Asymptotically Optimal Control of a Centralized Dynamic Matching Market with General Utilities

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Abstract

We consider a matching market where buyers and sellers arrive according to independent Poisson processes at the same rate and independently abandon the market if not matched after an exponential amount of time with the same mean. In this centralized market, the utility for the system manager from matching any buyer and any seller is a general random variable. We consider a sequence of systems indexed by n where the arrivals in the n^{th} system are sped up by a factor of n. We analyze two families of one-parameter policies: the population threshold policy immediately matches an arriving agent to its best available mate only if the number of mates in the system is above a threshold, and the utility threshold policy matches an arriving agent to its best available mate only if the corresponding utility is above a threshold. Using an asymptotic fluid analysis of the two-dimensional Markov process of buyers and sellers, we show that when the matching utility distribution is light-tailed, the population threshold policy with threshold $\frac{n}{\ln n}$ is asymptotically optimal among all policies that make matches only at agent arrival epochs. In the heavy-tailed case, we characterize the optimal threshold level for both policies. We also study the utility threshold policy in an unbalanced matching market with heavy-tailed matching utilities, and find that

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the buyers and sellers have the same asymptotically optimal utility threshold. To illustrate our theoretical results, we use extreme value theory to derive optimal thresholds when the matching utility distribution is exponential, uniform, Pareto, and correlated Pareto. In general, we find that as the right tail of the matching utility distribution gets heavier, the threshold level of each policy (and hence market thickness) increases, as does the magnitude by which the utility threshold policy outperforms the population threshold policy.

Keywords: Matching markets, queueing asymptotics, regularly varying functions, extreme value theory

1 Introduction

We consider a symmetric centralized dynamic matching market (the asymmetric case is also discussed for heavy-tailed utilities). Two types of agents, which we call buyers and sellers, arrive to the market according to independent Poisson processes with rate λ, and each agent abandons (i.e., exits) the market after an independent exponential amount of time with rate η if he has not yet been matched. The utility of a match between any buyer and any seller is a general random variable. In this centralized model, the agents make no explicit decisions, and at the time of an agent arrival, the system manager observes all matching utilities between the arrival and all potential mates (e.g., sellers if the arrival is a buyer) who are currently in the market. Using information about the number of buyers and sellers and their matching utilities, the system manager decides when to make matches and which agents to match.

Centralized dynamic matching markets occur in settings such as organ transplants, public housing, labor markets and various online platforms. In practice, matching utilities include information about tissue type matching and the geographical distance between the donor and the recipient for organ transplants; the location and desirability of the residence and the distance between the residence and the applicant's current residence in public housing; and the match between the needs of the employer and the experience and skills of the job applicant in the labor market. This information can lead to wide variations in the matching utilities between different buyers and sellers, and our goal is to understand how best to exploit this variation when managing the market. However, in our idealized model, the details about this information are suppressed (e.g., we do not use covariates describing the agents to help make decisions) and aggregated into the matching utility distribution between buyers and sellers.

A key issue in centralized dynamic matching markets is to find the optimal market thickness; i.e., rather than match a new agent upon its arrival, it may be preferable to place the arriving agent in the market and allow more agents to arrive in the hope of making a higher-utility match in the future. In our model, we aim to maximize the long-run expected average utility rate (i.e., utility of matches per unit time) of all matches. Although we do not include agent waiting costs, a strategy that forces agents to wait too long for the market to thicken can backfire because agents may abandon the market before they are matched.

Due to the challenging nature of this problem, we resort to asymptotic methods. We consider a sequence of systems where the arrival rates in the n^{th} system are multiplied by $n > 0$. In the absence of any matching, the number of agents of each type would be precisely the number of customers in a $M/M/\infty$ queue, which would be $O(n)$. We use two types of asymptotic methods: one is a fluid analysis of the two-dimensional Markov process for the number of buyers and sellers in the market when the arrival rates are large. The other is extreme value theory (Gumbel 1958, Galambos 1978) and regularly varying functions (Resnick 1987), which are used because the utility of a match under the policies we consider is the maximum of a (typically) large number of random variables.

In this asymptotic regime, we compute an upper bound on the utility rate of any policy that makes matches only at agent arrival epochs, and compare it to the utility rate of two families of threshold policies: the population threshold policy and the utility threshold policy. Under the population threshold policy, the system manager immediately matches an arriving agent to the available mate with the highest matching utility (at which point, the arriving agent and its matched mate exit the system and their matching utility is collected by the system manager) only if the number of available mates in the market exceeds a specified threshold; otherwise, the arriving agent is not immediately matched and is instead placed in the market. Under the utility threshold policy, the arriving agent is immediately matched to its best available mate only if the corresponding matching utility exceeds a specified threshold.

1.1 Preview of Results

In extreme value theory, the limiting distribution of the maximum of many random variables can be one of three types, loosely based on whether the underlying distribution of these random variables has an exponential right tail, has a heavier (e.g., power law) right tail, or is bounded from above, and our results are qualitatively different in each case. Although our main results are couched in terms of regularly varying functions, we preview our results with three canonical examples – one from each of the three domains of attraction – in the symmetric case, which are analyzed in §5 (Table 1). When matching utilities have an exponential distribution, the population threshold policy with a threshold of $\frac{n}{\ln n}$ is asymptotically optimal with a utility rate that is $O(n \ln n)$ and twice as large as the utility rate of the greedy policy – i.e., the population threshold policy with a threshold of zero – in the limit. When the matching utilities have a Pareto (c, β) distribution with shape parameter $\beta > 1$ (and hence a finite mean), the optimal population threshold is $\frac{\lambda}{\eta(1+\beta)}n$. Although the utility rate of this threshold policy does not converge to the loose upper bound in this case, the utility rate and the upper bound are both $O(n^{1+1/\beta})$, whereas the utility rate under the greedy policy is only $O(n^{1+1/(2\beta)})$. When the matching utilities have a uniform distribution, the greedy policy is asymptotically optimal and the optimal utility rate is $O(n)$.

In the Pareto case, the optimal utility threshold is $0.763\sqrt{\pi n}$ when $c = 1$ and $\beta = 2$, and the corresponding utility rate is $O(n^{1+1/\beta})$. This asymptotic utility rate is computed explicitly and it is shown to be larger than the utility rate of the optimal population threshold policy. In the exponential and uniform cases, where we have already identified an asymptotically optimal policy, we use heuristics to compute, in the pre-limit, utility threshold policies that are consistent with the asymptotically optimal descriptions, but outperform in simulation results the population threshold policy. We also consider a positively correlated Pareto case in §5.4, and show that the optimal population threshold is independent of the correlation, the optimal utility threshold decreases as the correlation increases, and the utility rates of both threshold policies decrease as the correlation increases. In §7, we consider an unbalanced market, where buyers have a different arrival rate and abandonment rate than sellers, and formally (i.e., without proofs) analyze the utility threshold policy in the heavy-tailed case. Surprisingly, although we allow the buyers and sellers to have a different utility threshold, we find that they have the same asymptotically optimal utility threshold.

Taken together, the optimal amount of patience – and hence market thickness - increases with the right tail of the matching utility distribution, as does the optimal utility rate and the performance gap between the utility threshold policy and the population threshold policy. Our limited analysis of an unbalanced market suggests that the optimal market thickness also increases with the amount of imbalance.

1.2 Related Work

Matching markets is a large and active area of research, and we restrict our review to centralized dynamic markets. Although our model lacks the contextual richness of some of the models for specific types of markets, the most distinctive feature of our model is the general matching utilities, which allows us to understand how the right tail of the matching utility distribution impacts the optimal thickness of the market (Table 1). In contrast, much of the recent work in dynamic (centralized or decentralized) matching markets, either via two-type agents (e.g., easy-to-match or hard-to-match agents, or matches that are preferred or non-preferred, Baccara *et al.* 2015, Ashlagi *et al.* 2019a, Ashlagi *et al.* 2019b) or a compatibility network (Ashlagi *et al.* 2013, Anderson *et al.* 2017, Akbarpour *et al.* 2019, Varma *et al.* 2019) essentially lead to dichotomous outcomes for a match. Exceptions include Ünver (2010), who considers blood type compatibility for a dynamic kidney exchange model, Emek *et al.* 2016 and Ashlagi *et al.* 2017a, who consider minimizing mismatch costs when agents arrive on a finite metric space in a non-bipartite and bipartite setting, respectively, and Ashlagi *et al.* 2018, who allow general matching utilities in a discrete time model with a constant time until abandonment. They perform a primal-dual analysis to derive competitive ratios for algorithms when there is no prior information about the match values or arrival times.

Some closely related studies are Hu and Zhou (2016), Büke and Chen (2017) and Liu *et al.* (2015). Hu and Zhou (2016) consider a discrete-time, multiclass, discounted variant of our problem that includes waiting costs. They show that the optimal policy is of threshold form under vertical and unidirectionally horizontal differentiated types. The other two studies consider fluid and diffusion limits of simplified versions of our model where either a match occurs with a certain probability for each buyer-seller pair (Büke and Chen 2017) or everyone matches when there is an available mate, which corresponds to our greedy policy, but with a deterministic utility (i.e., a matching utility distribution that is a point mass at one value). In both cases, the system state reduces to a one-dimensional quantity (the number of sellers minus the number of buyers), whereas our model requires a two-dimensional state space for a non-greedy policy.

Perhaps the most closely related paper is Mertikopoulos *et al.* (2020), which also considers a symmetric centralized dynamic matching market. Compared to our study, they assume independent exponential mismatch costs rather than general matching utilities, and consider waiting times rather than abandonment, and are interested in minimizing the sum of mismatch and waiting costs over a finite horizon. They consider a class of policies that make the k^{th} match (which has the lowest mismatch cost among possible matches) when the short side of the market grows to a certain one-parameter function of k. They analyze the performance of the policy (using the celebrated $\pi^2/6$ result for the expected minimum weight matching due to Mezard and Parisi (1987) and rigorously proved by Aldous (2001)) under various values of the parameter, and also identify a policy that balances the mismatch and waiting costs.

We briefly mention other work that is only peripherally related. Originally motivated by public housing (Kaplan 1988), Caldentey *et al.* (2009) and Adan and Weiss (2012) consider infinite bipartite matching of servers and customers under the first-come first-served policy. Ding *et al.* (2016) allows the matching utilities to depend on the class of buyer and seller, and performs a fluid analysis of a greedy policy, and Bŭsić and Meyn (2016) minimize linear holding costs in a system without class-dependent matching utilities or abandonment, but also find that matches are not made until there are a sufficient number of agents in the market; see Moyal and Perry (2017), where these systems are referred to as matching queues, for other references to these types of models. Gurvich and Ward (2014) study a control problem in a more general setting where arriving customers wait to be matched to agents of other classes. All these studies assume a finite number of classes. There is also a stream of work in online bipartite matching in an adversarial setting (Karp *et al.* 1990), where agents do not wait in the market if they are not matched immediately.

We formulate the model in §2 and state our main theoretical results in §3, which are proved in §9. After analyzing a greedy policy in §4, we apply our main results to specific matching utility distributions in §5 and assess the accuracy of these results in a simulation study in §6. The unbalanced case is analyzed in §7 and concluding remarks are offered in §8.

2 The Model

Dynamics. Buyers and sellers arrive to the market according to independent Poisson processes with rate λ. The agents are impatient, in that each buyer and each seller independently abandons the market after an independent and identically distributed (i.i.d.) exponential amount of time with rate η if they are not matched within this time. If an agent is matched prior to his abandonment then the agent leaves at the time of matching.

Let $B(t)$ and $S(t)$ be the number of buyers and sellers in the system at time t; these agents have arrived but have not yet abandoned or been matched. The utility of a match between any buyer and any seller is a random variable $V \geq 0$ with cumulative distribution function (CDF) $F(v)$. When a buyer (seller, respectively) arrives to this centralized system to find it in state $(B(t), S(t))$ then $S(t)$ ($B(t)$, respectively) instances of V are observed by the system manager, which represent the matching utilities of the arriving agent with all currently available potential mates. Thus, at any point in time the system manager knows the utility that would be generated by matching any buyer to any seller.

Policies. Our goal is to maximize the long run expected average rate of utility from matches, which we refer to as the utility rate. While the system manager could conceivably make matches at any point in time, we restrict our attention to *arrival-only policies*, where a match may occur only at the arrival epoch of one of the agents being matched. In particular, we consider the following two classes of arrival-only policies.

1. Population threshold policies: A buyer who arrives at time t is matched immediately to a seller if the number of sellers in the system satisfies $S(t) > z$; in this case, the

arriving buyer is matched to the seller who has the highest matching utility with the buyer, with ties broken arbitrarily. If $S(t) \leq z$, then the arriving buyer waits in the market, and leaves upon being matched to a later-arriving seller or upon abandonment. Similarly, a seller who arrives at time t is immediately matched to the highest-matching buyer if $B(t) > z$, and waits otherwise. We refer to the parameter z as the population threshold.

2. Utility threshold policies: A buyer who arrives at time t is matched immediately to the seller with matching value $\arg\max_{1 \leq i \leq S(t)} V_i$ if $\max_{1 \leq i \leq S(t)} V_i > v$ for some fixed $v \geq 0$, with ties broken arbitrarily. If $\max_{1 \leq i \leq S(t)} V_i \leq v$, then the arriving buyer waits in the market, and leaves upon being matched to a later-arriving seller or upon abandonment. Similarly, a seller who arrives at time t is immediately matched to the buyer with matching value $\arg\max_{1 \leq i \leq B(t)} V_i$ if $\max_{1 \leq i \leq B(t)} V_i > v$, and waits otherwise. We refer to the parameter v as the utility threshold.

Although possibly not optimal among all policies, these single-parameter policies are easy to implement and describe, and allow for quite explicit results. In fact, the population threshold policy can be implemented without ever calculating the utility of individual matches (although the probability distribution of matches is required to compute the optimal threshold): all that is required is a ranked ordering of the possible matches.

Utilities. In our model, the utilities of potential matches of a new arrival with agents on the other side of the market may be correlated. However, we make the following assumption.

Assumption 1 *There exists a sequence of distributions $F_1, F_2, \ldots$ such that $F_{k-1} \leq_{st} F_k$ for each k and, for an arriving agent who finds k agents on the other side of the market,* $\max\{V_1, \ldots, V_k\}$ *is independent of the past, and has distribution F_k.*

For example, if the utilities of different matches are i.i.d. with distribution F then

$F_k(x) = (F(x))^k$.

Let the random variable $M(k) \triangleq \max\{V_1, \dots, V_k\}$ have distribution F_k. We impose the following assumption on $M(k)$.

Assumption 2 *For each $x \in \mathbb{R}_+$, define*

$$m(x) = E(M(\lceil x \rceil)),$$

and suppose that $m(\cdot)$ is regularly varying with index $\alpha \in [0, 1)$. That is, for every $x > 0$,

$$\lim_{t \to \infty} \frac{m(tx)}{m(t)} = x^{\alpha}. \tag{1}$$

A regularly varying function with index $\alpha = 0$ is also known as slowly varying.

For the case of i.i.d. utilities, Assumption 2 covers every utility distribution such that $E\left(V^{1+\delta}\right) < \infty$ for some $\delta > 0$. All distributions that belong to the maximum domain of attraction of a generalized extreme value distribution – which unifies the Type I (Gumbel), Type II (Frechet) and Type III (Weibull) laws within a single parametric family – satisfy (1) (including, e.g., uniform, beta, gamma, lognormal, Pareto). There are also other distributions that do not belong to any domain of attraction in extreme value theory for which (1) holds; e.g., the geometric, negative binomial, and Poisson distributions satisfy (1) with $\alpha = 0$. For ease of reference, we collect some basic facts about extreme value theory and regularly varying functions in §10.

The case $\alpha = 0$ corresponds to distributions for which all moments exist (i.e., the tail of V decays faster than any polynomial), whereas $\alpha > 0$ corresponds to the case in which the tails of V decrease roughly like a polynomial with degree $1/\alpha$. The condition that $\alpha < 1$ is imposed to guarantee that $E\left(V^{1+\delta}\right) < \infty$ for some $\delta > 0$. We will refer to $\alpha = 0$ as the light-tailed case and $\alpha \in (0, 1)$ as the heavy-tailed case.

Scaling. To make further progress, we consider a sequence of systems indexed by $n = 1, 2, \dots$. The arrival rate in the n^{th} system is $n\lambda$, and the abandonment rate in the

n^{th} system is η. Alternatively and equivalently, we could leave the arrival rate unscaled and slow down the abandonment rate by a factor of n, as in Liu *et al.* (2015). The matching utilities are unscaled. In the n^{th} system, we denote the system state by $(B_n(t), S_n(t))$, the population threshold by z_n, the utility threshold by v_n, and the utility rate by U_n.

3 Main Results

Results for the population threshold policy and the utility threshold policy are given in Theorem 1 in §3.1 and in Theorem 2 in §3.2, respectively. Theorem 1 shows that the optimal population threshold policy is asymptotically optimal among the class of arrival-only policies when $\alpha = 0$, and derives the optimal population threshold when $\alpha \in (0,1)$. Theorem 2 derives the optimal utility threshold when $\alpha \in (0,1)$. The proofs of Theorems 1 and 2 appear in §9.

3.1 Population Threshold Policy

We begin by providing a dynamic description of the system using Poisson processes. Denote the indicator function of event x by $I_{\{x\}}$ and let $N_B^+(\cdot), N_B^-(\cdot), N_S^+(\cdot), N_S^-(\cdot)$ be independent Poisson processes with unit rate, which are used to construct buyer arrivals, buyer abandonments, seller arrivals and seller abandonments, respectively. Under the population threshold policy with threshold z_n, the system state (B_n, S_n) satisfies

$$\begin{aligned} B_n(t) = B_n(0) &+ \int_0^t I_{\{S_n(r_-)<z_n\}} dN_B^+(\lambda n r) - N_B^-\left(\eta \int_0^t B_n(r)\,dr\right) \\ &- \int_0^t I_{\{B_n(r_-)\geq z_n\}} dN_S^+(\lambda n r), \end{aligned} \tag{2}$$

$$\begin{aligned} S_n(t) = S_n(0) &+ \int_0^t I_{\{B_n(r_-)<z_n\}} dN_S^+(\lambda n r) - N_S^-\left(\eta \int_0^t S_n(r)\,dr\right) \\ &- \int_0^t I_{\{S_n(r_-)\geq z_n\}} dN_B^+(\lambda n r). \end{aligned} \tag{3}$$

By symmetry and because the process $\{B_n(t), S_n(t), t \geq 0\}$ is ergodic, the utility rate $U_n^p(z_n)$ of the population threshold policy with threshold z_n can be expressed as

$$\begin{aligned} U_n^p(z_n) &= \lambda n E\left[m\left(B_n\left(\infty\right)\right) I_{\{B_n(\infty)\geq z_n\}}\right] + \lambda n E\left[m\left(S_n\left(\infty\right)\right) I_{\{S_n(\infty)\geq z_n\}}\right], \\ &= 2\lambda n E\left[m\left(B_n\left(\infty\right)\right) I_{\{B_n(\infty)\geq z_n\}}\right]. \end{aligned} \tag{4}$$

The theorem below shows that for $\alpha = 0$, the population threshold policy is asymptotically optimal among the family of arrival-only policies. Also, for each $\alpha \in [0, 1)$, it characterizes the scaling of the optimal population threshold, i.e., the threshold z_n that maximizes the utility rate asymptotically as $n \to \infty$.

