EXACT SIMULATION OF MULTIDIMENSIONAL REFLECTED BROWNIAN MOTION

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Abstract. We present the first exact simulation method for multidimensional reflected Brownian motion (RBM). Exact simulation in this setting is challenging because of the presence of correlated local-time-like terms in the definition of RBM. We apply recently developed so-called ε-strong simulation techniques (also known as Tolerance-Enforced Simulation) which allow us to provide a piece-wise linear approximation to RBM with ε (deterministic) error in uniform norm. A novel conditional acceptance / rejection step is then used to eliminate the error. In particular, we condition on a suitably designed information structure so that a feasible proposal distribution can be applied.

1. Introduction

This paper is a contribution to the theory of exact simulation for stochastic differential equations (SDEs). In particular, we present the first exact simulation algorithm for multidimensional reflected Brownian motion (RBM).

Multidimensional RBM was introduced by Harrison and Reiman in [8] and it figures prominently in stochastic Operations Research. It turns out that RBM approximates the workload at each station in so-called generalized Jackson networks, which are comprised by single-server queues connected via Markovian routing. The approximation holds in heavy traffic (i.e. as the system approaches 100% utilization) and it is applicable in great generality (assuming only a functional central limit theorem for the arrival process and the service requirements at each station, see for example [11] and [5]).

Let us provide the specific definition of an RBM so that we can describe precisely our contribution and outline our strategy. Underlying the definition there is a matrix $R \in \mathbb{R}^{d\times d}$ of the form $R = I - Q^T$, where Q is substochastic and $Q^n \to 0$ as $n \to \infty$. The matrix Q is often referred as the routing matrix and R as the reflection matrix. For a given a continuous process $\mathbf{X}(\cdot)$ taking values in $\mathbb{R}^d$, the tuple $(\mathbf{Y}, \mathbf{L}) := (\mathbf{Y}(t), \mathbf{L}(t) : t \geq 0)$ is said to solve the *Skorokhod problem* if for a given initial value $\mathbf{Y}(0)$,

$$\mathbf{Y}(t) = \mathbf{Y}(0) + \mathbf{X}(t) + R\mathbf{L}(t), t \geq 0 \tag{1}$$

with $\mathbf{Y}(t) = (Y_1(t), \ldots, Y_d(t))^T$ and $\mathbf{L}(t) = (L_1(t), \ldots, L_d(t))^T$ satisfying:

a) $Y_i(t) \geq 0$ for all $t \geq 0$,
b) $L_i(t)$ is non-decreasing in t, and $L_i(0) = 0$,
c) $\int_0^t Y_i(s)dL_i(s) = 0$,

for each $t \geq 0$ and $i = 1, \ldots, d$. We call $\mathbf{X}(\cdot)$ the driving (or free) process, and $\mathbf{Y}(\cdot)$ the reflected process. The map S that takes $\mathbf{X}(\cdot)$ to $\mathbf{Y}(\cdot)$ in (1) is referred to as the Skorokhod map.

Multidimensional RBM is defined as the process $\mathbf{Y}(\cdot)$ obtained from the solution to the Skorokhod problem when the driving process $\mathbf{X}(\cdot)$ is a multidimensional Brownian motion (either standard or with a constant drift and diffusion matrix). As shown by Harrison and Reiman in [8], the pair $(\mathbf{Y}, \mathbf{L})$ is well defined (i.e. the solution of the Skorokhod problem exists and is unique). The Skorokhod problem has been further extended to more general reflection matrices R and non-continuous driving

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processes. Although much of our discussion here could be extended to some of these settings as well, here we concentrate on the standard setting introduced by Harrison and Reiman as this is enough to deal with multidimensional RBM.
This paper is dedicated to the proof of the following result.

