appendix

Proof of Proposition 2: We present the proof for the case of strategic complements. The proof for the case of strategic substitutes is analogous and omitted. Let $\{\sigma_k^*\}$ be the strategy played in a symmetric equilibrium of the network game. If $\{\sigma_k^*\}$ is a trivial strategy with all degrees choosing action 0 with probability 1, the claim follows directly. Therefore, from now on, we assume that the equilibrium strategy is non-trivial and that there is some k' and some $x' > 0$ such that $x' \in \mathbf{supp}(\sigma_{k'}^*)$.

Consider any $k \in \{0, 1, ..., n\}$ and let $x_k = \sup[\mathbf{supp}(\sigma_k^*)]$. If $x_k = 0$, it trivially follows that $x_{k'} \geq x_k$ for all $x_{k'} \in \mathbf{supp}(\sigma_{k'}^*)$ with $k' > k$. So let us assume that $x_k > 0$. Then, for any $x < x_k$, Property A and the assumption of (strict) strategic complements imply that

$$v_{k+1}(x_k, x_{l_1}, ..., x_{l_k}, x_s) - v_{k+1}(x, x_{l_1}, ..., x_{l_k}, x_s) \geq v_k(x_k, x_{l_1}, ..., x_{l_k}) - v_k(x, x_{l_1}, ..., x_{l_k})$$

for any x_s, with the inequality being strict if $x_s > 0$. Then, averaging over all types, the fact that the degrees of any two neighboring nodes are stochastically independent random variables together with the fact that there are some players with degree k who choose $x_k > 0$ implies that

$$U(x_k, \sigma^*, k+1) - U(x, \sigma^*, k+1) > U(x_k, \sigma^*, k) - U(x, \sigma^*, k).$$

On the other hand, note that from the choice of x_k,

$$U(x_k, \sigma^*, k) - U(x, \sigma^*, k) \geq 0$$

for all x. Combining the aforementioned considerations we conclude:

$$U(x_k, \sigma^*, k+1) - U(x, \sigma^*, k+1) > 0,$$

for all $x < x_k$. This in turn requires that if $x_{k+1} \in \mathbf{supp}(\sigma_{k+1}^*)$ then $x_{k+1} \geq x_k$, which of course implies that σ_{k+1}^* FOSD σ_k^*. Iterating the argument as needed, the desired conclusion follows, i.e., $\sigma_{k'}^*$ FOSD σ_k^* whenever $k' > k$. ■

Proof of Proposition 3: We present the proof for positive externalities. The proof for negative externalities is analogous and omitted. The claim is obviously true for a trivial equilibrium in which all players choose the action 0 with probability 1. So, let σ^* be a (non-trivial) equilibrium strategy. Suppose $x_k \in \mathbf{supp}(\sigma_k^*)$ and $x_{k+1} \in \mathbf{supp}\left(\sigma_{k+1}^*\right)$. Property A implies that

$$v_{k+1}(x_k, x_{l_1}, ..., x_{l_k}, 0) = v_k(x_k, x_{l_1}, ..., x_{l_k}),$$

for all $x_{l_1}, ..., x_{l_k}$. Since the payoff structure satisfies positive externalities, it follows that for any $x > 0$,

$$v_{k+1}(x_k, x_{l_1}, ..., x_{l_k}, x) \geq v_k(x_k, x_{l_1}, ..., x_{l_k}).$$

We now have to consider two cases. First, assume positive neighbor affiliation and let σ^* be a monotone increasing equilibrium. Then, looking at expected utilities, we obtain that:

$$U(x_k, \sigma^*, k+1) \geq U(x_k, \sigma^*, k).$$

Since σ_{k+1}^* is a best response in the network game being played and $x_{k+1} \in \mathbf{supp}\left(\sigma_{k+1}^*\right)$,

$$U(x_{k+1}, \sigma^*, k+1) \geq U(x_k, \sigma^*, k+1)$$

and the result follows. Second, observe that the case of negative neighbor affiliation and monotone decreasing equilibrium strategy can be proven using analogous arguments. ■

Proof of Proposition 4: We know from Subsection 3.3 that network games of substitutes exhibit the degree substitutes property. Proposition 1 then tells us that there exists a symmetric equilibrium which is non-increasing in degree. Fix the strategy σ in one such equilibrium. Suppose that for degree $k > 0$ there is positive probability $\sigma(1|k)$ of choosing action 1. We prove that $\sigma(1|l) = 1$, for all $l < k$. Consider first degrees $l = k - 1 < k$. Then, letting the same notation $v_k(\cdot, \cdot)$ stand for the usual mixed extension of the original payoff function, the marginal return to action 1 can be written as follows:

