Would You Take This Bet?

- We flip a (fair) coin once.
  - If it is heads, you win (at least) $2.
  - If it is tails, you win nothing.
- We keep on flipping the coin.
  - As long as the coin keeps landing on heads, your winnings keep doubling:
    - $4 on the second heads
    - $8 on the third heads
    - $16 on the fourth heads...
  - We stop flipping on the first tails.
- In exchange, you owe me $1 million.

Random Variable

- A random variable is a set of outcomes plus a probability associated with each outcome.
  - The probabilities must sum to one.
- Example: coin flip
  - Outcomes, probabilities: {Head, $\frac{1}{2}$; Tail, $\frac{1}{2}$}
- Usually (but not necessarily) the outcomes are numbers.
- Example: score on the midterm
  - {100, probability = $\frac{1}{4}$}
  - {90, probability = $\frac{1}{2}$}
  - {80, probability = $\frac{1}{4}$}

Expected Value

- The average outcome of a draw from a random variable (say, $X$).

\[ E[X] = \sum_{i=1}^{\infty} \pi_i x_i \]

Examples of Expected Values

- Example: coin flip – let heads = 1, tails = 0
  - $E[\text{coin flip}] = 1 \times \frac{1}{2} + 0 \times \frac{1}{2} = \frac{1}{2}$
- Example: midterm grades:
  - $E[\text{midterm grade}] = 100 \times \frac{1}{4} + 90 \times \frac{1}{2} + 80 \times \frac{1}{4} = 90$
- Example: California Lottery
  - "Half the money goes to the schools"
  - For a $1 bet, the expected return is 50 cents.

St. Petersburg Paradox

- To calculate the expected value of the bet from the beginning of the lecture, we need to find the probability of a long series of heads uninterrupted by a tails outcome:
  - 1 heads; outcome = $2; probability = \frac{1}{2}
  - 2 heads; outcome = $4; probability = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}
  - 3 heads; outcome = $8; probability = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}
  - $n$ heads; outcome = $2^n$; probability = $\frac{1}{2^n}$

Paradox (II)

- Your winnings in the bet are a random variable, say $W$.
- The expected value of $W$ is infinite:

\[ E[W] = 2 \times \frac{1}{2} + 4 \times \frac{1}{4} + 8 \times \frac{1}{8} + \ldots + 2^n \times \frac{1}{2^n} + \ldots \]

= 1 + 1 + 1 + \ldots = \infty

- Yet I am charging only $1 million for the bet—it’s a better deal than the California lottery.
- Paradox: no one will take this bet. Why?
Decreasing Marginal Utility of Wealth

Utility

$U(W)$ – Utility of Wealth

$U'(W) > 0$ – Marginal Utility of Wealth

$U''(W) < 0$ – Decreasing Marginal Utility of Wealth

Expected Utility

- Defined as the expected value of the utility function over all possible states of the world.
- Let:
  - $W_1$ be wealth in state 1; probability $\pi_1$
  - $W_2$ be wealth in state 2; probability $\pi_2$
  - ... 
  - $W_n$ be wealth in state $n$; probability $\pi_n$
- Expected utility is:
  $$E[U(W)] = \sum_{i=1}^{n} \pi_i U(W_i)$$

Explaining the Paradox

- Daniel Bernoulli’s explanation for the St. Petersburg paradox: Poor people value increments in wealth more than rich people do.
- The large upside potential of the St. Petersburg bet is valued less than the certain loss of $1$ million.

Explaining the Paradox (II)

- Suppose utility of wealth is given by:
  $$U(W) = \ln W$$
- The value the the bet’s payouts are:
  - 1 heads: $\ln 2$
  - 2 heads: $\ln 4$
  - $n$ heads: $\ln 2^n = n \ln 2$
- The expected utility gain from the bet is
  $$E[U(W)] = \frac{1}{2} \ln 2 + \frac{1}{4} \ln 4 + ... + \frac{1}{2^n} \ln 2^n + ...$$

Explaining the Paradox (III)

$$E[U(W)] = \frac{1}{2} \ln 2 + \frac{1}{4} \ln 4 + ... + \frac{1}{2^n} \ln 2^n + ... =$$

$$= (\ln 2) \left( \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ... + \frac{1}{2^n} + ... \right) =$$

$$= 2 \ln 2 = 1.39$$

- Expected utility loss from the bet:
  $$E[U(1\text{ million})]=\ln (1,000,000) = 13.8$$
- Clearly the St. Petersburg bet is not worthwhile, even with an infinite expected value.

Risk Aversion

- Definitions:
  - A risk averse individual exhibits decreasing marginal utility of wealth.
  - A risk neutral individual exhibits constant marginal utility of wealth.
  - A risk loving individual exhibits increasing marginal utility of wealth.
- Do these definitions make sense?
Accidents

Suppose there are two states of the world:
- In accident-free state #1, you earn income $W
- In state #2, you suffer a horrible accident and earn a paltry $P < $W

Let the probability of an accident be $\pi$

Expected income is $M = \pi P + (1-\pi)W$

Expected utility is:
$$E[U] = \pi U(P) + (1-\pi)U(W)$$

Utility at the expected income is $U(M)$

Accident Graph--Risk Averse

For a risk averse individual, a certain income of $M$ is preferable to an uncertain income which is $M$ on average, since:
$$U(M) > E[U]$$

Accident Graph--Risk Neutral

A risk neutral individual is indifferent between a certain income $M$ and an uncertain income which is $M$ on average, since:
$$U(M) = E[U]$$

Accident Graph--Risk Loving

For a risk loving individual, an uncertain income which is $M$ on average is preferable to a certain income of $M$, since:
$$U(M) < E[U]$$

How Much Should You Pay for Insurance?

