The Existence of Pairwise Stable Networks

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We examine networks connecting individuals, where the payoff to an individual from an economic or social activity depends on the full network of connections among individuals. Individuals can form and sever links connecting themselves to other individuals based on the improvement that the resulting network offers them relative to the current network. As individuals do this, we obtain sequences of networks called 'improving paths.' We study conditions under which such sequences cycle, and conditions under which such sequences lead to a stable network. Specifically, we give conditions necessary and sufficient to rule out cycles, which are in turn sufficient conditions for existence of pairwise stable networks.

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I. Introduction

Networks play a fundamental role in a wide variety of social and economic interactions, including such diverse applications as obtaining information about employment opportunities (*e.g.*, Boorman (1975); Montgomery (1991); Topa (2001); Arrow and Borzekowski

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(2000); and Calvo-Armengol and Jackson (2001)), to providing non-market insurance in developing countries (*e.g.*, Fafchamps and Lund (1997)), As researchers have become increasingly aware of the importance of networks in determining the outcome of economic relationships, the theoretical analysis of the determination of such networks has grown.[1]

One of the central issues addressed in this growing literature is the formation of networks. Here, we are primarily interested in two things: (i) understanding dynamic processes of network formation and when such processes might cycle, and (ii) understanding when there exist pairwise stable networks, where a dynamic process might come to rest. These two issues are closely related as the definitions underlying dynamic and pairwise stability are related. Pairwise stability follows the definition in Jackson and Wolinsky (1996): no two agents could gain from linking and no single agent could gain by severing one of his or her links. The basis for dynamic network formation is what are called improving paths by Jackson and Watts (1998): a sequence of networks, where each transition is obtained by either two individuals choosing to add a link or one individual deciding to delete a link. Thus, a pairwise stable network is one from which there is no improving path leaving it.

The improving paths emanating from any starting network must lead to either a pairwise stable network or a cycle (where a number of networks are repeatedly visited). It is easily seen that there always exists either a pairwise stable network or a cycle from which there is no exit. We show by example, that it is possible that only cycles exist and no pairwise stable networks exist, or that no cycles exist and only pairwise stable networks exist, or finally that both cycles and pairwise stable networks exist.

The notion of players becoming stuck in a cycle is disturbing as it conjures up images of a teenager's ever changing social network of friends. But most importantly from our perspective, cycles imply that one is stuck with chaotic dynamics while if cycles do not exist then one can make firm predictions about the dynamics of network formation. Thus a central question to ask is when can cycles be

[1]See Dutta and Jackson (2001) for an introductory account and references to some of the literature on the formation of social and economic networks.

ruled out? Theorem 1 gives the exact conditions, on primitives of the model, under which there are no cycles and so only pairwise stable networks exist. We also present some easy to verify sufficient conditions for no cycles. We illustrate these results in a series of economically well-motivated games and also show a general result regarding properties under the Myerson value allocation rules.

The next issue we address concerns identifying conditions under which dynamics should lead to efficient networks. Even in a game with no cycles it is possible that players will become stuck in an inefficient stable network. For instance, if the net benefit of forming an initial link is negative and if there are increasing returns to forming additional links, then players might become stuck in the empty network even though everyone would be better off in a connected network. Proposition 1 gives conditions under which improving paths starting from the empty network naturally lead to efficient and pairwise stable networks and so players do not become stuck in an empty inefficient network.

The work here is related to a number of papers in this growing literature. It provides new insights into the concepts of pairwise stability of Jackson and Wolinsky (1996), stable state of Watts (2001), and of improving path of Jackson and Watts (1998). The analysis here can be useful in studying the dynamic evolution of networks as in Jackson and Watts (1998, 1999), Goyal and Vega-Redondo (1999), Droste, Gilles and Johnson (2000) and Skyrms and Pemantle (2000). Finally, while we focus here on non-directed networks, directed networks are also of interest (*e.g.*, see Bala and Goyal (2000); and Dutta and Jackson (2000)), and the results here have analogs in that setting.

II. Networks

The model of social and economic networks that we consider is based on that of Jackson and Wolinsky (1996) and the notion of improving paths comes from Jackson and Watts (1998).

Players

Let $N=\{1,\cdots,n\}$ be the finite set of *players*. Depending on the application, a player may be a single individual, a firm, a country, or some other autonomous unit.

Networks

The network relations among the players are represented by graphs whose nodes or vertices represent the players and whose links (edges or arcs) capture the pairwise relations. We focus on non-directed networks where links are reciprocal. The complete network, denoted g^N, is the set of all subsets of N of size 2, The set of possible *networks or graphs* on N is $\{g \mid g \subset g^N\}$. The subset of N containing i and j is denoted ij and is referred to as the *link* ij. The interpretation is that if $ij \in g$, then nodes i and j are directly connected, while if $ij \notin g$, then nodes i and j are not directly connected.

