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ON EXACT SAMPLING OF STOCHASTIC PERPETUITIES

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Abstract

A stochastic perpetuity takes the form

$$D_\infty = \sum_{n=0}^{\infty} \exp\left(Y_1 + ... + Y_n\right) B_n,$$

where $(Y_n : n \geq 0)$ and $(B_n : n \geq 0)$ are two independent sequences of independent and identically distributed (i.i.d.) random variables (r.v.s). This is an expression for the stationary distribution of the Markov chain defined recursively by $D_{n+1} = A_n D_n + B_n,\ n \geq 0$, where $A_n = e^{Y_n}$; D_∞ then satisfies the stochastic fixed point equation $D_\infty =_d AD_\infty + B$, where A and B are independent copies of the A_n and B_n (and independent of D_∞ on the right-hand side).
In our framework, the quantity B_n, which represents a random reward at time n, is assumed to be positive, unbounded with $EB_n^p < \infty$ for some $p > 0$, and with a suitably regular continuous positive density. The quantity Y_n is assumed to be light-tailed and represents a discount rate from time n to $n-1$. The r.v. D_∞ then represents the net present value, in a stochastic economic environment, of an infinite stream of stochastic rewards. We provide an exact simulation algorithm for generating samples of D_∞. Our method is a variation of *dominated coupling from the past* (DCFTP) and it involves constructing a sequence of dominating processes.

Keywords: Perfect sampling; Coupling From The Past; Markov chains; Stochastic Perpetuities

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1. Introduction

Let $(Y_n : n \geq 0)$ and $(B_n : n \geq 0)$ be two independent sequences of i.i.d. r.v.'s, with Y and B denoting generic such copies. Suppose that the B_n are positive and denote the amount of reward obtained by running a system at time n, and the discount rate from time n to time $n-1$ is precisely Y_n, so that the present value of B_n at time zero is $B_n \exp\left(Y_n + Y_{n-1} + ... + Y_1\right)$. The net present value obtained by running the system over an infinite time horizon (starting with B_0 at time 0) is then given by the so-called stochastic perpetuity

$$D_\infty = \sum_{n=0}^{\infty} \exp\left(Y_1 + ... + Y_n\right) B_n. \tag{1}$$

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This is an expression for the stationary distribution of the Markov chain defined recursively by

$$D_{n+1} = A_n D_n + B_n, \ n \geq 0, \tag{2}$$

where in our set up, $A_n = e^{Y_n}$; the expression for D_∞ is then derived by setting $D_0 = 0$ and iterating the recursion (2) out to $n = \infty$ (while reversing the labeling of the rvs). Recursion (2) is an example of an ARCH(1) model, an important time-series model used in statistics and econometrics; see [3].

When $E \log A < 0$ and $E \log(1 + B) < \infty$, (see, for example, [18]), then this Markov chain has a proper stationary distribution limit D that satisfies the stochastic fixed point equation

$$D =_d AD + B, \tag{3}$$

where A and B are independent (and independent of D on the right-hand side).

We are interested in designing a simulation algorithm that allows us to obtain perfect samples of D_∞ under assumptions that allow us to accommodate a wide range of models of interest given the previous economic interpretation for D_∞. In particular, we assume that Y (not $A = e^Y$) has a finite moment generating function in a neighborhood of the origin and that $EY \in (-\infty, 0)$. (The latter assumption is assumed merely to ensure finiteness of D_∞.) We assume that B is positive with $EB^p < \infty$ for some $p > 0$. The most important assumption that we impose concerns the existence of a suitably regular density for B, which is to be positive, continuous on $[0, \infty)$, and with a tail decay that is not too light (see Section 2). The types of examples that are of most interest to us include situations in which B has a heavy-tailed distribution such as a Pareto. However, light-tailed distributions such as a mixture of exponentials are also tractable under our framework. We do not consider super-exponential tails but we believe that methods related to our development here could be adapted to the case in which B has bounded support.

A situation that can be easily handled (hence left out in the current paper) is that in which $p = P(Y = -\infty) > 0$ (equivalently $p = P(A = 0) > 0$). For then a simple direct *coupling from the past* (CFTP) algorithm (a general method introduced in [23]) applies as long as we can generate rvs distributed as A and B: Generate iid $A_n, \ n \geq 1$, until time $T = \min\{n \geq 0 : A_n = 0\}$ (geometric with success probability p), and then given $T = n$, generate n iid copies $B_1, \ldots, B_n$, and construct D_0 recursively from the past (from time $-n + 1$ up to 0), using (2) with $D_{-n+1} = B_n$ and using the $T - 1$ values (used to define T) $A_{n-1}, \ldots, A_1$ and $B_{n-1}, \ldots, B_1$. For example, if $T = 1$, then set $D_0 = B_1$. If $T = 2$, then set $D_0 = B_2 A_1 + B_1$, and so on. Then D_0 is distributed exactly as D_∞.

The above economic interpretation of D_∞ is useful in areas such as pension fund dynamics; in [8] a model is proposed based on stochastic perpetuities for the valuation of the pension fund. In the context of insurance risk theory, it is known that the distribution of D_∞ plays a crucial role in the evaluation of ruin probabilities with investments; see for instance [14], [21] and [20]. Explicit expressions for the distribution of D_∞ are, however, very challenging to obtain. Nevertheless, under very specific assumptions on the distributions of B and the Y such explicit expressions have been found in [24] and [12], see also [22]. D_∞ can also be viewed as the stationary distribution of a continuous-time "growth-collapse process" right before collapse epochs (see, for example, [16] and its many references). More general models in tree-like structures are discussed in [1] and [15].

As discussed in [4], D_∞ also plays a key role in applications arising in mathematical physics and finance. Applications to communication systems are given in [19]. Finally, [13], [9] also mention applications in the complexity analysis of algorithms related to the so called "Quickselect" algorithm and also in analytic number theory.

Applications to complexity analysis of sorting algorithms motivated the currently existing exact simulation methods for sampling D_∞. But in those cases, $B = 1$ and $A = U^{1/\beta}$, where $U \sim U[0, 1]$ and $\beta > 0$; these perpetuities are known as Vervaat perpetuities, [24]. The existing algorithms in this setting are constructed to sample from Vervaat perpetuities and related models. The first such sampling method based on a density approximation is given in [6]. It presents a sequence of upper and lower bounds for the density of D_∞ and then applied acceptance rejection; see also [7]. A recent paper by [11] develops a Dominated Coupling From The Past (DCFTP) based procedure to sample Vervaat perpetuities. DCFTP is in order to deal with the problem of sampling the steady-state distribution of unbounded Markov chains, see [17]. In turn, DCFTP was developed after the seminal paper [23], in which the CFTP protocol was introduced. A recent exposition on CFTP is given in [2]. A nice summary of DCFTP is given in [11].

A generic class of DCFTP algorithms have been developed in [17] and [5]. There it is shown that under certain ergodicity assumptions DCFTP can be constructed using a suitable Foster-Lyapunov function and a suitable subsample scheme. While these procedures are substantially general and are in principle applicable to our setting, there are important limitations that are outlined in p. 788 of [5]. In particular, their algorithm assumes that appropriate information is available in terms of the transition kernel of a Markov chain that is constructed based on k iterates of (2) – which is impractical in our setting. The value of k is found in order to ensure ergodicity of a suitable dominating process which turns out to be the workload system of a suitably defined $D/M/1$ queue.