To avoid a utility loss of $U(M) - E[U]$ at $M$ (due to uncertainty) you should be willing to accept a certain income of (at least) $M - y$. That is, you should be willing to pay at most $y$ for insurance.

Nicholson Example Problem

Ms. Fogg is planning an around the world trip on which she plans to spend $10,000. The utility from the trip is a function of how much she actually spends on it ($Y$), given by $U(Y) = \ln Y$.

(a) If there is a 25 percent probability that Ms. Fogg will lose $1,000 of her cash on the trip, what is the trip’s expected utility?
Ms. Fogg’s Expected Utility

- With probability 0.25, Ms. Fogg will spend $9,000 on the trip, gaining utility of $U(9,000) = \ln 9,000 \approx 9.10$
- With probability 1-0.25 = 0.75, Ms. Fogg will spend $10,000 on the trip, gaining utility of $U(10,000) = \ln 10,000 \approx 9.21$
- Her expected utility is:
  $$E[U(Y)] = 0.25 \times 9.10 + 0.75 \times 9.21 = 9.1825$$

Example Problem—Part (b)

- (b) Suppose that Ms. Fogg can buy insurance against losing the $1,000 (say, by purchasing traveller’s checks) at an actuarially fair premium of $250. Show that her expected utility is higher if she purchases this insurance than if she faces the chance of losing the $1,000 without insurance.

Actuarially Fair Insurance

- Actuarially fair insurance is the same thing as saying that the insurance is a fair bet.
- A competitive insurance industry will provide actuarially fair insurance.
  - If one company charges higher than actuarially fair prices, then it will lose all of its customers.
  - If one company charges lower than actuarially fair prices, it will lose money (on average) on each customer.
- For Ms. Fogg, actuarially fair insurance is $250, since that is the expected value of the loss.

Expected Utility Gain from Actuarially Fair Insurance

- For Ms. Fogg, if she buys insurance, she will spend $10,000 - $250 = $9,750 on the trip with certainty
  - Utility would be $U(9,750) = \ln 9,750 \approx 9.1850$
- Without insurance, we have already calculated that her expected utility would be 9.1825.
- Clearly, she is better off with insurance.

Example Problem—Part (c)

- (c) What would the maximum amount that Ms. Fogg would be willing to pay to insure her $1,000?
- Let $p$ be the maximum premium she would be willing to pay.
- Her utility when paying for this insurance is $U(10,000 - p)$
- But $p$ cannot be so high that it exceed the expected utility of the trip without insurance (9.1825), so
  $$U(10,000 - p) \geq 9.1825$$

Maximum Premium Calculation

- At the maximum $p$ that Ms. Fogg would be willing to pay, utility with insurance should equal utility without insurance:
  $$U(10,000 - p) = 9.1825$$
  $$\ln (10000 - p) = 9.1825$$
  $$10000 - p = 9725$$
  $$p = 275$$
Arrow-Pratt Measure of Risk Aversion

- The degree of risk aversion is closely related to the curvature of the utility function.
  - Utility of wealth curves that are close to straight exhibit less risk aversion
  - Utility of wealth curves that are “very” concave exhibit more risk aversion
- The Arrow-Pratt measure of risk aversion is:
  \[ r(W) = -\frac{U''(W)}{U'(W)} \]

Arrow-Pratt Measure and Willingness to Pay for Insurance

- Suppose you start with some level of wealth \( W \), and a utility function \( U(W) \).
- Consider a fair bet with an outcome \( h \) (which can be either positive or negative),
  \(-E[h] = 0 \) since the bet is fair.
- Expected utility after taking the bet is \( E[U(W+h)] < U(W) \)
- Let \( p \) be the maximum premium you would be willing to pay to avoid the bet. Then,
  \( E[U(W+h)] = U(W-p) \)

Arrow-Pratt (continued)

\[ E[U(W+h)] = U(W-p) \]

- Take a first order Taylor series approximation around \( p = 0 \) for the right hand side:
  \[ U(W-p) = U(W) - U'(W)p \]
- Take a second order Taylor series approximation around \( h = 0 \) for the left hand side:
  \[ E[U(W+h)] = U(W) + U'(W)h + \frac{U''(W)}{2}h^2 \]

Arrow-Pratt (continued)

\[ E[U(W+h)] = E[U(W) + U'(W)h + \frac{U''(W)}{2}h^2] \]

- Carrying the expectation through yields:
  \[ E[U(W+h)] = U(W) + U'(W)E[h] + \frac{U''(W)}{2}E[h^2] \]
- Setting the two approximations equal to each other yields: (note that \( E[h] = 0 \))
  \[ U(W) - U'(W)p = U(W) + \frac{U''(W)}{2}E[h^2] \]

Arrow-Pratt (continued)

\[ U(W) - U'(W)p = U(W) + \frac{U''(W)}{2}E[h^2] \]

- Rearranging terms yields:
  \[ p = -\frac{E[h^2]}{2}U'(W) = kr(W) \]
- The Arrow-Pratt measure is proportional to the maximum amount you would be willing to pay to avoid the actuarially fair bet.