Theorem 1. *Suppose that the input process* $\mathbf{X}(\cdot)$ *is Brownian motion and let* $T \in (0,1]$ *be any deterministic time. Then we can simulate* $\mathbf{Y}(T)$ *without any bias.*

Because of condition c) in the definition of RBM, it turns out that $L_i(\cdot)$ increases only at times t when $Y_i(t) = 0$ and so $L_i(\cdot)$ behaves like the local time of Brownian motion at the origin. So the term $R\mathbf{L}(t)$ appearing in (1) is not a standard "drift" term and cannot be dealt with using change of measure techniques as in [1], [2] or [9]. All the generic exact simulation techniques for SDEs are based on the use of acceptance / rejection, after applying Girsanov's transformation. The difficulty in applying acceptance / rejection in the multidimensional RBM setting is that there is no natural proposal distribution that can be used to "dominate" the target process directly. In particular, multidimensional RBM is not absolutely continuous with respect to any natural process that is easily simulatable. Note that in one dimension one can simulate RBM directly and so these challenging issues arise only in dimensions greater than one. Simulation techniques for one dimensional reflected processes have been studied in [7].
At the core of our strategy lies the following simple observation. Let us forget for the moment about the RBM exact sampling problem. Let Δ and Y be two independent random variables. Moreover, for simplicity let us assume that the density of Δ, $f_\Delta(\cdot)$, is continuous on its support which is given by the interval $[-a,a]$ for some $a \in (0,\infty)$. So, $\sup_{x\in[-a,a]} f_\Delta(x) \leq C$ for some $C \in (0,\infty)$. Let us assume that Y is also supported on $[-a,a]$ with an arbitrary distribution.
Say we want to simulate $W = Y + \Delta$. Imagine we can sample Y directly, so we might consider $Y = y$ simply as a parameter. Given $Y = y$, the density of W evaluated at w is simply $f_\Delta(w-y)$. Moreover, the support of such density is contained in the interval $[-2a, 2a]$. So, to simulate W conditional on y, we could propose W uniformly distributed in $[-2a, 2a]$, and accept if

$$V < C^{-1} f_\Delta(W - y), \tag{2}$$

where $V \sim U(0,1)$ (uniform in $(0,1)$), and independent of W and $Y = y$.
The key observation is that in order to accept W we simply need to decide if inequality (2) holds; *we do not actually need to know the value of* y. So, instead of having direct access to Y, we might instead know $\{Y^{\varepsilon_n}\}_{n\geq 1}$, independent of Δ, which converges to Y; say $\|Y^{\varepsilon_n} - Y\| \leq \varepsilon_n$ for some $\varepsilon_n \to 0$ as $n \to \infty$. Then under modest continuity properties of $f_\Delta(\cdot)$, for instance say $|f_\Delta(x) - f_\Delta(x')| \leq K\|x - x'\|$, we can accept W if

$$V \leq C^{-1} f_\Delta(W - Y^{\varepsilon_n}) - KC^{-1}\varepsilon_n, \tag{3}$$

or reject W if

$$V \geq C^{-1} f_\Delta(W - Y^{\varepsilon_n}) + KC^{-1}\varepsilon_n. \tag{4}$$

Since $\varepsilon_n \to 0$ and $V = C^{-1} f_\Delta(W - Y)$ has zero probability of occurring, one must be able to eventually decide whether to accept or reject. It is crucial to make sure, however, that the sequence $\{Y^{\varepsilon_n}\}_{n\geq 1}$ be independent of Δ.
Now, coming back to exact sampling of RBM, our plan is to apply the previous strategy by introducing a suitable conditioning. We use key facts about multidimensional RBM. First, the fact that if the driving process is Brownian motion then, for fixed T, $\mathbb{P}(Y_i(T) = 0) = 0$ for any $i \in \{1,...,d\}$. In addition, since $\mathbf{Y}(\cdot)$ is continuous, there exists a $\delta > 0$ and an interval $(T_{left}, T_{right}]$ which contains T, satisfying that $Y_i(s) > \delta$ for all $s \in (T_{left}, T_{right}]$ and all $i \in \{1,...,d\}$.