Let $g+ij$ denote the network obtained by adding link ij to the existing network g and let $g-ij$ denote the network obtained by deleting link ij from the existing network g (*i.e.*, $g+ij=g \cup \{ij\}$ and $g-ij=g \setminus \{ij\}$).

If $g'=g+ij$ or $g'=g-ij$, then we say that g and g' are *adjacent*.

Let $N(g)=\{i \mid \exists j \text{ s.t. } ij \in g\}$ be the set of players involved in at least one link and $n(g)$ be the cardinality of $N(g)$.

Paths and Components

A path in g connecting i_1 *and* i_n is a set of distinct nodes $\{i_1, i_2, \cdots, i_n\} \subset N(g)$ such that $\{i_1 i_2, i_2 i_3, \cdots, i_{n-1} i_n\} \subset g$.

A nonempty network $g' \subset g$ is a *component* of g, if for all $i \in N(g')$ and $j \in N(g')$, $i \neq j$, there exists a path in g' connecting i and j, and for any $i \in N(g')$ and $j \in N(g)$, $ij \in g$ implies $ij \in g'$.

Value Functions and Efficiency

The *value of a network* is represented by $v:\{g \mid g \subset g^N\} \rightarrow \mathcal{R}$, where $v(g)$ represents the total utility or production of the network. The set of all such functions is V. In some applications the value will be an aggregate of individual utilities or productions, $v(g)=\sum_i u_i(g)$, where $u_i:\{g \mid g \subset g^N\} \rightarrow \mathcal{R}$.

A network $g \subset g^N$ is *efficient* if $v(g) \geq v(g')$ for all $g' \subset g^N$.

Efficiency and Pareto efficiency coincide when value is transferable. For a more extensive discussion of various notions of efficiency in networks, see Jackson (2001).

Allocation Rules

An allocation rule $Y:\{g \mid g \subset g^N\} \times V \rightarrow \mathcal{R}^N$ describes how the value associated with each network is distributed to the individual players. $Y_i(g,v)$ may be thought of as the payoff to player i from network g under the value function v. For simplicity, if v is fixed, we will simply write $Y_i(g)$.

The allocation rule may represent the final payoff that an individual receives. For instance, in a purely social network, the allocation rule may represent the utility that each individual receives from the network and this utility might not be transferable. However, more generally, the allocation rule might also include transfers that are made either by the individuals through a bargaining process or imposed by some other party such as a government.

Pairwise Stability

The following concept of pairwise stability from Jackson and Wolinsky (1996) describes networks for which no player would benefit by severing an existing link, and no two players would benefit by forming a new link.

A network g is *pairwise stable* with respect to v and Y if

(i) for all $ij \in g$, $Y_i(g,v) \geq Y_i(g-ij;v)$ and $Y_j(g,v) \geq Y_j(g-ij;v)$, and

(ii) for all $ij \notin g$, if $Y_i(g,v) < Y_i(g+ij;v)$ then $Y_j(g,v) > Y_j(g+ij;v)$.

When a network g is not pairwise stable it is said to be *defeated* by g' if either $g'=g+ij$ and (ii) is violated for ij, or if $g'=g-ij$ and (i) is violated for ij.

There are variations on the notion of pairwise stability discussed in Jackson and Wolinsky (1996). Dutta and Mutuswami (1997) and Jackson and van den Nouweland (2001) discuss alternative approaches that capture coalitional deviations (and see Dutta and Jackson (2001) for references to other approaches of modeling network formation).

We refer to the following examples frequently in illustrating definitions and results.

***Example* 1:** *Connections Model* (Jackson and Wolinsky 1996)
The symmetric connections model is described as follows. Players form links with each other in order to exchange information. If player i is connected to player j, by a path of t links, then player i receives a payoff of δ^t from his indirect connection with player j. We assume $0<\delta<1$, and so the payoff δ^t decreases as the path connecting players i and j increases; thus information that travels a long distance becomes diluted and is less valuable than information obtained from a closer neighbor. Each direct link ij results in a cost c to both i and j. This cost can be interpreted as the time a player must spend with another player in order to maintain a direct link.