Theorem 1 *Suppose that Assumption 2 holds.*

i) If $\alpha = 0$ then there exists an $o(n)$ sequence of population thresholds z_n^* such that $\lim_{n\to\infty} \frac{m(n)}{m(z_n^*)} = 1$. For any such sequence of thresholds, the population threshold policy is asymptotically optimal in the following sense. Let $U_n^p(z_n^*)$ and U_n be the utility rates under the above policy and any other arrival-only policy, respectively. Then $\liminf_{n\to\infty} \frac{U_n^p(z_n^*)}{U_n} \geq 1$. The associated utility rate satisfies

$$\lim_{n\to\infty} \frac{U_n^p(z_n^*)}{nm(n)} = \lambda. \tag{5}$$

ii) If $\alpha \in (0, 1)$ then the population threshold policy with $z_n^* = z_* n$ where $z_* = \frac{\lambda\alpha}{\eta(1+\alpha)}$ is asymptotically optimal among the class of population threshold policies. The associated utility rate satisfies

$$\lim_{n\to\infty} \frac{U_n^p(z_n^*)}{nm(n)} = \lambda z_*^{\alpha} \left(1 - \frac{\eta z_*}{\lambda}\right). \tag{6}$$

For $\alpha = 0$, it remains to compute a $o(n)$ sequence of thresholds z_n^* such that $\lim_{n\to\infty} \frac{m(n)}{m(z_n^*)} = 1$. This is usually not difficult to do. For example, when utilities are i.i.d. with an exponential distribution, then $z_n^* = \frac{n}{\ln n}$ satisfies this property. More generally, as shown in Theorem 1 in Bojanic and Seneta (1971), for a large class of distributions, setting $z_n^* = \frac{n}{m(n)^\delta}$ for any positive real δ is sufficient. By setting z_n in this way (i.e., $o(n)$ but not too small), we simultaneously ensure the following: (1) the fraction of agents that abandon the system tends to 0, and (2) the market thickness, i.e. $B_n(\infty)$, is almost linear in n. In other words, almost all agents experience maximal utility. This can be seen most clearly in equation (5), where the utility rate under the optimal population threshold policy satisfies $U_n^p(z_n^*) \sim n\lambda m(n)$, which is the arrival rate of buyers times the expected value of the maximum of n matching utilities.

However, for heavy-tailed distributions in part ii of Theorem 1, $m(z_n)$ for any $o(n)$ sequence z_n is vanishingly small compared to $m(n)$. Thus, it is not possible to ensure that most users see maximal utility, implying that our simple upper bound is unachievable. Moreover, to maximize the utility rate, it is not obvious whether the system manager should set $z_n = o(n)$ to guarantee that most agents are matched instantly, or should set $z_n = O(n)$ to ensure that market thickness is maximal even if a nontrivial fraction of users abandon the system. Part ii of Theorem 1 implies that the latter option is the right choice under heavy-tailed distributions.

We conclude this subsection with a few comments about the proof of Theorem 1, which relies on a fluid analysis of equations (2)-(3). We define $\bar{B}_n(t) = n^{-1}B_n(t)$ and $\bar{S}_n(t) = n^{-1}S_n(t)$ and let $z_n = nz$, and show that $\{(\bar{B}_n(t), \bar{S}_n(t)), t \geq 0\}$ converges to $\{(\bar{B}(t), \bar{S}(t)), t \geq 0\}$ as $n \to \infty$, which satisfies a dynamical system that is studied as the solution to a certain Skorokhod problem. Finally, we show that the equilibrium point for this system is (z, z), and prove that $(\bar{B}_n(\infty), \bar{S}_n(\infty))$ converges to (z, z) as $n \to \infty$. This analysis, along with the following key lemma, which is proved in §9, allows us to compute the utility rate in (4) and derive z_* in part ii of Theorem 1.

Lemma 1 *Let $\{N_n\}_{n\geq 1}$ be a sequence of positive random variables taking values on the positive integers and let $\bar{N}_n = E\left(N_n\right) < \infty$. Assume that $\bar{N}_n \to \infty$, and that $P\left(\left|N_n - \bar{N}_n\right| > \varepsilon \bar{N}_n\right) \to$*

0. *Then* $E\left[m\left(N_n\right)\right] \sim m\left(\bar{N}_n\right)$ *as* $n \to \infty$.

Asymptotic optimality in part i of Theorem 1 is proved by constructing the following simple upper bound (see §9 for a proof of Lemma 2) on the performance of any arrival-only policy, which uses Lemma 1 and assumes that all agents are matched and that – when computing $B_n(\infty)$ in equation (4) – agents leave only upon abandonment (implying that $B_n(\infty) \sim \text{Poi}(\lambda n/\eta)$).

Lemma 2 *Let* U_n *be the utility rate for any arrival-only policy. Then an upper bound* U_n^+ *is given by*

$$U_n \leq U_n^+ = \lambda n m\left(\frac{\lambda n}{\eta}\right).$$

3.2 Utility Threshold Policy

Because the population threshold policy is asymptotically optimal within the class of arrival-only policies when $\alpha = 0$, we focus on the case $\alpha \in (0,1)$ in Theorem 2. In order to describe the dynamics of the utility threshold policy, we introduce two independent arrays of nonnegative i.i.d. random variables, $\left\{V_{i,j}^B : i \geq 1, j \geq 1\right\}$ and $\left\{V_{i,j}^S : i \geq 1, j \geq 1\right\}$ having CDF $F(\cdot)$. We let $\left\{A_j^B : j \geq 1\right\}$ be the sequence of arrival times associated with the process $N_B^+(n\lambda\cdot)$ and $\left\{A_j^S : j \geq 1\right\}$ be the sequence of arrival times associated with the process $N_S^+(n\lambda\cdot)$. The dynamics can be described path-by-path as follows:

$$\begin{aligned} B_n(t) &= B_n(0) + \sum_{j=1}^{N_B^+(n\lambda t)} I_{\left\{\max_{i=1}^{S_n\left(A_{j-}^B\right)} V_{i,j}^B \leq v\right\}} - \sum_{j=1}^{N_S^+(n\lambda t)} I_{\left\{\max_{i=1}^{B_n(A_{j-}^S)} V_{i,j}^S > v\right\}} \\ &\quad - N_B^-\left(\eta \int_0^t B_n(r_-)\, dr\right), \\ S_n(t) &= S_n(0) + \sum_{j=1}^{N_S^+(n\lambda t)} I_{\left\{\max_{i=1}^{B_n(A_{j-}^S)} V_{i,j}^S \leq v\right\}} - \sum_{j=1}^{N_B^+(n\lambda t)} I_{\left\{\max_{i=1}^{S_n(A_{j-}^B)} V_{i,j}^B > v\right\}} \\ &\quad - N_S^-\left(\eta \int_0^t S_n(r)\, dr\right). \end{aligned} \tag{7}$$

By symmetry and ergodicity, we can express the utility rate $U_n^u(v_n)$ for the utility threshold policy with threshold v_n as

$$U_n^u(v_n) = 2\lambda n E\left[E[M(B_n(\infty))I_{\{M(B_n(\infty))\geq v_n\}}|B_n(\infty)]\right]. \tag{8}$$

Because the analysis of the utility threshold policy considers the entire distribution of the maximum rather than only its expected value, we need to strengthen Assumption 2 by imposing the following additional assumption.

Assumption 3 *In addition to Assumption 2, suppose that $\alpha \in (0,1)$ and*

$$\frac{M(n)}{m(n)} \Rightarrow Z \quad \text{as } n \to \infty,$$

where $P(Z > t) = 1 - e^{-\kappa/t^{1/\alpha}}$ and κ is a normalizing constant such that $E(Z) = 1$.

Assumption 3 is satisfied if the utilities belong to the domain of attraction of the Frechet law, which in turn is equivalent, in the i.i.d. case, to requiring the distribution of utilities to be regularly varying with index $1/\alpha$.

Theorem 2 *Suppose that Assumption 3 holds. For $z \in [0, \lambda/\eta]$, define*

$$v(z) = \left(\frac{\kappa z}{\ln\left(\frac{2\lambda}{\eta z+\lambda}\right)}\right)^{\alpha}$$

and consider the unique solution $z_* \in (0, \lambda/\eta)$ satisfying

$$z_*^{1-\alpha} v(z_*) \frac{\eta}{2\lambda\alpha\kappa^{\alpha}} = \int_0^{\kappa z_*/v(z_*)^{1/\alpha}} t^{-\alpha}e^{-t}dt.$$

Then a threshold policy with utility threshold $v_n^* = v(z_*)\, m(n)$ is optimal among the class of utility threshold policies. The associated utility rate satisfies

$$\lim_{n\to\infty} \frac{U_n^u(v_n^*)}{nm(n)} = 2\lambda z_*^{\alpha} E\left[Z I_{\{Z \geq \frac{v(z_*)}{z_*^{\alpha}}\}}\right].$$

As in the population threshold policy, the above result shows that for heavy-tailed distributions it is beneficial to ensure that market thickness is maximal at the cost of abandonment of a nontrivial fraction of users in the system. Although we do not prove any results for the utility threshold policy in the $\alpha = 0$ case (since asymptotic optimality is already achieved for the population threshold policy), we show in §5 how heuristics inspired by Theorems 1 and 2 can lead to effective utility thresholds in the $\alpha = 0$ case.

The proof of Theorem 2 uses the same general approach as in the proof of Theorem 1 part ii. Although the results in Theorem 2 are less explicit than those in part ii of Theorem 1, the proof is slightly easier because one can work with the putative fluid limit directly and use results in Ethier and Kurtz (2005), since this system does not pose the degeneracies involving the Skorokhod map encountered in the case of Theorem 1 part ii. Readers interested in the basics of the analysis without the proof are referred to §7, where we provide an informal analysis of the utility threshold policy for unbalanced markets.

4 A Greedy Policy

In §5, we apply the results in Theorems 1 and 2 to several different matching utility distributions, and then assess the accuracy of these analyses in §6. To provide a natural benchmark for comparison, we first analyze the greedy policy, which corresponds to the population threshold policy with threshold $z_n = 0$. That is, under the greedy policy, each arriving agent is matched to the available mate with the highest matching utility, and waits in the market if there are no available mates.

Under the greedy policy, the state of the n^{th} system can be described by $B_n(t) - S_n(t)$ because there are never both buyers and sellers in the system at the same time. By Theorem 4.5 in Liu *et al.* (2015), the steady-state distribution of $\frac{B_n(t)-S_n(t)}{\sqrt{n}}$ converges to

$N(0, \lambda/\eta)$ as $n \to \infty$.

The probability that a buyer or seller abandons is the long-run expected number of abandonments per unit time divided by the total arrival rate of agents (i.e., buyers plus sellers), which can be approximated by

$$
\begin{aligned}
\frac{\sqrt{n}\eta E[|N(0,\lambda/\eta)|]}{2\lambda n} &= \frac{\sqrt{n}\eta\sqrt{\frac{2}{\pi}\frac{\lambda}{\eta}}}{2\lambda n}, \\
&= \frac{1}{\sqrt{2\pi n}}, && (9) \\
&\to 0. && (10)
\end{aligned}
$$

By (10), the match rate for the greedy policy converges to $n\lambda$ as $n \to \infty$.

When a match occurs (i.e., when there is at least one available mate upon an agent's arrival), the expected number of available mates when an agent arrives can be approximated by

$$
\begin{aligned}
& \sqrt{n}E[N(0,\lambda/\eta)|N(0,\lambda/\eta) > 0] \quad \text{by symmetry,} \\
= \; & \frac{\lambda}{\eta}\sqrt{\frac{2n}{\pi}}. && (11)
\end{aligned}
$$

By (10)-(11) and Lemma 1, the utility rate of the greedy policy, which is denoted by U_n^g, satisfies

$$
U_n^g \sim n\lambda m\left(\frac{\lambda}{\eta}\sqrt{\frac{2n}{\pi}}\right). \qquad (12)
$$

5 Examples

In §5.1-5.3, we consider one canonical matching utility distribution from each of the three domains of attraction (Weibull, Gumbel and Frechet), represented, respectively, by $\mathrm{U}(a,b)$, $\exp(\nu)$, and Pareto(c,β). In each of these examples, we compute the utility rate of the upper bound in Lemma 2, the utility rate under the greedy policy from (12), and the asymptotically optimal (or heuristic, in some cases) thresholds and corresponding utility rates for the population threshold policy and the utility threshold policy from Theorems 1 and 2, respectively. We continue to add the superscripts $+$, g, p and u to U to denote the utility rate of the upper bound, the greedy policy, the population threshold policy and the utility threshold policy, respectively. We briefly consider matching utilities that come from a correlated Pareto distribution in §5.4.

5.1 Matching Utilities are Exponential

Let the matching utilities be exponential with parameter ν and CDF $F(v) = 1 - e^{-\nu v}$ for $v \geq 0$, which falls under the $\alpha = 0$ case in Theorem 1. The exponential is in the domain of attraction of Type I, and so (see (109)-(110) in §10) $\mu = \gamma = 0.5772\ldots$, which is Euler's constant, $a_n = \frac{1}{\nu}$ and $b_n = \frac{\ln n}{\nu}$, and hence $m(n) \sim (\gamma + \ln n)/\nu$.

The utility rate of the upper bound is

$$\begin{aligned} U_n^+ &\sim \frac{\lambda n}{\nu}\left(\gamma + \ln\left(\frac{n\lambda}{\eta}\right)\right) \text{ by Lemma 2,} && (13) \\ &\sim \frac{\lambda}{\nu} n \ln n, && (14) \end{aligned}$$

and the utility rate of the greedy policy is

$$\begin{aligned} U_n^g &\sim \frac{\lambda n}{\nu}\left(\gamma + \ln\left(\frac{\lambda}{\eta}\sqrt{\frac{2n}{\pi}}\right)\right) \text{ by (12),} && (15) \\ &\sim \frac{\lambda}{2\nu} n \ln n. && (16) \end{aligned}$$

As noted below Theorem 1, a range of population thresholds are asymptotically optimal in the $\alpha = 0$ case. For concreteness, we consider $z_n^* = \frac{n}{\ln n}$, which has utility rate

$$\begin{aligned} U_n^p\left(\frac{n}{\ln n}\right) &\sim \frac{\lambda n}{\nu}\left(\gamma + \ln\left(\frac{n}{\ln n}\right)\right) \text{ by (5),} && (17) \\ &\sim \frac{\lambda}{\nu} n \ln n. && (18) \end{aligned}$$

By (53), (56) and (57), the population threshold policy with threshold $\frac{n}{\ln n}$ is asymptotically optimal and doubles the utility rate of the greedy policy in the limit.

Recall that Theorem 2 does not apply to the $\alpha = 0$ case. Nonetheless, we apply the ideas in Theorems 1 and 2 to derive a heuristic threshold level for the utility threshold policy. By considering the steady-state version of equation (94) and differentiating, we obtain

$$\frac{\eta B_n(\infty)}{\lambda n} = P(M(S_n(\infty)) \leq v_n) - (1 - P(M(B_n(\infty)) \leq v_n)),$$

which by symmetry yields

$$\frac{\eta B_n(\infty)}{\lambda n} = 2P(M(B_n(\infty)) \leq v_n) - 1. \tag{19}$$

Now we heuristically assume that the utility threshold v_n is such that it achieves a population level $B_n(\infty)$ that equals the optimal population threshold z_n^*, which for concreteness we again take to be $\frac{n}{\ln n}$. Substituting $\frac{n}{\ln n}$ for $B_n(\infty)$ in (19) and noting that $P(M(n) \leq v_n) = P(\mu M(n) - \ln n \leq \mu v_n - \ln n) \sim \exp(-\exp(-\mu v_n + \ln n))$ by (105), we get

$$\frac{\eta}{\lambda \ln n} = 2 \exp\left[-\exp\left(-\nu v + \ln\left(\frac{n}{\ln n}\right)\right)\right] - 1. \tag{20}$$

Solving equation (20) gives the proposed threshold level,

$$v_n^* = \frac{\ln n - \ln\ln n - \ln\ln\left(\frac{2\lambda \ln n}{\lambda \ln n + \eta}\right)}{\nu}. \tag{21}$$

This heuristic approach does not generate a corresponding utility rate.

