On the Weights of Nations: Assigning Voting Weights in a Heterogeneous Union*

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Abstract

Consider a voting procedure where countries, states, or districts comprising a union each elect representatives who later vote at the union level on their behalf. The countries, provinces, and states may vary in their populations and composition. If we wish to maximize the total expected utility of all agents in the union, how should we weight the votes of the representatives of the different districts at the union level? We provide a characterization of the efficient voting rule in terms of the weights assigned to different districts and the voting threshold (how large a majority is needed to induce change versus the status quo). In the context of a model of the correlation structure of agents preferences, we analyze how voting weights relate to the population size of a country. We then study the voting weights in Council of the European Union under the Nice Treaty and the recent constitution, and compare them to optimal ones based on our model.

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1 Introduction

Citizens vote occasionally, while their elected representatives vote frequently. This is sensible due to the burden of becoming informed on a myriad of issues and the cost of of involving full populations in all of the decisions that direct democracy would require. While indirect democracy is sensible and prevalent, it introduces distortions in the decision process due to the fact that a single vote by a representative does not completely represent the heterogeneity of votes that would be cast by that representative's constituency.

If districts are small, of similar size, and of similar degrees of heterogeneity, then weighting each representative's vote equally provides a system of indirect democracy that maximizes overall societal welfare. However, for a variety of reasons, there are many systems of indirect democracy that are not structured in this way. A particularly important and timely example is the Council of Ministers of the European Union, a critical decision making body of the EU. That council consists of a single representative from each country in the European Union. The countries differ widely in their population sizes and compositions. Similar examples include the United Nations, the US Senate, and a variety of state and local governments. In any democratic union where the districts are of different sizes and compositions, it makes sense to weight the votes of the representatives.[1] For instance, if districts differ in population and votes are not weighted, then small districts could impose decisions that a majority of citizens oppose.

In this paper, we take as given that a heterogeneous set of countries, states, or districts, each have one representative who votes on their behalf over collective decisions. We characterize the voting rule that maximizes total societal welfare, as measured by the sum of the utilities of all citizens of the union, subject to the constraint that the district structure is fixed exogenously and possibly heterogeneous. We examine votes over two alternatives: a status quo and change. We show that an optimal voting rule consists of a weight for each country's vote and a threshold, indicating how large the total weight of votes cast in favor of change must be in order for change to be enacted.

One important conclusion of our analysis is that the optimal voting weights and thresholds can be derived separately. The optimal weight of a country's vote depends on the size of the population and the distribution of preferences within a country relative to other countries. The threshold depends on the bias of preferences in terms of the intensity in favor of the status quo compared to change.

The efficient weights can be described intuitively as follows. Consider the vote by a given representative of a country. Suppose that he or she has voted "yes" on a given issue. We can then ask the following question. Given the vote of "yes", what is the surplus of people in the country who favor "yes" over "no"? For instance if 62 percent of the people favor "yes" and 38 percent favor "no", then 24 percent more of the population favor "yes" versus "no". Multiplying this percentage times the population gives us a measure of how much this country would benefit if we choose "yes" versus "no", and how much this country would suffer if we chose the reverse. The efficient voting weight is exactly this expected surplus.[2]

[1] Alternatively, one can think of adjusting the number of representatives that each country, state, or district has.

[2] Our model allows for heterogeneities in intensities of preferences among voters as well, and the weights adjust for that. Here we are simply describing a special case where intensity among voters is similar. The full characterization is provided below.

As the general characterization of efficient voting rules depends on the distribution of preferences within each country, we also explore a model that we refer to as the "block model," which allows us to derive optimal weights as a function of population size. This works by assuming that a country's population can be partitioned into blocks, where citizens within a block have perfectly correlated preferences, while citizens across blocks have independent preferences. This structure allows us to pinpoint the efficient voting weights and thresholds under two focal scenarios.

We also examine the model's implications for the voting system of the Council of Ministers of the European Union. The Nice treaty of 2000 and the Constitutional Convention of 2003, proposed different sets of weights and different voting thresholds. Under the Nice Treaty weights are less than proportional to population size and the threshold is relatively high (73.9 percent). The Constitutional Convention proposed weights that are directly proportional to population size and a lower threshold (65 percent).[3] We show that these two conflicting proposals coincide with the optimal weights under two polar cases of our "block model". Which weights are more efficient then boils down to an empirical question of preference patterns.

Contribution and Relation to the Literature

It is surprising that the previous literature has not considered the criterion of efficiency (total expected utility) as a guide to determine optimal voting rules for indirect democracy.[4] The literature on indirect democracy approaches the problem from other perspectives. For instance, there is a rich literature in cooperative game theory on weighted majority games. A main thread there has been to produce power indices, such as the Banzhaf (1965) and Shapley-Shubik (1954) indices, which measure things such as the relative probabilities that different voters are pivotal. While some researchers have built power measures based on satisfaction (that is, total utility) and contrasted them with power measures built on decisiveness (see, for instance, Dubey and Shapley (1979), Barry (1980) and Laruelle and Valenciano (2003)), our perspective is still quite different. Our aim is not to measure power or satisfaction or to compare rules under such measures, but instead to study the optimal design of voting rules. To the extent that the previous literature has thought about designing rules, it has focussed on equating the power of agents, rather than maximizing the total expected utilities of agents.[5] This dates to the seminal work of Penrose (1946). These two objectives can lead to quite different voting rules, and, as we show, maximizing total expected utility can result in large inequalities in the treatment of individuals across countries.