Formally, the payoff player i receives from network g is equal to $u_i(g)=\sum_{j\neq i}\delta^{t(ij)}-\sum_{j:ij\in g}c$, where $t(ij)$ is the number of links in the shortest path between i and j (setting $t(ij)=\infty$ if there is no path between i and j). Here the value of network g equals $v(g)=\sum_{i\in N}u_i(g)$ and $Y_i(g,v)=u_i(g)$. The incentives in forming links come from the consideration of direct costs and benefit, as well as the benefits of indirect connections.◇

***Example* 2:** *Co-Author Model* (Jackson and Wolinsky 1996)
Each player is a researcher who spends time writing papers. If two players are connected, then they are working on a paper together. The amount of time researcher i spends on a given project is inversely related to the number of projects, n_i, that he is involved in. Formally, player i's payoff is represented as

$$u_i(g)=\sum_{j:ij\in g}\frac{1}{n_i}+\frac{1}{n_j}+\frac{1}{n_in_j}$$

for $n_i>0$, For $n_i=0$ we assume that $u_i(g)=0$. Again, $v(g)=\sum_{i\in N}u_i(g)$, and $Y_i(v,g)=u_i(g)$. Here, the interesting tradeoffs from connection come from the benefit of gains from a co-author's time $(1/n_j)$, at the expense of diluting the synergy (interaction) term $1/(n_in_{jk})$ with other co-authors.◇

III. Improving Paths and Cycles

We now examine a notion due to Jackson and Watts (1998), which captures the sequences that might be followed as a network evolves. We emphasize that *a path represents a sequence of changes from one network to another*, rather than a path along links within a given network.

Improving Paths

An improving path is a sequence of networks that can emerge when individuals form or sever links based on the improvement the resulting network offers relative to the current network. Each network in the sequence differs by one link from the previous one. If a link is added, then the two players involved must both agree to its addition, with at least one of the two strictly benefiting from the addition of the link. If a link is deleted, then it must be that at least one of the two players involved in the link strictly benefits from its deletion.

Formally, an *improving path* from a network g to a network g' is a finite sequence of networks $g_1,\cdots,g_K$ with $g_1=g$ and $g_K=g'$ such that for any $k\in\{1,\cdots,K-1\}$ either:

(i) $g_{k+1}=g_k-ij$ for some ij such that $Y_i(g_k-ij)>Y_i(g_k)$, or
(ii) $g_{k+1}=g_k+ij$ for some ij such that $Y_i(g_k+ij)>Y_i(g_k)$ and $Y_j(g_k+ij)\geq Y_j(g_k)$.

An improving path is thus a sequence of networks that might be observed in a dynamic process where players are myopically adding and deleting links.

Note that a network is pairwise stable if and only if it has no improving paths emanating from it.

Let us say a few words about this myopic behavior. It is possible that under myopic behavior a player deletes a link making him or herself better off, but then this leads another player to delete another link which in turn leaves the first player worse off relative to the starting position, If the first player foresaw this, he or she might choose not to sever the link to begin with. This sort of consideration is not taken into account in the definition of improving paths, and may be important when there are relatively small numbers of forward-looking players who are well-informed about the value of the network and the motivations of others (see Watts (2002); and Page, Wooders, and Kamat (2001)). However, in larger networks and networks where players' information might be local and limited, or in networks where players significantly discount the future, myopic behavior is a natural assumption.

In addition to the assumption of myopic behavior, there are other implicit assumptions in the definition of improving path. For example, the definition only allows for the change of a single link at a time.

This limits the number of improving paths and makes the set of pairwise stable networks a rather generous one. Nevertheless, this still often results in fairly tight predictions and thus given its simple applicability offers a strong set of predictions simply based on necessary conditions. The concept can easily be adapted to allow for the simultaneous addition or deletion of several links at a time (see Jackson and Watts (1998)), and many of the results which follow have direct analogs for such variations.

The following example illustrates the concepts of improving path and pairwise stability.

***Example* 3:** *Improving Paths in the Symmetric Connections Model*
Consider the symmetric connections model of Example 1 with 4 players. The set of improving paths depend on the relative size of c and δ. If $\delta^2 < \delta - c$, then links are very cheap and players have an incentive to add every link and never to delete a link. Here, from any network that is not the fully connected network there exists an improving path leading to any larger network (*i.e.*, network whose links are a superset of the given network). If $0 < \delta - c < \delta^2$, then players are willing to add links to a player with whom they are not already connected (directly or indirectly), but are not willing to add (and are willing to delete) a link with someone who is also indirectly connected to them by an indirect path of length 2 (Whether or not the same holds for indirect connections of length 3 depends on the comparison of c to $\delta - \delta^3$). In this case, there are many improving paths leaving the empty network, some of which lead to the efficient network (a star[2]; see Jackson and Wolinsky

[2]A network is called a star if there is a central player, and all links are between that central person and each other person.