5.2 Matching Utilities are Pareto With Finite Mean

Let the matching utilities have CDF $F(v) = 1 - (cv)^{-\beta}$, for $\beta > 1, c > 0$ and $cv \geq 1$, so that the mean matching utility is finite and $\alpha = 1/\beta$ in Theorem 1. The Pareto distribution is in the domain of attraction of the Frechet distribution, and hence (see (109)-(110) in §10) $b_n = 0$, $a_n = (cn)^{1/\beta}$ and $\mu = \Gamma\Big(1 - \frac{1}{\beta}\Big)$, where $\Gamma(n)$ is the gamma function. It follows that

$$m(n) \sim (cn)^{1/\beta}\Gamma\Big(1 - \frac{1}{\beta}\Big). \tag{22}$$

By Lemma 2 and (12),

$$U_n^+ \sim \lambda\Big(\frac{c\lambda}{\eta}\Big)^{1/\beta}\Gamma\Big(1-\frac{1}{\beta}\Big)n^{1+1/\beta}, \tag{23}$$

and

$$U_n^g \sim \lambda\Big(\frac{c\lambda}{\eta}\sqrt{\frac{2}{\pi}}\Big)^{1/\beta}\Gamma\Big(1-\frac{1}{\beta}\Big)n^{1+1/(2\beta)}. \tag{24}$$

Because $\alpha = 1/\beta$, part ii of Theorem 1 implies that

$$z_n^* = \frac{\lambda}{\eta(1+\beta)}n. \tag{25}$$

That is, the optimal threshold equals the mean size of either side of the market in the absence of matching ($\lambda n/\eta$) times the factor $\frac{1}{1+\beta}$, which is less than 1/2. Substituting (62) into (6) gives the utility rate

$$U_n^p(z_n^*) \sim \lambda\Big(\frac{c\lambda}{\eta(1+\beta)}\Big)^{1/\beta}\left(\frac{\beta}{1+\beta}\right)\Gamma\Big(1-\frac{1}{\beta}\Big)n^{1+1/\beta}. \tag{26}$$

Comparing the utility rate under the optimal population threshold policy to the upper bound, we get

$$\frac{U_n^+}{U_n^p(z_n^*)} = \frac{1}{\left(\frac{1}{1+\beta}\right)^{1/\beta}\left(\frac{\beta}{1+\beta}\right)}, \tag{27}$$

which converges to 4 as $\beta \to 1$, and converges to 1 as $\beta \to \infty$.

Comparing the utility rate of the optimal population threshold policy to the utility rate of the greedy policy yields

$$\frac{U_n^p(z_n^*)}{U_n^g} = \left(\frac{\sqrt{\pi}}{\sqrt{2}(1+\beta)}\right)^{1/\beta}\left(\frac{\beta}{1+\beta}\right)n^{1/(2\beta)}, \tag{28}$$

which converges to $\sqrt{\frac{\pi n}{32}} \approx 0.3133\sqrt{n}$ as $\beta \to 1$, and converges to 1 as $\beta \to \infty$. For all finite values of β, the difference in performance between the two policies becomes unbounded as $n \to \infty$.

The optimal utility threshold needs to be computed numerically using the results in Theorem 2. To streamline the presentation, we consider the special case considered in the simulation experiments in §6, where $\lambda = \eta = 1$, $c = 1$ and $\beta = 2$, and hence $\alpha = 0.5$, $\kappa = 1/\pi$ and $m(n) = \sqrt{\pi n}$. Using the fact that the incomplete gamma function $\gamma(0.5, x) = \sqrt{\pi}\mathrm{erf}(\sqrt{x})$, by Theorem 2 we need to find the solution $z_* \in (0, \lambda/\eta)$ satisfying

$$\frac{z}{\sqrt{\pi \ln\left(\frac{2}{z+1}\right)}} = \mathrm{erf}\left(\sqrt{\ln\left(\frac{2}{z+1}\right)}\right). \tag{29}$$

Given the solution z_* to (29), Theorem 2 and its proof imply that

$$v_n^* = \sqrt{\frac{nz_*}{\ln\left(\frac{2}{z_*+1}\right)}} \tag{30}$$

and

$$U_n^u(v_n^*) \sim \frac{2(nz_*)^{3/2}}{\sqrt{\ln\left(\frac{2}{z_*+1}\right)}}. \tag{31}$$

5.3 Matching Utilities are Uniform

When the matching utilities are distributed as $\mathrm{U}(a, b)$ with $F(v) = \frac{v-a}{b-a}$ for $v \in [a, b]$, which is in the domain of attraction of the Weibull law, we have (see (109)-(110) in §10) $a_n = \frac{b-a}{n}$, $b_n = b$, $\mu = -\Gamma(2) = -1$, $m(n) \sim b - \frac{b-a}{n}$, and $\alpha = 0$ in Theorem 1. By Lemma 2 and (12),

$$\begin{aligned} U_n^+ &\sim n\lambda\Big(b - \frac{b-a}{\frac{\lambda}{\eta}n}\Big), && (32)\\ &\sim \lambda bn, && (33) \end{aligned}$$

and

$$\begin{aligned} U_n^g &\sim n\lambda\Big(b - \frac{b-a}{\frac{\lambda}{\eta}\sqrt{\frac{2n}{\pi}}}\Big), && (34)\\ &\sim \lambda bn. && (35) \end{aligned}$$

By (33) and (35), the greedy policy is asymptotically optimal, and so there is no need to consider a positive threshold level for the population threshold policy.

To heuristically analyze the utility threshold policy, we proceed as in §5.1, where equation (19) now becomes

$$\frac{\eta B_n(\infty)}{\lambda n} = 2\left(\frac{v_n - a}{b - a}\right)^{B_n(\infty)} - 1, \tag{36}$$

which can be rearranged as

$$v_n = a + (b-a)\left(\frac{\eta B_n(\infty)}{2\lambda n} + \frac{1}{2}\right)^{1/B_n(\infty)}.$$

Setting $B_n(\infty) = \frac{\lambda}{\eta}\sqrt{\frac{2n}{\pi}}$ from (11), which is the expected number of available mates for an arriving agent under the greedy policy, leads to

$$v_n^* = a + (b-a)\left(\frac{1}{\sqrt{2n\pi}} + \frac{1}{2}\right)^{\frac{\eta}{\lambda}\sqrt{\frac{\pi}{2n}}}. \tag{37}$$

5.4 Matching Utilities are Correlated

In this subsection, we demonstrate via an example how our analysis can be leveraged to study systems where the matching utilities associated with an arriving agent (one for each possible mate) are correlated. More specifically, suppose that when a seller (buyer) arrives and finds k buyers (sellers), the corresponding utilities $V_1, V_2, \ldots, V_k$ are given by

$$V_i = \rho U_0 + \sqrt{1-\rho^2}U_i \quad \text{for } i = 1, 2, \ldots, k, \tag{38}$$

where $\rho \in [0, 1)$, and $U_0, U_1, \ldots, U_k$ are i.i.d. with a Pareto$(\frac{2}{\sqrt{3}}, 3)$ distribution; i.e., for each i we have $P(U_i \le u) = 1 - (\frac{2}{\sqrt{3}u})^3$ for $u \ge \frac{2}{\sqrt{3}}$, and 0 otherwise. The variance of each V_i is 1, independent of ρ. However, for $\rho > 0$, $\mathrm{Cov}(V_i, V_j) = \rho^2$ for each $i \ne j$, and the utilities have correlation ρ^2.

Before computing the optimal thresholds and the corresponding utility rates for the population threshold policy and the utility threshold policy, we show that Assumptions 2-3 hold. We have $m(k) = \rho E[U_0] + \sqrt{1-\rho^2}E[\max_{i=1}^k U_i] \sim \sqrt{1-\rho^2}\Gamma(1-\frac{1}{\beta})k^{1/\beta}$, and

$\frac{M(k)}{m(k)} = \frac{\rho U_0 + \sqrt{1-\rho^2}\max_{i=1}^k U_i}{m(k)} \sim \frac{\max_{i=1}^k U_i}{E[\max_{i=1}^k U_i]}$ w.p.1, which implies that Assumptions 2 and 3 are satisfied with $\alpha = 1/3$.

By Theorem 1, the asymptotically optimal population threshold is $z_n^* = \frac{\lambda\alpha}{\eta(1+\alpha)}n$, independent of ρ, and the asymptotic utility rate is proportional to $\sqrt{1-\rho^2}$. By Theorem 2, the asymptotically optimal utility threshold is $v_n^* = \sqrt{1-\rho^2}\Gamma(1-\frac{1}{\beta})v_* n^{1/\beta}$ where v_* is as given in the theorem statement, and the corresponding utility rate is increasing in $\sqrt{1-\rho^2}$.

Hence, in both case, the utility rate decreases as the correlation ρ^2 increases. However, the optimal population threshold is independent of ρ^2 and the optimal utility threshold is decreasing in ρ^2. The optimal population threshold is independent of ρ^2 because it trades off the higher utility from additional thickness (i.e., increased z_n), which depends on the variation in $(U_1, \ldots, U_k)$, and the higher abandonment rate, which is independent of ρ^2.

It is also possible to apply our techniques to incorporate scenarios where the correlation scales with the number of potential matches, k. For example, suppose the matching utilities for an arriving agent are

$$V_i = 2^{-1/\beta}\max(c_k W, U_i) \quad \text{for } i = 1, 2, \ldots, k, \tag{39}$$

where $U_1, \ldots, U_k$ are i.i.d. with a Pareto$(1,\beta)$ distribution with $\beta > 1$, W is an independent Frechet(β) distributed random variable, and $c_k = k^{1/\beta}\Gamma(1-\frac{1}{\beta})$ for each k. Assumptions 2 and 3 are satisfied with $\alpha = 1/\beta$, and thus our results apply. In fact, even when W has a distribution that is different than Frechet(β), while Assumption 3 may not hold as is, our proof techniques may be leveraged to compute optimal thresholds. We omit the details for the sake of brevity.

6 Simulation Results

To assess the accuracy of our asymptotic results, we consider special cases of the three canonical examples in §5: exp(1), Pareto(1,2) and U[0,1]. For all cases, we let $\lambda = \eta = 1$ and $n = 1000$, so that the mean number of buyers and sellers in a match-free system is 1000. We initialize the system with 1000 buyers and 1000 sellers, simulate the system for 1500 time units, discarding the first 150 time units, and then repeat this procedure 100 times.

To find the optimal population threshold levels, we compute the utility rate for the population threshold policy for each integer threshold value in the range [0,1000], using the same set of random numbers for each threshold level. We repeat the same procedure for the utility threshold policy, and discretize the utility threshold values by 0.1 for the exp(1) and Pareto(1,2) cases, and by 0.01 for the U[0,1] case.

Exponential(1) case. In the exponential case, we predict that the optimal population threshold level is $z_n^* = \frac{1000}{\ln 1000} = 144.8$, and the utility rate under this threshold policy approaches the upper bound and is twice as large as the utility rate of the greedy policy (see §5.1). The optimal threshold level found via simulation is 148, and the suboptimality of the utility rate under the threshold 144.8 vs. the threshold 148 is 0.004% (Table 2). Our heuristic utility threshold is $v_n^* = 5.56$ from (21), which coincides with the optimal threshold found via simulation (with a discretization of 0.1) of 5.6.

However, the predicted utility rates are less accurate than our determination of the best threshold levels. By (17), our best estimate for the utility rate under the optimal population threshold policy is 5553, which is 14.9% higher than the simulated value in Table 2. By (9) and (15), our best estimate of the utility rate under the greedy policy is

$$\begin{aligned} U_n^g &\approx \frac{\lambda n}{\nu}\left(1 - \frac{1}{\sqrt{2\pi n}}\right)\left(\gamma + \ln\left(\frac{\lambda}{\eta}\sqrt{\frac{2n}{\pi}}\right)\right), \\ &= 3757, \end{aligned}$$

which is 8.5% higher than the simulated value in Table 2. Our best estimate of the upper bound is given in (13), which yields 7485. The optimal-to-greedy ratio of the simulated utility rates is $\frac{4833}{3462} = 1.40$ rather than 2. Further simulations reveal that convergence is very slow: this simulated ratio is 1.48 when $n = 10^4$ and 1.54 when $n = 10^5$. Most of the inaccuracy in estimating the optimal-to-greedy ratio is due to the fact that the simulated utility rate of the optimal threshold policy is not very close to the upper bound.

Finally, the utility rate of the optimal utility threshold policy is 5732 (Table 3). While still far from the upper bound, it is 18.6% higher than the utility rate achieved by the optimal population threshold policy.

Pareto(1,2) case. In the Pareto case, we predict that the optimal population threshold level is $\frac{1000}{3} = 333.3$. The optimal population threshold found via simulation is 347, and the utility suboptimality of the theoretical threshold is 0.03% (Table 2). The solution to (29) is $z^* = 0.512$. Hence, the optimal utility threshold level in (30) is $v_n^* = 42.8$, which is very close to the value of 42.0 found via simulation.

Our estimate of the utility rate under the optimal population threshold policy is 21,573 by (26), which is 2.4% less than the simulated value of 22,102 in Table 2. The optimal utility rate in (31) is U_n =43,756, which is nearly identical to the optimal simulated value of 43,750. Our best estimate for the utility rate of the greedy policy is $\left(1 - \frac{1}{\sqrt{2\pi n}}\right)$ times the right side of (60), or 8791, which is 6.4% larger than the simulated value in Table 2. Our estimate of the upper bound in (59) is 56,050. By (28), the predicted performance ratio between the optimal population threshold policy and the greedy policy is $\frac{2}{3}\left(\frac{1000\pi}{18}\right)^{1/4} = 2.42$, compared to the optimal-to-greedy simulated ratio of $\frac{22{,}102}{8259} = 2.68$ (Table 2). By (27), the ratio of the upper bound to the utility rate of the optimal population threshold policy is predicted to be $\frac{3\sqrt{3}}{2} = 2.60$, compared to the simulated value of $\frac{56{,}050}{22{,}102} = 2.54$.

The simulated utility rate of the optimal utility threshold policy is nearly twice as large as the simulated utility rate of the optimal population threshold policy (Table 3), although it is still 21.9% smaller than the predicted upper bound of 56,050.

Uniform(0,1) case. In the uniform case, we predict that the greedy policy is asymptotically optimal. The optimal population threshold level found via simulation is 22, and the resulting utility suboptimality of the greedy policy is 4.0% (Table 2). Note that other population thresholds aside from zero are also asymptotically optimal in this case, including $\ln(n) = \ln(1000) = 6.91$, which has a suboptimality of 2.0%. Our best estimate of the utility rate under the greedy policy is $\left(1 - \frac{1}{\sqrt{2\pi n}}\right)$ times the right side of (34), or 948.2, which is 4.4% larger than the simulated value in Table 2. The upper bound in (32) equals 987.4, which is 4.3% larger than the utility rate corresponding to the optimal population threshold level of 22. The predicted optimal utility threshold from equation (37) is $v_n^* = 0.974$, compared to the the value of 0.96 found via simulation, for a utility suboptimality of 0.24% (Table 3).

In summary, our analysis identifies the optimal threshold level within about 2% (considering the possible range of [0,1000]) and its suboptimality is no more than 2% for the population threshold policy in the uniform case, and is negligible in the other five cases. We also note that the predicted fraction of agents who abandon the market under the optimal population threshold policy, which is $\frac{2z_n}{2\lambda_n} = \frac{1}{\ln n} = 0.145$ (i.e., the total abandonment rate divided by the total arrival rate) in the exponential case, $\frac{z_n^*}{n} = \frac{1}{3}$ in the Pareto case by (62), and $1 - \frac{1}{\sqrt{2\pi n}} = 0.013$ in the uniform case by (9), are reasonably close to the simulated values in the fourth column of Table 3. As predicted by our analysis, the utility rate of the greedy policy – normalized by the mean of the matching distribution – increases with the right tail of the matching distribution (this quantity is 1817 for the uniform, 3462 for the exponential, and 4129 for the Pareto), as does the ratio of the utility rates between the best threshold policy and the greedy policy (1.04 for the uniform, 1.40 for the exponential, and 2.68 for the Pareto under the population threshold policy, and 1.06, 1.66 and 5.30 under the utility threshold policy). In addition, despite the asymptotic optimality result, there is a large gap between the utility rate of the best population threshold policy and the upper bound in the exponential case. The improvement of the utility threshold policy over the population threshold policy also increases with the right tail of the matching distribution, with the ratio of the utility rates equaling 1.02, 1.19 and 1.98 for the uniform, exponential and Pareto cases, respectively. This improvement is achieved by being more patient and allowing more agents to abandon the market, particularly in the Pareto case (last column in Table 3).

7 Unbalanced Markets

In this section, we consider unbalanced markets, where buyers and sellers arrive at rates $n\lambda_b$ and $n\lambda_s$ in the n^{th} system, and abandon at rates η_b and η_s, respectively. We restrict ourselves to the analysis of the utility threshold policy in the case $\alpha \in (0, 1)$, which is very similar to the corresponding analysis in the symmetric case, although we do not provide rigorous proofs in the unbalanced case. We also note that an analysis of the population threshold policy in the unbalanced case is likely to be quite challenging, due to the associated two-dimensional Skorokhod problem. Under the utility threshold policy in the n^{th} system, an arriving buyer is matched to the seller that yields the maximum utility if this utility exceeds the threshold $v_{n,s}$; similarly, an arriving seller is matched to its highest-matching buyer if the utility exceeds the threshold $v_{n,b}$. The dynamics are given by the equations

$$B_n(t) = B_n(0) + \sum_{j=1}^{N_B^+(n\lambda_b t)} I_{\left\{\max_{i=1}^{S_n\left(A_{j-}^B\right)} V_{i,j}^B \le v_{n,s}\right\}} - \sum_{j=1}^{N_S^+(n\lambda_s t)} I_{\left\{\max_{i=1}^{B_n(A_{j-}^S)} V_{i,j}^S > v_{n,b}\right\}} - N_B^-\left(\eta_b \int_0^t B_n(r_-)\, dr\right), \tag{40}$$

$$S_n(t) = S_n(0) + \sum_{j=1}^{N_S^+(n\lambda_s t)} I_{\left\{\max_{i=1}^{B_n(A_{j-}^S)} V_{i,j}^S \le v_{n,b}\right\}} - \sum_{j=1}^{N_B^+(n\lambda_b t)} I_{\left\{\max_{i=1}^{S_n(A_{j-}^B)} V_{i,j}^B > v_{n,s}\right\}} - N_S^-\left(\eta_s \int_0^t S_n(r)\, dr\right), \tag{41}$$

where $\left\{A_j^B : j \ge 1\right\}$ is the sequence of arrival times associated with $N_B^+(n\lambda_b \cdot)$, and $\left\{A_j^S : j \ge 1\right\}$ is the sequence of arrival times associated with $N_S^+(n\lambda_s \cdot)$. As in the symmetric case, the $V_{i,j}^B$s and $V_{i,j}^S$s are independent arrays of i.i.d. random variables with distribution $F(\cdot)$.