$$
\begin{aligned}
& U(1,\sigma,l)-U(0,\sigma,l) \\
= & \sum_{(k_1,\ldots,k_l)} P(k_1,\ldots,k_l|l)[v_l(1,\sigma(k_1),\ldots,\sigma(k_l))-v_l(0,\sigma(k_1),\ldots,\sigma(k_l))] \\
= & \sum_{(k_1,\ldots,k_{l+1})} P(k_1,\ldots,k_l|l))[v_k(1,\sigma(k_1),\ldots,\sigma(k_l),x_{l+1}=0)-v_k(0,\sigma(k_1),\ldots,\sigma(k_l),x_{l+1}=0)] \\
\geq & \sum_{(k_1,\ldots,k_k)} P(k_1,\ldots,k_k|k)[v_k(1,\sigma(k_1),\ldots,\sigma(k_{k-1}),x_k=0)-v_k(0,\sigma(k_1),\ldots,\sigma(k_{k-1}),x_k=0)] \\
> & \sum_{(k_1,\ldots,k_k)} P(k_1,\ldots,k_k|k)[v_k(1,\sigma(k_1),\ldots,\sigma(k_k))-v_k(0,\sigma(k_1),\ldots,\sigma(k_k))] \\
= & U(1,\sigma,k)-U(0,\sigma,k)\geq 0,
\end{aligned}
$$

where the second equality holds by Property A, the subsequent (weak) inequality holds because $\sigma(k-1)$ FOSD $\sigma(k)$, $\mathbf{P}$ exhibits negative neighbor affiliation and strict strategic substitutes holds, and the second (strict) inequality holds due to strict strategic substitutes and $\sigma(1|k)>0$. Finally, the last inequality simply reflects the hypothesis that σ constitutes an equilibrium. This argument can be repeated to establish that $\sigma(1|l)=1$, for all $l<k$. Analogous arguments, with a simple switching of inequality signs, shows that if $\sigma(0|k)>0$ then $\sigma(0|k')=1$, for all $k'>k$.

The above argument establishes that every non-increasing symmetric equilibrium strategy σ is defined by a threshold t. To complete the proof, we next show that this threshold is unique. Thus, for the sake of contradiction, suppose that there are two distinct thresholds, t and t' with $t'<t$, which induce strategies σ and σ' respectively. If the equilibrium σ' is played, a player with degree $t'+1$ (higher than the corresponding threshold t') strictly prefers action 0, i.e.

$$U(1,\sigma',t'+1)-U(0,\sigma',t'+1)<0, \tag{9}$$

while if equilibrium σ is played, a player with degree $t'+1$ (no higher than the corresponding threshold t) weakly prefers action 1, i.e.

$$U(1,\sigma,t'+1)-U(0,\sigma,t'+1)\geq 0. \tag{10}$$

We can then write:

$$\begin{aligned}
0 &\leq U(1,\sigma,t'+1)-U(0,\sigma,t'+1) \\
&= \sum_{(k_1,...,k_{t'+1})} P(k_1,...,k_{t'+1}|t'+1)[v_{t'+1}(1,\sigma(k_1),...,\sigma(k_{t'+1}))-v_{t'+1}(0,\sigma(k_1),...,\sigma(k_{t'+1}))] \\
&\leq \sum_{(k_1,...,k_{t'+1})} P(k_1,...,k_{t'+1}|t'+1)[v_{t'+1}(1,\sigma'(k_1),...,\sigma'(k_{t'+1}))-v_{t'+1}(0,\sigma'(k_1),...,\sigma'(k_{t'+1}))] \\
&= U(1,\sigma',t'+1)-U(0,\sigma',t'+1)<0,
\end{aligned}$$

where the first and third inequalities are simply (9) and (10), while the middle inequality is a consequence of strategic substitutes and the hypothesis that $\sigma(1|k) \geq \sigma'(1|k)$, for all $k \in \{0,1,...n-1\}$. This yields the contradiction that completes the proof. ■

Proof of Proposition 5: Under the maintained hypotheses there exists a unique non-increasing symmetric equilibrium with a threshold property under both degree distributions. Suppose that this equilibrium σ' has threshold t' under $\mathbf{P}'$. The assumptions that $\mathbf{P}$ dominates $\mathbf{P}'$ for all k and that players choose a non-increasing strategy imply that the equilibrium threshold under $\mathbf{P}$ cannot be lower than t'. To see this, suppose that in the non-increasing equilibrium under $\mathbf{P}$, σ, the threshold $t<t'$. We now show that this yields a contradiction. In equilibrium σ' under $\mathbf{P}'$, for the threshold degree t' the expected payoffs from action 1 are higher than the expected payoffs from action 0. Thus, again identifying each $v_k(\cdot,\cdot)$ with the mixed extension of the corresponding payoff function, we can write:

$$\begin{aligned}
0 &\leq U(1,\sigma',t')-U(0,\sigma',t') \\
&= \sum_{(k_1,...,k_{t'})} P'(k_1,...,k_{t'}|t')\,[v_{t'}(1,\sigma'(k_1),....,\sigma'(k_{t'}))-v_{t'}(0,\sigma'(k_1),....,\sigma'(k_{t'}))] \\
&\leq \sum_{(k_1,...,k_{t'})} P(k_1,...,k_{t'}|t')\,[v_{t'}(1,\sigma'(k_1),....,\sigma'(k_{t'}))-v_{t'}(0,\sigma'(k_1),....,\sigma'(k_{t'}))] \\
&< \sum_{(k_1,...,k_{t'})} P(k_1,...,k_{t'}|t')\,[v_{t'}(1,\sigma(k_1),....,\sigma(k_{t'}))-v_{t'}(0,\sigma(k_1),....,\sigma(k_{t'}))] \\
&= U(1,\sigma,t')-U(0,\sigma,t'),
\end{aligned}$$