In the present paper we shall use a variation of DCFTP to generate our exact samples of D_∞. Typically DCFTP requires the construction of a dominating stationary Markov chain that serves as a stochastic upper bound. Instead, we construct a sequence of upper bounds that do not form a stationary Markov chain per-se, but otherwise are used in the same way as the dominating chain in DCFTP. We point out why our sequence of processes do not directly induce a simulatable stationary Markov chain at the end of Section 2. For the construction of our stochastic upper bounds we also develop a simulation procedure to exactly sample from the steady-state distribution of a suitable $GI/G/1$ queue, relaxing some of the assumptions of an earlier algorithm in [10].

The rest of the paper is organized as follows. In Section 2, we introduce our assumptions and our basic strategy which is summarized at the end of the section. The stochastic upper bounds required to implement our strategy are given in Sections 3 and 4. Finally, the remaining proofs to guarantee finite termination time of our algorithm are given in Section 5.

2. Assumptions and Basic Strategy

We impose the following assumptions on the distributional properties of Y and B (generic copies of the Y_n's and the B_n's respectively). (Recall that $Y = \ln(A)$.)

A) $\psi(\theta) = \log E \exp(\theta Y) < \infty$, for some $\theta > 0$, and Y is not deterministic.

Furthermore, we assume that $\psi'(0) = EY < 0$; by the non-deterministic assumption we also have that $\psi''(0) > 0$.

B) B is unbounded ($P(B > x) > 0, x \geq 0$) with continuous positive density $f(\cdot)$ on $(0, \infty)$ satisfying that for each $\kappa \geq 1$ there exists $\lambda_\kappa \in (0, \infty)$ such that

$$\lambda_\kappa \leq \inf_{0 \leq z \leq 1, y \geq \kappa} \frac{f(y-z)}{f(y-\kappa)}. \tag{4}$$

C) $EB^p < \infty$ for some $p > 0$.

We assume that the tail, $\overline{F}(\cdot) := P(B > \cdot) = \int_\cdot^\infty f(s)\,ds$ is available in closed form to us. Finally, we assume that for each $\kappa > 0$ we can find a constant $C(\kappa) \in (0, \infty)$ such that $E[(B+\kappa)^p] < C(\kappa)$ for some $p > 1$. Moreover, note that assumptions A) and C) imply the existence of $\kappa_0 \in (0, \infty)$ satisfying

$$E \log \left(\frac{1 + \kappa_0 \exp(Y) + B}{1 + \kappa_0} \right) < \frac{1}{2} EY_1 < 0. \tag{5}$$

The previous observation will be used in the construction of a suitable Lyapunov bound to guarantee the finite termination time of our algorithm. In the sequel we will be interested in values of κ in B) that are larger than κ_0.

Remark 1. In A), we have assumed that Y has exponential moments and is non-deterministic. This is a mild assumption (recall that $A = e^Y$) because it allows us to accommodate virtually all models of interest rate dynamics used in practice. The case of deterministic Y is substantially easier and can be treated with methods similar to what we discuss here. The most important assumption we impose is on B. The bound (4) is naturally satisfied in applications in which the distribution of B is known and can be chosen by the modeler. It allows us to accommodate tails that are not too thin (typical tails that decrease at most exponentially fast satisfy (4)); for instance exponential, gamma, Pareto or Weibull. A tail decreasing like a Gaussian or faster, for example, will typically not satisfy (4). Finally, assumption C) is, we believe, also very mild and natural if one is concerned, as we are, with rewards that have unbounded support.

As we indicated earlier, our development is based on a slight variation of dominated coupling from the past in order to sample from the steady state distribution of the Markov chain

$$D_{n+1} = \exp(Y_n) D_n + B_n, \tag{6}$$

with $D_0 = 0$. In order to apply these techniques we need the following elements.

Elements of dominated coupling from the past for a monotone chain

1) We need to construct a recursion that preserves the monotonicity implied by recursion (6) but also that allows to detect coalescence via coupling. This construction will require a suitable minorization condition.

2) We need to construct a sequence of stochastic upper bounds for the steady-state distribution at subsequent deterministic times in the past assuming that one has simulated the process starting from an arbitrarily long time in the distant past. These stochastic upper bounds need to be constructed jointly.

Typically dominated coupling from the past requires constructing a coupling and a suitable Markov chain that dominates the Markov chain of interest. It also requires being able to sample the dominating Markov chain in stationarity and being able to simulate the chain backwards in time. We use a slight variation of dominated coupling from the past because we do not construct a dominating chain per-se but only a suitable sequence of stochastic upper bounds.

We will now construct the elements 1) and 2). Recursion (6) is convenient because it has a useful monotonicity structure. In particular, the mapping

$$\phi_0\left(d,y,b\right)=\exp\left(y\right)d+b \tag{7}$$

is monotone increasing in d. However, in addition to monotonicity, in our construction we will require to introduce a coupling as this is the tool that we will use to detect coalescence (as indicated in item 1). Therefore, we will take advantage of the following minorization which is applicable to the random variable B.

The minorization condition. Throughout our construction we will set $\kappa>0$. The selection of κ will be given according to our running time analysis in the last section depending on a Lyapunov inequality. Suppose that $z\in[0,\kappa]$, then, because of (4)

$$P\left(B+z\in y+dy\right)=f\left(y-z\right)I\left(y\geq z\right)dy\geq\lambda_{\kappa}f\left(y-\kappa\right)I\left(y\geq\kappa\right)dy.$$

Therefore, we have that

$$P\left(B+z\in y+dy\right)=\lambda_{k}P\left(B+\kappa\in y+dy\right)+(1-\lambda_{k})R\left(z,dy\right), \tag{8}$$

with

$$R\left(z,dy\right)=\frac{f\left(y-z\right)I\left(y\geq z\right)dy-\lambda_{k}f\left(y-\kappa\right)I\left(y\geq\kappa\right)dy}{(1-\lambda_{k})}.$$

Note that

$$\overline{H}_{z}\left(t\right):=\int_{t}^{\infty}R\left(z,dy\right)=\frac{\overline{F}\left(t-z\right)-\lambda_{k}\overline{F}\left(t-\kappa\right)}{1-\lambda_{k}}.$$

As a check note that $\overline{H}_{z}\left(z\right)=1$. In fact, in terms of the tail of the distribution of $B+z$ we see that the splitting (8) yields the obvious identity

$$P\left(B+z>t\right)=\lambda_{k}\overline{F}\left(t-\kappa\right)+(1-\lambda_{k})\frac{\overline{F}\left(t-z\right)-\lambda_{k}\overline{F}\left(t-\kappa\right)}{1-\lambda_{k}}=\lambda_{k}\overline{F}\left(t-\kappa\right)+(1-\lambda_{k})\overline{H}_{z}\left(t\right).$$