So, our plan is to first simulate enough information about $\mathbf{X}(\cdot)$ (i.e. the free Brownian motion) so that conditional on such information we have the representation

$$\mathbf{Y}(T) = \mathbf{Y}(T_{left}) + \mathbf{X}(T) - \mathbf{X}(T_{left}) := \mathbf{Y}(T_{left}) + \mathbf{\Delta}. \tag{5}$$

The objective is then to find a suitable conditioning satisfying, first, that $\Delta := \mathbf{X}(T) - \mathbf{X}(T_{left})$ and the density $f_{\Delta}(\cdot)$ can be evaluated, and second that $\mathbf{Y}(T_{left})$ is independent of Δ. Naturally, we identify $\mathbf{Y}(T)$ with the variable W introduced in the general setting discussed previously.

In order to condition on enough information so that we can guarantee $\mathbf{L}(T) - \mathbf{L}(T_{left}) = 0$, we use another important property of the Skorokhod map, S, namely, S is Lipschitz continuous as a function of the free process in the uniform norm over the time interval $[0, 1]$. Consequently, to detect the interval $(T_{left}, T_{right}]$ we use so-called ε-strong simulation techniques, also known as Tolerance-Enforced Simulation (TES), which allows us to simulate $\mathbf{X}^{\varepsilon}(\cdot)$ piecewise linear and guaranteed to be within ε-close in uniform norm to $\mathbf{X}(\cdot)$. This construction is, conceptually at least, not complicated. There are several methods that can be applied for the same: based on wavelets as in [4], localization using stopping times as in [6], or tracking jointly the maxima and minima on dyadic intervals as in [3]. We have chosen to use the latter construction, thereby ultimately obtaining $(T_{left}, T_{right}]$ a dyadic interval (i.e. $T_{left} = i2^{-N}$ and $T_{right} = j2^{-N}$ for some $0 \leq i < j \leq 2^{-N}$ and $N > 0$). The reason for choosing the construction in [3] is because it allows us to preserve the (conditional) independence of Δ and $\mathbf{Y}(T_{left})$ given all the information required to conclude that $\mathbf{L}(T_{right}) - \mathbf{L}(T_{left}) = 0$.

The Skorokhod problem is easy to solve for piecewise linear input $\mathbf{X}^{\varepsilon}$, because in such case one obtains a piecewise linear pair $(\mathbf{Y}^{\varepsilon}, \mathbf{L}^{\varepsilon})$ and so the gradients can be obtained by solving linear systems based on (1). Since, as it turns out, $\mathbf{Y}^{\varepsilon_n}$ can be computed explicitly for a computable $\{\varepsilon_n\}_{n\geq 1}$ such that $\varepsilon_n \to 0$ as $n \to \infty$, the Lipschitz continuity of $\mathbf{Y}$ as a function of $\mathbf{X}$, combined with the approximation $\mathbf{X}^{\varepsilon_n}$, and the fact that $\mathbf{Y}(T)$ must be strictly positive coordinate-wise, eventually can be used to detect $(T_{left}, T_{right}]$.

Our construction of $\mathbf{X}^{\varepsilon}(\cdot)$, as indicated earlier, based on [3] will give rise to a density for Δ which is expressed as an infinite series. So, the Lipschitz continuity of $f_{\Delta}(\cdot)$ used in (3) and (4) is obtained by means of some careful estimates. Since the value of $\mathbf{Y}(T_{left})$ depends on the information prior to time T_{left}, we are able to obtain the independence of Δ given all the information required to compute $\mathbf{Y}(T_{left})$, even including refinements $\mathbf{Y}^{\varepsilon_n}(T_{left})$ that we can keep making by increasing $n \geq 1$. Consequently, we will be able to implement the basic strategy underlying (2), (3), and (4).