Perhaps the closest predecessor to the theoretical part of our work is that of Felsenthal and Machover (1999), who also study the design of two-stage voting rules from an optimization perspective. Their objective is to minimize the expected difference between the size of the majority

[3]The Convention's proposal also includes a requirement that at 55 percent of the countries support a measure (Constitution, Title IV, Article I-25), which could also be binding, but less frequently; and that a blocking minority include at least four countries. (There are also special provisions for votes on issues that were not proposed through either the commission or the Union Minister for Foreign Affairs, where the requirement on countries is raised to 72 percent.) As a first (rough) approximation, we ignore these extra constraints in the discussion of the constitution's proposed weights. As we shall see in the discussion following Theorem 1, more complex voting systems can be optimal (see also Harstad (2005) for a rationalization of dual majority systems).

[4]Rae (1969) analyzed voting rules under this utilitarian perspective of maximizing expected utility or satisfaction rather than decisiveness (see also Badger (1972) and Curtis (1972)), but in the context of direct democracy.

[5]There are exceptions in the recent literature (e.g., Aghion and Bolton (2003), Barbera and Jackson (2004), Harstad (2004), Casella (2005)), but these approach the problem from very different perspectives.

and the number of supporters of the chosen alternative.[6] That objective differs from maximizing total expected utility as it does not account for the surplus of voters in favor of an alternative when the majoritarian alternative is selected, but only accounts for the deficit when the majoritarian alternative is not selected.[7]

Finally, researchers have also examined the European Union's decision-making and brought ideas from weighted games to assess the relative power of different countries under the Nice Treaty (e.g., see Laruelle (1998), Laruelle and Widgrén (1998), Sutter (2000), Baldwin, Berglöf, Giavazzi, and Widgrén (2001), Bräuninger and König (2001), Galloway (2001), Leech (2002), and some of the references cited there). As the foundations of our analysis of voting rules differ from the previous literature, so does our analysis of the Nice Treaty and the new Constitution. Among other things, we identify correlation structures of citizens' preferences that would justify the various rules that have been proposed, something which does not appear previously.

Since writing this, we have become aware of independent work by Feix, Lepelley, Merlin, and Rouet (2004), Bovens and Hartmann (2004), and Beisbart, Bovens, and Hartmann (2004) who examine efficiency as an issue in the definition of a voting rule. However, the works are (completely) complementary.[8]

2 An Example

A simple example gives a preview of some of the issues that arise in designing an efficient voting rule. The example shows why in some cases it will be efficient to use weights that are not proportional to population.

Example **1** *Non-Proportional versus Proportional Weights*

Consider a world with three countries. Countries 1 and 2 have populations of one agent each. Country 3 has a population of three agents. Each agent has an equal probability of supporting alternative a as alternative b, and preferences are independent across agents. An agent gets a payoff of 1 if their preferred alternative is chosen, and -1 if the other alternative is chosen. Thus, total utility can be deduced by keeping track of the number of agents who support each alternative.

First, consider a situation where countries are weighted in proportion to their populations and use a threshold of 50% of the total weight. That would result in weights of $(1, 1, 3)$ and a threshold of 2.5. This is equivalent to letting country 3 choose the alternative.

[6] See Felsenthal and Machover for an illuminating discussion of their objective, and some of the imprecisions in the previous literature.

[7] While these two perspectives differ, they lead to the same weights in the particular case of large countries of i.i.d. voters, where the weights are proportional to the square root of a country's population size (as originally suggested by Penrose (1946) from an even different perspective). The setting with a large number of i.i.d. voters is special and not so realistic - especially for applications such as to the European Union.

[8] Our characterization results, block models, and examination of the EU data differ from their analyses. Bovens and Hartmann examine combinations of maximin and utilitarian (efficient) measures and examine when degressive proportionality is justified, and Beisbart, Bovens, and Hartmann (2004) provide a Pareto ranking of several variations on potential rules for the EU council of ministers under various preference configurations. Feix, Lepelley, Merlin, and Rouet (2004) contrast two models of voter preferences: Impartial Culture and Impartial Anonymous Culture, and show through simulations that these can lead to different optimal voting rules.

Note that it is possible for a minority of agents to prefer an alternative and still have that be the outcome. For instance, if two agents in country 3 prefer a, and all other agents prefer b, then a is still chosen.

The "efficient" weights - that is, those that maximize the total expected utility - are (1,1,1.5), and the efficient threshold is 1.75. In this situation, the efficient voting rule is thus equivalent to one vote per country. The proof that this is the efficient rule comes from our characterization theorem below. However, to get a feeling for why it differs from the straight proportional rule, let us compare the expected utilities.

Under the efficient rule it is also still possible for a minority of agents to prefer a and a majority to prefer b, but to still have a selected. For instance, this happens if agents in countries 1 and 2 prefer a, but all agents in country 3 prefer b. Despite the fact that the efficient voting rule is not always making the correct choice in terms of maximizing the total utility, there is an important distinction between the efficient rule and the proportional rule here. Fewer configurations of preferences under the efficient voting rule lead to incorrect (minority-preferred) decisions, than under the proportional voting rule. This is seen as follows.

The configurations that are problematic in terms of agents' preferences are as follows (where the first two entries indicate the preferences of agents in the first two countries, and the last three entries correspond to the agents in country 3).

The only way that a can be the outcome and only be preferred by a minority under the efficient voting rule is when preferences are (a;a;b,b,b).

However, under the proportional voting rule there are three preference configurations that can lead to a being chosen when a majority prefers b. These are (b;b;a,a,b), (b;b;a,b,a) and (b;b;b,a,a).

When we compute the total expected utility (summed across all agents) it is 1.75 under the efficient voting rule compared to 1.5 under the proportional voting.

