The Theory of the Firm II

- Last lecture we covered:
  - – production functions
- Today:
  - – Cost minimization
  - – Firm's supply under cost minimization
  - – Short vs. long run cost curves

Firm Objectives

- Profit maximization: firms choose that level of output and that combination of inputs which yields the highest level of profits
- Cost minimization: firms choose that combination of inputs to produce a certain level of output at minimum cost.
- Cost minimization should hold even for non-profit firms.

Cost Minimization—Mathematical Approach

$$\min_{K,L} \; wL + rK$$
$$s.t. \;\; Q = F(K,L)$$

$$G = wL + rK - \lambda(Q - F(K,L))$$

First order conditions

$$\frac{\partial G}{\partial K} = r - \lambda\frac{\partial F}{\partial K} = 0 \Rightarrow r = \lambda MP_K$$
$$\frac{\partial G}{\partial L} = w - \lambda\frac{\partial F}{\partial L} = 0 \Rightarrow w = \lambda MP_L$$
$$\frac{\partial G}{\partial \lambda} = Q - F(K,L) = 0$$

$$\Rightarrow \frac{r}{w} = \frac{MP_K}{MP_L}$$

Input demand equations

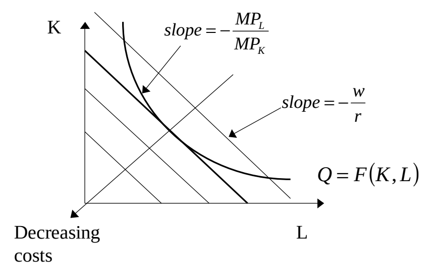
- Solving the three first order conditions simultaneously yields two input demand equations:

$$L^* = L(w,r,Q) \qquad K^* = K(w,r,Q)$$

- Plugging these back into the total input expenditure identity (Expend. = *wL* + *rK*) yields the minimum total cost curve:

$$TC = rK^* + wL^* = C(w,r,Q)$$

Cost Minimization—Graphical Approach



Firm Optimization

- Tangency Condition
  - increase use of an input until its marginal product is equal to its real price
  - if $MP_L/MP_K > w/r$ then increasing use of L (while decreasing use of K) will decrease costs
  - if $MP_L/MP_K < w/r$ then decreasing use of L (while increasing use of K) will decrease costs
- Technology Constraint

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Example Problem

- *For Cobb-Douglass Production $Q = K^a L^b$, what are the input demand functions for K and L? What is the minimum cost function?*
- Set up the Lagrangian:

$$G = rK + wL - \lambda\left(K^a L^b - Q\right)$$

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First Order Conditions

$$\frac{\partial G}{\partial K} = r - \lambda a K^{a-1} L^b = 0 \Rightarrow r = \lambda a K^{a-1} L^b$$

$$\frac{\partial G}{\partial L} = w - \lambda b K^a L^{b-1} = 0 \Rightarrow w = \lambda b K^a L^{b-1}$$

$$\Rightarrow \frac{r}{w} = \frac{aL}{bK}$$

$$\frac{\partial G}{\partial \lambda} = Q - K^a L^b = 0$$

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Input Demand

- Rearranging the FOC's yields:

$$L = \frac{br}{aw} K \qquad K^a L^b = Q$$

- Substituting in and solving yields:

$$K^a \left[\frac{br}{aw}\right]^b K^b = Q \Rightarrow K = \left[\frac{aw}{br}\right]^{\frac{b}{a+b}} Q^{\frac{1}{a+b}}$$

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Cost Function

- The demand for labor is: $L = \left[\frac{aw}{br}\right]^{\frac{-a}{a+b}} Q^{\frac{1}{a+b}}$
- Plugging the input demand functions into the expenditure function yields the cost function:

$$TC(w,r,Q) = rK + wL = r\left[\frac{aw}{br}\right]^{\frac{b}{a+b}} Q^{\frac{1}{a+b}} + w\left[\frac{aw}{br}\right]^{\frac{-a}{a+b}} Q^{\frac{1}{a+b}}$$

$$\Rightarrow TC(w,r,Q) = \left\{ r\left[\frac{aw}{br}\right]^{\frac{b}{a+b}} + w\left[\frac{aw}{br}\right]^{\frac{-a}{a+b}} \right\} Q^{\frac{1}{a+b}}$$

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Marginal and Average Cost

- Cost = $wL^* + rK^*$
  - The cost function relates costs to input prices and to quantity produced.
- Marginal Cost equals the extra cost required to produce a small amount for of the good:

$$MC = \frac{\partial TC}{\partial Q}$$

- Average Cost equals the cost per unit of good produced. $AC = \frac{TC}{Q}$

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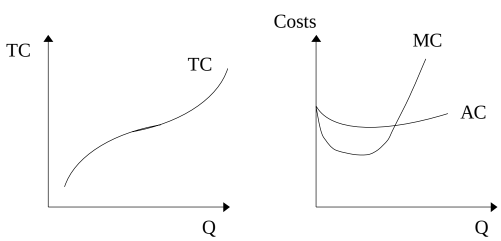
Relationship Between Marginal Cost and Marginal Product

- MC is the amount it costs to produce a small amount of extra output.
  - In principle, a firm might radically change its demand for inputs in order to increase output while minimizing costs.
- MP_X is the number of additional units of output you get from an additional unit of input, X.
  - In this thought experiment, the amount of all inputs other than X are held constant.

