# Stochastic models and control for electrical power line temperature

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**Abstract**

In this paper we present a rigorous analysis of the evolution of the temperature of a power line under stochastic exogenous factors such as ambient temperature. We present a solution to the resulting stochastic heat equation and we propose a number of control algorithms designed to maximize delivered power under chance constraints used to limit the probability that a line exceeds its critical temperature.

## 1 Introduction

A critical aspect in the secure operation of electrical power transmission systems concerns how to keep power lines at safe temperature levels. When a power line overheats it becomes exposed to a number of risk factors. If the overheating is severe the physical/mechanical attributes of the line may be compromised, rendering it unusable. Under less severe overheating the line may sag, thus bringing it into proximity with other objects, and thereby potentially causing a contact or arc which will trip the line. If overheating is determined, the line may be protectively tripped (be taken out of service). In any of these cases the line will become unavailable, and the power flow on that line will instead become rerouted, in a nontrivial fashion that obeys the laws of physics. This rerouting may possibly cause other lines to become overloaded. In a failure scenario of a transmission system, this sequence of events may result in a *cascade* resulting in a large-scale blackout. The Northeast U.S.-Canada blackout of 2013 produced precisely this type of event, see [23].

The temperature of a power line primarily depends on the amount of (active, or real) power flowing on that line (we refer the reader to [2], [3] or [12] for background on power engineering). However, high-voltage power lines are uninsulated and exposed to numerous exogenous factors, such as in particular wind and ambient temperature, among many. All of these factors can, and do, influence power line temperature. IEEE Standard 738 [11] amounts to a deterministic codification of the impact of a very large number of such factors, starting from a base model that relies on the classical heat equation. Even though this is a very thorough approach, an examination of the standard highlights the potential for misestimation due to erroneous, missing or variable data. The previously mentioned report [23]

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describes instances during the 2013 cascade where incorrect calibration of a power line led to unexpected tripping which contributed to system instability. A somewhat more nuanced analysis of power line temperature based on the heat equation is given in [1].

Power lines in transmission systems are quite long (on the order of 100 km long for shorter lines), and thus local and random variations in geographically-dependent exogenous factors (such as direction and strength of wind) leading to heating or cooling of lines can and do manifest themselves, and may well be persistent across multiple generator dispatch intervals (reviewed below). Various exogenous factors are in fact considered in IEEE 738, albeit in a simplified, deterministic manner. A comprehensive model that accounts for short-term (time dependent) exogenous variability over a large spatial domain may prove challenging; though in future work we plan to address this point.

In this paper we use a generic stochastic model to describe the impact of random exogenous factors which affect the power line temperature and to guide the selection of critical parameters in Optimal Power Flow (OPF) computations (as we shall explain in the next subsection). In a previous work [6] we focused on time-dependent stochasticity of exogenous factors. In this paper, instead, we assume that randomness is primarily of a spatial nature and ignore the time component. This viewpoint can be justified as follows. First, as discussed above, power lines are exposed to local spatial effects the precise nature of which is difficult to ascertain in real time. Furthermore, power engineering practice solves OPF equations every fifteen minutes (and even more frequently under proposed schemes) so as to set generator output levels, and, as a result, power-flows on all transmission lines during the next time window. Recourse (or feedback) is facilitated by observations obtained by the end of each window, and we ultimately capture this recourse situation in our modeling setting in Section 4. We thus view the impact of spatial randomness of exogenous factors (which may in fact be persistent across several time windows), as more significant as that due to time-related randomness.

### 1.1 Line limits in the context of Optimal Power Flow (OPF)

Here we briefly review how OPF is used in current power engineering practice and how our work here ultimately relates to guiding certain upper bound parameters in the evaluation of OPF equations. For a review see e.g. [15].

The OPF (or "*economic dispatch*") computation is now performed as often as every fifteen minutes, and is used to set the output of generators (in the next fifteen-minute window) so as to minimize cost while meeting estimates of demands in a safe manner, that is to say, without exceeding flow limits on the transmission lines in a systematic manner. Real-time, random fluctuations in demands are handled through automatic adjustment of generator output. Such automatic, short-term actions may cause power lines to exceed their flow limits in moderate form and for brief periods of time; however such overages are tolerated unless sustained over longer periods of time. See [2], [3].

Using the linearized "DC" model of power flows, which is common in the case of OPF, the optimization problem that is solved in power engineering practice is of the form:

$$\min \sum_{i \in G} w_i(P_i) \tag{1a}$$

$$\text{s.t. } B\theta \ - \ P \ = \ 0 \tag{1b}$$

$$|y_{ij}(\theta_i - \theta_j)| \ \leq \ u_{ij} \tag{1c}$$

$$P_i \ = \ \text{-(load at } i\text{), if } i \notin G \tag{1d}$$

$$\theta \ \in \ \mathbb{R}^n. \tag{1e}$$

In this formulation we consider a transmission system with $n$ buses in total. We denote by $G$ the set of buses attached to generators. The vector $P \in \mathbb{R}^n$ indicates net output at each bus. Thus when $i \in G$, $P_i$ is the output for the generator at $i$. When bus $i$ is a load bus (i.e., $i \notin G$) the quantity $P_i$ is fixed in equation (1d) to minus the load at $i$, possibly equal to zero. Matrix $B$ is the so-called *network susceptance matrix* and for a bus $i$, $\theta_i$ is the phase angle at bus $i$, a variable in the computation. Equation (1b) represents the (linearized) relationship between net power output levels and phase angles; under the DC model the power flow from $i$ to $j$ on line $ij$ equals

$$y_{ij}(\theta_i - \theta_j)$$

where $y_{ij} > 0$ is the *susceptance* of line $ij$, a physical parameter. Finally, the quantity $u_{ij}$ is the power transmission limit for line $ij$. Thus, equation (1c) states that the flow on $ij$, in absolute value does not exceed the limit on that line. Finally, the objective to be minimized is the overall cost of generation; at generator bus $i$ the function $w_i(\cdot)$ is a convex quadratic on the output $P_i$. Other constraints may be present, e.g. upper and lower bounds on the generator outputs $P_i$.