3 The Model

Decisions and Agents

A population of agents is divided into m countries.

Country i consists of n_i agents and this set is denoted by C_i. The total number of agents in the union is $n = \sum_i n_i$.

The agents make a decision between two alternatives labeled a and b. We refer to b as the status quo, and a as change.

A state of the world is a description of all agents' preferences over the two alternatives. Given that there are only two alternatives, we need only keep track of the difference in utility that an agent has for alternatives a and b. Therefor, without loss of generality we normalize things so that agent j gets a utility of u_j if a is chosen and a utility of 0 if b is chosen. So, if u_j is positive, then j prefers a.

A state of the world is thus a vector $u \in I\!R^n$, with element u_j being the difference between agent j's utilities for a and b. The uncertainty regarding the state is described by a probability distribution, and all expectations are with respect to that distribution.

A Two Stage Voting Procedure

The decision making process is described as follows.

The First Stage

In the first stage u is realized and each country's representative decides on whether to vote for a or vote for b. Generally, the representative's vote will be related to the preferences of the agents in the representative's country.

The representative's voting behavior is represented by a function $r_i : \mathbb{R}^n \to \{a, b\}$, which maps the state into a vote. The notation $r_i(u) = a$ indicates that the representative of country i votes for a and $r_i(u) = b$ indicates that the representative votes for b, given the state u. We assume that r_i depends only on the preferences of agents within country i.

It is important to emphasize that this formulation allows for many different ways in which the representative's vote could depend on the state of agents' preferences. It is conceivable that the representative is an existing politician who polls the population, or that the representative is a dictator, bureaucrat, etc., who might decide on how to vote quite differently. Later in the paper we consider the prominent case where the representative votes in accordance with a majority of the population.

The Second Stage

In the second stage, the votes of the representatives are aggregated according voting rule.

Let $v : \mathbb{R}^n \to \{0, \frac{1}{2}, 1\}$ denote the outcome of this two stage voting procedure as a function of the state. Here $v(u) = 1$ is interpreted as meaning that alternative a is chosen, $v(u) = 0$ means that alternative b is chosen, and $v(u) = \frac{1}{2}$ denotes that a tie has occurred and a coin is flipped.

Feasible voting rules are those which depend only on the information obtained through the votes of the representatives. The set of all feasible voting rules are thus those where $v(u) \neq v(u')$ implies that $r_i(u) \neq r_i(u')$ for at least one country i.

Given this coding of $v(u)$, the utility of agent j in state u can now be written as $v(u) \times u_j$. Thus the total utility summed across all agents in the union is

$$v(u) \sum_j u_j,$$

and the total expected utility of a union using a voting rule v is

$$E\left[\sum_j v(u) u_j\right].$$

Weighted Voting Rules

An important subclass of feasible voting rules are weighted voting rules. In such rules, the vote of the representative of country i is given a weight $w_i \in \mathbb{R}_+$. The tally of votes for a is the sum of the w_i's of the representatives who cast votes for a, and similarly for b. Alternative a is selected if its tally of weights exceeds the threshold (denoted $\tau \in [0, \sum_i w_i]$), alternative b is selected if the tally of weights for a is less than the threshold, and ties are broken by the flip of a fair coin.

Equivalent Voting Rules

When considering weighted voting rules, different weights and thresholds can lead to the same voting rule, and so weighted voting rules are only defined up to an equivalence class.[9] Instead of defining two different pairs of weights and thresholds to be equivalent if their induced voting rules always make the same choices, we need a coarser requirement for our main results due to the fact that tie-breaking is not completely tied down under efficient voting rules.

We say that a profile of voting weights and a threshold (w, τ) with induced weighted voting rule v is *equivalent up to ties* to a profile of voting weights and a threshold (w', τ') with induced weighted voting rule v' if $v(u) = v'(u)$ for all u such that $v'(u) \neq \frac{1}{2}$.

This is not quite an equivalence relationship, as it allows v to break ties differently from v'.[10]

To see why we define equivalence only up to ties consider a simple example. There are two countries and each consists of a single agent whose utilities take on values in $\{-1, 1\}$. Let w' be (1,1) and the threshold be 1. Note that the induced voting rule v' would be efficient for this example. When things are unanimous, v' picks the unanimous choice, but when u_1 and u_2 are of opposite signs, the rule flips a coin and so $v'(u) = \frac{1}{2}$. Alternative weights $w = (1 + \varepsilon, 1)$ with a threshold of $1 + \frac{\varepsilon}{2}$ would also be efficient, but would favor the first agent in the case of a tie. Thus, the induced voting rule v would be more resolute than v', but would make the same choices in any case where efficiency was at stake.

4 Efficient Voting Rules

Consider the problem maximizing the expected sum of the utilities of all agents in the union. The optimum would be to choose a when $\sum_j u_j > 0$ and b when $\sum_j u_j < 0$. This optimum will generally not be realized, since we lose information in a two stage procedure. In the second stage we see only the votes of the representatives, which includes only indirect information about the preferences of the agents.

Efficient Voting Rules

While we cannot always maximize the realized sum of utilities, we can still as which voting rule maximizes the expected total utility. An efficient voting rule v is one that has a maximal value of

$$E\left[\sum_j v(u)u_j\right],$$

across all feasible voting rules.

4.1 A Full Characterization of Efficient Voting Rules

Efficient voting rules work as follows. For each country assign two weights: one for a votes and one for b votes.[11] Country i's weight for a votes is

$$w_i^a = E\left[\sum_{k \in C_i} u_k \middle| r_i(u) = a\right],$$

[9] Equivalent voting rules can be rescalings of each other, but also might not be. For instance with three countries, $w = (3, 2, 2)$ with a threshold of 3.5 is equivalent to $w' = (1, 1, 1)$ with a threshold of 1.5 - they both select the alternative that at least two countries to voted for.