Marginal Costs Intersect Average Cost Curves at the Minimum Point

- If MC > AC at some point, then producing the extra unit will drive up average costs.
- If MC < AC at some point, then producing the extra unit will drive down average costs.
- If MC = AC, then producing the extra unit will reproduce average costs.
- Together, these three statements imply marginal cost curves intersect average costs at the minimum point.

Graphing Marginal and Average Costs



Proof

- AC=TC/Q
- To find the minimum point, find the first order condition:

$$\frac{\partial AC}{\partial Q} = \frac{1}{Q}\frac{\partial TC}{\partial Q} - \frac{1}{Q^2}TC = 0 \Rightarrow$$

$$MC = \frac{TC}{Q} = AC$$

- Homework: show that this is a minimum point.

Example Problem (continued)

- *What is the marginal cost curve for the Cobb-Douglass production function?*

$$TC(w,r,Q) = \left\{ r\left[\frac{aw}{br}\right]^{\frac{b}{a+b}} + w\left[\frac{aw}{br}\right]^{\frac{-a}{a+b}} \right\} Q^{\frac{1}{a+b}}$$

$$MC = \frac{\partial TC}{\partial Q} \Rightarrow$$

$$MC = \frac{1}{a+b}\left\{ r\left[\frac{aw}{br}\right]^{\frac{b}{a+b}} + w\left[\frac{aw}{br}\right]^{\frac{-a}{a+b}} \right\} Q^{\frac{1}{a+b}-1}$$

Costs and Returns to Scale

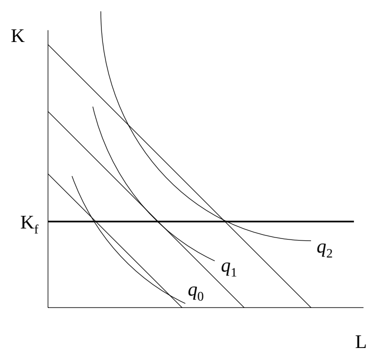
- Increasing returns to scale implies decreasing marginal cost functions.
- Constant returns to scale implies constant marginal cost functions.
- Decreasing returns to scale implies increasing marginal cost functions.
- This is easy to see in the Cobb-Douglass example.

Fixed Costs

- So far, we have only included “variable” costs--costs which vary with the amount of output produced
- Fixed costs do not vary with the amount of output produced.
  - e.g. a factory, a piece of equipment
- Total Cost(Q) = Fixed Cost + Variable Cost(Q)

Short Run vs. Long Run

- In the long run, firms can change all of their inputs in response to changes in input prices or Q.
- In the short run, some input may be difficult to adjust quickly. These inputs are fixed as far as the firm is concerned.
  - Under the additional constraint in the short run, the firm typically will not be able to choose the long run optimal input mix.



Average Total vs. Average Variable Cost

- Short-Run Average Total Cost (SATC) =

$$SATC = \frac{\text{Variable Costs}(Q) + \text{Fixed Costs}}{Q}$$

- Average Variable Cost (SVC) =

$$SVC = \frac{\text{Variable Costs}(Q)}{Q}$$

Properties of SMC, SATC, and SVC

- Short Run Marginal Cost (SMC) =

$$SMC = \frac{\partial SATC}{\partial Q} = \frac{\partial SVC}{\partial Q} = \frac{\partial \text{Variable Costs}(Q)}{\partial Q}$$

- Minimum SATC occurs to the right of minimum SVC
- MC curve cuts through the minimum of both ATC and AVC. Why?

Short Run Cost Curves for the Example

- *Derive the SATC, SMC, and SVC curves for the Cobb-Douglass production function,* $Q = K^a L^b$ *when K is fixed at* K_f *in the short run.*
- We are constrained to produce Q, while fixing the level of K_f. This leave no real choice as to the level of labor required:

$$L^b = \frac{Q}{K_f^a} \Rightarrow L^* = K_f^{-\frac{a}{b}} Q^{\frac{1}{b}}$$

Short Run Example (continued)

- Plug the labor demand curve and the fixed level of capital into the expenditure function to get short run total costs:

$$STC(Q,K_f)=rK_f+wL^*=rK_f+wK_f^{-\frac{a}{b}}Q^{\frac{1}{b}}$$

$$SATC(Q,K_f)=\frac{STC(Q,K_f)}{Q}=\frac{rK_f+wK_f^{-\frac{a}{b}}Q^{\frac{1}{b}}}{Q}$$

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SVC, SMC in the Example

- SVC does not count fixed costs:

$$SVC(Q,K_f)=wL^*=wK_f^{-\frac{a}{b}}Q^{\frac{1}{b}}$$

- SMC: $SMC(Q,K_f)=\frac{\partial SVC}{\partial Q}=\frac{w}{b}K_f^{-\frac{a}{b}}Q^{\frac{1}{b}-1}$
- You should verify that the other formulas for SMC yield the same outcome.

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Homework

- Read about the relationship between short and long run cost curves in Nicholson.
- Punchlines:
  - Short run cost curves are always greater than or equal to long run cost curves (since all inputs adjust in the long run).
  - Long and short run cost curves are equal only when the fixed inputs happen to be optimal for producing Q.

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