*An important part of formulation (1), of interest to us in the context of this paper, is the determination of the line limits $u_{ij}$.* If these are set to large values then lines may become overheated and at risk of tripping; on the other hand if the $u_{ij}$ are set too low the cost of operation (i.e. the objective value (1a) at optimality) may become too large or we may even obtain an infeasible optimization problem. In current power engineering practice a simple procedure is employed:

(a) Before each OPF period, if a line $u_{ij}$ is deemed to have become overheated (in the previous time period) or at risk of becoming overheated in the next time period, the $u_{ij}$ quantity is decreased to a “safe” value.

(b) The adjustment in (a) is typically done by reducing $u_{ij}$ to one of a (very) small set of precomputed values.

We stress that (a) and (b) are performed one line at a time. Effectively, one is setting the quantity $u_{ij}$ to act as a proxy for the maximum power flow that line $ij$ can safely sustain in the next time window; but of course, the computation in (1) may set the actual power flow to a lower value than $u_{ij}$.

At this point we note that there are several procedures that one could employ in order to incorporate into OPF the stochasticity of line temperatures. Let us denote by $\boldsymbol{B_{ij}} = \boldsymbol{B_{ij}}(y_{ij}(\theta_i - \theta_j))$ the (random) event that in the next time window line $ij$ will reach a critical temperature given its flow value $y_{ij}(\theta_i - \theta_j)$. This event depends on exogenous events; we consider a specific model below. Using $\mathcal{P}$ to denote probability, one can then consider the following variations of (1):

$$\min \sum_{i \in G} w_i(P_i) \tag{2a}$$

$$\text{s.t.constraints (1b) - (1e),} \tag{2b}$$

$$\mathcal{P}\left(\bigcup_{ij} \boldsymbol{B_{ij}}\right) \ \leq \ \epsilon, \tag{2c}$$

and

$$\min \sum_{i \in G} w_i(P_i) \tag{3a}$$

$$\text{s.t.constraints (1b) - (1e),} \tag{3b}$$

$$\mathcal{P}\left(\boldsymbol{B_{ij}}\right) \ \leq \ \epsilon, \quad \forall \ ij. \tag{3c}$$

Either (2) or (3) are risk-aware variants of OPF; with (2) likely much more conservative. Either one of these optimization problems is, likely, quite challenging due to the nature of the stochasticity (i.e. the "chance constraint" (3c) may be quite nontrivial). And in fact either model amounts to a significant departure from current practice.

*In this paper we follow a different approach that adjusts process (a)-(b). Our goal will simply be to refine the current practice of resetting the $u_{ij}$ with an intelligent computation that is risk-aware. In other words the OPF problem to be solved will still be of the form (1), albeit with carefully constructed quantities $u_{ij}$.*

Our approach will involve the computation of a general solution to a stochastic heat equation, with an explicit spatial dependence on stochastic effects, and suggest several control mechanisms relying on so-called "chance constraints" to maximize delivered power while maintaining an acceptable level of risk.

Our particular model for stochastics of line temperature and our specific analysis are motivated by the OPF application. The study of stochastic variants of the heat equation is a classical subject. See e.g. [21], [22] and citations therein.

This paper is organized as follows. Sections 2 and 3 contain our developments on the heat equation. Some numerical tests are presented in Section 3.1. Finally, in Section 4 we consider a two-stage adaptive control strategy that employs information acquired during the OPF interval in order to modify the line limits. This proposed scheme does deviate from current practice, albeit in a moderate manner.

## 2 Formulation

We now focus on a particular power line on the time domain $[0, \tau]$. The line is modeled as a one-dimensional object parameterized by $x$, $0 \leq x \leq L$, used to model the spatial dependence of temperature on an exogenous stochastic factor. Let

- $I = I(t)$ denote the current of that line at time $t$, with the dependence on $t$ highlighted so as to allow for control actions.
- $T(x, t)$ the temperature at location $x$ and at time $t$.

We note that, to first order (and under the DC model) the power flow $P$, and $I$ are related through the equation $P = I\ V$, where $V$ is the voltage, which under the so-called "per-unit system" will be assumed to be approximately 1.0 and constant. The *heat equation* states (see, for instance, [11], [1]):

$$\frac{\partial T(x,t)}{\partial t} = \kappa \frac{\partial^2 T(x,t)}{\partial x^2} + \alpha I^2(t) - \nu(T(x,t) - T^{ext}(x,t)), \tag{4}$$

where $\kappa \geq 0$, $\alpha \geq 0$ and $\nu \geq 0$ are (line dependent) constants, and $T^{ext}(x,t)$ is the ambient temperature at $(x,t)$.

In order to account for stochasticity, we model $T^{ext}(x,t) = G(\mathbf{h}(x))$ where $\mathbf{h}(\cdot)$ denotes a random process, in particular, for any given $x$, $\mathbf{h}(x)$ is a random variable with distribution that is either known or can be estimated (in what follows boldface will be used to denote uncertain quantities).

We shall assume that every realization of $\mathbf{h}(\cdot)$ is locally integrable (i.e. the integral of $|\mathbf{h}(\cdot)|$ exists and is finite on compact intervals).