[10] This is an asymmetric relationship: v can be equivalent up to ties with v' while the reverse might not hold.

[11] This is a feasible voting rule but not a weighted voting rule.

and its weight for b votes is

$$w_i^b = -E\left[\sum_{k\in C_i} u_k \middle| r_i(u) = b\right].$$

The efficient voting rule $v^E(u)$ is then defined by

$$v^E(u) = \begin{cases} 1 & \text{if } \sum_{i:r_i(u)=a} w_i^a > \sum_{i:r_i(u)=b} w_i^b, \\ 0 & \text{if } \sum_{i:r_i(u)=a} w_i^a < \sum_{i:r_i(u)=b} w_i^b, \\ \frac{1}{2} & \text{if } \sum_{i:r_i(u)=a} w_i^a = \sum_{i:r_i(u)=b} w_i^b. \end{cases}$$

THEOREM **1** *If preferences are independent across countries, then a voting rule is efficient if and only if it is equivalent up to ties to* v^E.

The proof appears in the appendix.

The intuition behind the theorem is straightforward. Conditional on a vote of $r_i(u) = a$, $w_i^a = E\left[\sum_{k\in C_i} u_k \middle| r_i(u) = a\right]$ is the estimate of the total utility support for a in country i. By weighting country's votes in proportion to these expectations, the voting rule chooses the alternative that will result in the highest total utility based on what can be inferred from the votes of the representatives.

To get a feeling for how such rules work, consider an example with three countries, where $1 = w_1^a = w_1^b = w_2^a = w_2^b = w_3^a$, but $w_3^b > 2$. Here all votes are equally informative, except when country 3 votes for b, which indicates stronger support for b.[12] In this case, the efficient rule is to choose b whenever country 3 votes for b, and otherwise to operate under majority rule.[13] Note that this rule cannot be represented as an ordinary weighted voting rule where each country is just given some weight and there is some threshold needed for change. There is an asymmetry between how country 3's vote for b is treated compared to all other votes.

Before turning to the application to the European Union, let us discuss a few of the implications of the formula. The assumption of the independence of voters' preferences across countries is restrictive and is important to the conclusions of the theorem. Without this assumption, the estimation of one country's utility for a given alternative would depend on the full profile of votes of all countries. In that case an optimal voting rule would no longer be a weighted rule, but a rule that was a much more complex mapping between vectors of votes and decisions, as each country's vote would convey information about the preferences of all countries' electorates.

4.2 Bias and Weighted Voting Rules

In many contexts, there might be some asymmetry in agents' preferences over alternatives.

We say that country i is *biased* with bias $\gamma_i > 0$ if

$$E\left[\sum_{k\in C_i} u_k \left| r_i(u) = b\right.\right] = -\gamma_i E\left[\sum_{k\in C_i} u_k \left| r_i(u) = a\right.\right].$$

[12]This could be due to asymmetries in intensities of preferences within country 3, or could be due to the correlation structure of preferences within country 3.

[13]This is reminiscent of features such as in the U.N. Security Council voting rule where core countries hold a veto.

A country's bias captures the difference in expectations concerning how much the country's voters prefer a over b when their representative votes for a, compared to our expectations about how much the country's voters prefer b over a when their representative votes for b.

In cases where there is a common bias factor γ across countries, Theorem 1 has the following corollary. In that case, $w_i^b = \gamma w_i^a$, and the efficient voting rule can be written as a weighted voting rule.

COROLLARY **1** *If preferences are independent across countries and each country has the same bias factor γ, then a voting rule is efficient if and only if it is equivalent up to ties to a weighted voting rule with weights*

$$w_i^* = E\left[\sum_{k \in C_i} u_k \middle| r_i(u) = a\right].$$

and a threshold of $\frac{\gamma \sum_i w_i^*}{\gamma+1}$.

A prominent case of interest is one where countries are unbiased ($\gamma = 1$). Then a voting rule is efficient if and only if it is equivalent up to ties to the weights w_i^* given above, and the 50% threshold of $\frac{\sum_i w_i^*}{2}$.

We make several remarks about Corollary 1

The threshold depends on the bias γ, while the weights are determined by the expectations that come from each country. Thus one can judge whether a voting rule's weights are optimal independently of the threshold, and vice versa.

The extent to which a country's representative's vote is tied to the utilities of the agents in the country has important consequences. For example, a large country with a representative who is a dictator whose vote is uncorrelated with his population's preferences receives a smaller weight than a smaller country with a representative whose vote is very responsive to his population's preferences. More generally, the larger support (in net utility) for an alternative that one infers based on a representative's vote for that alternative, the larger the weight that a country receives.

The weights are affected by the distribution of opinions inside a country. For instance, if a country's agents had perfectly correlated opinions (and the representative voted in accordance with them), then a vote for an alternative would indicate a strong surplus of utility in favor of that alternative. The more independent the population's opinions the lower the expected surplus of utility in any given situation. Thus, higher correlation among agents' utilities will generally lead to higher weights.

The efficient weights take into account the intensity of preferences. So, relatively larger utilities lead to relatively larger weights. Thus, a country that cares more intensely about issues is weighted more heavily than a country that cares less, all else held equal. Due to practical and philosophical difficulties with the appraisals of utilities, one might want to be agnostic on this dimension and just treat all u_j's equally in the sense of only assigning them values of +1 or -1. We do this in the following section.

The following example illustrates the relation between bias and the voting threshold, as well as the separability of weights and thresholds.