(1996) Proposition 1), but others which lead to lines, and in some cases circles. If $c > \delta$, then there are no improving paths emanating from the empty network. Provided c is not too large, there are other pairwise stable networks, but none of which have any "loose ends" (players with just one link connecting them to the network). For some intermediate networks, for instance a line connecting all 4 players, there exist improving paths that lead back to the empty network, but also improving paths leading to a circle (For more on pairwise stability, efficiency, and dynamic link formation in the connections model, see Jackson and Wolinsky (1996); Watts (2001); and Jackson and Watts (1998)). $\diamond$

The improving paths emanating from any starting network do not always need to lead to a pairwise stable network. They might also lead to a cycle, where a number of networks are repeatedly visited, as we now define.

Cycles

It is possible for a dynamic process, and improving paths in particular, to cycle among a set of networks. Let us examine this possibility in some detail.

A set of networks C, form a *cycle* if for any $g \in C$ and $g' \in C$ there exists an improving path connecting g to g'.

A cycle C is a *maximal cycle* if it is not a proper subset of a cycle.

A cycle C is a *closed cycle* if no network in C lies on an improving path leading to a network which is not in C.

Note that a closed cycle is necessarily a maximal cycle.

Example 4: *Asymmetric Connections Example—Existence of a Cycle*
Consider an asymmetric variation of the connections model of Example 2 with 5 players, where δ is player specific, denoted δ_i. Assume that $\delta_1 < c < \delta_1 + \delta_1^2 - \delta_1^3 - \delta_1^4$; so player 1 is willing to add a link to make a circle if player 1 is at the end of a line involving all players. But player 1 does not want to be directly linked to a player who is not linked to anyone else. Assume that the reverse is true for player 3, so $\delta_3 > c > \delta_3 + \delta_3^2 - \delta_3^3 - \delta_3^4$, (for example, let $\delta_1 <$

$(5^{1/2}-1)/2<\delta_3$ and set $c=(\delta_1+\delta_3)/2)$. Here, player 3 prefers to delete a link if he is in a circle with everyone else. Assume that $\delta_i>c>\delta_i-\delta_i^2$ for all other players. These players are willing to link with any player they are not directly or indirectly connected to, but these players do not wish to shorten an indirect connection of distance 2 (but may or may not wish to shorten longer paths).

In such a setting a cycle exists. Start with the circle {12,23,34, 45,15}. Player 3 wants to delete the link 23, and no one else is interested in deleting a link. So, we move to {12,34,45,15}. Player 1 now wants to delete the link 12, and no one else is interested in deleting a link. So, we move to {34,45,15}. Players 2 and 3 now want to add the link 23, and no one is interested in deleting a link. So, we move to {23,34,45,15}. Now, players 1 and 2 want to add the link 12. So, we move back to circle {12,23,34,45,15}.

Note that this cycle is reachable from an improving path from the empty network. For instance first add link 15, then 45, then 34, then 23, then 12. However, this cycle is not closed, since player 3 could start by severing 34 instead.

Note also that there is an asymmetry in payoffs in this example that allows for the cycle. If a network is completely symmetric and payoffs are symmetric (such as the circle in the symmetric connections model, where each link is similar to every other link in value) then there cannot be a cycle containing the network that consists of entirely subnetworks or entirely supernetworks, since no one should want to delete a link that they just added. ◇

The concept of improving path provides for an easy proof of the following existence result.

***Lemma* 1** (Jackson and Watts 1998). For any v and Y there exists at least one pairwise stable network or closed cycle of networks.

We include the short proof for completeness.

***Proof*:** Notice that a network is pairwise stable if and only if it does not lie on an improving path to any other network. So, start at any network. Either it is pairwise stable or it lies on an improving path to another network. In the first case the result is established so consider the second case. Follow the improving path. Given the finite number of possible networks, either the improving path ends at some network which has no improving paths leaving it, which then must be pairwise stable, or it can be continued through each network it hits. In the second case, the improving path must form a cycle. Thus, we have established that there always exists either a pairwise stable network or a cycle. So consider the case where there are no pairwise stable networks. We show that there must be a closed cycle. Since there must exist a cycle, given the finite number of networks there must exist a maximal cycle. Consider the collection of all maximal cycles. By the definition of maximal cycle, there must be at least one such cycle for which there is no improving path leaving the cycle (There can be improving paths leaving some of the maximal cycles, but these must lead to another maximal cycle. If all maximal cycles had improving paths leaving them, then there would be a larger cycle, contradicting maximality). Thus, there exists a closed cycle. $\diamond$

The following example shows that it is possible to have only closed cycles exist and have no pairwise stable networks (It is also possible to have the reverse, or have both exist).

***Example* 5:** *Trading Example—Non-Existence of a Pairwise Stable Network* (Jackson and Watts 1998)
Players benefit from trading with other players with whom they are linked, and trade can only flow along links. Players begin by forming a network. Subsequently, they receive random endowments and the players trade along paths of the network. Trade flows without friction along any path and each connected component trades to a Walrasian equilibrium.