We assume that the thresholds are suitably scaled so that

$$\frac{v_{n,b}}{m(n)} \to v_b \text{ and } \frac{v_{n,s}}{m(n)} \to v_s \text{ for some } v_b, v_s \ge 0. \tag{42}$$

Then, just as in the symmetric case, we consider the scaling $\bar{B}_n(t) = n^{-1}B_n(t)$ and $\bar{S}_n(t) = n^{-1}S_n(t)$ and send $n \to \infty$ in (40)-(41), obtaining the fluid limit

$$\bar{B}(t) = \bar{B}(0) + \lambda_b \int_0^t e^{-\kappa \bar{S}(r)/v_s^{1/\alpha}} - \eta_b \int_0^t \bar{B}(r)\, dr - \lambda_s \int_0^t \left(1 - e^{-\kappa \bar{B}(r)/v_b^{1/\alpha}}\right) dr, \tag{43}$$

$$\bar{S}(t) = \bar{S}(0) + \lambda_s \int_0^t e^{-\kappa \bar{B}(r)/v_b^{1/\alpha}} - \eta_s \int_0^t \bar{S}(r)\, dr - \lambda_b \int_0^t \left(1 - e^{-\kappa \bar{S}(r)/v_s^{1/\alpha}}\right) dr. \tag{44}$$

Any stationary solution to the fluid model in (43)-(44) must satisfy

$$\begin{aligned} \eta_b b + \lambda_s &= \lambda_b e^{-\kappa s/v_s^{1/\alpha}} + \lambda_s e^{-\kappa b/v_b^{1/\alpha}}, \\ \eta_s s + \lambda_b &= \lambda_b e^{-\kappa s/v_s^{1/\alpha}} + \lambda_s e^{-\kappa b/v_b^{1/\alpha}}. \end{aligned}$$

Moreover, following the development in the symmetric case (e.g. (8)), the utility rate takes the form

$$
\begin{aligned}
U_n^u(v_{n,b},v_{n,s}) &= n\lambda_b E\left[E[M\left(S_n\left(\infty\right)\right)I_{\{M(S_n(\infty))\geq v_{n,s}\}}|S_n\left(\infty\right)]\right] \\
&\quad +n\lambda_s E\left[E[M\left(B_n\left(\infty\right)\right)I_{\{M(B_n(\infty))\geq v_{n,b}\}}|B_n\left(\infty\right)]\right].
\end{aligned}
$$

Consequently,

$$
\frac{U_n^u(v_{n,b},v_{n,s})}{nm(n)} \sim \lambda_b s^\alpha E\left[ZI_{\{Z\geq v_s/s^\alpha\}}\right] + \lambda_s b^\alpha E\left[ZI_{\{Z\geq v_b/b^\alpha\}}\right] \quad \text{as} \quad n\to\infty. \tag{45}
$$

Hence, we need to solve the optimization problem

$$
\begin{aligned}
\max_{v_b\geq 0, v_s\geq 0} \quad & \lambda_b s^\alpha E\left[ZI_{\{Z\geq v_s/s^\alpha\}}\right] + \lambda_s b^\alpha E\left[ZI_{\{Z\geq v_b/b^\alpha\}}\right] && (46)\\
\text{subject to} \quad & \eta_b b + \lambda_s = \lambda_b e^{-\kappa s/v_s^{1/\alpha}} + \lambda_s e^{-\kappa b/v_b^{1/\alpha}}, && (47)\\
& \eta_s s + \lambda_b = \lambda_b e^{-\kappa s/v_s^{1/\alpha}} + \lambda_s e^{-\kappa b/v_b^{1/\alpha}}. && (48)
\end{aligned}
$$

We solve (46)-(48) by reducing it to a one-dimensional optimization problem over a compact interval using the following three-step procedure: make a change of variables, show that $v_b = v_s$ for any feasible pair (b,s), and solving for the optimal (b,s).

Step 1: The case of no matching, i.e., $v_b = v_s = \infty$, provides an upper bound on the system's steady-state population of $\rho_b = \frac{\lambda_b}{\eta_b}$ and $\rho_s = \frac{\lambda_s}{\eta_s}$. Hence, $b\in[0,\rho_b]$ and $s\in[0,\rho_s]$. Recall by Assumption 3 that $Z = \left(\kappa^{-1}T\right)^{-\alpha}$, where T is exponentially distributed with unit mean. Defining

$$
G\left(x\right) = \int_0^x t^{-\alpha}e^{-t}dt
$$

and using the change of variables

$$
x = \frac{\kappa s}{v_s^{1/\alpha}} \quad \text{and} \quad y = \frac{\kappa b}{v_b^{1/\alpha}}, \tag{49}
$$

we use (89)-(90) to express problem (46)-(48) as

$$
\begin{aligned}
\max_{s,b\in(0,\rho_s)\times(0,\rho_b),(x,y)\in R_+^2} \quad & \lambda_b s^\alpha G\left(x\right) + \lambda_s b^\alpha G\left(y\right) && (50)\\
\text{subject to} \quad & \lambda_b e^{-x} + \lambda_s e^{-y} = b\eta_b + \lambda_s = s\eta_s + \lambda_b. && (51)
\end{aligned}
$$

Step 2: Suppose for now that b and s are fixed (and feasible) and we are optimizing over (x, y) in (49). Because $G(\cdot)$ in (50) is strictly concave and increasing, problem (50)-(51) is given by

$$\max_{(x,y)\in R_+^2} \quad \lambda_b s^\alpha G(x) + \lambda_s b^\alpha G(y), \tag{52}$$

$$\text{subject to} \quad \lambda_b e^{-x} + \lambda_s e^{-y} \geq b\eta_b + \lambda_s = s\eta_s + \lambda_b, \tag{53}$$

which is a convex optimization problem because the constraints in (53) form a convex set. The optimality conditions for (52)-(53) are

$$\lambda_b s^\alpha x^{-\alpha} e^{-x} = \beta \lambda_b e^{-x},$$

$$\lambda_s b^\alpha y^{-\alpha} e^{-y} = \beta \lambda_s e^{-y},$$

where β is a Lagrange multiplier, which yields

$$\frac{b}{y} = \frac{s}{x} = \beta^{1/\alpha}.$$

The change of variables in (49) implies that

$$v_b = v_s. \tag{54}$$

Step 3: Finally, define the change of variable

$$\tau = \frac{\kappa}{v_b^{1/\alpha}}, \tag{55}$$

so that $x = \tau s$ and $y = \tau b$. Assume without loss of generality that $\lambda_b \geq \lambda_s$, which by (51) implies that

$$b(s) = \frac{s\eta_s + \lambda_b - \lambda_s}{\eta_b}. \tag{56}$$

Substituting (56) into (51), we define $\tau(s)$ to be the unique solution to

$$\lambda_b e^{-s\tau} + \lambda_s e^{-b(s)\tau} = s\eta_s + \lambda_b \tag{57}$$

for any $s \in (0, \rho_s)$ and $b \in (0, \rho_s)$. Note that $\tau(s)$ is well defined because the left side of (57) is decreasing in τ, given s and therefore $b(s)$. Moreover, the left side of (57) is larger than the right side when $\tau = 0$ and the right side is larger than the left side if $\tau = \infty$.

Then the optimization problem (50)-(51) takes the form

$$\max_{s\in[0,\rho_s]} \left\{ \lambda_b s^\alpha G\left(s\tau\left(s\right)\right) + \lambda_s \left(\frac{s\eta_s + \lambda_b - \lambda_s}{\eta_b}\right)^\alpha G\left(\left(\frac{s\eta_s + \lambda_b - \lambda_s}{\eta_b}\right)\tau\left(s\right)\right)\right\}. \tag{58}$$

Taken together, the asymptotically optimal thresholds in the n^{th} system are

$$v^*_{n,b} = v^*_{n,s} = \left(\frac{\kappa}{\tau(s^*)}\right)^\alpha m(n) \tag{59}$$

by (42), (54) and (55), where s^* is the solution to (58) and $\tau(s^*)$ is the unique solution to

$$\lambda_b e^{-s^*\tau} + \lambda_{s^*} e^{-b(s^*)\tau} = s^*\eta_{s^*} + \lambda_b. \tag{60}$$

By (45), (49), (59) and (90), the corresponding utility rate satisfies

$$U^u_n(v^*_{n,b}, v^*_{n,s}) \sim nm(n)\kappa^\alpha[\lambda_b[s^*]^\alpha G(s^*\tau(s^*)) + \lambda_s[b(s^*)]^\alpha G(b(s^*)\tau(s^*))] \text{ as } n \to \infty. \tag{61}$$

We conclude this section with a numerical example that is a variant of the one in §5.2: let $\lambda_b = 2$, $\lambda_s = 1$, $\eta_b = 1$, $\eta_s = 1$, $n = 1000$, and assume a Pareto(1,2) distribution, so that $\alpha = 1/2$, $\kappa = 1/\pi$ and $m(n) = \sqrt{1000\pi}$. Then $b(s) = s + 1$ in (56), and (57) reduces to

$$2e^{-s\tau} + e^{-(s+1)\tau} = s + 2. \tag{62}$$

The solution to (58) is $s^* = 0.365$ and $\tau(0.365) = 0.361$, which yields $v^*_{n,b} = v^*_{n,s} = \sqrt{\frac{1000}{0.361}} = 52.7$. Interestingly, this threshold level of 52.7 is higher than in the symmetric case, where $\lambda_b = 1$ and $v^*_n = 42.8$. Moreover, leaving all parameter values fixed except for λ_b, we numerically compute $v^*_{n,b}$ in (59) and find that it is increasing and concave in $\lambda_b \geq 1$.

With $\lambda_b = 2$, we simulate this system in the same manner as in §6. At a discretization of 0.1, a two-dimensional search of $(v_{n,b}, v_{n,s})$ space via simulation for the optimal thresholds yields (52.7,52.3), with a corresponding simulated utility rate of 71,046 and with abandonment fractions of 0.681 for buyers and 0.363 for sellers. The simulated utility rate at $(v^*_{n,b}, v^*_{n,s}) = (52.7, 52.7)$ is 71,010, which is suboptimal by 0.05%. The predicted utility rate, $U^u_n(v^*_{n,b}, v^*_{n,s})$ in (61), is 70,992, which is 0.03% less than the simulated value of 71,010.

8 Concluding Remarks

A fundamental tradeoff in centralized dynamic matching markets relates to market thickness: whether matches should be delayed – at the risk of antagonizing waiting agents – in the hope of obtaining better matches in the future. Very little is known about this issues when matching utilities are general. By combining queueing asymptotics (as an aside, we note that perhaps the most surprising part of our study is that rather than requiring a diffusion analysis, a fluid analysis is sufficient to analyze this problem) with extreme value theory, we obtain explicit results that shed light on this issue. For symmetric markets, as the right tail of the matching utility distribution gets heavier, it is optimal to become more patient and let the market thickness (and abandonment rate) increase. While empirical work on matching markets use more complicated covariate models than what we consider (e.g., Hitsch *et al.* 2010, Boyd *et al.* 2013, Agarwal 2015), it seems clear from these analyses that matching utilities typically are not in the domain of attraction of the Weibull law. Therefore, large centralized matching markets – whether balanced or unbalanced (see below) – are likely to benefit from allowing the market to thicken.

Our study appears to be the first to allow for correlated matching utilities, which is likely to be a common phenomenon in practice: an agent who is deemed objectively attractive in a labor, housing or school choice model is likely to have matching utilities with potential mates that are positively correlated rather than i.i.d. In §5.4, we find that positive correlation reduces the market thickness in the utility threshold policy but not the population threshold policy, and reduces the utility rate under both policies.

We note three limitations in our study. First, we restrict ourselves to arrival-only policies. In particular, it might be possible to do better by batching sets of agents and then matching them, as in Mertikopoulos *et al.* (2020). However, if there are many agents who abandon quickly after arrival, as in some call centers (e.g., Fig. 20 in Gans *et al.* 2003), a batching policy may not be very robust in practice. Moreover, generalizing their results to our setting is likely to be quite challenging, in that the $\pi^2/6$ result requires an exponential matching distribution and a minimum cost matching (they minimize mismatch plus waiting costs rather than maximizing utility in the presence of abandonment). While they generalize their results in §6 of their paper by positing a functional form for how the expected minimum mismatch costs decrease as a function of the number of agents in the market, this functional form does not appear to follow from any more primitive distributional assumptions.

Second, most of our analysis considers a symmetric market, with buyers and sellers having the same arrival and abandonment rates. While some markets, such as cadaveric organ transplants and public housing, tend to have chronic supply shortages, other markets have economic forces at play that tend to roughly balance supply and demand. In a static matching market, even a slight imbalance can give rise to a unique stable matching (Ashlagi *et al.* 2017b). We also note that a greedy policy is optimal in a somewhat different unbalanced market setting, where easy-to-match agents can match with all other agents in the market with a specified probability, but hard-to-match agents can match only with easy-to-match agents with a different specified probability (Ashlagi *et al.* 2019b). In our analysis of the utility threshold policy in the heavy-tailed case of an unbalanced market, we obtain the somewhat surprising result that the solution is symmetric: i.e., the utility threshold is the same for buyers and sellers. Moreover, we find (in our Pareto example) that the amount of patience increases with the amount of imbalance; i.e., the larger the imbalance, the more agents that are going to be turned away, and the more selective the matching becomes. However, we leave a complete analysis of the unbalanced problem for future work.

The final restriction is exponential abandonment. Relaxing this assumption would require a different approach, such as hazard rate scaling (Reed and Tezcan 2012), and would likely be much more difficult.

Finally, we note that there may be equity issues if a significant number of agents are allowed to abandon the market (Table 3). The consideration of a risk-sensitive objective function would likely require a diffusion approximation, which would be, e.g., a two-dimensional Ornstein-Uhlenbeck process with an unusual Skorokhod condition under a population threshold policy.

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Policy	Matching Utility Distribution		
	Exponential(ν)	Pareto (c, shape $\beta > 1$)	Uniform(a, b)
Upper Bound	$U_n^+ \sim \frac{\lambda}{\nu} n \ln n$	$U_n^+ = O(n^{1+1/\beta})$	$U_n^+ \sim \lambda b n$
Greedy Policy	$U_n^g \sim \frac{\lambda}{2\nu} n \ln n$	$U_n^g = O(n^{1+1/(2\beta)})$	asymptotically optimal
Population Threshold Policy with threshold z_n	$z_n^* = \frac{n}{\ln n}$ is asymptotically optimal	$z_n^* = \frac{\lambda}{\eta(1+\beta)} n$ $U_n^p(z_n^*) = O(n^{1+1/\beta})$ but no convergence to upper bound unless $\beta \to \infty$	$z_n^* = 0$ is asymptotically optimal
Utility Threshold Policy with threshold v_n	heuristic $v_n^* = \frac{\ln n - \ln \ln n - \ln \ln\left(\frac{2\lambda \ln n}{\lambda \ln n + \eta}\right)}{\nu}$	$v_n^* = 1.353\sqrt{n}$ when $c = 1$, $\beta = 2$; $U_n^u(v_n^*) = O(n^{1+1/\beta})$	heuristic $v_n^* = a + (b-a)\left(\frac{1}{\sqrt{2n\pi}} + \frac{1}{2}\right)^{\frac{\eta}{\lambda}\sqrt{\frac{\pi}{2n}}}$

Table 1: Summary of results for the three canonical cases in §5. U_n^+, U_n^g, $U_n^p(z_n)$ and $U_n^u(v_n)$ are the upper bound on the utility rate for any arrival-only policy, the utility rate for the greedy policy, the utility rate for the population threshold policy with threshold z_n, and the utility rate for the utility threshold policy with threshold v_n, all for the n^{th} system. We use $x_n \sim y_n$ as shorthand for $\frac{x_n}{y_n} \to 1$ as $n \to \infty$.

Utility Distribution	Optimal Population Threshold		Simulated Utility Rate [95% CI]		
	Theoretical	Simulation	Theoretical Threshold	Simulation Threshold	Greedy Policy
Exponential(1)	144.8	148	4833 [4824,4840]	4833 [4827,4841]	3462 [3425,3503]
Pareto(1,2)	333.3	347	22,095 [21,997,22,241]	22,102 [21,972,22,234]	8259 [8107,8428]
Uniform(0,1)	0	22	908.4 [906.0,911.3]	946.3 [945.1,947.7]	908.4 [906.0,911.3]

Table 2: Theoretical and simulation results for the population threshold policy.

Utility Distribution	Population Threshold Policy Optimal Threshold	Utility Rate	Fraction Abandoned	Utility Threshold Policy Optimal Threshold	Utility Rate	Fraction Abandoned
Exponential(1)	148	4833 [4827,4841]	0.140	5.6	5732 [5724,5740]	0.150
Pareto(1,2)	347	22,102 [21,972,22,234]	0.334	42.0	43,750 [43,541,43,960]	0.503
Uniform(0,1)	22	946.3 [945.1,947.7]	0.027	0.96	963.0 [961.7,964.2]	0.021

Table 3: Simulation results for both threshold policies. Columns 2 and 3 are taken from Table 2.

Appendix

9 Technical Proofs

Before we prove Theorems 1 and 2 in §9.2 and §9.3, we prove Lemmas 1 and 2, and state and prove Lemmas 3 and 4, in §9.1.