We wish to finish this introduction with a critical discussion of our main result. We do believe that the conditional acceptance / rejection strategy introduced here is of significant value as it addresses an important open problem (exact sampling of multidimensional RBM). Nevertheless, we must recognize that the algorithm, in its current form, is mostly of theoretical interest. Unfortunately, it is not difficult to see that the expected running time of the algorithm is infinite. One can see this already from the general procedure underlying (2), (3), and (4). Note that conditional on W and Y, the distance $D = \left|V - C^{-1} f_{\Delta}(W - Y)\right|$ is less than δ with probability $O(\delta)$ (because V is uniformly distributed). Thus, if the cost of generating Y^D (required to decide whether to accept or reject) is $C(D)$, the running time of the algorithm would be finite if $\int_0^1 C(u)\, du < \infty$. Unfortunately, however, the cost of producing an ε-strong approximation to Brownian motion $(\mathbf{X}^{\varepsilon})$ is roughly $O\left(1/\varepsilon^2\right)$ (see, for example, [3]) and therefore $C(D) \geq c/D^2$ for some $c > 0$, which yields an infinite expected running time. We are investigating strategies to mitigate this problem, for example exploring different proposal distributions. We feel that the practical implementation of our procedure here is a research topic in its own right and thus we prefer to leave implementation issues for future research.

In Section 2 we provide more specific details behind our sampling methodology and point to future relevant sections where details are fully fleshed out.

2. Sampling Methodology

We first introduce some notational conventions. Throughout the paper we consider the driving process $\mathbf{X}(\cdot)$ to be a standard Brownian motion in d-dimensions and thus we write $\mathbf{X}(\cdot) = \mathbf{B}(\cdot)$. The resulting solution $\mathbf{Y}(\cdot)$ in (1) is referred to as the Reflected Brownian motion (RBM). The extension of our development to the case in which $\mathbf{X}(\cdot)$ is a Brownian motion with constant drift and diffusion coefficients is straightforward. While all the variables and processes taking values in $\mathbb{R}^d$ for $d > 1$ are typeset in boldface, their 1-dimensional counterparts are not. For example, if $\mathbf{B}(\cdot)$ denotes the Brownian motion in multiple dimensions, then $B(\cdot)$ is to be understood as 1-dimensional Brownian motion.

Now let us introduce additional notation to clearly outline the major aspects of our algorithm and later we shall dwell on the details of simulation. As indicated in the introduction, we use conditional acceptance / rejection to obtain samples of $\mathbf{Y}(T)$, where the conditioning information is carefully chosen to be able to arrive at representation (5).

2.1. Overview of the Sampling scheme.

Consider the following uniform metric on the space of $\mathbb{R}^d$-valued continuous functions $C[0,1]^d$:

$$d(\mathbf{x}(\cdot), \mathbf{y}(\cdot)) := \max_{1 \le i \le d} \sup_{0 \le t \le 1} |x_i(t) - y_i(t)|,$$

for $\mathbf{x}(t) = (x_1(t), \ldots, x_d(t)), \mathbf{y}(t) = (y_1(t), \ldots, y_d(t)) \in C[0,1]^d$. Our algorithm consists of three major steps outlined in the following three subsections.

2.1.1. *Tolerance-Enforced Simulation of RBM.*

The strategy is to first generate a piecewise linear function on $[0,1]$ that approximates the underlying RBM path $\mathbf{Y}(\cdot)$ close enough in the uniform metric; that is, given any $\varepsilon > 0$, we aim to generate a process $(\mathbf{Y}^\varepsilon(t) : 0 \le t \le 1)$ such that,

$$d(\mathbf{Y}(\cdot), \mathbf{Y}^\varepsilon(\cdot)) < \varepsilon. \tag{6}$$

The fact that one can obtain a piecewise linear process $\mathbf{Y}^\varepsilon(\cdot)$ that does not differ from the RBM $\mathbf{Y}(\cdot)$ by more than ε (in the uniform topology on $C[0,1]$) is because of the following reasons:

1) Given $\varepsilon > 0$, it is possible to obtain piecewise linear approximations $\mathbf{B}^\varepsilon(\cdot)$ to the driving Brownian motion $\mathbf{B}(\cdot)$ such that $d(\mathbf{B}(\cdot), \mathbf{B}^\varepsilon(\cdot)) < \varepsilon$. This can be accomplished using the ε-strong algorithm of [3], the localization method of [6], or the wavelet approach of [4]. In this paper we use the approach in [3].
2) In the Skorokhod map S defined by (1), if the driving process $\mathbf{X}(\cdot)$ is piecewise linear, then the reflected process $\mathbf{Y}(\cdot)$ is also piecewise linear. The gradients determining $\mathbf{Y}(\cdot)$ are computable by solving a linear system of equations resulting from (1).
3) The Skorokhod map S is Lipschitz continuous under the uniform metric on $C([0,1], \mathbb{R}^d)$, and the Lipschitz constant K equals $1/(1-\alpha)$; here $0 \le \alpha < 1$ denotes the spectral radius of the matrix $Q = I - R$ (see [8]).