Consequently, (4) takes the form,

$$\frac{\partial T(x,t)}{\partial t} = \kappa \frac{\partial^2 T(x,t)}{\partial x^2} + \alpha I^2(t) - \nu(T(x,t) - G(\mathbf{h}(x))). \tag{5}$$

We shall further assume $\kappa = 0$. This is consistent with the use of the heat equation in [11], [1]; it is justified by noting that propagation in the time domain is much faster than in the spatial domain. We therefore obtain:

$$\frac{\partial T(x,t)}{\partial t} = \alpha I^2 - \nu(T(x,t) - G(\mathbf{h}(x)), \tag{6}$$

and we will account for the randomness of exogenous conditions in the spatial domain in an average, or aggregated manner. More precisely, integrating on both sides with respect to $x$, and dividing by $L$, we have

$$\begin{aligned} \frac{1}{L}\int_0^L \frac{\partial T(x,t)}{\partial t} dx &= \alpha I^2(t) - \frac{\nu}{L}\int_0^L T(x,t)dx \\ &+ \frac{\nu}{L}\int_0^L G(\mathbf{h}(x))dx. \end{aligned} \tag{7}$$

Denoting by $\mathbf{H}(t)$ average internal temperature along the line at time $t$, by $\mathbf{R}$ the average ambient temperature along the line, i.e.,

$$\mathbf{H}(t) \triangleq \frac{1}{L}\int_0^L T(x,t)dx, \quad \mathbf{R} \triangleq \frac{1}{L}\int_0^L G(\mathbf{h}(x))dx,$$

we therefore have

$$\frac{d\mathbf{H}(t)}{dt} = \frac{d}{dt}\frac{1}{L}\int_0^L T(x,t)dx = \frac{1}{L}\int_0^L \frac{\partial T(x,t)}{\partial t} dx.$$

Then (7) becomes:

$$\frac{d\mathbf{H}(t)}{dt} = \alpha I^2(t) - \nu \mathbf{H}(t) + \nu \mathbf{R}, \tag{8}$$

with solution

$$\mathbf{H}(t) = \int_0^t e^{-\nu(t-s)}(\alpha I^2(s) + \nu\mathbf{R})ds + Ce^{-\nu t} \tag{9}$$
$$= \int_0^t e^{-\nu(t-s)}\alpha I^2(s)ds + \mathbf{R}(1-e^{-\nu t}) + Ce^{-\nu t},$$

where

$$C = \mathbf{H}(0) = \frac{1}{L}\int_0^L T(x,0)dx.$$

We note that the quantity $\mathbf{R}$ is *not* observed – however we can assume that its distribution can be estimated. We are interested in control schemes that vary $I(t)$ in response to observed conditions. As criterion for stability, we will enforce the chance-constraint [7], [17]

$$\mathcal{P}\left(\max_{t\in[0,\tau]} \mathbf{H}(t) > k\right) \le \epsilon, \tag{10}$$

where $k > 0$ is a given limit and $\epsilon > 0$ is small.

# 3 Constant $I(t)$, $t \in [0,\tau]$

The case where $I(t)$ is constant in the time window of interest is of special interest because of its simplicity. We are interested in computing those values $\breve{I}$ such that setting $I(t) = \breve{I}$ for $0 \le t \le \tau$ satisfies (10). From the closed-form solution above we obtain

$$\mathbf{H}(t) = ((\alpha/\nu)\breve{I}^2 + \mathbf{R})(1-e^{-\nu t}) + Ce^{-\nu t}. \tag{11}$$

Now, let us assume that $G(x) \ge 0$, which implies that $\mathbf{R} \ge 0$.

It follows from (11) together with (8) that

$$\frac{d\mathbf{H}(t)}{dt} = \alpha I^2(t) - \nu((\alpha/\nu)\breve{I}^2 + \mathbf{R})(1-e^{-\nu t}) + vCe^{-\nu t} + \nu\mathbf{R},$$
$$= \alpha\breve{I}^2 e^{-\nu t} + v\mathbf{R}e^{-\nu t} + vCe^{-\nu t}.$$

In other words, $\mathbf{H}'(t) > 0$ if one assumes that $\breve{I}^2 + C \ge 0$, which in turn will always hold if $\mathbf{H}(0) = C \ge 0$. Consequently, if one assumes that $C = \mathbf{H}(0) < k$, it follows that (10) is equivalent to

$$\mathcal{P}(\mathbf{H}(\tau) > k) \le \epsilon.$$

Using (11) this implies

$$\breve{I}^2 \le \frac{\nu}{\alpha}\frac{k - Ce^{-\nu\tau} - r_\epsilon(1-e^{-\nu\tau})}{1-e^{-\nu\tau}}, \tag{12}$$

where $r_\epsilon$ is the $\epsilon$-quantile of $\mathbf{R}$ (i.e. $r_\epsilon$ is the smallest $x$ such that $\mathcal{P}(\mathbf{R} > x) \le \epsilon$). As discussed above we assume that the distribution of $\mathbf{R}$ is known, and consequently the bound in (12) is computable.

For future use, we use $L(\tau, k)$ to denote the right-hand side of expression (12).

Now consider an entire grid where several lines are assumed to be thermally stressed. We can then compute the upper bound on the current for each such line that is implied by computation in (12). These values can then be used the OPF computation (generator dispatch) at time $t = 0$.

### 3.1 Numerical examples

We next perform some numerical experiments designed to understand the behavior of the threshold expression (12) in a range of numerical values which are informed by actual IEEE Standards.

#### 3.1.1 Calibration of parameters

Expression (12) contains two line-dependent parameters, $\alpha$ and $\nu$, which need to be calibrated. Also the constant $C = \mathbf{H}(0)$ needs to be estimated. For the purpose of computing the line-dependent parameters, we use some formulas provided in the IEEE Standard 738 [11] applied to an ACSR (aluminum conductor steel reinforced) line. We include these parameters here because they give a sense of the physical measurements involved in ultimately evaluating the various parameters in the underlying model, and because we ultimately wish to show that the numerical values that we obtain make practical sense.

The formulas for $\alpha$ and $v$ require that we introduce some notation and parameters arising in the standard:

$\rho_f$: Density of air (1.029 $kg/m^3$).
$V_w$: Speed of air stream at conductor (0.61 $m/s$).
$\mu_f$: Dynamic viscosity of air (0.0000204 $Pa \cdot s$).
$k_f$: Thermal conductivity of air (0.0295 $W/(m -^o C)$).
$K_{angle}$: Wind direction factor ($90^o$ ).
$mC_{ACSR}$: Total heat capacity of conductor (ACSR in our case).
$q_c$: Convected heat loss rate per unit length per unit temp.
$q_r$: Radiated heat loss rate per unit length per unit temp.
$q_s$: Heat gain rate from the sun per unit temp.
$R_a$: AC resistance of conductor at average temperature ($5.34 \times 10^{-4}$ $\Omega/m$).
$mC_{ACSR}$: Total heat capacity of conductor (ACSR in our case).
$D$: Conductor diameter (.01 $m$).
$T^{ext}$: External temperature.