9.1 Preliminary Lemmas

Proof of Lemma 1

By uniform local convergence, we must have that for any $0 < a < b < \infty$ (Resnick 1987)

$$\lim_{x\to\infty} \sup_{a\le t\le b} \left| \frac{m(xt)}{m(x)} - t^{-\alpha} \right| = 0.$$

Therefore,

$$\begin{aligned} \overline{\lim}_{n\to\infty} E\left[\frac{\left| m(N_n) - m\left(\bar{N}_n\right) \right| I_{\left\{ \left| N_n - \bar{N}_n \right| \le \varepsilon \bar{N}_n \right\}}}{m\left(\bar{N}_n\right)} \right] &\le \overline{\lim}_{n\to\infty} \sup_{1-\varepsilon\le t\le 1+\varepsilon} \left| \frac{m\left(\bar{N}_n t\right)}{m\left(\bar{N}_n\right)} - 1 \right|, \\ &\le O(\varepsilon). \end{aligned}$$

On the other hand, note that

$$\begin{aligned} &E\left[\left| \frac{m(N_n)}{m\left(\bar{N}_n\right)} - 1 \right| I_{\left\{ \left| N_n - \bar{N}_n \right| > \varepsilon \bar{N}_n \right\}} \right] \\ &\le E\left[\left| \frac{m(N_n)}{m\left(\bar{N}_n\right)} - 1 \right| I_{\left\{ N_n - \bar{N}_n > \varepsilon \bar{N}_n \right\}} \right] + E\left[\left| \frac{m(N_n)}{m\left(\bar{N}_n\right)} - 1 \right| I_{\left\{ \bar{N}_n - N_n > \varepsilon \bar{N}_n \right\}} \right]. \end{aligned}$$

On the event, $\bar{N}_n(1-\varepsilon) > N_n$, since $m(\cdot)$ is nondecreasing, we have that

$$\left| \frac{m(N_n)}{m\left(\bar{N}_n\right)} - 1 \right| = 1 - \frac{m(N_n)}{m\left(\bar{N}_n\right)} \le 1,$$

and therefore

$$E\left[\left| \frac{m(N_n)}{m\left(\bar{N}_n\right)} - 1 \right| I_{\left\{ \bar{N}_n - N_n > \varepsilon \bar{N}_n \right\}} \right] \le P\left(\left| N_n - \bar{N}_n \right| > \varepsilon \bar{N}_n \right) = o(1) \quad \text{as} \quad n \to \infty.$$

On the other hand, again, because $m(\cdot)$ is nondecreasing,

$$E\left[\left|\frac{m(N_n)}{m(\bar{N}_n)}-1\right| I_{\{N_n-\bar{N}_n>\varepsilon\bar{N}_n\}}\right] \leq E\left[\frac{m(N_n)}{m(\bar{N}_n)} I_{\{N_n>\bar{N}_n(1+\varepsilon)\}}\right].$$

Applying Potter's bound (Bingham, Goldie and Teugels (1987), Theorem 1.5.6 part (iii)) for each $\delta > 0$, there exists $t > 0$ such that $m(y)/m(x) \leq 2(y/x)^{\alpha+\delta}$ if $y \geq x \geq t$. Hence, for any $\delta > 0$ there exists $n_0 > 0$ such that $\bar{N}_n \geq t$ for all $n \geq n_0$, and therefore

$$\begin{aligned} E\left[\frac{m(N_n)}{m(\bar{N}_n)} I_{\{N_n>\bar{N}_n(1+\varepsilon)\}}\right] &\leq 2E\left[\left(\frac{N_n}{\bar{N}_n}\right)^{\alpha+\delta} I_{\{N_n>\bar{N}_n(1+\varepsilon)\}}\right], \\ &\leq 2\left(E\left[\left(\frac{N_n}{\bar{N}_n}\right)^{(\alpha+\delta)r}\right]\right)^{1/r} P\left(\left|N_n-\bar{N}_n\right| > \varepsilon\bar{N}_n\right)^{1/s}, \end{aligned}$$

for any $r, s > 1$ such that $1/r + 1/s = 1$, by Hölder's inequality. Furthermore, because $\alpha < 1$, we can guarantee $(\alpha+\delta)r < 1$ by choosing $\delta > 0$ sufficiently small. Next, Jensen's inequality implies that

$$E\left[\left(\frac{N_n}{\bar{N}_n}\right)^{(\alpha+\delta)r}\right] \leq \left(E\left[\frac{N_n}{\bar{N}_n}\right]\right)^{(\alpha+\delta)r} = 1.$$

Therefore, we obtain that

$$E\left[\frac{m(N_n)}{m(\bar{N}_n)} I_{\{N_n-\bar{N}_n>\varepsilon\bar{N}_n\}}\right] \to 0 \quad \text{as} \quad n \to \infty,$$

which completes the proof.

Proof of Lemma 2

For any arrivals-only policy, $B_n(\infty)$ is stochastically bounded by a system where no matches occur, i.e., agents leave only upon abandonment. In this case, each side of the market can be modeled as a $M/M/\infty$ queue, which has a Poisson $(\lambda n/\eta)$ stationary queue length distribution. Because the total arrival rate of agents is $2\lambda n$ and two agents exit upon each match, the maximum long-run rate for matches under any policy is λn. Hence, for any arrival-only policy we have that $U_n \leq \lambda n E[m(P_n)]$, where P_n is a Poisson $(\lambda n/\eta)$ random variable. Because Lemma 1 applies for P_n, it follows that $U_n \leq \lambda n m(\lambda n/\eta)$.

Lemma 3 *For $\alpha = 0$, there exists an $o(n)$ sequence z_n such that* $\lim_{n\to\infty} \frac{m(z_n)}{m(n)} = 1$.

Proof of Lemma 3

Recall that $m(t) = 0$ for $t \in [0,1)$ and let us define

$$x_n = \inf\left(x \in (0,1] : 1 - \frac{m(xn)}{m(n)} \leq x\right).$$

We show that x_n is $o(1)$. Suppose that this is not true. Then, we have that $\limsup_n x_n > 0$. Let $\epsilon = \limsup_n x_n$. By Assumption 2, for all $a \in (0,1]$ and for all $\epsilon' > 0$ there exists $n_0(a, \epsilon')$ such that for all $n \geq n_0$ we have $1 - \frac{m(an)}{m(n)} \leq \epsilon'$. Now, we pick $a = \epsilon/2$ and $\epsilon' = \epsilon/2$. Then, for each $n \geq n_0(\epsilon/2, \epsilon/2)$, we have that $1 - \frac{m(\epsilon n/2)}{m(n)} \leq \epsilon/2$. Hence, $x_n \leq \epsilon/2$ for each $n \geq n_0(\epsilon/2, \epsilon/2)$. This implies $\limsup_n x_n \leq \epsilon/2$ which is a contradiction.

Let $z_n = nx_n$. Because x_n is $o(1)$, it follows that z_n is $o(n)$. Further, by construction, $\lim_{n\to\infty} \frac{m(z_n)}{m(n)} = 1$. This completes the proof of the lemma.

Lemma 4 *Suppose that $\{X_n(\cdot)\}_{n\geq 1}$ is a sequence of stochastic processes in $\mathbb{R}^d$ and define $X_n^*(t) = \sup_{0\leq s\leq t} |X_n(s)|$. Assume that for each $t > 0$, the sequence $\{X_n^*(t)\}_{n\geq 1}$ is tight. Moreover, suppose that*

$$X_n(t) = X_n(0) + \int_0^t b_n(X_n(s))\,ds + M_n(t),$$

where $\{b_n(\cdot)\}_{n\geq 0}$ is a sequence of continuous functions such that $b_n(\cdot) \to b(\cdot)$ uniformly on compact sets and $b(\cdot)$ is locally Lipschitz, and $M_n(\cdot)$ is a martingale with quadratic variation $[M_n](\cdot)$ satisfying

$$E[[M_n](t)] \to 0$$

as $n \to \infty$ for each $t > 0$. Finally, suppose that $X_n(0) \to X(0)$ in probability as $n \to \infty$. Then $X_n(\cdot) \to X(\cdot)$ in probability in the uniform topology on compact sets, where $X(\cdot)$ is the unique solution to

$$X(t) = X(0) + \int_0^t b(X(s))\,ds.$$

Proof of Lemma 4

We have that for any $c > 0$, on the set $X_n^* (1) \leq c < \infty$, the process

$$W_n (t) = \int_0^t b_n (X_n (s)) \, ds$$

satisfies the following: there exists $n_0 := n_0 (c) < \infty$ such that for each $n \geq n_0$ and for any $0 \leq s < t \leq 1$, we have

$$\begin{aligned} |W_n (t) - W_n (s)| &\leq \int_s^t \|b_n (X_n (r))\| \, dr, \\ &\leq \sup_{0 \leq r \leq 1} \|b_n (X_n (r))\| \, |t - s| \,, \\ &\leq (1 + \sup_{0 \leq r \leq 1} \|b (X_n (r))\|)(t - s). \end{aligned}$$

The last inequality follows because $b_n (\cdot) \to b (\cdot)$ uniformly on compact sets and because of the tightness of $X_n^* (1)$. The Arzela-Ascoli theorem implies that $W_n (\cdot)$ is tight in the uniform topology. The Burkholder-Davis-Gundy inequality implies that the martingale sequence converges to zero uniformly on compact sets and therefore is tight. We conclude that $X_n (\cdot)$ must be tight in the uniform topology. Consequently, every subsequence contains a further sub-subsequence converging to the solution to the dynamical system

$$X (t) = X (0) + \int_0^t b (X (s)) \, ds,$$

which in turn has a unique solution because $b (\cdot)$ is locally Lipschitz.

9.2 Proof of Theorem 1

For now, let us assume that there exists a real $z > 0$ such that $z_n = zn$.

We define $\bar{B}_n (t) = n^{-1} B_n (t)$ and $\bar{S}_n (t) = n^{-1} S_n (t)$. By (2)-(3) in the main text, we can write

$$\begin{aligned}\bar{B}_n(t) &= \bar{B}_n(0) + \frac{N_B^+(n\lambda t)}{n} - \frac{N_B^-\left(\eta \int_0^t B_n(r)\,dr\right)}{n} \\ &\quad - \frac{\int_0^t I_{\{\bar{B}_n(r_-)\geq z\}} dN_S^+(\lambda n r)}{n} - \frac{\int_0^t I_{\{\bar{S}_n(r_-)\geq z\}} dN_B^+(n\lambda r)}{n}, \qquad (63)\\ \bar{S}_n(t) &= \bar{S}_n(0) + \frac{N_S^+(n\lambda r)}{n} - \frac{N_S^-\left(\eta \int_0^t S_n(r)\,dr\right)}{n} \\ &\quad - \frac{\int_0^t I_{\{\bar{S}_n(r_-)\geq z\}} d\tilde{N}_B^+(\lambda n r)}{n} - \frac{\int_0^t I_{\{\bar{B}_n(r_-)\geq z\}} dN_S^+(\lambda n r)}{n}. \qquad (64)\end{aligned}$$

We assume that $\bar{B}_n(0) \to \bar{B}(0)$ and $\bar{S}_n(0) \to S(0)$. Although formally the process in (63)-(64) converges to

$$\begin{aligned}\bar{B}(t) &= \bar{B}(0) + \lambda t - \eta \int_0^t \bar{B}(r)\,dr - \lambda \int_0^t I_{\{\bar{B}(r)\geq z\}} dr - \lambda \int_0^t I_{\{\bar{S}(r)\geq z\}} dr,\\ \bar{S}(t) &= \bar{S}(0) + \lambda t - \eta \int_0^t \bar{S}(r)\,dr - \lambda \int_0^t I_{\{\bar{B}(r)\geq z\}} dr - \lambda \int_0^t I_{\{\bar{S}(r)\geq z\}} dr,\end{aligned}$$

this dynamical system is non-standard because the indicator functions are not continuous. Hence, we need to study this system as the solution to a certain Skorokhod problem. In particular, we can write

$$\bar{B}(t) = \bar{B}(0) + \lambda t - \eta \int_0^t \bar{B}(r)\,dr - L_z^{\bar{B}}(t) - L_z^{\bar{S}}(t), \qquad (65)$$

$$\bar{S}(t) = \bar{S}(0) + \lambda t - \eta \int_0^t \bar{S}(r)\,dr - L_z^{\bar{B}}(t) - L_z^{\bar{S}}(t), \qquad (66)$$

where $L_z^{\bar{B}}(\cdot)$, $L_z^{\bar{S}}(\cdot)$ are nondecreasing processes such that $L_z^{\bar{B}}(0) = L_z^{\bar{S}}(0) = 0$ and

$$\int_0^t \left(\bar{B}(r) - z\right) dL_z^{\bar{B}}(r) = \int_0^t \left(\bar{S}(r) - z\right) dL_z^{\bar{S}}(r) = 0,$$

and $\bar{B}(t), \bar{S}(t) \leq z$. Hence, $L_z^{\bar{B}}(\cdot)$ and $L_z^{\bar{S}}(\cdot)$ are minimal nondecreasing processes that constrain the dynamics of $\bar{B}(\cdot)$ and $\bar{S}(\cdot)$ to stay below z.

The existence and uniqueness of the solution to the dynamical system in (65)-(66) is studied in §9.2.1, which appears at the end of the proof of this theorem. Also, we see that if $z < \lambda/\eta$, the equilibrium point of (65)-(66) is $\bar{Y}(\infty) = (z, z)^T$.

Note that

$$
\begin{aligned}
\bar{B}_n(t) &= \bar{B}_n(0) + \lambda t - \eta \int_0^t \bar{B}_n(r)\,dr \\
&\quad - \frac{\int_0^t I_{\{\bar{B}_n(r_-)\geq z\}} dN_B^+(\lambda n r)}{n} - \frac{\int_0^t I_{\{\bar{S}_n(r_-)\geq z\}} N_S^+(\lambda n r)}{n} \\
&\quad + \frac{N_B^+(n\lambda t) - n\lambda t}{n} + \frac{\eta \int_0^t n\bar{B}_n(r)\,dr - N_B^-\left(\eta \int_0^t n\bar{B}_n(r)\,dr\right)}{n} \\
&\quad + \frac{\int_0^t I_{\{\bar{B}_n(r_-)\geq z\}} dN_B^+(\lambda n r)}{n} - \frac{\int_0^t I_{\{\bar{B}_n(r_-)\geq z\}} dN_S^+(\lambda n r)}{n} \\
&\quad + \frac{\int_0^t I_{\{\bar{S}_n(r_-)\geq z\}} dN_S^+(\lambda n r)}{n} - \frac{\int_0^t I_{\{\bar{S}_n(r_-)\geq z\}} dN_B^+(n\lambda r)}{n}, \\
&= \bar{B}_n(0) + \lambda t - \eta \int_0^t \bar{B}_n(r)\,dr - L_z^{\bar{B}_n}(t) - L_z^{\bar{S}_n}(t) \\
&\quad + M_{n,1}^B(t) + M_{n,2}^B(t) + M_{n,3}^B(t) + M_{n,4}^B(t), \qquad (67)
\end{aligned}
$$

where

$$
\begin{aligned}
M_{n,1}^B(t) &= \frac{N_B^+(n\lambda t) - n\lambda t}{n}, \\
M_{n,2}^B(t) &= \frac{\eta \int_0^t n\bar{B}_n(r)\,dr - N_B^-\left(\eta \int_0^t n\bar{B}_n(r)\,dr\right)}{n}, \\
M_{n,3}^B(t) &= \frac{\int_0^t I_{\left\{\bar{B}_n(r_-)\geq z\right\}} dN_B^+(\lambda n r)}{n} - \frac{\int_0^t I_{\left\{\bar{B}_n(r_-)\geq z\right\}} dN_S^+(\lambda n r)}{n}, \\
M_{n,4}^B(t) &= \frac{\int_0^t I_{\left\{\bar{S}_n(r_-)\geq z\right\}} dN_S^+(\lambda n r)}{n} - \frac{\int_0^t I_{\left\{\bar{S}_n(r_-)\geq z\right\}} dN_B^+(n\lambda r)}{n}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\bar{S}_n(t) &= \bar{S}_n(0) + \lambda t - \eta \int_0^t \bar{S}_n(r)\,dr - L_z^{\bar{B}_n}(t) - L_z^{\bar{S}_n}(t) \\
&\quad + M_{n,1}^S(t) + M_{n,2}^S(t) + M_{n,3}^S(t) + M_{n,4}^S(t). \qquad (68)
\end{aligned}
$$

In (67)-(68), the processes $\left\{M_{n,i}^B\right\}_{n\geq 1}$ and $\left\{M_{n,i}^S\right\}_{n\geq 1}$ are martingales for $i = 1,2,3,4$ such that

$$
E\left[\sup_{0\leq r\leq t} \left|M_{n,i}^B(r)\right|^2\right] + E\left[\sup_{0\leq r\leq t} \left|M_{n,i}^S(r)\right|^2\right] = O\left(n^{-1}\right),
$$

which is obtained by upper bounding $\bar{B}_n(\cdot)$ by that of a system where no matches occur, i.e., agents leave only upon abandonment so that either side of the market can be modeled as an $M/M/\infty$ queue, and then applying Doob's maximal inequalities. Therefore, we have that

$$M_{n,i}^{B}, M_{n,i}^{S} \to 0$$

uniformly on compact sets as $n \to \infty$ in probability for $i = 1, 2, 3, 4$.

The Lipschitz continuity property of the Skorokhod map, which is established in Section §9.2.1, implies that

$$\bar{S}_n(\cdot) \to \bar{S}(\cdot), \ \bar{B}_n(\cdot) \to \bar{B}(\cdot)$$

uniformly on compact sets in probability.

The dynamical system describing $(\bar{B}, \bar{S})$ has a unique attractor, which is the point (z, z) if $\lambda/\eta \geq z$, given the initial condition $\bar{B}(0) \leq z$, $\bar{S}(0) \leq z$.

To show that the limit interchange ($t \to \infty$ and $n \to \infty$) holds, we begin by upper bounding with a system with no matching, which implies that the steady-state number of buyers in the system is less than or equal to a Poisson random variable with rate $\lambda n/\eta$. It follows that $E\left[\bar{B}_n(\infty)\right] \leq \lambda/\eta$, which in turn implies the uniformity property,

$$\sup_{n\geq 1} E\left[\bar{B}_n(\infty)\right] < \infty. \tag{69}$$

Next, we have that $\bar{B}_n(\cdot) \to \bar{B}(\cdot)$ on compact sets if $\bar{B}_n(0) \to Z$ as $n \to \infty$. Let us select $\bar{B}_n(0)$ in stationarity (i.e. equal in distribution to $\bar{B}_n(\infty)$). Tightness now follows from (69). Therefore, by choosing a subsequence, we may assume that $\bar{B}_n(0) \to Z$ (by Skorokhod embedding, we can assume that this convergence occurs almost surely). By stationarity, $\bar{B}_n(1) \to \bar{B}(1) = Z$ and therefore Z must be a stationary distribution of the dynamic system $\bar{B}(\cdot)$. But the stability point is unique and $Z = z$. Thus, we have that $\left(\bar{B}_n(\infty), \bar{S}_n(\infty)\right) \to (z, z)$ almost surely as $n \to \infty$ and we can exchange limits and expectations.