There are two goods. All players have identical utility functions for the two goods which are symmetric Cobb-Douglas of the form $U(x,y)=xy$. Each player has a random endowment, which is independently and identically distributed. A player's endowment is either (1,0) or (0,1), each with probability 1/2. Links are formed before players' endowments are realized. For a given network, Walrasian equilibria occur on each connected component, regardless of the configuration of links. For instance, three players a line have the same trades as three players in a circle (triangle), but with a lower total cost of links. Let the cost of a link be equal to 5/96 (for each player).

Let us show that if n is a least 4, then there does not exist a pairwise stable network.

The utility of being alone is 0. Not accounting for the cost of links, the expected utility for a player of being connected to one other is 1/8 (There is a 1/2 probability that the realized endowments will differ, in which case the players will trade to an allocation of (1/2,1/2) which results in a utility of 1/4 for each of the two players. There is also 1/2 probability that the realized endowments will be identical in which case the utility will be 0 for each player). Similar calculations show that, not accounting for the cost of links, the expected utility for a player of being connected (directly or indirectly) to two other players is 1/6; and of being connected to three other players is 3/16. Most importantly, the expected utility of a player is strictly concave in the number of other players that he is directly or indirectly connected to. Thus the marginal gain of being connected to an additional player is decreasing in the number of players that one is already connected to.

Accounting for the cost of a link, it becomes clear that if k players are in a component, then there must be exactly $k-1$ links. If there are more than $k-1$ links, then there is at least one link that could be severed without changing the component structure of the network. Thus, some player can sever a link thereby saving the cost of the link but not losing any expected utility from trading.

Note that if g is pairwise stable, then any component with 3 or more players cannot contain a player who has just one link. This result follows from the fact that a player connected to another player, who is not connected to anyone else, loses at most $1/6-1/8=1/24$ in expected utility by severing the link, but saves the cost of 5/96 and so should sever this link.

From these two observations it follows that if there were to exist a pairwise stable network, then it would have to consist of pairs of connected players (as two completely unconnected players benefit from forming a link), and one unconnected player if n is odd. If n is at least 4, then there must exist at least two pairs. However, such a network is not pairwise stable, since any two players in opposite pairs gain from forming a link. Thus, there is no pairwise stable network. From Lemma 1, we know that there exists a closed cycle. An instance of a cycle in this trading example is {12,34} to {12,23,34} to {12,23} to {12} to {12,34}. A closed cycle would include many more networks, as there are a number of alternative improving paths from each of these networks. ◇

IV. Ruling out Cycles

Let us explore conditions on Y and v that rule out the existence of cycles. If there are no cycles, then all improving paths will necessarily lead to a pairwise stable network by Lemma 1.

Fix Y and v. If there exists an improving path from g to g', then let us use the symbol $g \to g'$. Given the transitivity of $\to$, there are no cycles if and only if $\to$ is asymmetric. Although this provides a direct characterization of the existence of cycles, Theorem 1 provides what turns out to be a more useful characterization.

The following definitions are used in Theorem 1.

Two networks g and g' are *adjacent* if they differ by one link.

Y and v *exhibit no indifference* if for any g and g' that are adjacent either g defeats g' or g' defeats g.

***Theorem* 1:** Fix v and Y. If there exists a function, $w{:}\{g \mid g \subset g^N\} \to \mathcal{R}$, such that [$g'$ defeats g] $\Leftrightarrow$ [$w(g') > w(g)$ and g' and g are adjacent], then there are no cycles. Conversely, if Y and v exhibit no indifference, then there are no cycles only if there exists a function, $w{:}\{g \mid g \subset g^N\} \to \mathcal{R}$, such that [$g'$ defeats g] $\Leftrightarrow$ [$w(g') > w(g)$ and g' and g are adjacent].

Note that w is independent of the players involved in adding or severing the link. Thus, in a rough sense, w is similar to a potential function. Theorem 1 shows that the existence of a cycle is tied to the existence of a single function (that is player independent) that represents the incentives of players with regards to any adjacent changes. The existence of such a function is precisely what is needed to rule out cycles.

Note that the supposition in the last part of Theorem 1, that Y and v exhibit no indifference is critical to the existence of such a w. To see this, consider the following example with $n=3$. Suppose that {12,23,13} defeats {12,23} defeats {12} defeats {12,13}, but that players 2 and 3 are both indifferent between {12,23,13} and {12,13}. Suppose also, that no other network defeats any other. Here there are no cycles, and yet the existence of such a w would require that $w(\{12,23,13\}) > w(\{12,13\})$, while {12,23,13} does not defeat {12,13}.