Thus (as per the standard) we have that

$$\text{low wind speed per temp.}: q_{c1} = \left(1.01 + 0.0372\left(\frac{D\rho_f V_w}{\mu_f}\right)^{0.52}\right) k_f K_{angle},$$

$$\text{high wind speed per temp.}: q_{c2} = 0.0119\left(\frac{D\rho_f V_w}{\mu_f}\right)^{0.6} k_f K_{angle},$$

$$q_c = \max(q_{c1}, q_{c_2}).$$

The IEEE Standard 738 [11] provides equations for $q_r$ and $q_s$, but these two quantities typically approximately offset each other (CITATION) and hence $q_r \approx q_s$, thereby obtaining that

$$\alpha = \frac{R_a}{mC_{ACSR}}, \tag{13}$$

and

$$\nu = \frac{q_r + q_c - q_s}{mC_{ACSR}} \approx \frac{\max(q_{c1}, q_{c_2})}{mC_{ACSR}}. \tag{14}$$

We use equations (13) and (14) to estimate $\alpha = 3.99 \times 10^{-6}$ and $\nu = 2.96 \times 10^{-4}$.

In addition, we assume that the initial average external temperature is $C := \mathbf{H}(0) = 70^o C$, we perform our analysis in a time window of $\tau = 900s$ (approximately 15 minutes). Finally, we shall set the line temperature limit to $k = 110^o C$.

Figure 1 indicates how $\sqrt{L(\tau, k)}$ (namely, the square root of the right hand side in (12)) changes with $r_\epsilon$.

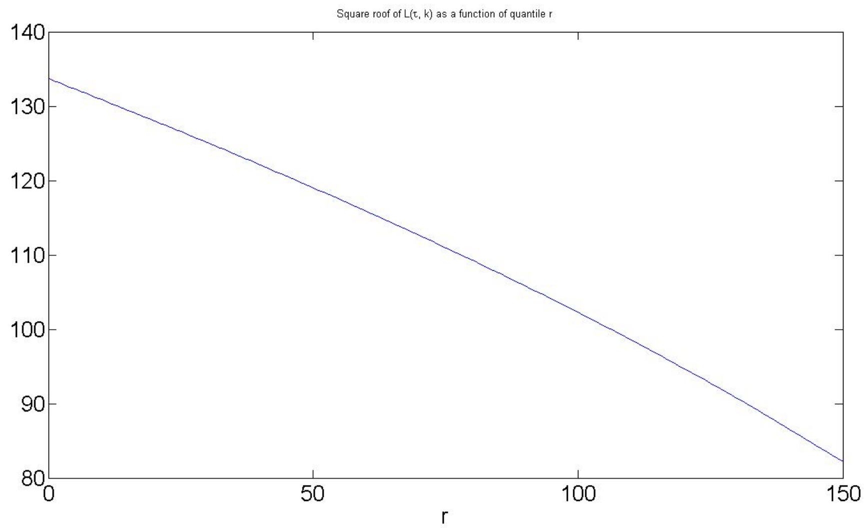



Figure 1: Upper Bound - $\sqrt{L(\tau, k)}$ as a function of $r_\epsilon$

#### 3.1.2 Applying control to a power grid

Next we describe experiments where the methodology presented above for setting upper bounds on power flows is used in a system-wide fashion for OPF calculations. In the computations described in this section we will rely on formulation (1) given above.

In the tests performed below, we use varying choices of the parameter of $r_\epsilon$ to set line limits as indicated above. More precisely, we assume that $u_{i,j}$ in (1) is set equal to

$$u_{ij} = \sqrt{L(\tau, k)}.$$

Of course, for line $ij$, the evaluation of $\sqrt{L(\tau, k)}$ depends on a wide range of line-dependent characteristics as indicated in Section 3.1.1. For the purpose of this exercise we assume that all the lines have the same characteristics and therefore $u_{ij}$ is constant across lines.

We shall study the resulting total "carrying capacity" for the network as measured by the optimal value of the objective function in (1), and we will choose below two different objective functions (linear and quadratic). Moreover, a question of central importance for any procedure that sets line limits in protective fashion is precisely how much such procedure may constrain (or over-constrain) operations. So, an important part of the outcome of our study is to understand when the problem (1) becomes unfeasible as we change a parameter which reflects underlying stochastic fluctuations, and we will choose $r_\epsilon$ to perform this parametric study. As we will see, we might obtain a critical threshold $r_\epsilon^*$ such that if $r_\epsilon > r_\epsilon^*$ then (1) becomes unfeasible, whereas if $r_\epsilon \leq r_\epsilon^*$, (1) is feasible.

For our testing we used the 9-bus system in [25], with three buses attached to generators (buses 1, 2, and 3). We used the following procedure:

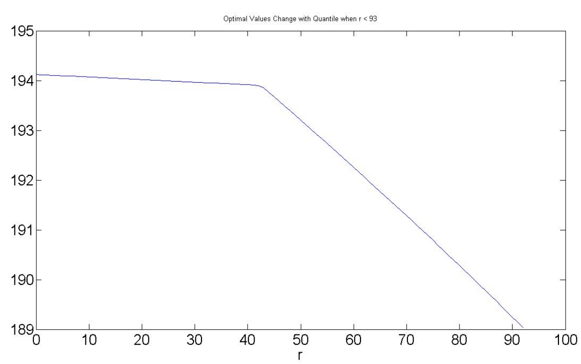
- For $r_\epsilon \le r_\epsilon^*$ (that is, if (1) is feasible) we study how the optimal value of (1) changes as we vary $r_\epsilon$.
- For $r_\epsilon > r_\epsilon^*$ (that is, if (1) is unfeasible), then at least one of the constrains represented in (1c) is violated. In this case, we introduce for each line, $ij$, a deficit variable $z_{ij} \ge 0$, and replace each of the constrains (1c) by the constrains