EXAMPLE **2** *Bias and Thresholds*

Consider three countries. Countries 1 and 2 have 1 voter each. Country 3 has n_3 voters.

Voters' preferences are biased, with bias factor γ. All voters are equally likely to prefer a or b, and have u_j's take on values either 1 or $-\gamma$ with equal probability. Thus, when a voter prefers b, he or she cares more intensely than when he or she prefers a, by a factor γ. Country 3's voters have perfectly correlated preferences so that either they all prefer a or all prefer b. So, country 3 is just a scaled up version of countries 1 and 2.

Theorem 1 tells us that vector of voting weights should be $(1, 1, n_3)$, and the voting threshold should be a fraction of $\frac{\gamma}{\gamma+1}$ of the total weight. As γ becomes large, unanimity for a is required to overturn the status quo b. If $\gamma = 1$, then the threshold is 50 percent of the weighted votes.

As we vary n_3 and γ, the efficient voting rule takes some interesting forms. For instance, suppose that $n_3 = 3$. Then country 3 has three times as many voters and relatively favored in terms of weights. However, 3's "power" still depends on the voting threshold. If $\gamma = 1$, then the threshold is 50 percent of the weighted votes, and then country 3 is the only country that has a nontrivial vote and dictates. However, if $\gamma = 2$, then the threshold is 2/3 of voting weights. Then, a passes if and only if country 3 and at least one of 1 or 2 votes for a. Either 3, or 1 and 2 together, can block a and keep the status quo.

This example shows the separability of how the weights and thresholds are determined. The weights depend on the relative utilities (and thus populations) represented by each of the countries, while the threshold depends on the underlying preference structure in terms of a bias for change versus the status quo. It also shows that the overall voting rule can still depending in subtle ways on both the threshold and weights.

5 A Block Model

In order to apply the theory and calculate weights as a function of a country's population, we now introduce a model that is more specific about the distribution of agents' preferences and how representatives vote. We call this stylized model the "block model," and it works as follows.

First, we treat agents' utilities equally, in the sense that we only account for them as +1 or -1, and disregard personal intensities. This may be defended on grounds of practicality, but also more philosophically as an equal treatment condition. We also examine a case where each agent has an equal probability of supporting either alternative.

Second, we assume that representatives vote for the alternative that has majority support in their country.

Third, the distribution of the utilities of agents are as follows. Each country is made up of some number of blocks of agents, where agents within each block have perfectly correlated preferences, and where preferences across blocks are independent. The blocks within a country are of equal size.

These assumptions reflect the fact that countries are often made up of some variety of constituencies, within which agents tend to have correlated preferences. For instance, the farmers in a country might have similar opinions on a wide variety of issues, as will union members, intellectuals, etc. The block model is a stylized but useful way to introduce correlation among voters preferences, and simple variations of it provide interesting and pointed specifications of optimal voting rules.

Efficient Weights in the Block Model

Let N_i be the number of blocks in country i. Letting p_i be the size of each block, then we obtain the following expression for the efficient weight of country i.

$$w_i = p_i 2^{-N_i} \sum_{x > \frac{N_i}{2}} (2x - N_i) \frac{N_i!}{x!(N_i - x)!}. \tag{1}$$

There are two prominent variations on the block model that we focus on in what follows.

We call the first variation the *fixed-size-block model*. In this variation, blocks are of a fixed size across all countries. In this case, a country's population can be measured in blocks, and a larger country has more blocks than a smaller one. Here the p_i's are the same across all countries.

We call the second variation the *fixed-number-of-blocks model*. In this variation, all countries have the same number of blocks, and the size of the blocks in a given country adjusts according the country's population size. Here the N_i's are the same across all countries.

We obtain the following expressions for the efficient weights in the two specializations of the block model.

Efficient Weights in the Fixed-size-block Model

Given that the population size of a block (p_i) is the same across all countries, these can be cancelled out, and the weights in the fixed-size block model reduce to:

$$w_i^{FS} = 2^{-N_i} \sum_{x > \frac{N_i}{2}} (2x - N_i) \frac{N_i!}{x!(N_i - x)!}. \tag{2}$$

These weights are graphed as follows, as a function of the number of blocks in the country.[14]

Figure 1 here

For large numbers of blocks, the weights vary with the square root of the number of blocks, which is consistent with weights originally proposed by Penrose (1946), while for small numbers of blocks the weights diverge from this.

Efficient Weights in the Fixed-number-of-blocks model

In the fixed-number-of-blocks model, as the number of blocks (N_i) are the same in all countries, the difference in the weights then comes only in how many agents are represented in a block. The weights are directly proportional to the population size of the countries:

$$w_i^{FN} = p_i. \tag{3}$$

Asymmetries and Non-Monotonicities in Expected Utilities

Our perspective has been to maximize the sum of expected utilities, which is quite different from trying to equalize expected utilities across agents. We now illustrate this difference explicitly in the context of the block models. Efficient rules necessarily treat agents asymmetrically, depending on the size of the country they live. In the following proposition, we compare the expected utilities of agents living in two countries of different population size, under the efficient voting rule in the two variations of the block model.

[14] Note that the weights are the same for 1 and 2 blocks, 3 and 4 blocks, etc. This is reflective of the expression in (2).