Our next goal is to compute the utility rate. Note that if $\left(\bar{B}_n(0), \bar{S}_n(0)\right)$ follows the stationary distribution then taking expectations on both sides of equation (2) of the main text yields

$$\eta E\left(\bar{B}_n(\infty)\right) = \lambda\{P\left(\bar{S}_n(\infty) < z\right) - P\left(\bar{B}_n(\infty) \geq z\right)\}. \tag{70}$$

Observe that

$$\begin{aligned} I_{\{\bar{S}_n(\infty)<z\}} &= I_{\{\bar{S}_n(\infty)<z, \bar{B}_n(\infty)\geq z\}} + I_{\{\bar{S}_n(\infty)<z, \bar{B}_n(\infty)<z\}}, \\ &= I_{\{\bar{B}_n(\infty)\geq z\}} + I_{\{\bar{B}_n(\infty)<z, \bar{S}_n(\infty)<z\}}. \end{aligned} \tag{71}$$

Equations (70)-(71) imply that

$$\eta E\left(\bar{B}_n(\infty)\right) = \lambda P\left(\bar{B}_n(\infty) < z, \bar{S}_n(\infty) < z\right). \tag{72}$$

Taking the limit in (72) as $n \to \infty$, we conclude that

$$\frac{\eta z}{\lambda} = \lim_{n\to\infty} P\left(\bar{B}_n(\infty) < z, \bar{S}_n(\infty) < z\right). \tag{73}$$

Equation (71) also implies that

$$I_{\{\bar{B}_n(\infty)\geq z\}} + I_{\{\bar{S}_n(\infty)\geq z\}} = 1 - I_{\{\bar{B}_n(\infty)<z, \bar{S}_n(\infty)<z\}}. \tag{74}$$

By symmetry, we conclude from (73)-(74) that

$$\lim_{n\to\infty} P\left(\bar{B}_n(\infty) \geq z\right) = \frac{1}{2}\left(1 - \frac{\eta z}{\lambda}\right). \tag{75}$$

Equation (75) allows us to compute the utility rate:

$$\begin{aligned} U_n^p(z_n) &= 2\lambda n E\left[m\left(B_n(\infty)\right) I_{\{B_n(\infty)\geq zn\}}\right], \\ &= 2\lambda n E\left[m\left(B_n(\infty)\right) | B_n(\infty) \geq zn\right] P\left(B_n(\infty) \geq zn\right), \\ &= \lambda n m(zn)\left(1 - \frac{\eta z}{\lambda}\right)(1 + o(1)) \end{aligned} \tag{76}$$

as $n \to \infty$, where the last equality follows from the use of Lemma 1 and the observation that

$$P\left(|B_n(\infty) - zn| \geq \varepsilon n | B_n(\infty) \geq zn\right) \to 0 \quad \text{as} \quad n \to \infty.$$

Thus, by equation (1) in the main text and (76),

$$\frac{U_n^p(z_n)}{nm(n)} = \lambda z^\alpha \left(1 - \frac{\eta z}{\lambda}\right)(1 + o(1)). \tag{77}$$

We first consider the case where $\alpha > 0$. Then (77) implies that among all policies such that $z_n = \Omega(n)$, setting $z_n = z_* n$ is asymptotically optimal. Further, for $z_n = o(n)$, using the same technique based on the fluid analysis, one can show that $\frac{U_n^p(z_n)}{nm(n)} = o(t)$; details are omitted for brevity. This completes the proof for $\alpha > 0$.

Finally, we consider the case where $\alpha = 0$. To make the dependence of $B_n(\infty)$ on z_n more explicit, for the rest of the proof we denote $B_n(\infty)$ as $B_n^{(z_n)}(\infty)$. Using the fluid limit analysis similar to above, for any sequence of thresholds z_n that is $o(n)$, we can show that $E[B_n^{(z_n)}(\infty)]$ is $o(n)$, as follows. In the pre-limit, in particular in equations (63)-(64) and (67)-(68), we replace z with z_n/n, and then using essentially the same arguments as above we obtain the fluid limit where $\left(\bar{B}_n^{(z_n)}(\infty), \bar{S}_n^{(z_n)}(\infty)\right) \to (0, 0)$ almost surely. Further, by using the same arguments as those used to obtain (75), we can show that

$$\lim_{n \to \infty} P\left(B_n^{(z_n)}(\infty) \geq z_n\right) = \frac{1}{2}. \tag{78}$$

By symmetry, we also have

$$\lim_{n \to \infty} P\left(S_n^{(z_n)}(\infty) \geq z_n\right) = \frac{1}{2}. \tag{79}$$

Now consider the n^{th} system, i.e., the system with the arrival rate of buyers equal to λn. For $k = 1, 2, \ldots$, define $I_k^{(n)}$ as follows: $I_k^{(n)}$ is equal to 1 if the k^{th} arrival of buyers sees at least z_n sellers upon arrival and is equal to 0 otherwise. Then PASTA (Poisson Arrivals

See Time Averages) implies that

$$\lim_{k\to\infty}\frac{1}{k}\sum_{j=1}^{k} I_j^{(n)} = \frac{1}{2}+o(1).$$

For each $k=1,2,\ldots,$ and for each $n=1,2,\ldots$, let $R_k^{(n)}$ be an independent random variable with distribution F_{z_n}. Then, again by PASTA, we have

$$\lim_{k\to\infty}\frac{1}{k}\sum_{j=1}^{k} I_j^{(n)} R_j^{(n)} = m(z_n)\left(\frac{1}{2}+o(1)\right).$$

However, by Assumption 1 and by symmetry, with probability 1 we have

$$U_n^p(z_n) \geq 2\lambda n \lim_{k\to\infty}\frac{1}{k}\sum_{j=1}^{k} I_j^{(n)} R_j^{(n)}. \tag{80}$$

Thus, we have $U_n^p(z_n) \geq \lambda n m(z_n)\left(1+o(1)\right)$. Consequently, for any sequence $z_n = o(n)$ such that

$$\lim_{n\to\infty}\frac{m(z_n)}{m(n)} = 1,$$

we would have that $U_n^p(z_n) \geq \lambda n m(n)\left(1+o(1)\right)$. Lemma 3 guarantees that such a sequence exists.

Combining these results with the upper bound in Lemma 2 completes the proof for $\alpha = 0$, and thus also the overall proof of Theorem 1.

9.2.1 Skorokhod Problem

In this subsection, we consider the existence and uniqueness of the system of differential equations

$$\begin{aligned}\bar{B}\left(t\right) &= \bar{B}\left(0\right) - \eta\int_0^t \left(\bar{B}\left(r\right) - \frac{\lambda}{\eta}\right)dr - L_z^{\bar{B}}\left(t\right) - L_z^{\bar{S}}\left(t\right),\\ \bar{S}\left(t\right) &= \bar{S}\left(0\right) - \eta\int_0^t \left(\bar{S}\left(r\right) - \frac{\lambda}{\eta}\right)dr - L_z^{\bar{B}}\left(t\right) - L_z^{\bar{S}}\left(t\right),\end{aligned}$$

where $L_z^{\bar{B}}\left(\cdot\right), L_z^{\bar{S}}\left(\cdot\right)$ are nondecreasing processes such that $L_z^{\bar{B}}\left(0\right) = L_z^{\bar{S}}\left(0\right) = 0$,

$$\int_0^t \left(\bar{B}\left(r\right) - z\right) dL_z^{\bar{B}}\left(r\right) = \int_0^t \left(\bar{S}\left(r\right) - z\right) dL_z^{\bar{S}}\left(r\right) = 0,$$

and $\bar{B}(t), \bar{S}(t) \leq z$. Hence, $L_z^{\bar{B}}(\cdot)$ and $L_z^{\bar{S}}(\cdot)$ are minimal nondecreasing processes that constrain the dynamics of $\bar{B}(\cdot)$ and $\bar{S}(\cdot)$ to stay below z.

In order to use explicit expressions for Skorokhod problems studied in the positive orthant, we introduce a change of coordinates. Defining $\bar{B}_z(t) = z - \bar{B}(t)$, $\bar{S}_z(t) = z - \bar{S}(t)$ and $\bar{\lambda}_z = \lambda/\eta - z \geq 0$, we have that

$$\bar{B}_z(t) = \bar{B}_z(0) - \eta \int_0^t \left(\bar{B}_z(r) + \bar{\lambda}_z\right) dr + L(t) \geq 0,$$
$$\bar{S}_z(t) = \bar{S}_z(0) - \eta \int_0^t \left(\bar{S}_z(r) + \bar{\lambda}_z\right) dr + L(t) \geq 0,$$

where

$$\int_0^t \min\left(\bar{B}_z(r), \bar{S}_z(r)\right) dL(r) = 0, \quad L(0) = 0.$$

Further, if we define

$$Z(t) = \min\left(\bar{B}_z(0) - \eta \int_0^t \left(\bar{B}_z(r) + \bar{\lambda}_z\right) dr, \bar{S}_z(0) - \eta \int_0^t \left(\bar{S}_z(r) + \bar{\lambda}_z\right) dr\right),$$

then

$$Y(t) := \min\left(\bar{B}_z(t), \bar{S}_z(t)\right)$$
$$= Z(t) + L(t),$$

which implies that

$$Y(t) = Z(t) + \max_{0 \leq s \leq t} \{-Z(s), 0\},$$
$$L(t) = \max_{0 \leq s \leq t} \left\{-\bar{B}_z(0) + \eta \int_0^s \left(\bar{B}_z(r) + \bar{\lambda}_z\right) dr, -\bar{S}_z(0) + \eta \int_0^s \left(\bar{S}_z(r) + \bar{\lambda}_z\right) dr, 0\right\}.$$

We then obtain $L\left(t; \bar{B}_z, \bar{S}_z\right) := L(t)$, to emphasize the dependence on $\left(\bar{B}_z, \bar{S}_z\right)$, yielding

$$\bar{B}_z(t) = \bar{B}_z(0) + \eta \int_0^t \left(\bar{B}_z(r) + \bar{\lambda}_z\right) dr + L\left(t; \bar{B}_z, \bar{S}_z\right), \tag{81}$$
$$\bar{S}_z(t) = \bar{S}_z(0) + \eta \int_0^t \left(\bar{S}_z(r) + \bar{\lambda}_z\right) dr + L\left(t; \bar{B}_z, \bar{S}_z\right). \tag{82}$$

We need to show that (81)-(82) has a unique solution. We first argue uniqueness. Assume that there exists another solution that we shall denote as $\left(\bar{B}_z', \bar{S}_z'\right)$ and consider $\Delta_B = \bar{B}_z - \bar{B}_z'$ and $\Delta_S = \bar{S}_z - \bar{S}_z'$. Suppose that $\Delta_S(0) = \Delta_B(0) = 0$. Then

$$\Delta_B(t) = \eta \int_0^t \Delta_B(r)\,dr + L\left(t; \bar{B}_z, \bar{S}_z\right) - L\left(t; \bar{B}_z', \bar{S}_z'\right),$$
$$\Delta_S(t) = \eta \int_0^t \Delta_S(r)\,dr + L\left(t; \bar{B}_z, \bar{S}_z\right) - L\left(t; \bar{B}_z', \bar{S}_z'\right).$$

Now, consider any real numbers a, b, c, a', b', c'. Suppose without loss of generality that $a = \max(a, b, c) \geq \max(a', b', c')$. Then, since $\max(a', b', c') \geq a'$, we conclude that

$$0 \leq a - \max(a', b', c') \leq a - a',$$

which implies

$$|\max(a, b, c) - \max(a', b', c')| \leq |a - a'| + |b - b'| + |c - c'|.$$

Consequently, if $D(t) = |\Delta_B(t)| + |\Delta_S(t)|$, we conclude

$$D(t) \leq 2\eta \int_0^t D(r)\,dr.$$

Because $D(t) = 0$, a direct application of Gronwall's inequality yields that $D(t) = 0$ for all $t > 0$, and uniqueness follows.

Now we argue the existence of a solution to (81)-(82). The construction follows by applying a standard Pickard iteration. Let

$$\mathfrak{B}_t(B, S) = B(0) + \eta \int_0^t \left(B(r) + \bar{\lambda}_z\right) dr + L(t; B, S),$$
$$\mathfrak{S}_t(B, S) = S(0) + \eta \int_0^t \left(S(r) + \bar{\lambda}_z\right) dr + L(t; B, S).$$

Observe that the map $(B, S) \to (\mathfrak{B}, \mathfrak{S})$ is Lipschitz continuous with respect to the uniform topology over any compact time interval $[0, T]$, using the corresponding uniform metric

$$\|(B, S)\|_{[0,T]} = \sup_{0 \leq t \leq T} \left(|B(t)| + |S(t)|\right).$$

Define $B_z^{(0)}(t) = B_z^{(0)}(0)$, $S_z^{(0)}(t) = S_z^{(0)}(0)$, and iteratively, for $m \geq 1$,

$$B_z^{(m)}(t) = \mathfrak{B}_t(B_z^{(m-1)}, S_z^{(m-1)}); \;\; S_z^{(m)}(t) = \mathfrak{S}_t(B_z^{(m-1)}, S_z^{(m-1)}).$$

Noting that $B_z^{(m)}(0) = B_z^{(m-1)}(0)$ and $S_z^{(m)}(0) = S_z^{(m-1)}(0)$, we have

$$\begin{aligned} &|B_z^{(m)}(t) - B_z^{(m-1)}(t)| + |S_z^{(m)}(t) - S_z^{(m-1)}(t)| \\ &\leq 3\eta \int_0^t |B_z^{(m-1)}(r) - B_z^{(m-2)}(r)| + |S_z^{(m-1)}(r) - S_z^{(m-2)}(r)| dr. \end{aligned}$$

Consequently, using $\|\cdot\|_{[0,T]}$ to denote the uniform norm over the interval $[0,T]$, we conclude that

$$\left\|(B_z^{(m)}, S_z^{(m)}) - (B_z^{(m-1)}, S_z^{(m-1)})\right\|_{[0,T]} \leq 3\eta T \left\|(B_z^{(m-1)}, S_z^{(m-1)}) - (B_z^{(m-2)}, S_z^{(m-2)})\right\|_{[0,T]}.$$

Choosing $0 < T < 1/(3\eta)$ we can deduce – by applying successive iterations and the triangle inequality – that $\{(B_z^{(m)}, S_z^{(m)}) : m \geq 1\}$ forms a Cauchy sequence in the space of continuous functions endowed with the uniform topology, which is a complete separable metric space. Therefore, by continuity of the map $(B, S) \to (\mathfrak{B}, \mathfrak{S})$, the limiting sequence must satisfy (81). The construction can be applied sequentially to consecutive intervals of size less than $1/(3\eta)$.

9.3 Proof of Theorem 2

For now, we assume that the thresholds satisfy

$$\frac{v_n}{m(n)} \to v \text{ for some } v \geq 0.$$

We consider a Poisson-flow representation of the scaled utility-based dynamics, whose validity is demonstrated in §9.3.1. In particular, it suffices to study the scaled processes

$\bar{B}_n(t) = n^{-1}B_n(t)$ and $\bar{S}_n(t) = n^{-1}S_n(t)$, which give rise to the representation

$$\begin{aligned}\bar{B}_n(t) &= \bar{B}_n(0) + \frac{N_B^+\left(\lambda n \int_0^t F_{S_n(r)}(v_n)\,dr\right)}{n} - \frac{N_B^-\left(\eta \int_0^t B_n(r)\,dr\right)}{n} \\ &\quad - \frac{\tilde{N}_S^+\left(\lambda n \int_0^t \left(1 - F_{B_n(r)}(v_n)\right)dr\right)}{n},\end{aligned} \tag{83}$$

$$\begin{aligned}\bar{S}_n(t) &= \bar{S}_n(0) + \frac{N_S^+\left(\lambda n \int_0^t F_{B_n(r)}(v_n)\,dr\right)}{n} - \frac{N_S^-\left(\eta \int_0^t S_n(r)\,dr\right)}{n} \\ &\quad - \frac{\tilde{N}_B^+\left(\lambda n \int_0^t \left(1 - F_{S_n(r)}(v_n)\right)dr\right)}{n},\end{aligned} \tag{84}$$

where $N_B^+(\cdot), \tilde{N}_B^+(\cdot), N_B^-(\cdot), N_S^+(\cdot), \tilde{N}_S^+(\cdot), N_S^-(\cdot)$ are independent Poisson processes with unit mean.

Note that under Assumption 3, we have

$$\begin{aligned}F_{nx}(m(n)z) &= P(M(nx) \le m(n)z), \\ &= P(Zm(nx) \le m(n)z)(1 + o(1)), \\ &= e^{-\kappa x/z^{1/\alpha}}(1 + o(1)).\end{aligned}$$

Using this result, the putative fluid limit of (83)-(84) is given by

$$\bar{B}(t) = \bar{B}(0) + \lambda \int_0^t e^{-\kappa \bar{S}(r)/v^{1/\alpha}} - \eta \int_0^t \bar{B}(r)\,dr - \lambda \int_0^t \left(1 - e^{-\kappa \bar{B}(r)/v^{1/\alpha}}\right)dr,$$

$$\bar{S}(t) = \bar{S}(0) + \lambda \int_0^t e^{-\kappa \bar{B}(r)/v^{1/\alpha}} - \eta \int_0^t \bar{S}(r)\,dr - \lambda \int_0^t \left(1 - e^{-\kappa \bar{S}(r)/v^{1/\alpha}}\right)dr.$$

We proceed in four steps. The first step is to obtain a martingale decomposition similar to that given in the proof of Theorem 1 part ii. The martingales will converge to zero on compact sets. The second step is to show that $\left\{\left(\bar{B}_n(\cdot), \bar{S}_n(\cdot)\right)\right\}_{n\ge1}$ is tight in the Skorokhod topology using the technique developed in Ethier and Kurtz (2005). The third step is to show that the putative fluid limit has a unique solution. The fourth step is to show that the ordinary differential equation (ODE) describing the fluid limit has a unique stationary point.

Together, these four steps imply that any subsequence of the sequence $\left\{\left(\bar{B}_n(\cdot), \bar{S}_n(\cdot)\right)\right\}_{n\geq 1}$ will contain a subsequence that converges (by tightness) to the unique solution of the above ODE, and therefore the fluid limit convergence holds. Further, since $\sup_n E[\bar{B}_n(\infty)] < \infty$ follows easily from the upper bound where no matching happens, we have that any subsequence of $\left\{(\bar{B}_n(\infty), \bar{S}_n(\infty))\right\}_{n\geq 1}$ will contain a subsequence that converges.

Now, consider the stationary versions of the process $(B_n(.), S_n(.))$. By stationarity, we have that if any subsequence of $\left\{(\bar{B}_n(\infty), \bar{S}_n(\infty))\right\}_{n\geq 1}$ converges then it has to converge to $(\bar{z}, \bar{z})$, which is the unique stationary point of the dynamical system describing $(\bar{B}(t), \bar{S}(t))$. Thus, $\left\{(\bar{B}_n(\infty), \bar{S}_n(\infty))\right\}_{n\geq 1}$ converges to $(\bar{z}, \bar{z})$.