The proof of the first part of the theorem, that the existence of such a w precludes the existence of cycles, is direct. The proof of the second part of the theorem, that the existence of such a w is necessary for the absence of cycles, is more involved. It uses a result from decision theory that a binary relation on a finite set (like $\rightarrow$) has a representation by such a w if and only if it is negatively transitive and asymmetric.[3] Here, the binary relation of improving paths ($\rightarrow$) is transitive and asymmetric when there are no cycles, but may fail to be negatively transitive. We then show that the relation of improving paths may be extended to a more complete relation, that is negatively transitive, asymmetric and still agrees with $\rightarrow$ on adjacent networks.

***Proof of Theorem* 1:** First, suppose that there exists a cycle, so that there is some g such that $g \rightarrow g$. We show that there cannot exist such a w. Suppose, to the contrary, that there exists such a w. By transitivity of $>$, w satisfies $w(g) > w(g)$, which is impossible. So, if there is a cycle there cannot exist such a w, and so the existence of such a w precludes any cycles.

Next, assume there are no cycles and that for g and g' (that are adjacent) either g defeats g' or g' defeats g. We show that there exists such a w, The following Lemma (see Kreps (1988), Proposition 3.2) is helpful.

***Lemma*:** If X is a finite set and b is a binary relation, then there exists $w: X \rightarrow \mathcal{R}$ such that $w(x) > w(y) \Leftrightarrow x\ b\ y$, if and only if b is asymmetric and negatively transitive.

Since there are no cycles, our binary relation $\rightarrow$ is acyclic, and thus asymmetric. Also, $\rightarrow$ is transitive by the definition of improving path. However $\rightarrow$ is not necessarily negatively transitive (For an easy example, consider the connections model with $n=3$, $c=0.1$, and $\delta=0.9$). Thus, we construct a binary relation b over the set of networks such that (i) $g \rightarrow g'$ implies $g'\ b\ g$, (ii) if g and g' are adjacent, then $g \rightarrow g'$ iff $g'\ b\ g$, and (iii) b is asymmetric and negatively transitive. Then, by (iii) we can apply the Lemma to obtain w, and Theorem 1 follows from (ii).

Construct b as follows.

***Case* 1**. For every distinct g and g' at least one of the following holds: $g \rightarrow g'$ or $g' \rightarrow g$.

[3]Negative transitivity is the transitivity of the “not $\rightarrow$” relation.

Set b by $g'\ b\ g$ iff $g \to g'$. We show that b is negatively transitive. Write $g''\ nb\ g'''$ if it is not the case that $g''\ b\ g'''$. Suppose that $g\ nb\ g'$ and $g'\ nb\ g''$. This implies that $g'\ b\ g$ and $g''\ b\ g'$. Thus, by transitivity, it follows that $g''\ b\ g$, and so by asymmetry $g\ nb\ g''$. Thus negative transitivity is satisfied.

***Case* 2**. There exist distinct g and g' (which are not adjacent) such that g not $\to g'$ and g' not $\to g$.

Define the binary relation b_1 as follows. Let $g''\ b_1\ g'''$ iff $g''' \to g''$, except on g and g' where we arbitrarily set $g'\ b_1\ g$. Note that by construction, (i) and (ii) are true of b_1. Note also that b_1 is acyclic (and hence asymmetric). To see the acyclicity of b_1, note that if there were a cycle that it would have to include g and g', as this is the only place that b_1 and $\to$ disagree. However, the existence of such a cycle would imply that $g' \to g$, which is a contradiction. Next, define b_2 by taking all of the transitive implications of b_1. Again, (i) and (ii) are true of b_2. By construction b_2 is transitive. We argue that b_2 is acyclic. Let us show this by constructing b_2: we will add one implication from b_1 and transitivity at a time, and we will verify acyclicity at each step. Consider the first new implication that is added and suppose that there exists a cycle. Let g''' and g'' be the networks in question. So $g''\ b_1\ g'''$ and $g'''\ nb_1\ g''$, but $g'''\ b_2\ g''$, and there exist a sequence of networks $\{g_0, g_1, \cdots, g_r\}$ such that $g'''\ b_1\ g_0\ b_1\ g_1\ \cdots\ b_1\ g_r\ b_1\ g''$. This implies that there is a cycle under b_1, which is a contradiction. Iterating this logic implies that b_2 is acyclic.

Now, reconsider Cases 1 and 2 when b_2 is substituted for $\to$. Iterating on this process, we will eventually come to a case where we have constructed b_k, and relative to b_k we are in Case 1. Iterating on the argument under Case 2, it follows that (i) and (ii) will be true of b_k, and b_k will be transitive and asymmetric. Then, by the argument under Case 1, b_k will be negatively transitive. Let $b = b_k$ and the proof is complete. $\diamond$

We illustrate Theorem 1 in the context of the following coordination example.