The Markov chain representation based on the splitting induced by the minorization condition. If we let $z=\exp\left(y\right)d$, representation (8) induces a mapping $\phi_1\left(z,U,B\right)$ such that

$$\phi_1\left(z,U,B\right)\overset{D}{=}\phi_0\left(d,y,B\right), \tag{9}$$

where U, B are independent with U uniformly distributed over $[0,1]$. In particular we let

$$\phi_1\left(z,U,B\right)=\begin{cases} B+z & \text{if } z:=\exp\left(y\right)d>\kappa \\ I\left(U\leq\lambda_k\right)\left(B+\kappa\right)+\overline{H}_{z}^{-1}\left(\overline{F}\left(B\right)\right)I\left(U>\lambda_k\right) & \text{if } z:=\exp\left(y\right)d\leq\kappa \end{cases}$$

and define $B'\left(z,U,B\right)=\phi_1\left(z,U,B\right)-z$. We then can write

$$\phi_1\left(z,U,B\right)=z+B'\left(z\right)=\exp\left(y\right)d+B'\left(\exp\left(y\right)d\right).$$

The previous representation will allow us to deal with item 1). In order to see this we need to introduce some notation and verify monotonicity properties of the mapping $\phi_1(\cdot, U, B)$. Assume that the random variables $\{(U_n, B_n) : n \geq 0\}$ (with the iid uniforms (U_n) independent of all else) are given together with the sequence $\{Y_n : n \geq 0\}$. In fact we will require to set the B_n according to equation (12) and the Y_n's will be simulated according to a suitable random walk construction as explained in Section 4. For $j = 0, 1, ..., n$, set

$$\begin{aligned}
D_0(n, w) &= w, \\
D_1(n, w) &= \phi_1(\exp(Y_n) D_0(n, w), U_{n-1}, B_{n-1}) \\
&= \exp(Y_n) D_0(n, w) + B'_{n-1}(\exp(Y_n) w, U_{n-1}, B_{n-1}), \\
D_2(n, w) &= \phi_1(\exp(Y_{n-1}) D_1(n, w), U_{n-1}, B_{n-1}) \\
&= \exp(Y_{n-1}) D_1(n, w) + B'_{n-2}(\exp(Y_{n-1}) D_1(n, w), U_{n-2}, B_{n-2}), \\
&... \\
D_j(n, w) &= \exp(Y_{n-j+1}) D_{j-1}(n, w) + B'_{n-j}(\exp(Y_{n-j+1}) D_{j-1}(n, w), U_{n-j}, B_{n-j}).
\end{aligned}$$

In simple words, the previous recursions are interpreted as follows. The value $D_0(n, w) = w$ indicates *an initial value equal to w at n units of time in the past*. Then, $D_j(n, w)$ represents the value at $n - j$ units of time in the past given that at n units of time in the past the position was w. Note that the value at $n - j$ units of time in the past given the position at $n - j + 1$ of units of time in the past depends only on U_{n-j} and B_{n-j} and that the driving sequence $\{(U_i, B_i) : i \geq 0\}$ is kept fixed even if we start the iterations at arbitrary long times n in the past. We clearly have that if $n \geq n_0$ and $0 \leq j \leq n_0$,

$$D_{n-j}(n, w) = D_{n_0-j}(n_0, D_{n-n_0}(n, w)).$$

We now show the following useful monotonicity property.

Proposition 1. *If $w \leq v$ then*

$$D_j(n, w) \leq D_j(n, v)$$

for $j = 0, ..., n$. Moreover,

$$z + B'(z, U, B) \leq (B + \kappa) + z.$$

Proof. Initially we verify that if $z_0 \leq z_1$ then

$$\phi_1(z_0, U, B) \leq \phi_1(z_1, U, B).$$

First, if $z_0 \leq z_1 \leq \kappa$ or $\kappa \leq z_0 \leq z_1$ the result is clear because $\overline{H}_{z_0}(t) \leq \overline{H}_{z_1}(t)$ for all $t \geq 0$. Now, if $z_0 \leq \kappa \leq z_1$ then we have two cases. If $U \leq \lambda_k$ then clearly the inequality holds because $B + z_1 \geq B + \kappa$. If $U > \lambda_k$, then we need to show that

$$B + z_1 \geq \overline{H}_{z_0}^{-1}\left(\overline{F}(B)\right).$$

Now, since $\overline{H}_{z_0}(\cdot)$ is decreasing $\overline{H}_{z_0}^{-1}(\cdot)$ is also decreasing and therefore

$$\overline{H}_{z_0}(B + z_1) \leq \overline{F}(B).$$

In addition, we have noted that $\overline{H}_{z_0}(t) \leq \overline{H}_{z_1}(t)$. Therefore, we have that

$$\overline{H}_{z_0}(B+z_1) \leq \overline{H}_1(B+z_1) \leq \overline{H}_1(B+\kappa) = \frac{\overline{F}(B)-\lambda_k}{1-\lambda_k} \leq \overline{F}(B).$$

Therefore, the claim holds true. Now we proceed with the statement of the proposition using induction. For $j=0$ the claim holds by definition. Assuming that the inequality holds for $j-1$, then by induction and by monotonicity of $\phi_1(\cdot, U_{n-j}, B_{n-j})$ we obtain

$$\begin{aligned} D_j(n,w) &= \exp(Y_{n-j+1}) D_{j-1}(n,w) + B'_{n-j}(D_{j-1}(n,w)) \\ &= \phi_1(\exp(Y_{n-j+1}) D_{j-1}(n,w), U_{n-j}, B_{n-j}) \\ &\leq \phi_1(\exp(Y_{n-j+1}) D_{j-1}(n,v), U_{n-j}, B_{n-j}) \\ &= \exp(Y_{n-j+1}) D_{j-1}(n,v) + B'_{n-j}(D_{j-1}(n,v)), \end{aligned}$$

verifying the claim for j. The second part of the proposition follows similar steps. If $z \leq \kappa$ then $z + B'(z,U,B) \leq B+\kappa$ and $z \geq \kappa$ implies $z + B'(z,U,B) = B + z$. In any case, $z + B'(z,U,B) \leq (B+\kappa) + z$.

To complete the construction of the basic elements behind our algorithm, let us write $S_n = Y_1 + ... + Y_n$ $(S_0 = 0)$. It follows from (9) that for any w,

$$D_n(n,w) \stackrel{D}{=} \exp(S_n) w + \sum_{j=0}^{n-1} \exp(S_j) B_j$$

and therefore, since $\exp(S_n) w \to 0$ almost surely as $n \nearrow \infty$,

$$X \stackrel{D}{=} \lim_{n \longrightarrow \infty} D_n(n,w),$$

where

$$X = \sum_{n=0}^{\infty} \exp(S_n) B_n.$$

Now let us define

$$\begin{aligned} B_n^+ &= B_n + \kappa, \\ W_n^+ &= B_n^+ + \exp(Y_{n+1}) B_{n+1}^+ + \exp(Y_{n+1}+Y_{n+2}) B_{n+2}^+ + ..., \\ W_n &= B_n + \exp(Y_{n+1}) B_{n+1} + \exp(Y_{n+1}+Y_{n+2}) B_{n+2} + ... \end{aligned} \tag{10}$$

We actually have

$$X \stackrel{D}{=} D_n(n, W_n).$$

Now assume that we can find a sequence of random variables $(V_k^+ : k \geq 0)$ such that $V_k^+ \geq W_k^+$; this is precisely item 2) in our list of basic elements. The next basic result allows us to detect coalescence.