where the second inequality follows from the hypotheses that $\mathbf{P}$ dominates $\mathbf{P}'$, σ' is non-increasing and $v_{t'}(\cdot,\cdot)$ satisfies the strategic substitutes property, while the third inequality follows from the hypothesis that $t<t'$ and $v_{t'}(\cdot,\cdot)$ satisfies the strict strategic substitutes property. This however

implies that for degree t' action 1 yields strictly higher expected payoffs than action 0 under equilibrium σ, a contradiction with $t < t'$. ■

Proof of Proposition 6: Suppose that $\tilde{F}(t') \leq \tilde{F}'(t'-1)$ but, contrary to what is claimed, $t < t'$. Then, under $\mathbf{P}$, the probability that any of the neighbors chooses action 1 is bounded above by $\tilde{F}(t')$ and, therefore, by $\tilde{F}'(t'-1)$. Given the hypothesis that t' is the threshold under $\mathbf{P}'$, the assumption of strategic substitutes now generates a contradiction with the optimality of actions of degree t' in an equilibrium under $\mathbf{P}$, and this completes the proof. ■

Proof of Proposition 10: Let us consider first the case of strategic complements and denote by $\sum^m$ the set of monotone strategies. The proof is based on the following two claims:

Claim 1: For any player i, if all other players $j \neq i$ use a common strategy $\sigma \in \sum^m$ there is always a strategy $\sigma_i \in \sum^m$ that is a best response to it.

Claim 2: A symmetric equilibrium exists in the strategic-form game where players' strategies are taken from $\sum^m$.

To establish Claim 1, consider a player i and let $t_i, t_i' \in \mathcal{T}$ such that $t_i' \succeq t_i$, where $\succeq$ is the partial order applicable to the case of strategic complements (see Section 6). For any $\sigma \in \sum^m$ chosen by every $j \neq i$, let $BR(\sigma, t_i)$ be the set of best-response strategies of player i to σ when his type is t_i. Let us assume that $\sigma(t_j) \neq 0$ for some $t_j \in \mathcal{T}$. (Otherwise, the desired conclusion follows even more directly, since the best-response correspondence is unaffected by being connected to a player whose strategy chooses action 0 uniformly.) By definition, for every $x_{t_i} \in BR(\sigma, t_i)$, we must have that

$$\forall x \in X, \qquad U(x_{t_i}, \sigma, t_i) - U(x, \sigma, t_i) \geq 0.$$

Then, since $t_i' \succeq t_i$, the assumption of (strict) strategic complements implies that

$$\forall x \leq x_{t_i}, \qquad U(x_{t_i}, \sigma, t_i') - U(x, \sigma, t_i') > 0. \tag{11}$$

This follows from a two-fold observation:

(i) From Property A, if $t_i = (k, \ell_1, \ell_2, ..., \ell_k)$ and $t_i' = (k', \ell_1', \ell_2', ..., \ell_k')$ and $t_i' \succeq t_i$ we can think of t_i involving k' neighbors with all neighbors indexed from $k+1$ to k' (if any) choosing the action 0;

(ii) From strict strategic complements, since $\ell_u' \geq \ell_u$ the probability distribution over actions corresponding to each of his neighbors under t_i, $u = 1, 2, ..., k$, is dominated in the FOSD sense by the corresponding neighbor under t_i'. This follows from the fact the beliefs applying separately to each of the ℓ_u and ℓ_u' second-neighbors under consideration in each case are identical and stochastically independent.

Let us now make use of (11) in the case where x_{t_i} is the highest best response by type t_i to σ. Then, it follows that any $x_{t_i'} \in BR(\sigma, t_i')$ must satisfy:

$$x_{t_i'} \geq \sup\{x_{t_i} : x_{t_i} \in BR(\sigma, t_i)\},$$

which establishes Claim 1.

To prove Claim 2, we can simply invoke that, for any given $x \in X^k$, the function $v_k(\cdot, x)$ in own action either has a discrete domain or is concave, combined with the fact that the set of monotone strategies is compact and convex. To see the latter point, note that the monotonicity of a strategy σ is characterized by the condition:

$$\forall t_i, t_i' \in \mathcal{T}, \qquad t_i' \succeq t_i \Rightarrow \sigma(t_i') \text{ FOSD } \sigma(t_i). \tag{12}$$

Clearly, if two different strategies σ and σ' satisfy (12), then any convex combination $\hat{\sigma} = \lambda\sigma + (1-\lambda)\sigma'$ also satisfies it.

Finally, to prove the result for the case of strategic substitutes, note that the above line of arguments can be applied unchanged, with the suitable adaptation of the partial order used to define monotonicity. In this second case, as explained in Section 6, we say that $t \succeq t'$ if and only if $k \geq k'$ and $\ell_u \leq \ell_u'$ for all $u = 1, 2, ..., k'$. ■

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