***Example* 6:** *Coordination Example* (*Application of Theorem* 1)
A group of players (perhaps a group of students) are each endowed with a language, either A or B. Players interact only with those players they are directly connected to. Players prefer to interact with a player who speaks the same language as they do, although all players prefer interaction with someone to interaction with no one.

If a player interacts with just one other player, then he receives a payoff from the matrix:

	A	B
A	a,a	c,d
B	d,c	b,b

where $a>c>0$ and $b>d>0$. The first payoff in a cell is the row player's payoff and the second payoff is the column player's: the payoffs are a function of the languages the players speak.

The payoff matrix is from a simple coordination game. However, here the players' actions in the game are fixed and their choice instead concerns whom they interact with.[4] If a player interacts with several other players, then he receives the average payoff from all such interactions (and 0 if there is no interaction). Thus if an A player has direct links to two A players and one B player, he would receive a payoff of $(2/3)a+(1/3)c$. In addition to the average payoff, a player also receives an infinitesimally small payoff for each direct link that he has. Thus, all else held equal, a player prefers to be connected to more rather than fewer players.

Theorem 1 can easily be applied to this example. Notice that if a player is unlinked, then he always wants to add a link with someone else. But if an A (B) player is linked to another A (B) player, then he does not want to have any B (A) links. The following function $w(g)$ meets the conditions of Theorem 1; therefore the coordination example has no cycles. Let $w(g)=$(the number of links in g between players of the same type)$+\delta\cdot$(the number of

[4]This payoff matrix is examined by Kandori, Mailath and Rob (1993), Young (1993), Ellison (1993), among others, in the context of evolutionary processes where players interaction structure is fixed but their actions may vary. See Jackson and Watts (1999) for an analysis of situations where both the actions and network structure are choice variables.

links between A and B players such that the A player is not linked to another A player and the B player is not linked to another B player) $-\ \delta\ \cdot$ (the number of links between A and B players such that either the A player is also linked to another A player or the B player is linked to another B player); where δ satisfies $0 < \delta\ \cdot$ (the number of links in $g^N) < 1/2$. In fact, in this example (provided there are at least three players), from every network there is an improving path leading to the network that has two disjoint fully connected components, one with all A players and the other with all B players. This network is thus the unique pairwise stable network. The theorem tells us that there are no cycles in the improving paths. ◇

***Example* 7:** *Existence of Pairwise Stable Networks under the Myerson Value* (Jackson 2001)
An allocation rule satisfies equal bargaining power if for any component additive value function v and network g, $Y_i(g) - Y_i(g-ij) = Y_j(g) - Y_j(g-ij)$. Equal bargaining power does not require that individuals split the marginal value of a link. It just requires that they equally benefit or suffer from its addition. Myerson (1977) showed in the context of communication games that such a condition leads to an allocation that is a variation on the Shapley value. This rule was subsequently referred to as the Myerson value (*e.g.*, see Aumann and Myerson (1988)). The Myerson value also has a corresponding allocation rule in the context of networks beyond communication games, as shown by Jackson and Wolinsky (1996). That allocation rule is expressed as follows.

Let $g|_S = \{ij\colon ij \in g \text{ and } i \in S,\ j \in S\}$. Thus $g|_S$ is the network found deleting all links except those that are between individuals in S.

$$Y_i^{MV}(g,v) = \textstyle\sum_{S \subset N/\{i\}} \ (v(g|_{S\cup i})) - (v(g|_S))(\#S!(n - \#S - 1)!)/(n!).$$

Let $w(g) = \sum_{T \subset N} v(g|_T)((|T| - 1)!(n - |T|)!)/(n!)$.

Straightforward calculations verify that for any g, i and $ij \in g$

$$Y_i^{MV}(g,v) - Y_i^{MV}(g - ij, v) = w(g) - w(g - ij).$$

Thus, applying Theorem 1, we find that all improving paths (relative to the Myerson value allocation rule) emanating from any network (under any v) lead to pairwise stable networks. Thus, there are no cycles under the Myerson value allocation rule. ◇

Exact Pairwise Monotonicity

The existence of a function satisfying the role of w is sometimes difficult to check, but in some situations there is a natural candidate for w, which is simply v. This is captured in the following condition, which is a slight modification of the pairwise monotonicity condition of Jackson and Wolinsky (1996).

Y is *exactly pairwise monotonic* relative to v if g' defeats g if and only if $v(g')>v(g)$ (and g' is adjacent to g).

Exact pairwise monotonicity provides a nice alignment of individual incentives and overall value. It implies that efficient networks are pairwise stable (but not necessarily that all pairwise stable networks are efficient). By Theorem 1, exact pairwise monotonicity rules out cycles.