$$-u_{ij} - z_{ij} \le y_{ij}(\theta_i - \theta_j) \le \ u_{ij} + z_{ij},$$

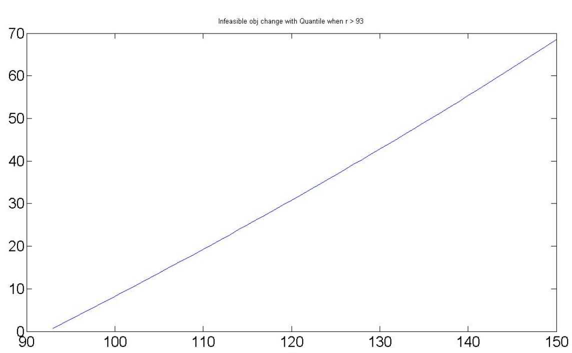
  we also replace the objective function (1) by minimizing $\sum_{ij} z_{ij}$. We then study how the optimal sum of deficits changes as we vary $r_\epsilon > r_\epsilon^*$.

We performed two tests with two different objective functions:

1. Linear function: $\max 0.5P_1 + 0.6P_2 + 0.7P_3$.
   The goal of this experiment is to study how a weighted sum of generation adapts to a change in risk aversion.




(a) Linear Obj Function: feasible, when $r < 93$




(b) Linear Obj Function: infeasible, when $r > 93$

Figure 2: Adding upper bound to a power grid DC problem - linear objective function

Figure 2 shows that when $r_\epsilon > r_\epsilon^* = 93$, the LP problem becomes infeasible (as a result of power flow limits), and thereafter the infeasibility increases monotonically because $u_{ij}$ is monotone in $r_\epsilon$.

2. Quadratic function: $\min 0.2P_1^2 + 0.3P_2^2 + 0.5P_3^2$.
   This objective function is of the typical type in an OPF computation. See Figure 3.

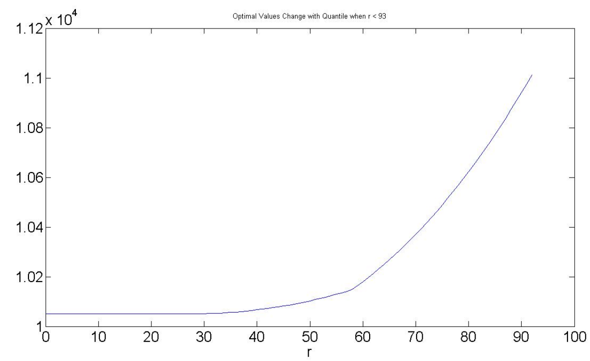



Figure 3: Quadratic Obj function: feasible, when $r < 93$

# 4 Adaptive Control

As an extension to the above analysis above, we consider a setup where it is known at time $t = 0$ that uncertainty will be resolved at later point in the time window $[0, \tau]$. We wish to make use of this fact by designing a better line limit than that obtained through e.g. (12); this improvement will come about by incorporating recourse actions that will be deployed when uncertainty is resolved. We present our specific approach below; first we will motivate our approach through some examples. These examples deviate from current power engineering practice, but only to a small degree.

We also note that the problem studied in this section falls under the broad category of stochastic programming with recourse, which has received much attention, with many elegant and powerful results that are progressively finding more applications in power engineering. See [20]. More to the point, we will be solving optimization problems subject to chance constraints. See, e.g. [9], [13], [14]. The specific setting we consider can be viewed as a variant of two-stage stochastic programming with recourse, a topic for which a broad literature is available. We furthermore rely on chance constraints. See [18], [10]. For recent results and literature see [19].

The particular approach we develop below takes advantage of the structure of the problem at hand, together with a reasonable model for resolution of the uncertainty underlying exogenous factors in the temperature of a power line, and we demonstrate in experiments the speed of our approach. Consider, first, the OPF setting, with generators output levels computed so as to cover the time interval $[0, \tau]$. As discussed above, if some line is known to be thermally stressed at time $t = 0$, special attention should (and will) be devoted to setting its limit in the OPF computation. Using (12) provides a risk-aware methodology for doing so.

Suppose, however, that at time $t = 0$ it is known that the numerical value of $\mathbf{W}$ will be known at $t = \tau/2$, and, further, that a second generator dispatch (possibly in limited form) can be performed at $t = \tau/2$. Then additional efficiencies might be attained by resetting the line limit at $t = \tau/2$. We thus have a two-stage problem: set a line limit to be applied over $[0, \tau/2]$ and another limit over $[\tau/2, \tau]$ contingent on the value of $\mathbf{W}$ observed at $\tau/2$. The combined action must be done in risk-aware fashion (i.e. guaranteeing that line temperature will exceed its critical value with probability $\leq \epsilon$) while maximizing some weighted average of the two line limits. In addition, we would also want to guarantee that with probability 1, some *higher* critical temperature is not exceeded.

A second justification refers to a method already found, to some degree, in OPF practice. This is the idea that a given OPF computation should be performed in such a way to also incorporate information from the time window *after* the current one, i.e. the time window $[\tau, 2\tau]$. In our setting, suppose that at time $t = 0$ the line is thermally stressed, with uncertainty, but that the random variable $\mathbf{W}$ will become known at time $t = \tau$. The computation used to set the line limit over $[0, \tau]$ should be performed aware of the fact that the limit should be set again over $[\tau, 2\tau]$ because there is hysteresis in the thermal process. The computation should also be done so as to take advantage of the fact that uncertainty will disappear at time $t = \tau$.

A third justification involves the *Unit Commitment* problem, which covers a much larger span of time (e.g. 24 hours). However, if thermal stresses are expected over this longer time period, line limits will have to be accordingly reduced. Again we have a setting where the resolution of uncertainty at some intermediate point permits a better line limit setting (while relying on a second generator output computation).