Proposition 1 *In the fixed-number-of-blocks model, agents living in a larger country have expected utilities which are at least as large as agents living in a smaller country; and whenever the two countries weights are not equivalent*[15] *then the agents in the larger country have a strictly higher expected utility. In the fixed-size-block model, the ordering of expected utilities of agents across countries can in either direction relative to the ordering of the countries' population sizes.*

The proof of the proposition is straightforward. We offer a simple argument for the fixed-number-of-blocks model, and an example showing the ambiguity for the fixed-size-block model. In the fixed-number-of-blocks model, any agent's block in any country has exactly the same probability of agreeing with the agent's representative's vote. Thus, the expected utilities of agents in different countries differ only to the extent that their representatives receive different weights. As larger countries have larger weights, the claim in the proposition follows directly.[16]

To see the ambiguity in the fixed-size-block model, consider a union of three countries. The following are the expected utilities of the agents as we vary the number of blocks in the various countries.[17]

Table 1

Populations of Countries in Blocks	Efficient Voting Weights	Expected Utility of a Agent in Country 1 or 2	Expected Utility of a Agent in Country 3
(1,1,1)	(1,1,1)	.25	.25
(1,1,3)	(1,1,1.5)∼(1,1,1)	.25	.125
(1,1,5)	(1,1,1.875)∼(1,1,1)	.25	.09375
(1,1,7)	(1,1,2.186)∼(0,0,1)	0	.15625
(2,2,7)	(1,1,2.186)∼(0,0,1)	0	15625
(3,3,7)	(1.5,1.5,2.186)∼(1,1,1)	.125	.078125

The changes in voting weights result in non-monotonicities in expected utilities in several ways. In the cases of (1,1,3) and (1,1,5), a agent in country 1 or 2 has a higher utility than a agent in country 3. However, once country 3 hits a population of 7, then its weight is such that the votes from countries 1 and 2 are irrelevant. Thus, a agent would rather be in the larger country when the configuration is (1,1,7), while a agent would prefer to be in a smaller country when the configuration is (1,1,3) or (1,1,5). Also, as we increase country 3's population from 3 to 5, its agents' utilities fall, but then increasing the population from 5 to 7 leads to an increase in its agents' utilities. This contrasts with decreases in utilities of agents in the other countries.

[15] Two countries weights are equivalent if there exists a set of weights that lead in an equivalent voting rule where the two countries weights are identical.

[16] As pointed out by a referee, in the extreme where countries are actually formed as fairly homogeneous entities (as suggested, for instance by Alesina and Spalaore (1997)), we would be in a situation where each country was a single block. This would convey the maximal information possible from a representative's vote, and the overall voting rule would end up being completely efficient.

[17] If an agent prefers a, then he or she gets a payoff 1 when his or her preferred outcome is chosen and 0 otherwise. Whereas if an agent prefers b, then he or she gets a payoff of 0 when b is chosen and -1 if a is chosen. Then, for example, for an agent in country 1 in the (1,1,1), (1,1,3), and (1,1,5) cases, there is a 3/4 chance at least one of the other countries will prefer the agent's preferred alternative and a 1/4 chance that the other two countries will both favor the other alternative. So, in the 1/2 probability case that the agent prefers a, his or her expected utility is 3/4 (1) + 1/ 4 (0), and in the 1/2 probability case that the agent prefers b it is 3/4 (0) + 1/4 (-1). Overall this sums to .25. The calculations in each other case are similarly direct.

This example shows that there are no regularities concerning agents' utilities in the fixed-size-block model. The difficulty is that changes in population can dilute a given agents' impact within a country, but can also lead to a relative increase of that country's voting weight. As these two factors move against each other, changes can lead to varying effects.

Another question is how does the total expected utility vary under efficient voting rules as we change the division of a given population into different districts or countries. This issue is also generally ambiguous, regardless of which version of the block model one considers. For instance, one might conjecture that if we start with one division of a population into districts, and then further subdivide the population into finer districts, we would enhance efficiency since agents would become closer to their representatives. However, this is not always the case. To see this note that with a union of just one district or country, then we essentially have direct democracy. This is the most efficient possible. But then dividing this into several districts or countries would lead to a lower total expected utility under the efficient rule, than having just one district. Now, if we continue to further subdivide the districts, we eventually reach a point where each agent resides in a district of one, which brings us back to direct democracy and full efficiency! Generally, subdivisions lead to conflicting changes: on the one hand having a smaller number of agents within a district gives them a better say in the determination of their representative's vote, but on the other hand their representative is now just one among many. This leads to non-monotonicities and ambiguities of the types discussed above.[18]

6 The European Union

We now examine the voting rules of the Council of Ministers of the European Union under the Nice Treaty and under the draft of the Constitution produced by the Constitutional Convention in June of 2003 (Article 24). Given the stylized nature of the block model, and the fact that the overall decision making process of the European Union goes far beyond votes by the Council of Ministers, this is more of an exploratory exercise than a hard commentary on the EU voting process. This is also not a positive exercise, but rather a normative one. Our analysis provides a normative description of how voting systems should be designed. In examining the various proposals for EU weights, we are not presuming that they are optimal systems. Instead, we discuss which variation of the block model would justify a given proposal.

The voting rule for the European Council of Ministers under the Nice Treaty is by weighted voting. At least 255 of the 345 weighted votes (73.9%) must be cast in approval of a proposal

[18]This leads to a basic tradeoff in structuring an indirect democracy. There are tradeoffs between the cost of involving more voters, and having more precise representation. There has been little exploration of such tradeoffs, either theoretically or empirically, and they might help explain why one sees attempts at more centralization in some cases (e.g., the EU) and yet more decentralization within some states.

for it to pass.[19,20] The relative voting weights appear in Figure 2, below. We also examine the efficient voting under the two block models. The efficient weights in the fixed-size-block model are calculated for a block size of 1 million. So for instance, Germany has 83 blocks, France has 59, and Italy has 58, etc. This leads to efficient voting weights of 7.3, 6.2 and 6.1 for these countries, respectively. The efficient weights in the fixed-number-of-blocks model are simply proportional to population. These also directly correspond to those proposed under the constitution. In order to make comparisons, we rescale the weights so that they sum to 1.