Proposition 2. *If there exists $N_0 < \infty$ with probability one such that for some $1 \leq j \leq N_0$*

$$\exp(Y_j) D_{N_0-j}(N_0, V_{N_0}^+) \leq 1 \text{ and } U_{j-1} \leq \lambda_k$$

then $D_n(n, W_n^+) = D_{N_0}(N_0, V_{N_0}^+)$ for all $n \geq N_0$. Moreover, $D_{N_0}(N_0, V_{N_0}^+) \stackrel{D}{=} X$.

Proof. Note that if $n \geq N_0$

$$D_{n-N_0}\left(n, W_n^+\right) \leq D_{n-N_0}^+\left(n, W_n^+\right) = W_{N_0}^+ \leq V_{N_0}^+.$$

Therefore, for each $1 \leq j \leq N_0$ we have that

$$\begin{aligned} D_{n-j}\left(n, W_n^+\right) &= D_{N_0-j}\left(N_0, D_{n-N_0}\left(n, W_n^+\right)\right) \\ &\leq D_{N_0-j}\left(N_0, D_{n-N_0}^+\left(n, W_n^+\right)\right) = D_{N_0-j}\left(N_0, W_{N_0}^+\right) \leq D_{N_0-j}\left(N_0, V_{N_0}^+\right). \end{aligned}$$

So we have that

$$\exp\left(Y_j\right) D_{n-j}\left(n, W_n^+\right) \leq \exp\left(Y_j\right) D_{N_0-j}\left(N_0, V_{N_0}^+\right) \leq 1$$

and $U_{j-1} \leq 1$. This implies that the coalescence (coupling) occurs and therefore we must have that $D_n\left(n, W_n^+\right) = D_{N_0}\left(N_0, V_{N_0}^+\right)$. To show that indeed $D_{N_0}\left(N_0, V_{N_0}^+\right) \stackrel{D}{=} X$ we simply observe that

$$D_n\left(n, W_n^+\right) \stackrel{D}{=} D_n\left(n, X^+\right) \stackrel{D}{=} \exp\left(S_n\right) X^+ + \sum_{j=0}^{n-1} \exp\left(S_j\right) B_j,$$

where X^+ is a copy of W_n^+ which is independent of all the B_j's and Y_j's. The right hand side converges to X almost surely as $n \nearrow \infty$ and the left hand side equals $D_{N_0}\left(N_0, V_{N_0}^+\right)$ for $n \geq N_0$; the result follows.

The previous result is not very useful unless we are able to find a sequence of stochastic upper bounds V_k^+'s and ensure that $P(N_0 < \infty) = 1$. The construction of these upper bounds will require first dealing with the maximum of an appropriate random walk and second simulating the B_n's in a suitable fashion. We will study the construction of the V_k^+'s in the next sections. Assuming that such construction is in place and that $N_0 < \infty$ the basic algorithm takes the following form. The proof that $P(N_0 < \infty) = 1$ will be given in the last section of the paper.

Algorithm for exact simulation of X

We set $\kappa > \kappa_0$, where κ_0 satisfies (5).

Step 1: At iteration $l \geq 1$ set $k = 2l$. Sample V_k^+ and let $D_0\left(k, V_k^+\right) = V_k^+$. (The definition of V_k^+ is given in (13).)

Step 2: Obtain $D_j\left(k, V_k^+\right)$ for $j = 1, 2, ..., k$.

Step 3: If there exists j such that $\exp\left(Y_j\right) D_{N_0-j}\left(N_0, V_{N_0}^+\right) \leq \kappa$ and $U_{j-1} \leq \lambda_k$ then let $X = D_k\left(k, V_k^+\right)$ and stop, otherwise let $m \longleftarrow m + 1$ and go to Step 1.

Remark 2. Note that we are using a Markov chain that has the same structure as D_n in order to construct our DCFTP-type algorithm, namely, one in which the B_n is replaced by B_n^+. The standard application of DCFTP would therefore involve simulating a stationary version of the the dominating Markov chain. This problem, however, is basically equivalent to the original problem. Nevertheless, we note that to carry-over the basic ideas behind DCFTP, all we need is the construction of stochastic upper bound for the steady distribution of our dominating chain; this is the role played by the V_k^+'s and this is why our processes do not directly induce a single stationary Markov chain.

3. Simulatable Stochastic Upper Bounds for the Steady-state Distribution

For any $a \in (0,1)$, let us define

$$S_n(a) = Y_1(a) + ... + Y_n(a),$$

where $Y_j(a) = Y_j + a$ and write

$$\begin{aligned} W_k^+ &= \sum_{n=k}^{\infty} \exp\left(S_n(a) - S_k(a)\right) \exp\left(-(n-k)a\right) B_n^+. \\ &\leq \exp(ka) \exp\left(M_k(a)\right) \sum_{n=k}^{\infty} \exp(-na) B_n^+, \end{aligned} \tag{11a}$$

where

$$M_k(a) = -S_k(a) + \max_{n \geq k} S_n(a).$$

Our strategy for constructing and simulating a suitable upper bound V_k^+ takes advantage of the representation (11a). We need to explain how to simulate subsequent elements of the sequence $\{M_k(a) : k \geq 0\}$. We also need to sample an upper bound for the infinite sum that appears multiplying $\exp(M_k)$ in (11a); we first deal with this infinite sum and discuss the simulation of the $M_k(a)$'s in the next section.

A useful observation is that, by assumption, for each $a \in (0,1)$

$$P\left(B_n^+ > \exp(na)\right) \leq C(\kappa) \exp(-nap).$$

Since $p > 1$, Borel-Cantelli ensures that the event $\{B_n^+ > na\}$ occurs just finitely many times. Let us define $T_0 = 0$ and $T_j = \inf\{n > T_{j-1} : B_n + \kappa > \exp(na)\}$ for $j = 1, 2, ...$ We note that if $J = \max\{j \geq 0 : T_j < \infty\}$ then, as indicated earlier, $J < \infty$ almost surely and $1 \leq \chi := \max\{n \geq 0 : B_n + \kappa > \exp(na)\} < \infty = T_J$.