While 2) above talks about the computability of the approximating process $\mathbf{Y}^\varepsilon(\cdot)$ from $\mathbf{B}^{\varepsilon/K}(\cdot)$, 3) renders that such an approximation indeed satisfies (6).

2.1.2. *Proposal for Y(T).*

Since the time spent by the RBM $\mathbf{Y}(\cdot)$ at the reflecting boundary is zero, it follows that $\mathbf{Y}(T)$ lies in the interior of positive orthant with probability one. The next objective is to identify a time interval $(T_{left}, T_{right}]$ containing T such that the underlying path of the RBM $\mathbf{Y}(\cdot)$ remains strictly inside the positive orthant during $(T_{left}, T_{right}]$. Such an interval exists almost surely for every path because of the continuity properties of RBM. Since the RBM $\mathbf{Y}(\cdot)$ does not hit the reflecting boundary anywhere in $(T_{left}, T_{right}]$, it is easy to see that $\mathbf{L}(T_{right}) - \mathbf{L}(T_{left}) = \mathbf{0}$,

and the dynamics of RBM $\mathbf{Y}(\cdot)$ matches locally with that of the driving Brownian motion $\mathbf{B}(\cdot)$ in the following sense:

$$\mathbf{B}(t) - \mathbf{B}(T_{left}) = \mathbf{Y}(t) - \mathbf{Y}(T_{left}), \text{ for } t \in (T_{left}, T_{right}]. \tag{7}$$

In Section 2.2, we shall see that it is algorithmically possible to identify a $\delta > 0$ and simulate a piece-wise linear process $\mathbf{Y}^\delta(\cdot)$ satisfying $d(\mathbf{Y}(\cdot), \mathbf{Y}^\delta(\cdot)) < \delta$ from which we can determine such an interval $(T_{left}, T_{right}]$. For accomplishing this, it becomes necessary to simulate certain information about the driving Brownian motion (for example, realized values of the underlying Brownian path at certain discrete set of points, extrema of the Brownian path, etc.). Let $(\mathcal{F}_t : t \geq 0)$ denote the standard Brownian filtration and $\mathcal{I}$ denote the collection of all the random variables required to arrive at the approximation $\mathbf{Y}^\delta(\cdot)$. We make the following assumptions on $\mathcal{I}$, which shall be shown to hold in Section 2.3 after introducing Algorithm 1 and the contents of $\mathcal{I}$:

Assumption 1. *Conditional on $\mathcal{I}$, the Brownian increment $\mathbf{\Delta} := \mathbf{B}(T) - \mathbf{B}(T_{left})$ remains independent of the past, $\mathcal{F}_{T_{left}}$, and its law is computable. In other words,*

(1) $Law(\mathbf{B}(T) - \mathbf{B}(T_{left})|\mathcal{F}_{T_{left}}, \mathcal{I}) = Law(\mathbf{B}(T) - \mathbf{B}(T_{left})|\mathcal{I})$.
(2) $f_{\mathbf{\Delta}}(x)d\mathbf{x} := \mathbb{P}\{\mathbf{B}(T) - \mathbf{B}(T_{left}) \in d\mathbf{x}|\mathcal{I}\}$ is computable.