Step 1: Consider the martingales

$$
\begin{aligned}
M_{n,1}^B(t) &= \frac{N_B^+\left(\lambda n \int_0^t F_{S_n(r)}\left(v_n\right) dr\right)}{n} - \lambda \int_0^t F_{S_n(r)}\left(v_n\right) dr, \\
M_{n,2}^B(t) &= \frac{N_B^-\left(\eta \int_0^t B_n\left(r\right) dr\right)}{n} - \eta \int_0^t \bar{B}_n\left(r\right) dr, \\
M_{n,3}^B(t) &= \frac{\tilde{N}_S^+\left(\lambda n \int_0^t \left(1 - F_{B_n(r)}\left(v_n\right)\right) dr\right)}{n} - \lambda \int_0^t \left(1 - F_{B_n(r)}\left(v_n\right)\right) dr,
\end{aligned}
$$

so that (83) can be expressed as

$$
\begin{aligned}
\bar{B}_n(t) &= \bar{B}_n(0) + \lambda \int_0^t F_{S_n(r)}\left(v_n\right) - \eta \int_0^t \bar{B}_n\left(r\right) dr - \lambda \int_0^t \left(1 - F_{B_n(r)}\left(v_n\right)\right) dr \\
&\quad + M_{n,1}^B(t) - M_{n,2}^B(t) - M_{n,3}^B(t).
\end{aligned}
$$

As argued in the proof of Theorem 1 part ii, using the upper bound on $B_n(t)$ vis-a-vis no matching and Doob's maximal inequality, we get

$$E\left[\sup_{0\leq r\leq t}|M_{n,i}(r)|^2\right]=O\left(n^{-1}\right).$$

Thus, for $i=1,2,3$, $M^B_{n,i}\to 0$ as $n\to\infty$ uniformly on compact sets.

Further, because $F_{S_n(r)}$ lies in $[0,1]$ w.p. 1, from Assumption 3 the process $\int_0^t F_{S_n(r)}(v_n)\,dr-\int_0^t\exp\left(-\kappa\bar{S}_n(r)/v^{1/\alpha}\right)dr$ converges to 0 uniformly over compact sets. Similarly,

$$\int_0^t\left(1-F_{B_n(r)}(v_n)\right)dr-\int_0^t\left(1-e^{-\kappa\bar{B}_n(r)/v^{1/\alpha}}\right)dr$$

converges to 0 uniformly over compact sets.

Analogously define martingales $M^S_{n,i}$ for $i=1,2,3$ for $\bar{S}_n$. Again, for $i=1,2,3$, $M^S_{n,i}\to 0$ as $n\to\infty$ uniformly on compact sets.

Because $F_{S_n(r)}$ lies in $[0,1]$ w.p.1, the quadratic variations $[M_{n,1}(t)]$ and $[M_{n,3}(t)]$ are bounded from above by the quadratic variation of $n^{-1}N_B^+(\lambda nt)$, which tends to 0 as $n\to\infty$. We now show that $[M_{n,2}(t)]$ also tends to 0 as $n\to\infty$ for a given t. Note that $B_n(.)$ is bounded from above by the process corresponding to no matching, which has $O(n)$ mean. Thus, the mean number of jumps in $n^{-1}N_B^-\left(\eta\int_0^t B_n(r)\,dr\right)$ is $O(n)$. Further, the size of each jump is $1/n$ with probability 1. Hence, the quadratic variation $[M_{n,2}(t)]\to 0$ as $n\to\infty$.

Similarly, the quadratic variations of $M^S_{n,i}(t)$ for $i=1,2,3$ also tend to 0 as $n\to\infty$.

Thus, from Lemma 4 in §9.1 we have that

$$\bar{B}_n(\cdot)\to\bar{B}(\cdot)\text{ and }\bar{S}_n(\cdot)\to\bar{S}(\cdot)$$

uniformly on compact sets in probability.

Step 2: Using Theorem 7.2 of Chapter 3 in Ethier and Kurtz (2005), we show that the family $\left\{\left((\bar{B}_n(t):t\geq 0),(\bar{S}_n(t):t\geq 0)\right)\right\}_{n\geq 1}$ is tight in the Skorokhod topology. The first condition of Theorem 7.2 of Ethier and Kurtz (2005) holds easily since for each t we have that $\bar{B}_n(t)$ and $\bar{S}_n(t)$ are both stochastically bounded from above by $1/n$ times Poisson distributed random variables with mean λnt, each of which concentrates as $n\to\infty$.

We now show that the second condition of Theorem 7.2 of Chapter 3 in Ethier and Kurtz (2005) holds as well. Note that, for each n, the times of positive jumps in B_n are a subset of jump times in a Poisson process of rate λn. Also, the departures from B_n may occur either when a customer in B_n abandons or when a customer arrives in S_n. Further, the times of customer abandonment is a subset of departure times in a $M/M/\infty$ queue, which – due to the time-reversibility of the $M/M/\infty$ queue – form a Poisson process with rate λn (plus a finite number of departures due to finite initial conditions). Thus, the times of negative jumps in B_n are a subset of jump times in a Poisson process of rate λn. Also, w.p.1, each jump is of size 1.

Thus, the modulus of continuity (see page 122 of Ethier and Kurtz 2005) of $\bar{B}_n$ is less than that of $n^{-1}A_n^B$ where A_n^B is a Poisson process of rate $3\lambda n$. Similarly, the modulus of continuity of $\bar{S}_n$ is less than that of $n^{-1}A_n^S$ where A_n^S is a Poisson process of rate $3\lambda n$. Therefore, it is enough to verify the second condition of Theorem 7.2 of Chapter 3 in Ethier and Kurtz (2005) for $\left\{\left((n^{-1}\bar{A}_n^B(t) : t \geq 0), (n^{-1}\bar{A}_n^S(t) : t \geq 0)\right)\right\}_{n\geq 1}$, which is easy to do.

Step 3: Recall that $\bar{B}_n(0) \to \bar{B}(0)$ and $\bar{S}_n(0) \to S(0)$. Because the dynamical system describing $\bar{B}(t)$ and $\bar{S}(t)$ is an ODE with Lipschitz coefficients, it has a unique solution.

Step 4: Using the fluid limit characterization, the fundamental theorem of calculus, and symmetry, for each stationary solution to the ODE we have

$$\begin{aligned} 0 &= \lambda e^{-\kappa\bar{z}/v^{1/\alpha}} - \eta\bar{z} - \lambda(1 - e^{-\kappa\bar{z}/v^{1/\alpha}}), \\ &= -\eta\bar{z} - \lambda + 2\lambda e^{-\kappa\bar{z}/v^{1/\alpha}}. \end{aligned} \tag{85}$$

Solving (85) for v, we define for $\bar{z} \in (0, \lambda/\eta)$,

$$v(\bar{z}) = \left(\frac{\kappa\bar{z}}{\ln\left(\frac{2\lambda}{\eta\bar{z}+\lambda}\right)}\right)^{\alpha}. \tag{86}$$

We can also uniquely solve for $\bar{z}(v)$ for $v \in (0,\infty)$ by finding the inverse of (86), which yields

$$\bar{z}(v) = -\frac{\lambda}{\eta} + \frac{v^{1/\alpha}}{\kappa} W\left(\frac{2\lambda\kappa}{\eta v^{1/\alpha}} \exp\left(\frac{\lambda\kappa}{\eta v^{1/\alpha}}\right)\right), \tag{87}$$

where $W(x)$ is the Lambert W function. Although we can work with either $v \in (0,\infty)$ or $\bar{z} \in (0, \lambda/\eta)$ when optimizing the asymptotic utility rate, it will be more convenient to optimize in terms of $\bar{z}$ and then find the optimal utility threshold using (86), as we now explain.

For each $v \geq 0$, we have established that $\left\{(\bar{B}_n(\infty), \bar{S}_n(\infty))\right\}_{n\geq 1}$ converges to a unique $(\bar{z}, \bar{z})$, which can be characterized as above. Observe that $\bar{z}(v) \to \lambda/\eta$ as $v \to \infty$. Also, again using (85), we have that $\bar{z}(v) \to 0$ as $v \to 0$, which is same as the limit under the greedy policy. Thus, if v_n is $o(n)$ then $\bar{B}_n(\infty)$ is also $o(n)$.

Assumption 3 implies that

$$\begin{aligned}
&E\left[E[M(B_n(\infty)) I_{\{M(B_n(\infty))\geq v_n\}} | B_n(\infty)]\right] \\
&\sim E\left[E[m(B_n(\infty)) Z I_{\{m(B_n(\infty))Z\geq v_n\}} | B_n(\infty)]\right], \\
&\sim m(n\bar{z}(v)) E\left[Z I_{\{m(n\bar{z}(v))Z\geq v_n\}}\right], \\
&\sim m(n\bar{z}(v)) E\left[Z I_{\{\bar{z}(v)^\alpha Z\geq v\}}\right].
\end{aligned}$$

Thus, by (8) in the main text, the utility rate satisfies

$$\begin{aligned}
U_n^u(v_n) &= 2\lambda n E\left[E[M(B_n(\infty)) I_{\{M(B_n(\infty))\geq v_n\}} | B_n(\infty)]\right], \\
&= 2\lambda n m(n\bar{z}(v)) E\left[Z I_{\left\{Z\geq \frac{v}{\bar{z}(v)^\alpha}\right\}}\right](1 + o(1)).
\end{aligned}$$

Assumption 2 implies that $U_n^u(v_n)/m(n)$ is $o(n)$ if $v_n = o(n)$, and is $\Theta(n)$ if v_n is $\Theta(n)$. Furthermore, $E\left[Z I_{\{Z\geq v/\bar{z}(v)^\alpha\}}\right] \to 0$ if $v \to \infty$. Thus, $U_n^u(v_n)/m(n)$ is $o(n)$ if $v_n = \omega(n)$. Consequently, the optimal policy can be computed either as

$$\sup_{v\in(0,\infty)} 2\lambda \bar{z}(v)^\alpha E\left[Z I_{\left\{Z\geq \frac{v}{\bar{z}(v)^\alpha}\right\}}\right],$$

or, in terms of $\bar{z}$, as

$$\sup_{\bar{z}\in(0,\lambda/\eta)} 2\lambda\bar{z}^{\alpha} E\left[ZI_{\left\{Z\geq\frac{v(\bar{z})}{\bar{z}^{\alpha}}\right\}}\right], \tag{88}$$

and we will solve (88).

Note that by Assumption 3, $Z = \left(\kappa^{-1}T\right)^{-\alpha}$, where T is an exponential random variable with mean one. It follows that

$$\begin{aligned} E\left[ZI_{\left\{Z\geq\frac{v(\bar{z})}{\bar{z}^{\alpha}}\right\}}\right] &= E\left[\left(\kappa^{-1}T\right)^{-\alpha} I_{\left\{\frac{\gamma\bar{z}}{v(\bar{z})^{1/\alpha}}\geq T\right\}}\right], && (89)\\ &= \kappa^{\alpha}\int_0^{\kappa\bar{z}/v(\bar{z})^{1/\alpha}} t^{-\alpha}e^{-t}dt, && (90)\end{aligned}$$

where by (86) the upper integration limit in (90) is

$$\frac{\kappa\bar{z}}{v\left(\bar{z}\right)^{1/\alpha}} = \ln\left(\frac{2\lambda}{\eta\bar{z}+\lambda}\right). \tag{91}$$

Note that the objective function of (88) is zero at $\bar{z} = 0$ and at $\bar{z} = \lambda/\eta$ (taking the limits from the left and right, respectively). Therefore, since the right side of (88) is positive for $\bar{z} \in (0, \lambda/\eta)$, any global maximizer in (88) must be a stationary point. This, in turn, implies that any global maximizer must satisfy

$$\begin{aligned} 0 &= \frac{d}{d\bar{z}}\bar{z}^{\alpha}\int_0^{\kappa\bar{z}/v(\bar{z})^{1/\alpha}} t^{-\alpha}e^{-t}dt,\\ &= \bar{z}^{\alpha}\frac{d}{d\bar{z}}\int_0^{\kappa\bar{z}/v(\bar{z})^{1/\alpha}} t^{-\alpha}e^{-t}dt + \alpha\bar{z}^{\alpha-1}\int_0^{\kappa\bar{z}/v(\bar{z})^{1/\alpha}} t^{-\alpha}e^{-t}dt.\end{aligned}$$

Equation (91) implies that

$$\frac{d}{d\bar{z}}\frac{\kappa\bar{z}}{v\left(\bar{z}\right)^{1/\alpha}} = \frac{d}{d\bar{z}}\ln\left(\frac{2\lambda}{\eta\bar{z}+\lambda}\right) = -\frac{\eta}{(\eta\bar{z}+\lambda)}. \tag{92}$$

Therefore,

$$\begin{aligned} \frac{d}{d\bar{z}}\int_0^{\kappa\bar{z}/v(\bar{z})^{1/\alpha}} t^{-\alpha}e^{-t}dt &= \left(\frac{\kappa\bar{z}}{v\left(\bar{z}\right)^{1/\alpha}}\right)^{-\alpha}\exp\left(\ln\left(\frac{\eta\bar{z}+\lambda}{2\lambda}\right)\right)\frac{d}{d\bar{z}}\frac{\kappa\bar{z}}{v\left(\bar{z}\right)^{1/\alpha}},\\ &= -\frac{v(\bar{z})\eta}{2\lambda(\kappa\bar{z})^{\alpha}},\end{aligned}$$

which implies that

$$\alpha\bar{z}^{-1}\int_0^{\kappa\bar{z}/v(\bar{z})^{1/\alpha}} t^{-\alpha}e^{-t}dt = \frac{d}{d\bar{z}}\int_0^{\kappa\bar{z}/v(\bar{z})^{1/\alpha}} t^{-\alpha}e^{-t}dt, \\ = \frac{v(\bar{z})\eta}{2\lambda(\kappa\bar{z})^{\alpha}},$$

or equivalently,

$$\frac{v(\bar{z})\,\eta}{2\lambda\kappa^{\alpha}} = \alpha\bar{z}^{\alpha-1}\int_0^{\kappa\bar{z}/v(\bar{z})^{1/\alpha}} t^{-\alpha}e^{-t}dt. \tag{93}$$

Because $\alpha \in (0,1)$, the right side of (93) is decreasing and continuous in $(0,\lambda/\eta)$, whereas by (92) the left side of (93) is increasing in the same range. Moreover, the left side of (93) vanishes at zero and the right side vanishes at λ/η. We conclude that there a unique solution z_* to (93). Finally, the optimal policy is given by $v(z_*)$ in (86) and the conclusion of the theorem follows directly by substituting in the expresson for z_* into (88).

9.3.1 Markov Dynamics of Utility-Based Process

In this subsection, we show that the actual dynamics of the utility threshold policy are equivalent to the Poisson-flow representation given by

$$\begin{aligned} B_n(t) &= B_n(0) + N_B^+\left(\lambda n\int_0^t F_{S_n(r)}(v_n)\,dr\right) - N_B^-\left(\eta\int_0^t B_n(r)\,dr\right) \\ &\quad -\tilde{N}_S^+\left(\lambda n\int_0^t \left(1-F_{B_n(r)}(v_n)\right)dr\right), \end{aligned} \tag{94}$$

$$\begin{aligned} S_n(t) &= S_n(0) + N_S^+\left(\lambda n\int_0^t F_{B_n(r)}(v_n)\,dr\right) - N_S^-\left(\eta\int_0^t S_n(r)\,dr\right) \\ &\quad -\tilde{N}_B^+\left(\lambda n\int_0^t \left(1-F_{S_n(r)}(v_n)\right)dr\right), \end{aligned} \tag{95}$$

where $N_B^+, \tilde{N}_B^+, N_S^+, \tilde{N}_S^+, N_B^-, N_S^-$ are all independent Poisson processes with unit mean. For simplicity, we shall let $n=1$ and $\lambda=\eta=1$.

Recall that the actual dynamics under the utility threshold policy are governed by the equations

$$\widetilde{B}(t) = \widetilde{B}(0) + \sum_{j=1}^{N_B^+(t)} I_{\left\{\max_{i=1}^{\widetilde{S}\left(A_{j-}^B\right)} V_{i,j}^B \le v\right\}} - \sum_{j=1}^{N_S^+(t)} I_{\left\{\max_{i=1}^{\widetilde{B}(A_{j-}^S)} V_{i,j}^S > v\right\}} \tag{96}$$

$$- N_B^- \left(\int_0^t \widetilde{B}(r_-)\, dr \right),$$

$$\widetilde{S}(t) = \widetilde{S}(0) + \sum_{j=1}^{N_S^+(t)} I_{\left\{\max_{i=1}^{\widetilde{B}(A_{j-}^S)} V_{i,j}^S \le v\right\}} - \sum_{j=1}^{N_B^+(t)} I_{\left\{\max_{i=1}^{\widetilde{S}(A_{j-}^S)} V_{i,j}^B > v\right\}}$$

$$- N_S^- \left(\int_0^t \widetilde{S}(r)\, dr \right), \tag{97}$$

where $\left\{V_{i,j}^B : i \ge 1, j \ge 1\right\}$ and $\left\{V_{i,j}^S : i \ge 1, j \ge 1\right\}$ form two independent arrays of i.i.d. random variables with CDF $F(\cdot)$. In addition, $\left\{A_j^B : j \ge 1\right\}$ is the sequence of arrival times associated with N_B^+ and $\left\{A_j^S : j \ge 1\right\}$ is the sequence of arrival times associated with N_S^+. Because of the mutual independence among the $V_{i,j}^B$s, the $V_{i,j}^S$s and all of the unit rate Poisson processes, $N_B^+, N_S^+, N_B^-, N_S^-$, it follows that the process $\left(\widetilde{B}, \widetilde{S}\right)$ is Markovian and is non-explosive because each of its coordinates (i.e. $\widetilde{B}$ and $\widetilde{S}$, respectively) can be bounded by independent infinite server queues, simply by setting $v = \infty$. Consequently, this Markov process is well defined.

We have introduced a slight inconsistency in the notation in this subsection only, since we are now using $(\tilde{B}, \tilde{S})$ to denote the actual dynamics. Ultimately, this is not important because, as our analysis in this subsection demonstrates, these are equivalent representations. The strategy consists of showing that the generators (or rate matrices) of the processes coincide.