We will explain how to simulate the B_n's jointly with the T_j's. To do this first define two sequences of independent random variables, namely $(\widehat{B}_n : n \geq 0)$ and $(\overline{B}_n : n \geq 0)$. The corresponding distributions are as follows. $\widehat{B}_n$ follows the distribution of B_n given that $B_n + \kappa > \exp(na)$ and $\overline{B}_n$ follows the distribution of B_n given that $B_n + \kappa \leq \exp(na)$. We also define $(I_n : n \geq 0)$ to be a sequence of independent Bernoulli random variables (independent of the $\widehat{B}_n$'s and the $\overline{B}_n$'s) such that $P(I_n = 1) = p(n) := P(B_n + \kappa \geq \exp(an))$. We then can write

$$B_n = \widehat{B}_n I_n + \overline{B}_n (1 - I_n), \tag{12}$$

and $T_j = \inf\{n > T_{j-1} : I_n = 1\}$ for $j = 1, 2, ...$with $T_0 = 0$. Moreover, we have that

$$\begin{aligned} \sum_{n=k}^{\infty} \exp(-na) B_n^+ &\leq \sum_{n=k}^{\infty} \exp(-na) \widehat{B}_n I_n + \frac{\exp(-ka)}{1 - \exp(-a)} \\ &= \sum_{n=k}^{\chi} \exp(-na) \widehat{B}_n I_n + \frac{\exp(-ka)}{1 - \exp(-a)}. \end{aligned}$$

Therefore, if we define

$$V_k^+ = \exp(ka)\exp(M_k(a))\sum_{n=k}^{\chi}\exp(-na)\widehat{B}_n I_n + \frac{\exp(M_k(a))}{1-\exp(-a)}, \tag{13}$$

we clearly have from (11a) that $V_k^+ \geq W_k^+$.

We explain how to sample $T_1, T_2, T_3 ....$ We shall consider T_1 only as sampling T_j given the T_{j-1} is entirely analogous. Note that $T_1 > T_0 = 0$ and

$$P(T_1 = k) = p(k)\prod_{j=1}^{k-1}(1-p(j)),$$

for $k \geq 1$. Moreover, we have that $P(T_1 = \infty) = \prod_{j=1}^{\infty}(1-p(j)) \in (0,1)$. We note that by assumption, using Chebyshev's inequality,

$$p(n) \leq \min(C(\kappa)\exp(-nap), 1).$$

Therefore, we have that if $1 > C(\kappa)\exp(-map)$

$$\prod_{j=1}^{m-1}(1-p(j)) \geq P(T_1 = \infty) = \prod_{j=1}^{\infty}(1-p(j)) \geq \prod_{j=1}^{m-1}(1-p(j))\exp\left(-\sum_{j=m}^{\infty}\frac{C(\kappa)}{\exp(map)}\right) \tag{14}$$

$$= \prod_{j=0}^{m-1}(1-p(j))\exp\left(-\frac{C(\kappa)}{\exp(map)(1-\exp(ap))}\right).$$

Consequently, in order to sample a Bernoulli r.v. Z with parameter $P(T_1 = \infty)$ we can simply let

$$Z = I(U \leq P(T_1 = \infty)),$$

where U is uniformly distributed in $[0,1]$. Note that, with probability one, the condition $U \leq P(T_1 = \infty)$ can be obtained from the bounds of (14) by making m sufficiently large without computing the infinite product in the definition of $P(T_1 = \infty)$.

Now, if $T_1 < \infty$, we need to simulate a random variable with probability mass function

$$P(T_1 = k | T_1 < \infty) = p(k)\frac{\prod_{j=1}^{k-1}(1-p(j))}{\prod_{j=1}^{\infty}(1-p(j))} \leq \frac{1}{\prod_{j=0}^{\infty}(1-p(j))}\min(C(\kappa)\exp(-kap), 1)$$

Once again, we apply an acceptance rejection procedure. A suitable proposal random variable, K, with probability mass function

$$P(K = k) = \exp(-[k-1]ap)(1-\exp(ap)),$$

for $k \geq 1$ works in this case. This type of procedure allows to simulate the sequence $(I_n : n \geq 0)$. Simulating the sequences $(\widehat{B}_n : n \geq 0)$ and $(\overline{B}_n : n \geq 0)$ is immediate.

4. The Maxima of a Negative Drift Random Walk

Our goal is to simulate $M_k(a)$'s jointly with the random walk $(S_n(a) : n \leq k)$. The design of our algorithm is based on importance sampling. We first need the next lemma which follows easily from the strict convexity of $\psi(\cdot)$ and so its proof is omitted.

Lemma 1. *Suppose the moment generating function of the non-degenerate random variable Y is finite in a neighborhood of the origin, so that $\psi'(0) < 0$ and $\psi''(0) > 0$. Define $\psi_a(\theta) = \log E \exp(\theta Y(a)) = \psi(\theta) + a\theta$. Then we can always find $a > 0$ and $\eta = \eta(a) > 0$ such that $\psi_a(\eta) = 0$.*

Remark 3. In the so-called Cramer case, that is when there exists $\theta_* > 0$ such that $\psi(\theta_*) > 0$, then for any $a \in (0, |\psi'(0)|)$ one can find the required $\eta(a)$.

The previous lemma guarantees that there exists $a > 0$ and $\eta > 0$ such that $\psi_a(\eta) = 0$, $\psi'_a(0) < 0$ and $\psi'_a(\eta) > 0$. The root η allows to define a convenient change-of-measure which we shall use repeatedly in our sampling strategy. In particular, we have that if $L_n = \exp(\eta S_n(a))$ then $(L_n : n \geq 0)$ is a positive martingale and induces a probability measure P_η defined for each $A \in \sigma(S_j(a) : j \leq k)$ (the σ-field generated by $S_1(a), ..., S_k(a)$) via $P_\eta(A) = E[\exp(-\eta S_k(a)); A]$. It is well known that under $P_\eta(\cdot)$ the random walk has positive drift equal to $\psi'_a(0) > 0$. In fact, if we let $\xi > 0$ and put $T_\xi = \inf\{n \geq 0 : S_n(a) > \xi\}$ then we have that

$$P(T_\xi < \infty) = E_\eta[\exp\left(-\eta S_{T_\xi}\right)].$$

Moreover, if $\xi_1 > \xi_0$ then

$$P(T_{\xi_0} < \infty, T_{\xi_1} = \infty) = E_\eta[\exp(-\eta S_{T_{\xi_0}}) P_{S_{T_{\xi_0}}}(T_{\xi_1} = \infty)].$$

If all we wanted is to simulate $M_0(a)$ we could take advantage of the following idea due to [10]. They observe that if one introduces an artificial random variable τ, exponentially distributed with unit mean and independent of the random walk under P_η, then we obtain

$$P(M_0(a) > x) = E_\eta[\exp(-\eta S_{T_x})] = P_\eta(\tau/\eta > S_{T_x}).$$

Then, if one defines the (random) function $G(u) = \inf\{x \geq 0 : S_{T_x} > u\}$ we obtain that $G(S_{T_x}) = x$ for almost every x with respect to the Lebesgue measure and therefore we conclude that

$$P(M_0(a) > x) = P_\eta(G(\tau/\eta) > x). \tag{15}$$

In other words, we have that $M_0(a) \overset{D}{=} G(\tau/\eta)$ and therefore we can sample $M_0(a)$ in finite time by sampling τ and then computing $G(\tau/\eta)$, which requires simulating $S_1(a), ..., S_{T_{\tau/\eta}}(a)$ under $P_\eta(\cdot)$. Our problem, however, is to simulate jointly the $M_k(a)$'s and the underlying random walk and for this reason we will require a sequential procedure.