Define $\tilde{\mathbf{\Delta}} := \mathbf{Y}(T) - \mathbf{Y}^\delta(T_{left})$. Since $\mathbf{\Delta} = \mathbf{Y}(T) - \mathbf{Y}(T_{left})$ because of (7), $\mathbf{\Delta}$ and $\tilde{\mathbf{\Delta}}$ are related as below:

$$\tilde{\mathbf{\Delta}} = \mathbf{\Delta} - \mathbf{Y}^\delta(T_{left}) + \mathbf{Y}(T_{left}).$$

Now consider the following two step procedure for proposing for $\mathbf{Y}(T)$:

(1) Choose a density $g(\cdot)$ which is easy to sample from. Draw a sample $\tilde{\mathbf{Z}}$ from the density $g(\cdot)$ and propose $\tilde{\mathbf{Z}}$ for $\tilde{\mathbf{\Delta}}$.
(2) Since we know the approximation $\mathbf{Y}^\delta(\cdot)$ pointwise, propose $\mathbf{Z}$ for $\mathbf{Y}(T)$ as below:

$$\mathbf{Z} = \mathbf{Y}^\delta(T_{left}) + \tilde{\mathbf{Z}}.$$

Since $\tilde{\mathbf{\Delta}} = \mathbf{\Delta} - \mathbf{Y}^\delta(T_{left}) + \mathbf{Y}(T_{left})$, likelihood of the proposal $\mathbf{Z}$ from the actual measure with respect to the proposal measure is given by,

$$L(\mathbf{Z}) = \frac{f_{\mathbf{\Delta}}\left(\tilde{\mathbf{Z}} + \mathbf{Y}^\delta(T_{left}) - \mathbf{Y}(T_{left})\right)}{g(\tilde{\mathbf{Z}})} = \frac{f_{\mathbf{\Delta}}\left(\mathbf{Z} - \mathbf{Y}(T_{left})\right)}{g\left(\mathbf{Z} - \mathbf{Y}^\delta(T_{left})\right)}. \tag{8}$$

2.1.3. *Accept/Reject* $\mathbf{Z}$. If $g(\cdot)$ is chosen in a way that the likelihood ratio $L(\cdot)$ is bounded from above by some constant c, then accepting $\mathbf{Z}$ with probability $L(\mathbf{Z})/c$ will yield an exact sample of $\mathbf{Y}(T)$. This is equivalent to drawing V uniformly from $[0, 1]$ and accepting $\mathbf{Z}$ only if

$$cV < L(\mathbf{Z}).$$

From (8), we see that performing this comparison requires the knowledge of $\mathbf{Y}(T_{left})$ which we do not possess. To overcome this difficulty, consider that the following are true:

Assumption 2.

(1) The proposal density $g(\cdot)$ is such that the resulting likelihood ratio $L(\cdot)$ is Lipschitz continuous with some Lipschitz constant K.
(2) Conditional on $\mathcal{I}$, which yields the current piecewise linear approximation $\mathbf{Y}^\delta(t)$, we can obtain a further refinement $(\mathbf{Y}^\varepsilon(t) : 0 \leq t \leq T_{left})$ such that

$$\max_{1 \leq i \leq d} \sup_{0 \leq t \leq T_{left}} |\mathbf{Y}_i(t) - \mathbf{Y}_i^\varepsilon(t)| < \varepsilon$$

for any desired $\varepsilon < \delta$.

These assumptions are later verified in Section 2.3. To continue with the development of our rejection sampler, we use the Lipschitz continuity of $L(\cdot)$ mentioned in (a) as below:

$$|L(\mathbf{Z}) - L(\mathbf{Z} - \mathbf{Y}^{\varepsilon}(T_{left}) + \mathbf{Y}(T_{left}))| \leq Kd(\mathbf{Y}^{\varepsilon}(T_{left}), \mathbf{Y}(T_{left})) < K\varepsilon.$$