For simplicity, in what follows we will assume that uncertainty is resolved at time $t = \tau/2$. The specific details of our model are as follows: Define the random variable

$$\mathbf{W} \triangleq \mathbf{R} \cdot (1 - e^{-\nu\tau}),$$

and to simplify the analysis we assume that $\mathbf{W}$ is discretely distributed, $\mathcal{P}(\mathbf{W} = w_i) = p_i, i = 1, 2, \ldots, m$ with a *known* distribution (i.e., the distribution of $\mathbf{R}$ is known). We are interested in a control scheme with the following characteristics

(1) At time $t = 0$, we compute values $I_1$, and $I_{2,i}$ for $i = 1, 2, \ldots, m$. These values are used as follows:

(2) In the time window $[0, \tau/2]$ the upper bound on current is set to the value $I_1$.

(3) At time $\tau/2$, we observe the value of $\mathbf{R}$ and thus of $\mathbf{W}$. Assuming $\mathbf{W} = w_i$ then the upper bound on current is set, in the interval $[\tau/2, \tau]$, to the value $I_{2,i}$.

The values $I_1$, and $I_{2,i}$, $1 \leq i \leq m$ are computed according to the following criteria

(a) $\mathcal{P}(\mathbf{H}(\tau) > k) < \epsilon$.

(b) $I_1 \leq L(\tau/2, k)$.

(c) Let $k$ be the critical line temperature. For a given value $k^* > k$ we guarantee, with probability 1, that line temperature will not exceed $k^*$.

(d) Where $F : \mathbb{R}_+^2 \to \mathbb{R}_+$ is coordinate-wise monotonic increasing, we want to maximize, in expectation

$$\sum_{i=1}^{m} F(I_1, I_{2,i}) p_i$$

We now discuss these modeling features. First, as dicussed above our control scheme broadly falls within the class of problems termed “two-stage stochastic optimization with recourse”İn the initial time window we use a (computed) current value $I_1$. At the midway point we observe prevailing conditions and we adjust our policy as per criterion (3). In the setting of interest, we expect that a moderately large choice for $m$ represents a reasonable compromise between mathematical accuracy of the underlying model, and realistic expectations as to what could be observed in actual practice.

Note that as per the analysis in the previous section, for any choice of the values $I_1$, $I_{2,i}$ we will generally have that $\mathbf{H}(t)$ is monotonely increasing or decreasing for $t \in [0, \tau/2]$ and also for $t \in [\tau/2, \tau]$.

Thus, our control scheme may possibly result in a realization where e.g. $\mathbf{H}(\tau/2) > k$. However, (a) guarantees that even if this were to be the case, the line temperature will have reached a safe value by time $\tau$. And of course (b) helps ensure that the probability of this event is small.

Regarding item (d), many examples are reasonable, keeping in mind that a central goal is to attain “high” values for $I_1$ or $I_2$ without incurring risk. As discussed at the end of Section 3 and in the Introduction, a choice of safe value for the current parameter $I$ imputes an effective value for the line limit and thus is a central parameter to be used in computing generator dispatch (OPF). For example, we could set $F(I_1, I_2) = \pi_1 I_1 + \pi_2 I_2$ where $\pi_1, \pi_2 \geq 0$. Or, $F(I_1, I_2) = \pi_1 I_1^2 + \pi_2 I_2^2$. Below we discuss several cases.

We will now cast the choice of the values $I_1$, $I_{2,i}$ as an optimization problem. To do so, suppose that $\mathbf{W} = w_i$. Then, using (9),

$$\mathbf{H}(\tau) = v_1 I_1^2 \ + \ v_2 I_2^2(i) \ + \ w_i \ + \ Ce^{-\nu\tau}, \tag{15}$$

where

$$v_1 \ \triangleq \ \int_0^{\tau/2} e^{-\nu(\tau - s)} \alpha \, ds \tag{16}$$

and

$$v_2 \ \triangleq \ \int_{\tau/2}^{\tau} e^{-\nu(\tau - s)} \alpha \, ds. \tag{17}$$

Define

$$z_1 \triangleq v_1 I_1^2, \quad z_2(i) \triangleq v_2 I_2^2(i), \quad 1 \leq i \leq n, \tag{18}$$

and, for $1 \leq i \leq n$

$$\bar{k}_i \triangleq k - Ce^{-\nu\tau} - w_i,$$

Using this notation we have that, when $\mathbf{W} = w_i$

$$\mathbf{H}(\tau) \ > \ k \quad \text{is equivalent to:} \quad z_1 + z_2(i) > \bar{k}_i.$$

Likewise, if we define $u_i \triangleq k^* - Ce^{-\nu\tau} - w_i$, if $\mathbf{W} = w_i$, then $\mathbf{H}(\tau) \ > \ k^*$ is equivalent to $z_1 + z_2(i) > u_i$. Finally, let us define

$$\tilde{F}(z_1, z_2) \triangleq F(\sqrt{z_1/v_1}, \sqrt{z_2(i)/v_2}), \tag{19}$$

which is simply recasting function $F$ in terms of the $z$ variables. It follows that we can write our optimal control problem as

$$\mathcal{P}_1: \quad \max_{z_1, z_2} \sum_{i=1}^{m} \tilde{F}(z_1, z_2(i)) p_i,$$

$$\text{s.t.} \sum_{i=1}^{m} \mathcal{I}\{z_1 + z_2(i) > \bar{k}_i\} p_i \ \leq \ \epsilon \tag{20}$$

$$z_1 \ \leq \ v_1 L(\tau/2, k) \tag{21}$$

$$z_1 + z_2(i) \ \leq \ u_i, \quad \forall i, \tag{22}$$

$$z_1 \geq 0, \quad z_2(i) \geq 0, \ \forall i. \tag{23}$$

In (20), $\mathcal{I}$ is the indicator function; the square roots in the objective, and the bound (21), arise from our definition of the $z$ variables. Constraint (22) guarantees that the line temperature at time $\tau$ not exceed $k^*$, with probability 1. Of course, this constraint may render the problem above infeasible – however assuming that $\mathbf{H}(0)$ is "safe" the problem *will* be feasible (if necessary by setting $z_1 = z_2(i) = 0$ for all $i$) assuming realistic $\mathbf{R}$. As per the above discussions we have that $u_i > \bar{k}_i$.