Fix $m \geq 1$ so that $\exp(-3\eta m) < 1/2$. This is a technical constraint on m whose nature will become evident momentarily. Define the sequence of times $\Delta_1 = \inf\{n \geq 0 : S_n(a) < -2m\}$, $\Gamma_1 = \inf\{n \geq \Delta_1 : S_n(a) - S_{\Delta_1}(a) > m\}$ and for $j \geq 2$, $\Delta_j = \inf\{n \geq \Gamma_{j-1} I(\Gamma_{j-1} < \infty) \vee \Delta_{j-1} : S_n < S_{\Delta_{j-1}} - 2m\}$ and $\Gamma_j = \inf\{n \geq \Delta_j : S_n - S_{\Delta_j} > m\}$. We use the convention that if $\Gamma_{j-1} = \infty$, then $\Gamma_{j-1} I(\Gamma_{j-1} < \infty) = 0$ so we have that

$\Gamma_{j-1} I(\Gamma_{j-1} < \infty) > \Delta_{j-1}$ if and only if $\Gamma_{j-1} < \infty$. We will sequentially simulate the random walk at the times $\Delta_1, \Delta_2, \ldots$ jointly with the sequence $\Gamma_1, \Gamma_2, \ldots$. Note that $P(\Gamma_1 = \infty | S_{\Delta_1}(a)) > 0$ so simulating Δ_2 sequentially given Γ_1 requires being able to simulate the random walk conditional on $\Gamma_1 = \infty$. Similarly for subsequent Δ_j's. We will explain how to simulate the Δ_j's and Γ_j's sequentially jointly with the underlying random walk. However, first we note that indeed this sequential simulation procedure is all that is needed to simulate the $M_k(a)$'s jointly with the random walk $(S_n(a) : n \geq 0)$:

Proposition 3. *We have that $\Delta_n < \infty$ with probability one for each $n \geq 1$ and that $\Delta_n \nearrow \infty$ as $n \nearrow \infty$ also with probability one. Furthermore, if $\exp(-3\eta m) < 1/2$ then the event $P(\Gamma_n = \infty$ i.o.$) = 1$. Consequently, for each $k \geq 0$ we can find $N_0(k) = \inf\{n \geq 1 : \Delta_n \geq k\}$ and $\mathcal{T}(k) = \inf\{j \geq N_0(k) + 1 : \Gamma_j = \infty\}$ both finite random variables such that $M_k(a) = -S_k(a) + \max_{k \leq n \leq \cdot_{\mathcal{T}(k)}} S_n(a)$.*

Proof. The first statement of the proposition follows easily from the law of large numbers since $EY_1(a) < 0$. Now we show that $P(\Gamma_n = \infty \text{ i.o.}) = 1$. First it follows from the definition of Γ_1 that $P(\Gamma_1 = \infty | S_{\Delta_1}(a)) = P(T_m = \infty) > 0$. We claim that for $j \geq 2$ we can find $\delta > 0$ such that

$$P\left(\Gamma_j = \infty | S_1, \ldots, S_{\Delta_j}, \Gamma_1, \ldots, \Gamma_{j-1}\right) \geq \delta > 0.$$

To see this first suppose that $\Gamma_l < \infty$ for each $l = 1, 2, \ldots, j-1$. Then, by the strong Markov property we have that

$$P\left(\Gamma_j = \infty | S_1, \ldots, S_{\Delta_j}, \Gamma_1, \ldots, \Gamma_{j-1}\right) = P(T_m = \infty) > 0.$$

Now suppose that $\Gamma_l = \infty$ for some $l \leq j-1$ and let $l^* = \max\{l \leq j-1 : \Gamma_l = \infty\}$. Define $K = S_{\Delta_{l^*}} + m - S_{\Delta_j} \geq 3m$ and note that

$$P\left(\Gamma_j < \infty | S_1, \ldots, S_{\Delta_j}, \Gamma_1, \ldots, \Gamma_{j-1}\right) = P(T_m < \infty | T_K = \infty).$$

Keep in mind that in the conditional probability that appears in the right hand side we regard K as a deterministic constant. Now we have that

$$P(T_m < \infty | T_K < \infty) = \frac{P(T_m < \infty, T_K = \infty)}{1 - P(T_K < \infty)} = \frac{E_\eta[\exp(-\eta S_{T_m}) P_{S_{T_m}}(T_K = \infty)]}{1 - P(T_K < \infty)}.$$

Since $K \geq 3m$, we have that $P(T_K = \infty) = 1 - P(T_K < \infty) \geq 1 - \exp(-3\eta m)$. Therefore, the previous expression implies that

$$P\left(\Gamma_j = \infty | S_1, \ldots, S_{\Delta_j}, \Gamma_1, \ldots, \Gamma_{j-1}\right) \geq 1 - \frac{\exp(-3\eta m)}{1 - \exp(-3\eta m)}.$$

The right hand side is strictly positive if $\exp(-3\eta m) < 1/2$. Since the right hand side is non-random it follows from the Borel-Cantelli lemma that $P(\Gamma_n = \infty \text{ i.o.}) = 1$. Finally, the fact that $M_k(a) = -S_k(a) + \max_{k \leq n \leq \cdot_{\mathcal{T}(k)}} S_n(a)$ follows easily by construction. Note that is important, however, to define $\mathcal{T}(k) \geq N_0(k) + 1$ so that $\Delta_{N_0(k)+1}$ is computed first and we can make sure that the maximum of the sequence $\{S_n(a) : n \geq k\}$ is achieved between k and $\cdot_{\mathcal{T}(k)}$.

The next lemmas provide the basis to simulate the random walk $(S_n(a) : n \geq 0)$ jointly with the Δ_j's and the Γ_j's. First, provide a representation that allows to simulate a Bernoulli random variable with success parameter $P(T_{\xi_0} < \infty | T_{\xi_1} = \infty)$. The result is straightforward so the proof is omitted.

Lemma 2. *Let $0 < \xi_0 < \xi_1 \leq \infty$ then we have*

$$P(M_0(a) > \xi_0 | M_0(a) \leq \xi_1) = P(T_{\xi_0} < \infty | T_{\xi_1} = \infty).$$

In particular, we can simulate a Bernoulli with parameter $P(T_{\xi_0} < \infty | T_{\xi_1} = \infty)$ if we just sample $M_0(a)$ given that $M_0(a) \leq \xi_1$ and then output $I(M_0(a) > \xi_0)$.

Next we describe how to simulate the random walk conditional on $T_{\xi_0} < \infty$ and $T_{\xi_1} = \infty$.