Now if ε is such that $K\varepsilon < |cV - L(\mathbf{Y}^{\delta}(T_{left}) + \mathbf{\Delta} - \mathbf{Y}^{\varepsilon}(T_{left}))|$, then checking whether $cV < L(\mathbf{Z})$ or not is equivalent to performing the following comparison:

$$cV < L(\mathbf{Z} - \mathbf{Y}^{\varepsilon}(T_{left}) + \mathbf{Y}(T_{left})) = \frac{f_{\mathbf{\Delta}}(\mathbf{Z} - \mathbf{Y}^{\varepsilon}(T_{left}))}{g(\mathbf{Z} - \mathbf{Y}^{\varepsilon}(T_{left}) + \mathbf{Y}(T_{left}) - \mathbf{Y}^{\delta}(T_{left}))}. \tag{9}$$

In Section 2.3, we show that $g(\cdot)$ can be chosen simply to be a uniform distribution on some bounded support, in which case the denominator in (9) ceases to depend on $\mathbf{Y}(T_{left})$. Then the comparison (9) can be accomplished unambiguously (because we know all the quantities involved), and accepting $\mathbf{Z}$ based on this will yield an exact sample of $\mathbf{Y}(T)$. This completes the overview of our sampling procedure. In Section 2.2, we explain how to algorithmically obtain the approximations $\mathbf{B}^{\varepsilon}(\cdot), \mathbf{Y}^{\varepsilon}(\cdot)$ for any $\varepsilon > 0$ and other relevant quantities such as δ, T_{left} and T_{right}. In Section 2.3, we describe the rejection sampler in detail and justify Assumptions 1 and 2.

2.2. **Generating ε-Strong Simulation.** In this section, we first provide a brief description of the ε-strong algorithm of [3] that simulates a piecewise linear approximation to 1-dimensional standard Brownian motion $B(\cdot)$. The algorithm iteratively generates a sequence of pairs of dominating processes, $\{B_n^{\uparrow}(t) : t \in [0,1]\}$ and $\{B_n^{\downarrow}(t) : t \in [0,1]\}$, that satisfy the following properties: For all $t \in [0,1]$,

$$B_n^{\downarrow}(t) \leq B_{n+1}^{\downarrow}(t) \leq B(u) \leq B_{n+1}^{\uparrow}(t) \leq B_n^{\uparrow}(t), \text{ and}$$

$$\sup_{t\in[0,1]} |B_n^{\uparrow}(t) - B_n^{\downarrow}(t)| \searrow 0, \text{ a.s. as } n \nearrow \infty.$$

At every step $n \geq 1$, the algorithm generates information about the Brownian motion $B(\cdot)$ in dyadic intervals $\{((j-1)2^{-n}, j2^{-n}] : j = 1, \ldots, 2^n\}$ conditional on the information available on dyadic intervals from the $(n-1)^{th}$ step. Let $m_{j,n}$ and $M_{j,n}$ denote the extrema of $B(\cdot)$:

$$m_{j,n} := \inf\{B(t) : t \in ((j-1)2^{-n}, j2^{-n}]\} \text{ and } M_{j,n} = \sup\{B(t) : t \in ((j-1)2^{-n}, j2^{-n}]\}.$$

During n^{th} iteration, the ε-strong algorithm simulates the following random quantities for each dyadic interval (indexed by $j = 1, \ldots, 2^n$):

1) an interval that contains the minimum: $L_{j,n}^{\downarrow}$ and $L_{j,n}^{\uparrow}$ such that $m_{j,n} \in [L_{j,n}^{\downarrow}, L_{j,n}^{\uparrow}]$ and $L_{j,n}^{\uparrow} - L_{j,n}^{\downarrow} < 2^{-(n+1)/2}$,
2) an interval that contains the maximum: $U_{j,n}^{\downarrow}$ and $U_{j,n}^{\uparrow}$ such that $M_{j,n} \in [U_{j,n}^{\downarrow}, U_{j,n}^{\uparrow}]$ and $U_{j,n}^{\uparrow} - U_{j,n}^{\downarrow} < 2^{-(n+1)/2}$, and
3) the end-points of Brownian motion: $B((j-1)2^{-n})$ and $B(j2^{-n})$.