**Remark 1** *Let $z_1^*$, $z_2^*(i)$ $(1 \leq i \leq m)$ be an optimal solution to problem $\mathcal{P}_1$. Then, for each $1 \leq i \leq m$*

$$z_1^* + z_2^*(i) \ = \ \bar{k}_i \ \text{ or } \ u_i.$$

*Proof.* Suppose that for some $i$, $z_1^* + z_2^*(i) < \bar{k}_i$. Then increasing $z_2^*(i)$ maintains feasibility, and the monotonicity assumption on $F$ implies that the objective improves. The same reasoning applies if $u_i > z_1^* + z_2^*(i) > \bar{k}_i$. ■

Using this observation we can simplify the optimization problem. Define, for $1 \leq i \leq m$ a binary variable $y_i$ such that

$$y_i = \begin{cases} 0 & \text{when } z_1 + z_2(i) = \bar{k}_i \\ 1 & \text{when } z_1 + z_2(i) = u_i \end{cases} \tag{24}$$

Then the above optimization problem can be recast as:

$$\mathcal{P}_2: \quad \max_{z_1, y} \sum_{i=1}^{m} \tilde{F}(z_1, \bar{k}_i - z_1) p_i (1 - y_i) + \tilde{F}(z_1, u_i - z_i) p_i y_i$$

$$\text{s.t.} \sum_{i=1}^{m} p_i y_i \ \leq \ \epsilon$$

$$0 \ \leq \ z_1 \ \leq \ \min\{v_1 L(\tau/2, k), \min_i \{u_i\}\} \tag{25}$$

$$y_i \ = \ 0 \text{ or } 1, \ \text{all } i.$$

Problem $\mathcal{P}_2$ is a nonlinear, binary optimization problem. We are interested in methodologies and special cases where a near optimal solution can be obtained in practicable form. Here we will present two results:

- We will describe a general approximation method that should prove effective when the parameter $m$ is moderately large, say $m < 10^4$.

- We also describe a provably good approximation algorithm for the case where $m$ is very large, which however only applies in a special case of the function $F$.

As an aside, the issue of the magnitude of $m$ concerns several practical questions, primarily with regards with how accurate a representation of the random variable $\boldsymbol{R}$ can be constructed in "real time". Adequate sensorization of power lines should help in this regard, however there is a larger issue of how data uncertainty can arise in this context (e.g., the spatial distribution of exogenous temperatures over a small time window).

### 4.1 General approach for moderately large $m$

Suppose that in problem $\mathcal{P}_2$ we were to *fix* the variable $z_1$ to a value $\hat{z}_1$ satisfying (25). The remaining part of problem $\mathcal{P}_2$ has the following general structure:

$$\mathcal{P}_2(\hat{z}_1): \quad \max \sum_{i=1}^{m} \tilde{f}_i(\hat{z}_1) y_i$$
$$\text{s.t.} \sum_{i=1}^{m} p_i y_i \ \leq \ \epsilon,$$
$$y_i \ = 0 \text{ or } 1, \ \text{all } i.$$

Problem $\mathcal{P}_2(\hat{z}_1)$ is a *linear* (binary) *knapsack* problem. Knapsack problems are NP-hard – however in this case we are dealing with values of $m$ that are not very large. In fact, it is fair to say that knapsack problems are the easiest of the NP-hard problems, and, more to the point, commercial mixed-integer program solvers can handle such problems with ease even for large $m$.

These observations suggest the following (grid- or mesh-parameterization) approach:

(1) Enumerate equally spaced values of $\hat{z}_1$ between the two bounds in (25).

(2) For each enumerated value, solve $\mathcal{P}_2(\hat{z}_1)$.

While suffering from an enumerative component, this approach does have the attribute of handling any objective function $F$ in the definition of our problem.

A separate issue regarding the small $n$ case concerns the robustness of the computed answers with respect to e.g. the (necessarily estimated) parameters $p_i$ and $w_i$. Using a small $n$ has the effect of accumulating more probability mass into fewer values, with a resulting increase in numerical sensitivity (to the choices for the $p_i$ and $w_i$).

### 4.2 Very large $m$

Suppose now that $m$ is very large. As stated above we expect that even in this case a good mixed-integer programming solver should be able to solve the problems $\mathcal{P}_2(\hat{z}_1)$. Nevertheless, we would like to discuss a case where a solution methodology with sound theoretical foundations is available.

Recall the formula (15) for $\mathbf{H}(\tau)$ as well as (18), and note that $z_1 + z_2(i)$ appears in $\mathbf{H}(\tau)$ and thus, in the chance constraint (20). Consider the special case where

$$F(I_1, I_2) \;=\; v_1^2 I_1^2 \;+\; v_2^2 I_2^2 \tag{26}$$

for all $I_1$, $I_2$. Below we will discuss the implications of this selection of $F(I_1, I_2)$. Using (26), it will follow that using (19),

$$\tilde{F}(z_1, z_2) = z_1 + z_2, \quad \text{for all } z_1, z_2.$$

Thus, problem $\mathcal{P}_2$ can be equivalently restated as:

$$\begin{aligned}
\mathcal{P}_3 : \max \sum_{i=1}^{m} \bar{k}_i p_i (1 - y_i) + u_i p_i y_i & \\
\text{s.t.} \sum_{i=1}^{m} p_i y_i \;\leq\; \epsilon & \\
0 \;\leq\; z_1 \;\leq\; \min\{v_1 L(\tau/2, k)\,,\, \min_i\{u_i\}\} & \qquad (27)\\
y_i \;=\; 0 \text{ or } 1,\; \text{all } i. &
\end{aligned}$$

We can see that constraint (27) is not needed. In short, $\mathcal{P}_3$ can be rewritten in the form:

$$\begin{aligned}
\mathcal{P}_3' : \max \; \sum_{i=1}^{m} f_i \, y_i & \\
\text{s.t. } \sum_{i=1}^{m} p_i y_i \;\leq\; \epsilon & \\
y_i \;=\; 0 \text{ or } 1,\; \text{all } i, &
\end{aligned}$$

for appropriate quantities $f_i > 0$ and $q_i > 0$; this problem constitutes a standard, linear $0-1$ knapsack problem.