Lemma 3. *Let $0 < \xi_0 < \xi_1 \leq \infty$ and consider any sequence of bounded positive measurable functions $f_{k+1} : \mathbb{R}^{k+1} \longrightarrow [0, \infty)$ and define $\zeta(\xi_0, \xi_1) = \exp(-\eta S_{T_{\xi_0}}) P_{S_{T_{\xi_0}}}(T_{\xi_1} = \infty)$. Then we obtain that*

$$\begin{aligned} &E[f_{T_{\xi_0}}(S_0(a), ..., S_{T_{\xi_0}}(a)) | T_{\xi_0} < \infty, T_{\xi_1} = \infty] \\ &= \frac{E_\eta[f_{T_{\xi_0}}(S_0(a), ..., S_{T_{\xi_0}}(a)) \zeta(\xi_0, \xi_1)]}{E_\eta[\zeta(\xi_0, \xi_1)]}. \end{aligned}$$

So, if $P^*(\cdot) = P(\cdot | T_{\xi_0} < \infty, T_{\xi_1} = \infty)$ we conclude that

$$\frac{dP^*}{dP_\eta} = \frac{\zeta(\xi_0, \xi_1)}{E_\eta[\zeta(\xi_0, \xi_1)]} \leq \frac{1}{E_\eta[\zeta(\xi_0, \xi_1)]}. \tag{16}$$

Consequently, we can apply acceptance rejection. In particular, we propose a sample $S_1(a), ..., S_{T_{\xi_0}}(a)$ from $P_\eta(\cdot)$ and accept with probability

$$\exp(-\eta S_{T_{\xi_0}}) P_{S_{T_{\xi_0}}}(T_{\xi_1} = \infty).$$

Finally, we observe that acceptance occurs with probability precisely equal to $P(T_{\xi_0} < \infty, T_{\xi_1} = \infty)$. In particular, if $\xi_1 = \infty$ we have that $P_{S_{T_{\xi_0}}}(T_{\xi_1} = \infty) = 1$ and $P(T_{\xi_0} < \infty, T_{\xi_1} = \infty) = P(T_{\xi_0} < \infty)$, so in this case the acceptance step yields a Bernoulli with parameter $P(T_{\xi_0} < \infty)$ and if the Bernoulli is successful the sample path follows the law $S_1(a), ..., S_{T_{\xi_0}}(a)$ given that $T_{\xi_0} < \infty$.

Proof. Martingale theory and the strong Markov property yield

$$\begin{aligned} &E[f_{T_{\xi_0}}(S_0(a), ..., S_{T_{\xi_0}}(a)), T_{\xi_0} < \infty, T_{\xi_1} = \infty] \\ &= E_\eta[f_{T_{\xi_0}}(S_0(a), ..., S_{T_{\xi_0}}(a)) \exp(-\eta S_{T_{\xi_0}}) P_{S_{T_{\xi_0}}}(T_{\xi_1} = \infty)]. \end{aligned}$$

Letting $f_k = 1$ we conclude that

$$P(T_{\xi_0} < \infty, T_{\xi_1} = \infty) = E_\eta[\zeta(\xi_0, \xi_1)]$$

and therefore we arrive at the likelihood ratio in (16). The rest of the result by standard result on acceptance rejection algorithms.

Finally, given $\xi_0 \in (0,\infty)$ denote $T_{-\xi_0} = \inf\{n \geq 0 : S_n < -\xi_0\}$. We shall explain how to simulate a path up to time $T_{-\xi_0}$ conditional on $T_{\xi_1} = \infty$ for any $\xi_1 \in (0,\infty]$. The result

Lemma 4. *Let $0 < \xi_0 < \xi_1 \leq \infty$ and consider any sequence of bounded positive measurable functions $f_{k+1} : \mathbb{R}^{k+1} \longrightarrow [0,\infty)$.*

$$\begin{aligned} & E[f_{T_{\xi_0}}(S_0(a), ..., S_{T_{-\xi_0}}(a)) | T_{\xi_1} = \infty] \\ & = \frac{E[f_{T_{\xi_0}}(S_0(a), ..., S_{T_{\xi_0}}(a)) P_{S_{T_{-\xi_0}}}(T_{\xi_1} = \infty)]}{P(T_{\xi_1} = \infty)}. \end{aligned}$$

So, if $P^*(\cdot) = P(\cdot | T_{\xi_1} = \infty)$ we conclude that

$$\frac{dP^*}{dP} = \frac{P_{S_{T_{-\xi_0}}}(T_{\xi_1} = \infty)}{P(T_{\xi_1} = \infty)} \leq \frac{1}{P(T_{\xi_1} = \infty)}. \tag{17}$$

Consequently, we can apply acceptance rejection to sample $S_0(a), ..., S_{T_{-\xi_0}}(a)$ given $T_{\xi_1} = \infty$. In particular, we propose a sample $S_1(a), ..., S_{T_{-\xi_0}}(a)$ from $S_0(a), ..., S_{T_{-\xi_0}}(a)$ under the nominal (unconditional probability) and accept the path with probability $P_{S_{T_{-\xi_0}}(a)}(T_{\xi_1} = \infty)$.

Proof. The result follows directly from the strong Markov property and basic facts of acceptance rejection.

We conclude the section with a summary of the sequential procedure for the random walk.

Sequential Simulation of the Random Walk

Select m such that $\exp(-3\eta m) < 1/2$ and pick κ such that Set $s_0 = s = 0$, $\xi_1 = \infty$, $t = 0$.

Output the random walk $(s_0, s_1, ...)$ and the times $(\Delta_1, \Delta_2, ...)$, $(\Gamma_1, \Gamma_2, ...)$

At iteration $k \geq 1$ proceed as follows

Step 1: Sample $S_1(a), ..., S_{T_{-2m}}(a), T_{-2m}$ given that $T_{\xi_1} = \infty$ and $S_0(a) = 0$ (apply Lemma 4 and (15)).

Step 2: Let $\Delta_k = t + T_{-2m}$, $s_{t+1} = s + S_1(a), ..., s_{\Delta_k} = s + S_{T_{-2m}}(a)$, $t \longleftarrow \Delta_k$, $s \longleftarrow s_{\Delta_k}$

Step 3: Simulate a Bernoulli J with parameter $P(T_m < \infty | T_{\xi_1} = \infty)$ (apply Lemma 2).

Step 4: IF $J = 1$ then sample $S_1(a), ..., S_{T_m}(a), T_m$ given $T_m < \infty$ and $T_{\xi_1} = \infty$ (apply Lemma 3). Let $\Gamma_k = t + T_m$, $s_{t+1} = s + S_1(a), ..., s_{\Gamma_k} = s + S_{T_m}(a)$, $t \longleftarrow \Delta_k$, $s \longleftarrow s_{\Gamma_k}$. ELSE $(J = 0)$ let $\xi_1 = s + m$ and $\Gamma_k = \infty$.

Step 5: Set $k \longleftarrow k + 1$ and go to Step 1.

5. Finite Termination Time

All the elements of our algorithm for the exact simulation of X are in place now. The simulation of the B_n's has been discussed in Section 3, we have discussed how to simulate $Y_n = Y_n(a) - a$ for $n \geq 1$ in Section 4 and also in these two sections we have indicated how to construct and simulate the V_n^+'s. The only remaining issue is to make sure that $N_0 < \infty$ with probability one.

Let ϱ be the number of iterations that are required to terminate the algorithm. In order to show that $E\varrho < \infty$ we will take advantage of the following bound

$$V_k^+ = \exp(ka)\exp(M_k(a))\sum_{n=k}^{\chi}\exp(-na)\widehat{B}_n I_n + \frac{\exp(M_k(a))}{1-\exp(-a)}$$
$$\leq D^+ := \exp(\chi a)\exp(M_k(a))\sum_{n=0}^{\chi}\exp(-na)\widehat{B}_n + \frac{\exp(M_k(a))}{1-\exp(-a)}.$$

The following lemma allows us to provide a bound on $E\varrho$ based on a bound on the mean of $\Theta_\kappa = \inf\{n \geq 0 : D_n < \kappa\}$ given that D_0 is selected as an independent copy of D^+. The fact that $N_0 < \infty$ with probability one clearly follows from the fact that $E\varrho < \infty$.

