Theory of the Consumer

• What we have done so far:
• Derived “downward” sloping demand curves from underlying principles
  – budgets
  – preferences
  – optimizing behavior

Theory of the Firm

• We will derive “upward sloping” supply curves from underlying principles
  – Technology
  – Firm objectives (profit maximization / cost minimization)
  – Firm optimization

Technology

We represent technology by a production function:

\[ y = f(K, L) \]

For example: Cobb-Douglass production function:

\[ y = AK^a L^\beta \]

Production Functions

Simple Case: 1 input, 1 output

\[ y = f(x) \]

Properties of Production Functions

1. Slope of the production function is equal to the marginal product of the input
   – The marginal physical product of the input \( (MP_x) \) is defined as the incremental increase in output associated with an increase in the input.
2. Marginal physical products of inputs are positive

\[ MP_X > 0 \]

3. The average physical product of an input is the ratio of output to the amount of input used

\[ AP_X = \frac{f(x)}{x} \]

4. If \( MP_X > AP_X \), then \( AP_X \) is increasing
   – \( MP_X > AP_X \) means that the last unit of input will produce more than the average level of output up to that point.
   – Thus this last unit of input will increase the average product produced.

5. If \( MP_X < AP_X \), then \( AP_X \) is decreasing
   – The last unit of input will produce less than the average level of output up to that point, driving down the average

**Proof that \( AP_X \) is maximized when \( AP_X = MP_X \)**

- Given: \( AP_X = \frac{f(x)}{x} \)
- The necessary first order condition to maximize \( AP_X \) is:

\[ \frac{dAP_X}{dx} = 0 \]

**Law of Diminishing Returns**

- Eventually, each input has decreasing marginal product
- ‘S-shaped’ production functions
  – increasing marginal product at low levels of \( x \)
  – decreasing marginal product at high levels of \( x \)
More than One Input

• Typically, descriptions of technology will require the transformation of several inputs into one (or more outputs).
• Multiple output production technologies are beyond the scope of this class.
• We will look at production technologies with two inputs: e.g. capital and labor.

Marginal Physical Product with Two Inputs \( Y = F(K, L) \)

• MP is defined in exactly the same way as with one input, except now you need partial derivatives:
  \[
  MP_K = \frac{\partial F}{\partial K}; \quad MP_L = \frac{\partial F}{\partial L}
  \]

Cobb-Douglas Example

\[
Y = AK^\alpha L^\beta
\]

\[
MP_K = \frac{\partial Y}{\partial K} = \alpha AK^{\alpha-1}L^\beta; \quad AP_K = \frac{Y}{K} = AK^{\alpha-1}L^\beta
\]

\[
MP_L = \frac{\partial Y}{\partial L} = \beta AK^\alpha L^{\beta-1}; \quad AP_L = \frac{Y}{L} = AK^\alpha L^{\beta-1}
\]

Returns to Scale

• When there is only 1 input, marginal returns to the input exactly equal returns to scale!
• Increasing returns to scale: if you ‘double’ all inputs, output more than ‘doubles’
• Decreasing returns to scale: if you ‘double’ all inputs, output less than ‘doubles’
• Constant returns to scale: if you ‘double’ inputs, then output ‘doubles’

Homogeneity and Returns to Scale

• Recall the definition of kth degree homogeneity.
  – A function is homogenous of degree k if
  \[
f(t(x_1, x_2, \ldots, x_n)) = t^k f(x_1, x_2, \ldots, x_n) \quad \forall t > 0
  \]
• A production function which is 1st degree homogenous exhibits constant returns to scale.
Cobb-Douglass and Returns to Scale

\[ Y = F(K, L) = AK^\alpha L^\beta \]

\[ F(tK, tL) = A(tK)^\alpha (tL)^\beta = t^{\alpha + \beta} AK^\alpha L^\beta = t^{\alpha + \beta} Y \]

So Cobb-Douglass production functions are constant returns to scale if \( \alpha + \beta = 1 \). They are increasing returns to scale if \( \alpha + \beta > 1 \). They are decreasing returns to scale if \( \alpha + \beta < 1 \).

Isoquants

- These are defined as the set of all combinations of inputs that produce a given level of output.
- Notice the similarity between the definition of isquants and the definition of indifference curves—the set of all consumption combinations that produce a given level of utility.

Isoquant—Cobb-Douglass Case

\[ Y = AK^\alpha L^\beta \]

Increasing production

- \( Y_1 < Y_2 < Y_3 \)

Rate of Technical Substitution

- The marginal rate of technical substitution is defined as the rate at which one input can be substituted for another along an an isoquant.
- It is defined to be \((-1\) times\) the slope of the isoquant:

\[ \text{RTS}_{L,K} = -\frac{dK}{dL} \text{ if constant} \]

Nicholson Example Problem

- The production of barstools \( (q) \) is characterized by a production function of the form \( q = \sqrt{KL} \).
  - What is \( APL \) and \( APK \)?

\[ APL = \frac{q}{L} = \frac{K}{\sqrt{L}} \quad APK = \frac{q}{K} = \sqrt{\frac{L}{K}} \]

- Graph the \( APL \) curve for \( K = 100 \).

AP\(_L\) graph for \( K = 100 \)
Nicholson Example (continued)

- For this particular function, show that $MP_L = 0.5 AP_L$ and $MP_K = 0.5 AP_K$. Using that information, add a graph of the MPL function to the graph in part (b) (at $K = 100$). What is unusual about this curve?

\[
MP_L = \frac{\partial q}{\partial L} = \frac{1}{2} \sqrt{\frac{K}{L}} = \frac{1}{2} AP_L \quad MP_K = \frac{\partial q}{\partial K} = \frac{1}{2} \sqrt{\frac{L}{K}} = \frac{1}{2} AP_K
\]

Nicholson Example (Continued)

- This graph is unusual because $AP_L$ never intersects $MP_L$. Since $MP_L < AP_L$, every extra worker hired decreases average productivity!

- Sketch the $q = 10$ isoquant for this production function.

Nicholson Example (continued)

- What is the rate of technical substitution (RTS) on the $q=10$ isoquant at the points:
  - $K = L = 10$
  - $K = 25; L = 4$
  - $L = 25; K = 4$

- Does this function exhibit a diminishing RTS?

RTS in the Example Problem

- Recall that RTS is (-1) times the slope of the isoquant.
- To derive this in terms of what we already know, take the production function and totally differentiate:

\[
q = \sqrt{KL}
\]
RTS (continued)

\[ dq = MP_K dK + MP_L dL = 0 \Rightarrow \]
\[ \frac{dK}{dL} \left|_{\text{constant}} \right. = -\frac{MP_L}{MP_K} = -\frac{\sqrt{K/L}}{\sqrt{L/K}} = -\frac{K}{L} \Rightarrow \]
\[ \text{RTS}_{L,K} = -\frac{dK}{dL} \left|_{\text{constant}} \right. = \frac{K}{L} \]

RTS is Diminishing in L

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