The relationship between the four different weighted voting rules is pictured as follows.

Figure 2 here

A regression of the Nice Treaty weights on the efficient weights under the fixed-size-block model provides an R^2 of 96% for the case of 1 million sized blocks (and 95% for the case of 2 million sized blocks, with F-statistics in each case over 600). As a comparison, the fit using weights proportional to population is only 81% (with an F-statistic of 102), and so the efficient weights under the fixed-size block model provide a much closer match to the Nice Treaty weights. The reverse is true for the proposed voting rule in the Constitution, with weights which are proportional to population. That rule would not be very efficient if the world is well approximated by the fixed-size-block model, but would be a perfect fit under the fixed-number-of-blocks model.

Thus, we are left with an empirical question. If the world is a good match to the fixed-size-block model then the Nice Treaty weights are (approximately) efficient, while if the world is a good match to the fixed-number-of-blocks model then the new Constitution's weights are efficient. Of course, these are highly stylized models and it is likely that the world does not conform to either. While it seems clear that countries such as Luxembourg and Malta consist of more than one block, it also seems clear that the smallest countries have fewer voting blocks than the largest ones. This suggests that the weights should be nonlinear, although perhaps not quite to the level suggested by the fixed-size-block model. A detailed empirical investigation of voting patterns within the countries of the EU is beyond the scope of this article.[21]

Let us also comment briefly on the voting thresholds. The threshold under the Nice treaty is 73.9% of the total weight - which would be efficient if countries have a bias of roughly $\gamma = 3$. This indicates a strong bias for the status quo. In contrast, the threshold of 65% under the Constitution would be efficient if countries have a bias of roughly $\gamma = 1.86$. This is also a bias for the status quo, but a less pronounced one.

[19]There are two other qualifications as well: (i) that the votes represent at least 14 of the 27 countries and (ii) that the votes represent at least 62% of the total population. Calculations by Bräuninger and König (2001) suggest that there are relatively few scenarios in which the weighted vote threshold of 255 votes would be met while one of the other two criteria would fail. It appears that the only impact will be from the population threshold and that this will only involve a few configurations of votes providing a very slight boost in power to Germany and slight decrease in power to Malta. Thus, for practical purposes, these additional considerations are relatively unimportant and the voting weights themselves are the main component of the voting procedure.

[20]There are discrepancies in the Nice Treaty in that some statements imply a threshold of 258 votes and others a threshold of 255 votes. It appears that the correct number is the 255.

[21]See Barbera and Jackson (2004a) for a preliminary analysis of estimated optimal voting weights based on poll data from the Eurobarometer (2003ab).

7 Concluding Remarks

We have provided a framework for designing and analyzing efficient voting rules in the context of indirect democracy. We have shown that the model can be applied to analyzing voting rules such as those of the European Union, and that the relative merits of different rules reduce to readily identifiable hypotheses that are amenable to empirical testing. A careful analysis of the EU voting rules will require richer data and applying our results beyond the case of the block model. Nevertheless, our theoretical results provide a framework with intuitive characterizations of optimal voting rules, that appears to lend itself well to such an exercise.

There are other considerations that should be explored in further studies. In making decisions, it may be that countries can sometimes include side payments or logroll so that multiple decisions can be made at once. These possibilities can further enhance the efficiency of the decision making and might influence the structure of the voting system that is to be used.[22] One can also consider a voting system's stability. As the rules can be amended, considerations other than efficiency enter the long-run picture, as only certain rules will survive. [23] Another is the issue of fairness or equality. As we have shown, efficient weights do not necessarily lead to the same expected utilities for agents in different countries. For instance Proposition 1 showed that larger countries are favored under proportional weights in the fixed-number-of-blocks model . There are other issues that can be considered, such as votes over more than two alternatives, private information, agenda formation, and risk aversion, to highlight a few of the more obvious ones.

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[22]For example, see Harstad (2004, 2005) who considers side payments and the ability to invest in projects, Casella (2005) who considers rules specifically designed for votes over sequences of decisions, and Jackson and Sonnenschein (2003) who show that efficiency can be obtained if multiple decisions can be bundled and voted over in a linked manner.

[23]See Barbera and Jackson (2000) and Sosnowska (2002) for an examination of the stability of voting rules.

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Appendix: Proof of Theorem 1

An efficient voting rule is a feasible voting rule that maximizes

$$E\left[\sum_k v(u)u_k\right].$$

Let $(r_1(u),\ldots,r_m(u)=r_1,\ldots,r_m)$ be an event where the realization of representatives (i.e., votes of the countries) is $(r_1,\ldots,r_m)\in\{a,b\}^m$. Under feasibility, we can then write $v(u)$ as a function of $(r_1,\ldots,r_m)$ instead of u. Hence, the total expected utility is written as

$$\sum_{r_1,\ldots,r_m} E\left[\sum_k v(r_1,\ldots,r_m)u_k\left|r_1(u),\ldots,r_m(u)=r_1,\ldots,r_m\right.\right]P(r_1(u),\ldots,r_m(u)=r_1,\ldots,r_m),$$

Given the independence across countries, we can write this as

$$\sum_{r_1,\ldots,r_m} v(r_1,\ldots,r_m)\left(\sum_i E\left[\sum_{k\in C_i} u_k\,|r_i(u)=r_i\right]\right)P(r_1(u),\ldots,r_m(u)=r_1,\ldots,r_m).$$

It then follows that if we can find a voting rule that maximizes

$$v(r_1,\ldots,r_m)\sum_i E\left[\sum_{k\in C_i} u_k\,|r_i\right] \tag{4}$$

pointwise for each $(r_1,\ldots,r_m)$, then it must be efficient. Moreover, if we find one that leads to a 0 whenever there is indifference between a and b, then and all efficient voting rules must be equivalent to it up to ties.

