Asymptotic Expansion for the Cumulants of $GI/GI/1$ Queue under Heavy Traffic

Jose Blanchet and Henry Lam

We provide an asymptotic expansion for the cumulants of the steady-state waiting time distribution of a $GI/GI/1$ queue under heavy traffic, up to the highest available moment allowed by the service time and interarrival time distributions. This problem is also of interest in risk theory, given that it is well known (see for example Asmussen (2003)) that the expression of the tail distribution of the steady-state delay in a $GI/GI/1$ coincides precisely with the ruin probability of an insurance portfolio under the so-called renewal model. The tail of the maximum of a random walk is also related to sequential analysis and statistics (see for example Siegmund (1985)) because of its equivalence to a one sided sequential probability ratio test.

Indeed, the equivalence between the steady-state distribution of the $GI/GI/1$ queue and the maximum of a random walk is the first step towards our expansion. We let T_n be the service time of the n-th customer, A_n be the interarrival time between the n-th and $(n+1)$-th customers, where T_n and A_n are i.i.d. with $ET_n < EA_n$. The waiting time of the n-th customer, denoted by W_n, converges weakly to $M = \max_{n\geq 0} S_n$ where $S_n = \sum_{k=0}^{n} V_k$ is a negatively drifted random walk with increment $V_n = T_n - A_n$. Let N be the highest integer moment available for V_n (i.e. $E\left|V_n\right|^N < \infty$ but $E\left|V_n\right|^{N+1} = \infty$).

Our main goal is to find an asymptotic expansion for all the available cumulants of M as $\mu := EV_n \nearrow 0$. Because of both its applications and theoretical interest, there have been substantial studies on this problem as well as a closely related counterpart on the limiting overshoot $R(\infty) := \lim_{b\to\infty} R(b)$, where $R(b) = S_{\tau(b)} - b$ and $\tau(b) = \inf\{n > 0 : S_n > b\}$, defined for positively drifted random walk. Siegmund (1979) first obtained an expansion for the first moment of M in terms of the amount of exponential tilting from the zero-drift random walk, assuming finite exponential moment, up to the first-order term. The coefficients involve the moments of ladder heights, and using a result of Lai (1976) he found an integral representation for the coefficients. Woodroofe (1979) and Siegmund (1985) then obtained an integral representation for the moment generating function of $R(\infty)$. Chang (1992) presented an asymptotic refinement of the moments of ladder height by studying the covariance between $\tau(b)$ and $R(b)$, which led to obtaining another term in the expansion of the first moment of M. Chang and Peres (1997) used complex analysis to get a complete expansion of the first moment of ladder height as the drift goes to 0 for Gaussian random walks; the coefficients are expressed in terms of evaluations of the Riemann zeta function. Blanchet and Glynn (2006) considered light-tailed random walk and generated a full expansion for the moment generating function of $R(\infty)$ in terms of the amount of exponential tilting from the zero-drifted walk. They did so by modifying the integral obtained by Woodroofe (1979) and Siegmund (1985) and using short-time asymptotics for a suitably defined Cauchy process in the analysis. Finally, Janssen and Leeuwaarden (2007) used Bateman series to get the expansion for all the cumulants of M for Gaussian random walks. The result of our paper is a generalization of Janssen and Leeuwaarden (2007) and Blanchet and Glynn (2006) to all random walks with finite moment conditions.

We rely heavily on the Spitzer-Baxter identity (see for example Asmussen (2003)):

$$\log Ee^{-\alpha M} = \sum_{n=1}^{\infty} \frac{1}{n}(Ee^{-\alpha S_n^+} - 1) \tag{1}$$

Our strategy is to expand $\log Ee^{-\alpha M}$ as a "double series" in α and μ. Looking at the power series in α, the n-th cumulant of M is equal to $(-1)^n n! a_n$ where a_n is the coefficient of α^n. If we let the system approaches heavy traffic by letting $\mu \nearrow 0$. a_n are then expressed as power series (with possibly negative powers) of μ.

To do so we first derive an integral representation for (1) in the spirit of Woodroofe (1979), Siegmund (1985) and Blanchet and Glynn (2006):

Theorem 1. *Suppose* V_k *are i.i.d. satisfying* $\mu = EV_k < 0$ *and* $\sigma^2 = var(V_k) < \infty$. *Also let* $g(\lambda) = Ee^{i\lambda V_k}$ *be the characteristic function of* V_k. *Spitzer's identity can be written in the form*

$$\log Ee^{-\alpha M} = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{1}{\alpha+i\lambda} - \frac{1}{i\lambda}\right)\log\left(\frac{1-g(\lambda)}{-i\mu\lambda+\sigma^2\lambda^2/2}\right)d\lambda - \log\left(1-\frac{\sigma^2\alpha}{2\mu}\right) \tag{2}$$

for $\alpha \geq 0$.