***Corollary* 1:** If Y is exactly pairwise monotonic relative to v, then there are no cycles.

***Example* 8:** *Coordination Example* (*Application of Corollary* 1)
Consider the coordination example (Example 6). Change the payoffs such that $a>0>c$ and $b>0>d$. Now players prefer no links to being linked with someone of a different type. Thus if two players decide to form a link, they must be of the same type. Thus both players gain from forming the link, while every other player's payoff remains the same: so v must increase. If a player decides to sever a link, then he must be linked to a player of the opposite type. Thus both players gain from having the link severed, while every other player's payoff remains the same: so v must increase. In this example, Y is exactly pairwise monotonic relative to v, and so by Corollary 1, there are no cycles. ◇

***Example* 9:** *Connections Example*
The connections model (Example 1) satisfies exact pairwise monotonicity for certain values of c and δ. For instance, if $\delta>c>n(\delta-\delta^{n-1})$, then the connections model satisfies exact pairwise monotonicity. Here a player only wants to add a link if it is to a player who is not already in his or her component, and a player wants to sever any links whose deletion would not change the component:

both of which are value increasing operations. The connections model also satisfies pairwise monotonicity if c is very small or very large. Yet for other choices of c and δ, for example when $c > \delta$ and c is low enough for a star to be efficient, the connections model fails to satisfy exact pairwise monotonicity. ◇

Stable States

A network g is *a stable state* if it is pairwise stable and there exists an improving path connecting the empty network to g.

The notion of a stable state is from Watts (2001) (although with a different process), and here is tied directly to the idea of an improving path. This notion identifies the networks that may be reached by a process where players act to improve their situation starting from the empty network.

***Example* 10:** *Co-Author Example*
In the co-author model (Example 2), all pairwise stable networks are stable states. A straightforward set of calculations show that there is at least one improving path leading from the empty network to any specific pairwise stable network. ◇

While ruling out cycles implies that a dynamic process which follows improving paths will come to rest at a pairwise stable network, this condition does not imply that any pairwise stable network, and in particular that any efficient network, is reachable if we start at the empty network. The following example illustrates the issue.

***Example* 11:** *A Connections Model with Increasing Returns*
Consider a variation on the connections model (Example 1) where the payoff to any individual is scaled by δ times n, where n is the number of direct links that the given individual has. The value of a link increases as connectedness increases; so this model exhibits increasing returns (Example, substituting $n\delta$ where δ was before, the middle person in a three-player line receives a payoff of $4\delta - 2c$, while the end players receive $\delta + \delta^2 - c$).

Suppose that $c > \delta$, so that starting from the empty network initial inertia exists, as in the symmetric connections model; and so the unique stable state is the empty network even though the complete network is efficient and pairwise stable. ◇

If the following condition is satisfied, then the problem indicated in Example 11 is precluded.

Single Peakedness

A value function v is *single peaked* if for any $g \subset g' \subset g''$ we have that $v(g) > v(g')$ implies $v(g') > v(g'')$ and $v(g'') > v(g')$ implies $v(g') > v(g)$.

The idea of a single peaked value function is as follows: consider growing a network by adding links one by one. Suppose that adding links adds value initially. The value function is 'single peaked' if once you add a link that lessens value, then continuing to add links will continue to lessen value.

A path is an *increasing path* if it only involves adding links.

***Proposition* 1**
Suppose that $v(g+ij) \neq v(g)$ for any g and $ij \notin g$. If Y is exactly pairwise monotonic relative to a v that is single peaked, then for every pairwise stable network there exists an increasing improving path leading from the empty network to that pairwise stable network. Therefore, in such a case the set of stable states coincides with the set of pairwise stable networks, and every efficient network is a stable state.

***Proof*:** Consider any pairwise stable network g and some $g-ij$. By exact pairwise monotonicity it follows that $v(g-ij) < v(g)$. Thus by single peakedness, $v(g-ij-kl) < v(g-ij)$ and so on for any order of removal of links. By exact pairwise monotonicity any order of addition of links to get to $v(g)$ defines a increasing (ordered) improving path (note that no two players want to delete a link at any point since the value of the resulting network must be lower and the severing player's payoff could be no higher by exact pairwise monotonicity). $\diamond$

Proposition 1 rules out cases where adjacent networks have identical values. That case requires a significant complication of the pairwise monotonicity condition, with little gain in insight.

***Example* 12:** *The Connections Model*
For some values of δ and c the connections model satisfies exact pairwise monotonicity and single peakedness (for instance if c is very small or very large). If c is very small then adding an

additional link always helps both players involved in the addition and always increases overall value thus both exact pairwise monotonicity and single peakedness are satisfied and the complete network is the unique stable state and the unique pairwise stable network. If c is very large then adding an additional link always harms both players involved and always decreases overall value and so the empty network is the unique stable state and the unique pairwise stable network. $\diamond$

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