Let $f : \mathbb{Z}^{\mathbb{N}} \times \mathbb{Z}^{\mathbb{N}} \to \mathbb{R}$ be any bounded function and note that (by standard properties of the Poisson process),

$$
\begin{aligned}
&E\left[f\left(B\left(h\right),S\left(h\right)\right)-f\left(B\left(0\right),S\left(0\right)\right)|B\left(0\right),S\left(0\right)\right]\\
&=E\left[\int_0^h\left[f\left(B\left(t_-\right)+1,S\left(t_-\right)\right)-f\left(B\left(t_-\right),S\left(t_-\right)\right)\right]dN_B^+\left(\int_0^t F_{S(r)}\left(v\right)dr\right)|B\left(0\right),S\left(0\right)\right]\\
&+E\left[\int_0^h\left[f\left(B\left(t_-\right)-1,S\left(t_-\right)\right)-f\left(B\left(t_-\right),S\left(t_-\right)\right)\right]dN_B^-\left(\int_0^t B\left(r\right)dr\right)|B\left(0\right),S\left(0\right)\right]\\
&+E\left[\int_0^h\left[f\left(B\left(t_-\right)-1,S\left(t_-\right)\right)-f\left(B\left(t_-\right),S\left(t_-\right)\right)\right]d\tilde{N}_S^+\left(\int_0^t \left(1-F_{B(r)}\left(v\right)\right)dr\right)|B\left(0\right),S\left(0\right)\right]\\
&+E\left[\int_0^h\left[f\left(B\left(t_-\right),S\left(t_-\right)+1\right)-f\left(B\left(t_-\right),S\left(t_-\right)\right)\right]dN_S^+\left(\int_0^t F_{B(r)}\left(v\right)dr\right)|B\left(0\right),S\left(0\right)\right]\\
&+E\left[\int_0^h\left[f\left(B\left(t_-\right),S\left(t_-\right)-1\right)-f\left(B\left(t_-\right),S\left(t_-\right)\right)\right]dN_S^-\left(\int_0^t S\left(r\right)dr\right)|B\left(0\right),S\left(0\right)\right]\\
&+E\left[\int_0^h\left[f\left(B\left(t_-\right),S\left(t_-\right)-1\right)-f\left(B\left(t_-\right),S\left(t_-\right)\right)\right]d\tilde{N}_B^+\left(\int_0^t \left(1-F_{S(r)}\left(v\right)\right)dr\right)|B\left(0\right),S\left(0\right)\right]\\
&+o\left(h\right)\quad\text{as}\quad h\to 0. \qquad (98)
\end{aligned}
$$

Since

$$\mathfrak{M}_0\left(t\right)=N_B^+\left(\int_0^t F_{S(r)}\left(v\right)dr\right)-\int_0^t F_{S(r)}\left(v\right)dr, \qquad (99)$$

is a martingale, we have that

$$
\begin{aligned}
&E\left[\int_0^h\left[f\left(B\left(t_-\right)+1,S\left(t_-\right)\right)-f\left(B\left(t_-\right),S\left(t_-\right)\right)\right]dN_B^+\left(\int_0^t F_{S(r)}\left(v\right)dr\right)|B\left(0\right),S(0)\right]\\
&=E\left[\int_0^h\left[f\left(B\left(t_-\right)+1,S\left(t_-\right)\right)-f\left(B\left(t_-\right),S\left(t_-\right)\right)\right]F_{S(t)}\left(v\right)dt|B\left(0\right),S(0)\right],\\
&=\left[f\left(B\left(0\right)+1,S\left(0\right)\right)-f\left(B\left(0\right),S\left(0\right)\right)\right]F_{S(0)}\left(v\right)h+o\left(h\right)\quad\text{as}\quad h\to 0.
\end{aligned}
$$

Similarly, we can evaluate each of the expectations appearing in the right side of (98); e.g., the second and third expectations are

$$
\begin{aligned}
&E\left[\int_0^h\left[f\left(B\left(t_-\right)-1,S\left(t_-\right)\right)-f\left(B\left(t_-\right),S\left(t_-\right)\right)\right]dN_B^-\left(\int_0^t B\left(r\right)dr\right)|B\left(0\right),S(0)\right] \qquad (100)\\
&+E\left[\int_0^h\left[f\left(B\left(t_-\right)-1,S\left(t_-\right)\right)-f\left(B\left(t_-\right),S\left(t_-\right)\right)\right]d\tilde{N}_S^+\left(\int_0^t \left(1-F_{B(r)}\left(v\right)\right)dr\right)|B\left(0\right),S(0)\right]\\
&=\left[f\left(B\left(0\right)-1,S\left(0\right)\right)-f\left(B\left(0\right),S\left(0\right)\right)\right]\left[B\left(0\right)+\left(1-F_{B(0)}\left(v\right)\right)\right]h+o\left(h\right)\quad\text{as}\quad h\to 0.
\end{aligned}
$$

The above calculations show that the key to verifying that two well-defined Markov jump processes with unit-size jumps are identical in law (i.e., have the same generator) is showing that the corresponding compensators of the associated point processes agree (i.e., they depend on the associated processes in the same way). The corresponding martingales that identify the compensators are, in addition to $\mathfrak{M}_0$ in (99),

$$\begin{aligned}
\mathfrak{M}_1(t) &= N_B^-\left(\int_0^t B(r)\,dr\right) - \int_0^t B(r)\,dr,\\
\mathfrak{M}_2(t) &= \tilde{N}_S^+\left(\int_0^t \left(1 - F_{B(r)}(v)\right) dr\right) - \int_0^t \left(1 - F_{B(r)}(v)\right) dr,\\
\mathfrak{M}_3(t) &= N_S^+\left(\int_0^t F_{B(r)}(v)\,dr\right) - \int_0^t F_{B(r)}(v)\,dr,\\
\mathfrak{M}_4(t) &= N_S^-\left(\int_0^t S(r)\,dr\right) - \int_0^t S(r)\,dr,\\
\mathfrak{M}_5(t) &= \tilde{N}_B^+\left(\int_0^t \left(1 - F_{S(r)}(v)\right) dr\right) - \int_0^t \left(1 - F_{S(r)}(v)\right) dr.
\end{aligned}$$

To identify the corresponding compensators of the actual dynamics of the utility threshold policy, we express these dynamics by means of a point process representation and then compute the corresponding compensators with respect to the σ-field generated by the population processes (buyers and sellers).

We need to study the compensator of the point processes

$$\sum_{j=1}^{N_B^+(t)} I_{\left\{\max_{i=1}^{\tilde{S}\left(A_{j-}^B\right)} V_{i,j}^B \le v\right\}}, \tag{101}$$

$$\sum_{j=1}^{N_S^+(t)} I_{\left\{\max_{i=1}^{\tilde{B}(A_{j-}^S)} V_{i,j}^S > v\right\}}, \tag{102}$$

$$\sum_{j=1}^{N_S^+(t)} I_{\left\{\max_{i=1}^{\tilde{B}(A_j^S-)} V_{i,j}^S \le v\right\}}, \tag{103}$$

$$\sum_{j=1}^{N_B^+(t)} I_{\left\{\max_{i=1}^{\tilde{S}(A_j^B-)} V_{i,j}^B > v\right\}}, \tag{104}$$

with respect to the filtration generated by the processes $\widetilde{B}(\cdot)$ and $\widetilde{S}(\cdot)$, which we shall denote as $\mathcal{G} = \{\mathcal{G}_t : t \geq 0\}$. We claim that for any $t, r > 0$, the conditional expectation of (101) can be expressed as

$$E\left[\sum_{j=N_B^+(t)+1}^{N_B^+(t+r)} I_{\left\{\max_{i=1}^{\widetilde{S}\left(A_j^B-\right)} V_{i,j}^B \leq v\right\}} |\mathcal{G}_t\right] = E\left[\int_t^{t+r} F_{\widetilde{S}(u)}(v)\, du|\mathcal{G}_t\right].$$

By Fubini's theorem, because every term is nonnegative in the second equality in the following display, we have that

$$\begin{aligned}
E\left[\sum_{j=N_B^+(t)+1}^{N_B^+(t+r)} I_{\left\{\max_{i=1}^{\widetilde{S}\left(A_j^B-\right)} V_{i,j}^B \leq v\right\}} |\mathcal{G}_t\right] &= E\left[\sum_{j=1}^{\infty} I_{\left\{t<A_j^B\leq t+r\right\}} I_{\left\{\max_{i=1}^{\widetilde{S}\left(A_j^B-\right)} V_{i,j}^B \leq v\right\}} |\mathcal{G}_t\right], \\
&= \sum_{j=1}^{\infty} E\left[I_{\left\{t<A_j^B\leq t+r\right\}} I_{\left\{\max_{i=1}^{\widetilde{S}\left(A_j^B-\right)} V_{i,j}^B \leq v\right\}} |\mathcal{G}_t\right].
\end{aligned}$$

Note that for each $j \geq 1$, A_j^B is a stopping time, and therefore, by the tower property in the second equality,

$$\begin{aligned}
&E\left[I_{\left\{t<A_j^B\leq t+r\right\}} I_{\left\{\max_{i=1}^{\widetilde{S}\left(A_j^B-\right)} V_{i,j}^B \leq v\right\}} |\mathcal{G}_t\right] \\
&= \int_t^{t+r} E\left[I_{\left\{\max_{i=1}^{\widetilde{S}(u-)} V_{i,j}^B \leq v\right\}} |\mathcal{G}_t, A_j^B = u\right] P\left(A_j^B \in du|\mathcal{G}_t\right), \\
&= \int_t^{t+r} E\left[E\left[I_{\left\{\max_{i=1}^{\widetilde{S}(u-)} V_{i,j}^B \leq v\right\}} |\mathcal{G}_{u-}, A_j^B = u\right] |\mathcal{G}_t, A_j^B = u\right] P\left(A_j^B \in du|\mathcal{G}_t\right), \\
&= \int_t^{t+r} E\left[F_{\widetilde{S}(u_-)}(v)\, |\mathcal{G}_t, A_j^B = u\right] P\left(A_j^B \in du|\mathcal{G}_t\right), \\
&= E\left[I_{\left\{t<A_j^B\leq t+r\right\}} F_{\widetilde{S}\left(A_{j-}^B\right)}(v)\, |\mathcal{G}_t\right].
\end{aligned}$$

Applying Fubini's theorem again and summing over j, we conclude that

$$E\left[\sum_{j=N_B^+(t)+1}^{N_B^+(t+r)} I_{\left\{\max_{i=1}^{\widetilde{S}\left(A_j^B-\right)} V_{i,j}^B \leq v\right\}} |\mathcal{G}_t\right] = E\left[\int_t^{t+r} F_{\widetilde{S}(u_-)}(v)\, dN_B^+(u)\, |\mathcal{G}_t\right].$$

However, we have that the compensator of $N_B^+(\cdot)$ is the identity and therefore

$$E\left[\sum_{j=N_B^+(t)+1}^{N_B^+(t+r)} I_{\left\{\max_{i=1}^{\widetilde{S}\left(A_j^B-\right)} V_{i,j}^B \le v\right\}} |\mathcal{G}_t\right] = E\left[\int_t^{t+r} F_{\widetilde{S}(u_-)}(v)\, du|\mathcal{G}_t\right].$$

We conclude that

$$\widetilde{\mathfrak{M}}_0(t) = \sum_{j=1}^{N_B^+(t)} I_{\left\{\max_{i=1}^{\widetilde{S}\left(A_{j-}^B\right)} V_{i,j}^B \le v\right\}} - \int_0^t F_{\widetilde{S}(u)}(v)\, du$$

is a martingale and it proves the corresponding compensator with respect to $\mathcal{G}$. A completely analogous development can be obtained for the point processes in (102)-(104), resulting in the martingales

$$\widetilde{\mathfrak{M}}_1(t) = N_B^-\left(\int_0^t \widetilde{B}(r)\, dr\right) - \int_0^t \widetilde{B}(r)\, dr,$$

$$\widetilde{\mathfrak{M}}_2(t) = \sum_{j=1}^{N_S^+(t)} I_{\left\{\max_{i=1}^{\widetilde{B}(A_{j-}^S)} V_{i,j}^S > v\right\}} - \int_0^t \left(1 - F_{\widetilde{B}(r)}(v)\right) dr,$$

$$\widetilde{\mathfrak{M}}_3(t) = \sum_{j=1}^{N_S^+(t)} I_{\left\{\max_{i=1}^{\widetilde{B}(A_{j-}^S)} V_{i,j}^S \le v\right\}} - \int_0^t F_{\widetilde{B}(r)}(v)\, dr,$$

$$\widetilde{\mathfrak{M}}_4(t) = N_S^-\left(\int_0^t \widetilde{S}(r)\, dr\right) - \int_0^t \widetilde{S}(r)\, dr,$$

$$\widetilde{\mathfrak{M}}_5(t) = \sum_{j=1}^{N_B^+(t)} I_{\left\{\max_{i=1}^{\widetilde{S}(A_{j-}^B)} V_{i,j}^B > v\right\}} - \int_0^t \left(1 - F_{\widetilde{S}(r)}(v)\right) dr.$$

This implies, by the reasoning given right after (100) and comparing $\widetilde{\mathfrak{M}}_i$ vs $\mathfrak{M}_i$ for $i \in \{0, 1, ..., 5\}$, that (94)-(95) and (96)-(97) are equivalent.

10 Extreme Value Theory and Regularly Varying Functions

In this section, we collect some useful facts about extreme value theory and show that Assumption 2 is satisfied by distributions that are subject to the application of extreme value theory. Throughout this section we assume that the V_is are i.i.d. random variables.

The central result in extreme value theory is that, for certain distributions $F(v)$, the CDF of a properly normalized version of $M(n)$ converges to a limiting CDF that is known as the generalized Pareto distribution. More precisely,

$$P\left(\frac{M(n)-b_n}{a_n}\leq x\right)\rightarrow \Xi(ax+b;\xi) \quad \text{as} \quad n\rightarrow\infty, \tag{105}$$

where

$$\Xi(x;\xi):=\exp\left(-\left(1+\xi x\right)^{-1/\xi}\right), \qquad 1+\xi x>0$$

and $\xi\in\mathcal{R}$. The case $\xi=0$ is interpreted as $G(x;0)=e^{-e^{-x}}$.

There are three domains of attraction: $\xi<0$ (Weibull), $\xi=0$ (the Gumbel), and $\xi>0$ (the Frechet). Distributions with bounded support (e.g., uniform, beta) typically belong to the Weibull domain of attraction. Distributions with finite moments of every order (often, but not always, with unbounded support) belong to the domain of attraction of the Gumbel distribution (e.g., exponential, gamma, normal, lognormal). Distributions with power-law-like decaying tails belong to the domain of attraction of the Frechet distribution (e.g., Pareto, Cauchy). The CDFs corresponding to the Weibull, Gumbel and Frechet domains of attraction are readily available by evaluating the corresponding values of ξ in $G(x;\xi)$.

Define $\bar{F}(x)=1-F(x)$ and let $w(F)=\sup\{x:F(x)<1\}$ be the upper endpoint of the support of F. If convergence to a generalized Pareto distribution with parameter ξ holds (i.e. if extreme value theory applies), then the corresponding constants can be computed as follows:

$$\text{Weibull}(\xi<0): \ a_n=w(F)-\bar{F}^{-1}(n^{-1}),\ b_n=w(F),\ a=-1/\xi,\ \text{and}\ b=1/\xi, \tag{106}$$

$$\text{Gumbel}(\xi=0): \ a_n=\frac{1}{\bar{F}(b_n)}\int_{b_n}^{w(F)}\bar{F}(t)\,dt,\ b_n=\bar{F}^{-1}(n^{-1}),\ a=0,\ \text{and}\ b=0, \tag{107}$$

$$\text{Frechet}(\xi>0): \ a_n=\bar{F}^{-1}(n^{-1}),\ b_n=0,\ a=1/\xi,\ \text{and}\ b=-1/\xi, \tag{108}$$

where $\bar{F}^{-1}(\cdot)$ is the inverse of $\bar{F}(\cdot)$.

We now switch our attention to $E[M(n)]$. Theorem 2.1 in Pickands (1968) shows that the first moment converges as long as $E[V^{1+\delta}] < \infty$ for some $\delta > 0$, which is satisfied for all the concrete examples explored in this paper. This result implies that

$$E[M(n)] \sim b_n + a_n \mu \ , \tag{109}$$

where

$$\mu = \int_{-\infty}^{\infty} x \ d\Xi(ax + b; \xi) \text{ for } \xi \in (-\infty, \infty) \tag{110}$$

is the mean of the distribution $\Xi(\cdot; \xi)$ computed according to the limiting value of the domain of attraction. The symbol $x_n \sim y_n$ is shorthand for $\frac{x_n}{y_n} \to 1$ as $n \to \infty$. The mean in (110) is $\mu = \gamma = 0.5772\ldots$ if $\xi = 0$, which is Euler's constant; $\mu = \Gamma(1-\xi)$, where $\Gamma(a)$ is the gamma function evaluated at $a > 0$, if $\xi \in (0, 1)$, which holds in the Frechet case if $E(V^{1+\delta}) < \infty$; and $\mu = -\Gamma(1 - \xi)$ if $\xi < 0$.

We conclude this section by stating the relationship between extreme value theory and slowly varying functions, thereby showing that the extreme value distributions satisfy Assumption 2.

Fact 1. If F belongs to the domain of attraction of the Gumbel distribution (i.e. $\xi = 0$) then b_n is slowly varying at infinity.

Fact 1 follows from Proposition 0.10 of Resnick (1987), combined with the first exercise on p. 35 of Resnick (1987).

Fact 2. (Resnick 1987, p. 52) If F belongs to the domain of the Gumbel law then $a_n = o(b_n)$ as $n \to \infty$.

Fact 3. (Resnick 1987, p. 54) F belongs to the domain of attraction of the Frechet distribution (i.e. $\xi > 0$) if and only if $a_n = 0$ and $b_n = \bar{F}^{-1}(1/n)$ as $n \to \infty$ and $F(\cdot)$ is regularly varying with index $-\xi$.

Fact 4. (Resnick 1987, p. 59) F belongs to the domain of attraction of a Weibull distribution if and only if $w(F) < \infty$ and $\bar{F}\left(w(F) - x^{-1}\right)$ is regularly varying with index ξ.

Facts 1 and 2 imply that $m(\cdot)$ is slowly varying (i.e. regularly varying with index 0) for distributions in the Gumbel domain of attraction. Fact 3 implies that $m(\cdot)$ is regularly varying with index $\xi \in (0,1)$ for distributions in the domain of attraction of the Frechet law. Finally, Fact 4 implies that $m(\cdot)$ is slowly varying for distributions in the domain of attraction of the Weibull law.