Remarkably, there is a duality relation between M and $R(\infty)$ in the sense that the integral representation we obtain for M has exactly the same analytical form as $R(\infty)$. The sole difference is the sign of the drift: for $Ee^{-\alpha M}$ one has to impose the condition $\mu < 0$ while for $Ee^{-\alpha R(\infty)}$ one has $\mu > 0$. Simple complex analysis shows that the second term disappears for $\mu > 0$ (whence recovering the expression obtained in Blanchet and Glynn (2006)).

We let $\mu \nearrow 0$ through a parametrization $V_n^{(\mu)} = X_n + \mu$, where X_n are i.i.d. r.v. with mean zero. From now on we let $g(\cdot)$ be the characteristic function of X_n and $g_\mu(\lambda) := g(\lambda)e^{i\lambda\mu}$ be the characteristic function of $V_n^{(\mu)}$. Rewriting (2) gives

$$\log Ee^{-\alpha M} = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{1}{\alpha+i\lambda} - \frac{1}{i\lambda}\right)\log\left(\frac{1-g_\mu(\lambda)}{-i\mu\lambda+\sigma^2\lambda^2/2}\right)d\lambda - \log\left(1-\frac{\sigma^2\alpha}{2\mu}\right) \tag{3}$$

for $\alpha \geq 0$. We assume that $\inf_{|\lambda|>\delta}|1-g(\lambda)| > 0$ for any $\delta > 0$ i.e. X_n is nonlattice.

Our expansion methodology is outlined as follows:

Step 1: Expand the second term in (3) around $\alpha = 0$ to get

$$-\log\left(1-\frac{\sigma^2\alpha}{2\mu}\right) = \sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{\sigma^2}{2\mu}\right)^n \alpha^n$$

Step 2: From now till Step 4 consider $g(\cdot)$ with super-polynomial decay (see conditions in Lemma 1). Center the first integral in (3) at $\mu = 0$:

$$\begin{aligned}
& -\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{1}{\alpha+i\lambda} - \frac{1}{i\lambda}\right)\log\left(\frac{1-g_\mu(\lambda)}{-i\mu\lambda+\sigma^2\lambda^2/2}\right)d\lambda \\
= & -\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{1}{\alpha+i\lambda} - \frac{1}{i\lambda}\right)\log\left(\frac{1-g_\mu(\lambda)}{1-g(\lambda)}\frac{\sigma^2\lambda^2/2}{-i\mu\lambda+\sigma^2\lambda^2/2}\right)d\lambda \\
& -\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{1}{\alpha+i\lambda} - \frac{1}{i\lambda}\right)\log\left(\frac{1-g(\lambda)}{-i\mu\lambda+\sigma^2\lambda^2/2}\right)d\lambda
\end{aligned}$$

and justify the following representation:

$$\begin{aligned} & -\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{1}{\alpha+i\lambda}-\frac{1}{i\lambda}\right)\log\left(\frac{1-g_\mu(\lambda)}{1-g(\lambda)}\frac{\sigma^2\lambda^2/2}{-i\mu\lambda+\sigma^2\lambda^2/2}\right)d\lambda \\ = & -\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{1}{\alpha+i\lambda}-\frac{1}{i\lambda}\right)\log\left(1-\left(\frac{i\lambda}{\Psi(\mu)+i\lambda}\right)H(\mu,\lambda)\right)d\lambda \\ \sim & -\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{1}{\alpha+i\lambda}-\frac{1}{i\lambda}\right)\sum_{j=1}^{\infty}\left(\frac{i\lambda}{\Psi(\mu)+i\lambda}\right)^j\sum_{k=1}^{N}h_{k,j}(\lambda)\mu^k d\lambda \end{aligned}$$

where $\Psi(\mu)=2\mu/\sigma^2$,

$$H(\mu,\lambda)=-\frac{2i\mu}{\sigma^2\lambda}+1-\frac{1-g_\mu(\lambda)}{1-g(\lambda)}$$

and

$$H(\mu,\lambda)^j=\sum_{k=j}^{N}h_{k,j}(\lambda)\mu^k+o(\mu^N)$$

The following lemma provides conditions for such an expansion:

Lemma 1. *Suppose* $|g(\lambda)|=o(\lambda^{-p})$ *for any* $p>0$ *as* $\lambda\to\infty$ *(this condition is satisfied if the* V_k*'s have an infinitely differentiable density). Then* $H(\mu,\lambda)$ *possesses the expansion*

$$H(\mu,\lambda)=H(\mu,\lambda)=\sum_{k=1}^{\infty}l_k(\lambda)\frac{\mu^j}{j!}$$

where $l_k(\lambda), k=1,\dots$ *are bounded on* $\mathbb{R}/\{0\}$*. In particular, if we define*

$$H_n(\mu,\lambda)=\sum_{k=1}^{n}l_k(\lambda)\frac{\mu^j}{j!}$$

Then $\sup_{\lambda\in\mathbb{R}/\{0\}}|H_n(\mu,\lambda)|\to 0$ *as* $\mu\to 0$*.*