Note that for any given $(r_1,\ldots,r_m)$, maximizing expression (4) requires setting $v(r_1,\ldots,r_m) = 1$ when

$$\sum_i E\left[\sum_{k\in C_i} u_k\,|r_i\right] > 0 \tag{5}$$

and $v(r_1,\ldots,r_m) = 0$ when

$$\sum_i E\left[\sum_{k\in C_i} u_k\,|r_i\right] < 0, \tag{6}$$

and does not have any requirement in the case that this expression is equal to 0.

Given the definitions of w_i^a and w_i^b, we can then rewrite (5) and (6) as $v(r_1,\ldots,r_m) = 1$ when

$$\sum_{i:r_i=a} w_i^a - \sum_{i:r_i=b} w_i^b > 0 \tag{7}$$

and $v(r_1,\ldots,r_m) = 0$ when

$$\sum_{i:r_i=a} w_i^a - \sum_{i:r_i=b} w_i^b < 0. \tag{8}$$

This is as defined in v^E, where we flip a coin in the case of a tie. Any efficient voting rule must agree with this one except in the case where this rule results in an expression equal to 0. This concludes the proof of the theorem. ∎

Figure 1: Weights under the Fixed Number of Blocks Model

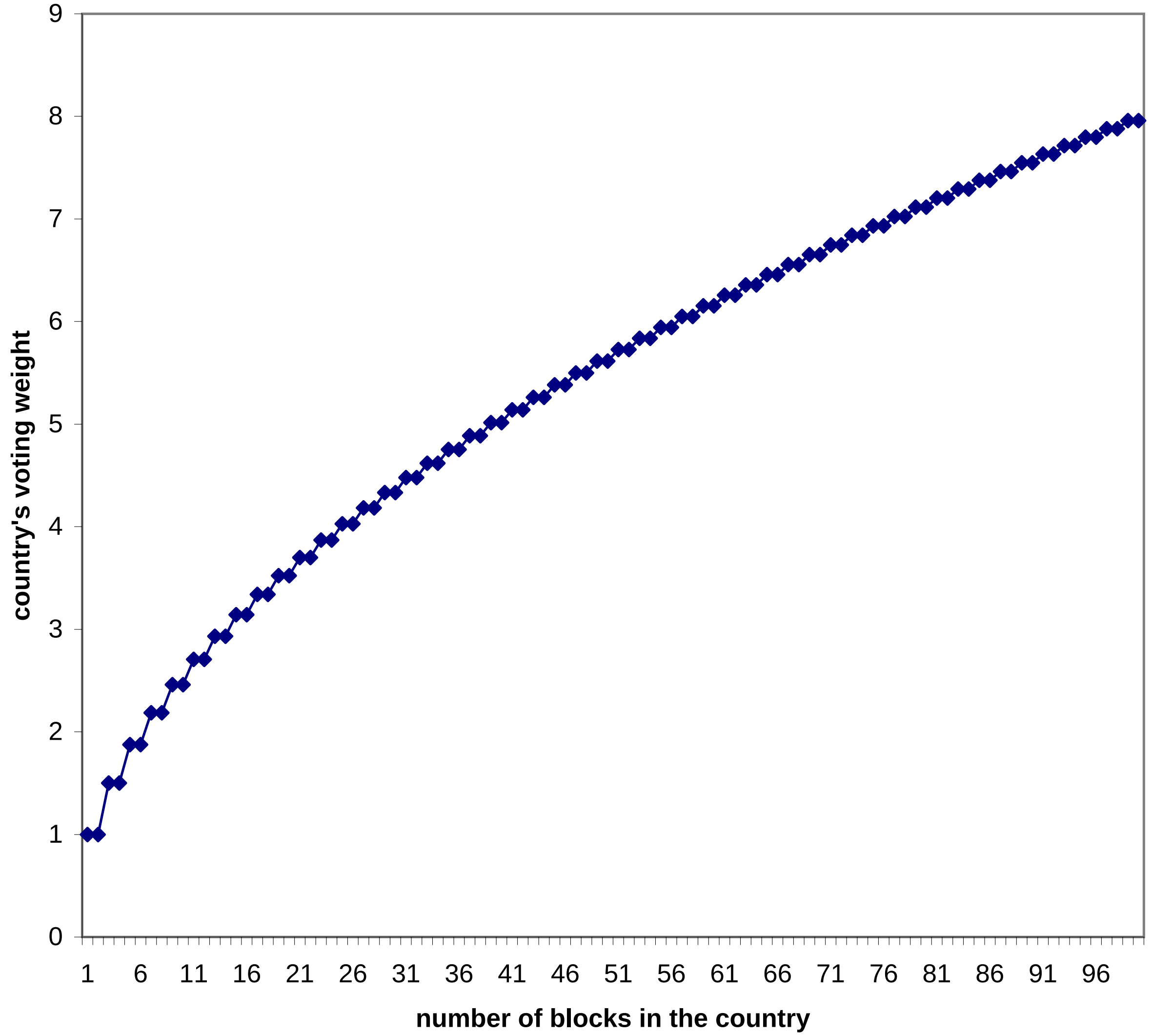

Figure 2: Comparison of weights from the block models and EU Council of Ministers