In comparing this approach to that used for the small $n$ case, we can see that we have simplified the problem (no enumeration over the $\hat{z}_1$ values). Of course, we do have to solve the possibly large knapsack problem $\mathcal{P}_3$. As we discussed before, this should prove routine (and very fast) even for $n$ in the thousands. However, even though the knapsack problem is NP-hard, there is a very large literature regarding rigorous algorithms for obtaining nearly-optimal solutions to a knapsack problem, within arbitrary precision, in an efficient manner. For a broad survey, see [24]. Also see [4] and references therein. An alternative procedure would rely on branch-and-cut to directly tackle the underlying chance-constrained problem; see [16]. Another alternative would rely on affine controls, see e.g. [8].

Finally we comment on (26). We would argue that this is a "reasonable" functional form for $F(I_1, I_2)$ in that it amounts to a weighted sum. Of course the weights are not flexibly chosen. Nevertheless, note (see (16), (17)) that $v_2 > v_1$. Thus, (26) places more emphasis on what happens in the time interval $[\tau/2, \tau]$. We would argue that this is a reasonable approach, in the sense that we focus in the later time interval, where, coincidentally, we are able to make more precise decisions since randomness has been resolved.

### 4.3 Numerical example

We implement a numerical example for moderately large $m = 50$ using the methodology in Section 4.1. In the test,

- The distribution of $W$ is shown in Figure 4.
- $k^* = k + 10 = 120^o C$.
- The number of enumerated values of $\hat{z}_1$ is 1000.
- The remaining parameters are as in Section 3.1. This implies that $L(0, \tau) \approx 110.80$.

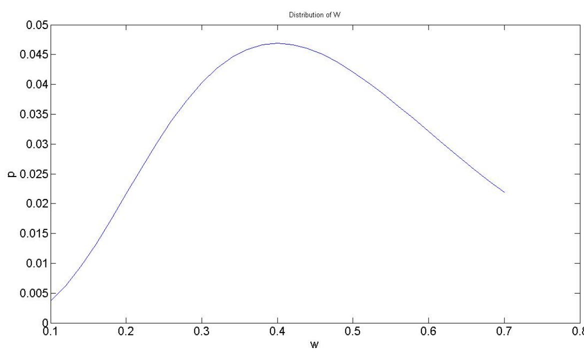



Figure 4: Distribution of W

In the experiment we performed, $F(I_1, I_2) = I_1 + \sum_{i=1}^{m} p_i I_{2,i}$, i.e. we seek to maximize the average line limit (averaged over first and second stages and across all realizations). Solution of the resulting ensemble of knapsack problems yields $I_1 = 120.98$, and the values of $I_{2,i}, i = 1, 2, \ldots, m$, as shown in Figure 5. The entire computation (1000 knapsacks) required approximately one CPU second on a current workstation.

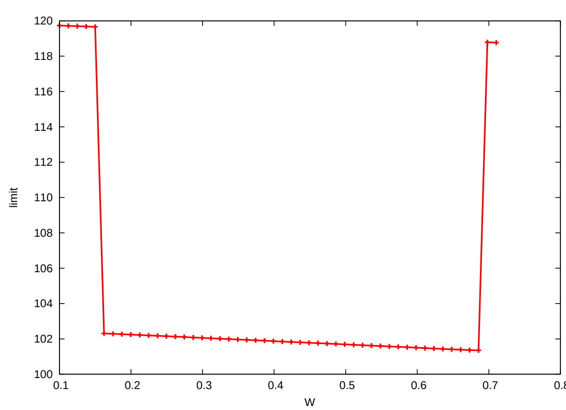



Figure 5: Values of $I_2$ for first 2-stage experiment

The figure displays a compelling behavior: when $W$ is large then $I_2$ is small (because it should be) and when $W$ is small then $I_2$ is large (because it can be). Moreover, this intelligent second-stage behavior is leveraged in the first stage by setting $I_1$ quite large; in fact large compared to $L(0, \tau)$. Hence the main benefit of the two-stage approach is that during the first stage the line limit is approximately 9% higher; moreover the average of first- and second-stage limits is approximately 111.83, again larger than $L(0, \tau)$, and all of this under the much more conservative additional constraint that guarantees that line temperature will not exceed the value $k^*$ with probability one.

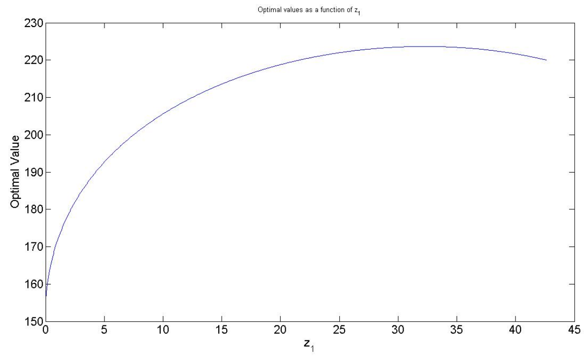



Figure 6: Optimal Values change with $z_1$

# 5 Conclusion

In this paper we have presented a guiding methodology for risk-sensitive selection of power line limits under thermal uncertainty. Our methodology is based on the use of the heat equation with random input. In particular, we introduce a stochastic process which governs the variations of exogenous factors such as wind, and their impact in the external temperature which surrounds the line. Our analysis leads to accurate chance-constrained policies for setting line limits (safety levels), obtained as the (approximate) solution to an appropriate optimization problem. We also consider a two-stage stochastic optimization problem that leads to an adaptive control mechanism, where observations made at an intermediate point in a time window provide a means to adjust line limits.

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