Step 3: Analyze

$$J_m(\alpha,\theta,f)=-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\alpha}{(\alpha+i\lambda)i\lambda}\left(\frac{i\lambda}{\theta+i\lambda}\right)^m f(\lambda)d\lambda$$

and in particular

$$J_0(\alpha,f)=-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\alpha}{(\alpha+i\lambda)i\lambda}f(\lambda)d\lambda$$

The following provides an expansion of J_m:

Lemma 2. *If* $f(\cdot)$ *is* $(m+1)$*-th differentiable at 0, then*

$$J_m(\alpha,\theta,f)=J_0\left(\alpha,\sum_{s=0}^{n}(1-(1+\theta T)^m)^s f\right)+I_m\left(\alpha,\theta,\sum_{s=0}^{n}(1-(1+\theta T)^m)^s f\right)+R_{m,n}(\alpha,\theta)$$

for $n\le m-1$*, where* $R_{m,n}(\alpha,\theta)$ *can be expanded as*

$$R_{m,n}(\alpha,\theta)=\sum_{t=1}^{m}r_{m,n,t}(\theta)\alpha^t+r_{m,n+1}(\alpha,\theta)$$

$r_{m,n,t}(\theta) = o(\theta^{n-t+1})$ as $\theta \to 0$ and $r_{m,n+1}(\alpha, \theta) = o(\alpha^m)$ as $\alpha \to 0$ uniformly in $\theta < 0$. Here T is defined as the operator

$$T_j f(\lambda) := \frac{f(\lambda) - \sum_{r=0}^{k-1} f^{(2r)}(0)\lambda^{2r}/(2r)!}{\lambda^{2k}}$$

for symmetric function $f(\cdot)$ that is $(2k)$-th differentiable at 0, and I_m is defined by

$$I_m(\alpha, \theta, f) = \sum_{k=1}^{m} \left(-\frac{\alpha}{\theta}\right)^{m-k} \left(1 - \frac{\alpha}{\theta}\right)^{-m} \sum_{l=0}^{k} \frac{1}{l!} f^{(l)}(0)(-\alpha)^l$$

for function $f(\cdot)$ that is k-th differentiable at 0.

Step 4: With the expression in terms of series of J_0, differentiate with respect to α to obtain the expansion using Cauchy process short time asymptotic (see Blanchet and Glynn (2006)).

Step 5: For X_n that possesses general characteristic function and is nonlattice, smooth its characteristic function by letting $Y_n = X_n + \delta Z_n$ where Z_n is a standard Gaussian r.v.. Expand cumulants of Y_n using Steps 1 to 4. Let $\delta \searrow 0$.

From an analytic point of view we believe that the expansion we obtain in this paper is close to the best possible, since as in the case of Edgeworth expansion for i.i.d. sum, the moment condition on the random walk increments limits the available terms in the expansion of M. Moreover, we will show that for stable-law random walk even the first moment of M does not exist. This provides a consistent transition to the finite variance case.

References

1. Asmussen, S. (2003), *Applied Probability and Queues*, Springer.
2. Blanchet, J. and Glynn, P. (2006), Complete Corrected Diffusion Approximations for the Maximum of Random Walk, *Ann. Appl. Prob.*, **16**(2), 951-953.
3. Chang, J. (1992), On moments of the first ladder height of random walks with small drift, *Ann. Appl. Prob.*, **2**(3), 714-738.
4. Chang, J., and Peres, Y. (1997) Ladder heights, Gaussian random walks and the Riemann zeta function, *Ann. Prob.*, **25**(2), 787-802.
5. Janssen, A. J. E. M. and van Leeuwaarden, J. S. H. (2007), On Lerch's transcendent and the Gaussian random walk, *Ann. Appl. Prob.*, **17**(2), 421-439.
6. Janssen, A. J. E. M. and van Leeuwaarden, J. S. H. (2007), Cumulants of the maximum of the Gaussian random walk, *Stoc. Proc. Appl.*, **117**(12), 1928-1959.
7. Lai, T. (1976), Asymptotic moments of random walks with applications to ladder variables and renewal theory, *Ann. Prob.*, **4**(1), 51-66.
8. Siegmund D. (1979), Corrected diffusion approximations in certain random walk problems, *Adv. Appl. Prob.*, **11**(4), 701-719.
9. Siegmund D. (1985), *Sequential Analysis*, Springer-Verlag, New York.
10. Woodroofe, M. (1979), Repeated likelihood ratio tests, *Biometrika*, **66**(3), 453-463.