Lemma 5. *Let $g_\kappa(d) = E_d\Theta_\kappa$, then*

$$Eg_\kappa(D^+) \leq \frac{1}{1-\lambda_\kappa}\{Eg_\kappa(D^+) + \sup_{0\leq x\leq 1} E_x[g_\kappa(\exp(Y)x + B)]\}.$$

Proof. First, for convenience let us extend the construction $\{D_j(2l, V_{2l}^+) : 0 \leq j \leq 2l\}$ to values of $j > 2l$ using the dynamics of the Markov chain with an independent sequence of B_n's, Y_n's and U_n's. Now let $A_{2l}(1, V_{2l}^+) = \inf\{j \geq 0 : D_j(2l, V_{2l}^+) \leq 1\}$ and set

$$A_{2l}(i+1, V_{2l}^+) = \inf\{j > A_{2l}(i, V_{2l}^+) : D_j(2l, V_{2l}^+) \leq 1\}.$$

We then define $N(k, V_{2l}^+) = \max\{m \geq 0 : A_{2l}(m, V_{2l}^+) \leq k\}$. We note that

$$P(\varrho > l) \leq E[(1-\lambda)^{N(2l, V_{2l}^+)}].$$

The previous inequality simply says that $\varrho > l$ implies that the process $\{D_j(2l, V_{2l}^+) : 0 \leq j \leq 2l\}$ either does not visit the interval $[0, \kappa]$ or, when it does visit $[0,\kappa]$, a successful coupling does not occur. Now we introduce an artificial random variable τ, geometrically distributed with success parameter λ, then

$$E[(1-\lambda)^{N(2l, V_{2l}^+)}] = P(\tau > N(2l, V_{2l}^+)) \leq P(A_{2l}(\tau, V_{2l}^+) > 2l).$$

Consequently,

$$E\varrho \leq EA_{2l}(\tau, V_{2l}^+).$$

By monotonicity and a submartingale argument we obtain that

$$EA_{2l}(\tau, V_{2l}^+) \leq \frac{1}{1-\lambda_\kappa}\{Eg_\kappa(D^+) + \sup_{0\leq x\leq 1} E_x[g_\kappa(\exp(Y)x + B)]\},$$

thereby concluding the result.

In order to obtain a bound on $g_\kappa(d)$ we will take advantage of the following well known Foster-Lyapunov criterion.

Proposition 4. *Suppose that we can find a non-negative function $h_\kappa(\cdot)$ such that*

$$E[h_\kappa(d\exp(Y_1)+B_1)]-h_\kappa(d)\leq -1$$

for all $d\geq\kappa$. Then $g_\kappa(d)\leq h_\kappa(d)$.

The following lemma provides the construction of a suitable Lyapunov function from the previous proposition.

Lemma 6. *Given $\kappa>\kappa_0$ we can find $c\in(0,\infty)$ such that $h(x)=c\log(1+x)$ is an appropriate Lyapunov function. In particular, for each $d\geq\kappa$ we have that $E_d\Theta_\kappa\leq h_\kappa(d)$.*

Proof. Note that

$$0\leq\log\left(\frac{1+d\exp(Y_1)+B_1}{1+d}\right)\leq\log(1+B_1+\exp(Y_1)).$$

In addition, for each $\delta>0$ we can find $C_\delta\in(0,\infty)$ such that $\log(1+r)\leq C_\delta r^\delta$. Therefore,

$$\log\left(\frac{1+B_1+\exp(Y_1)}{1+B_1}\right)=\log\left(1+\frac{\exp(Y_1)}{1+B_1}\right)\leq C_\delta\exp(\delta Y_1)$$

and we conclude that

$$0\leq\log\left(\frac{1+d\exp(Y_1)+B_1}{1+d}\right)\leq\log(1+B_1)+C_\delta\exp(\delta Y_1).$$

We then recall bound (5) and the fact that $\kappa>\kappa_0$ to conclude that if $c>2/|EY_1|$ and $d\leq\kappa$ then

$$E[h_\kappa(d\exp(Y_1)+B_1)]-h_\kappa(d)=cE\log\left(\frac{1+d\exp(Y_1)+B_1}{1+d}\right)\leq cEY_1/2\leq -1,$$

thereby concluding the result.

Lemma 7. *We have that*

$$Eg_\kappa\left(D^+\right)<\infty\text{ and }\sup_{0\leq x\leq 1}E_x[g_\kappa(\exp(Y)x+B)]\}<\infty.$$

Therefore, $E\varrho<\infty$ and thus the algorithm terminates in finite time with probability one.

Proof. It suffices to show that $E\log(1+D^+)<\infty$ to show the first part of the result. The second part is straightforward following a similar argument as in the proof of Lemma 6. Note that

$$\begin{aligned}E\log\left(1+D^+\right)&\leq EM_0(a)+E\{\log[1+\exp(\chi a)\sum_{n=0}^{\chi}\exp(-na)\widehat{B}_n+1/(1-\exp(-a))]\}\\&\leq EM_0(a)+\log(3/[1-\exp(-a)])+aE\chi+E\log(\chi\max_{n=0}^{\chi}\exp(-na)\widehat{B}_n).\end{aligned}$$

It is well know that $M_0(a)$ has exponentially decaying tails, so in particular $EM_0(a) < \infty$. Now we have that

$$P(\chi = k) \leq P(\kappa + B_k \geq \exp(ka)),$$

since we are assuming that $EB_k^p < \infty$ for some $p > 0$ we clearly obtain that $E\chi < \infty$. Observe that

$$E \log(\max_{n=0}^{\chi} \exp(-na) \widehat{B}_n) \leq E[\sum_{n=0}^{\chi} \{-na + \log(\widehat{B}_n)\}]$$

and also note that for each $\delta > 0$ there exists a constant $C(\delta, n)$ such that $C(\delta, n) \longrightarrow 0$ as $n \nearrow \infty$

$$\begin{aligned} & E[-na + \log(B_n + \kappa) \,|B_n + \kappa \geq \exp(na)] \\ & = \int_{na}^{\infty} \frac{P(\log(B_n + \kappa) > t)}{P(\log(B_n + \kappa) > na)} dt \leq C(\delta, n) P(\log(B_n + \kappa) > na)^{-\delta}. \end{aligned}$$

Therefore, if $a \in (0, 1)$,

$$E[\sum_{n=0}^{\chi} \{-na + \log(\widehat{B}_n)\}] \leq E\{\chi P(\log(B_n + \kappa) > \chi|\chi)^{-\delta}\}.$$

We then conclude

$$E\{\chi P(\log(B_n + \kappa) > \chi|\chi)^{-\delta}\} \leq \sum_{k=1}^{\infty} kP(\log(B_n + \kappa) > k|)^{1-\delta} < \infty,$$

where the last inequality follows from Chebyshev's bound again using the fact that $E(B_k + \kappa)^p < \infty$ for some $p > 0$. The result then follows from Lemma 5.

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