Let $\mathcal{I}_{j,n}$ denote the collective information,

$$\mathcal{I}_{j,n} := \{L_{j,n}^{\downarrow}, L_{j,n}^{\uparrow}, U_{j,n}^{\downarrow}, U_{j,n}^{\uparrow}, B((j-1)2^{-n}), B(j2^{-n})\},$$

which is referred to as *intersection layer* in [3]. Let $\mathcal{I}$ denote the collection of all the intersection layers; at the end of n^{th} iteration, the collection $\mathcal{I}$ is updated as below:

$$\mathcal{I} := \{\mathcal{I}_{j,n} : j = 1, \ldots, 2^n\}.$$

The $(n+1)^{th}$ step makes use of $\mathcal{I}$ generated in the n^{th} step to arrive at $\{\mathcal{I}_{j,n+1} : j = 1, \ldots, 2^{n+1}\}$. Specific details of how these random quantities are simulated can be found in [3]. From the intersection layers $\mathcal{I}_{j,n}$ generated by the algorithm at n^{th} step, the dominating piecewise constant processes $B_n^{\uparrow}(\cdot)$ and $B_n^{\downarrow}(\cdot)$ can be formed as below:

$$B_n^{\uparrow}(t) = \sum_{j=1}^{2^n} U_{j,n}^{\uparrow} \mathbf{1}(t \in ((j-1)2^{-n}, j2^{-n}]), \text{ and}$$
$$B_n^{\downarrow}(t) = \sum_{j=1}^{2^n} L_{j,n}^{\downarrow} \mathbf{1}(t \in ((j-1)2^{-n}, j2^{-n}]).$$

Further define the following piecewise linear process which shall serve as our approximation for $B(\cdot)$:

$$B_n(t) = \sum_{j=1}^{2^n} \left[B((j-1)2^{-n}) + 2^n (B(j2^{-n}) - B((j-1)2^{-n}))(t - (j-1)2^{-n}) \right], \tag{10}$$

which is just a linear interpolation of the points $\{B((j-1)2^{-n}) : j = 1, \ldots, 2^n\}$ over the dyadic intervals in $[0,1]$. Note that all the random variables used in the construction of $B_n^{\uparrow}(\cdot)$, $B_n^{\downarrow}(\cdot)$ and $B_n(\cdot)$ are available in $\mathcal{I}$, and can be simulated on a personal computer without any discretisation error. It is proved in [3] that the dominating processes $B_n^{\uparrow}(\cdot)$ and $B_n^{\downarrow}(\cdot)$ have the following convergence behavior:

$$\varepsilon_n := \sup_{t\in[0,1]} |B_n^{\uparrow}(t) - B_n^{\downarrow}(t)| = \max_{1\le j\le 2^n} |U_{j,n}^{\uparrow} - L_{j,n}^{\downarrow}| \searrow 0, \text{ and} \tag{11}$$
$$\mathbb{E}\left[\int_0^1 |B_n^{\uparrow}(t) - B_n^{\downarrow}(t)| dt \right] = O(2^{-n/2}),$$

as $n \nearrow \infty$. Now the following observations are in order:

A) Recall that $S : C[0,1] \to C[0,1]$ denotes the Skorokhod map given by (1). Since S is Lipschitz continuous with respect to the uniform topology on $C[0,1]$ (with Lipschitz constant $K := 1/(1-\alpha)$), if we solve (1) with $X(\cdot) = B_n(\cdot)$, the corresponding reflected process $S(B_n)$ satisfies that,
$$\sup_{t\in[0,1]} |S(B_n)(t) - Y(t)| < K\varepsilon_n.$$
This is because $\sup\{|B_n(t) - B(t)| : t \in [0,1]\} < \varepsilon_n$, which follows from (11)

B) Consider the case of 1-dimensional Brownian motion. Since $U_{j,n}^{\uparrow} - L_{j,n}^{\downarrow} < \varepsilon_n$ in every dyadic interval at step n, the following is true for $j = 1, \ldots, 2^n$:
$$|B(t) - B(s)| < \varepsilon_n, \text{ for all } t, s \in \left[(j-1)2^{-n}, j2^{-n}\right].$$
This means that if $Y(T) > \varepsilon_n$, the RBM $Y(\cdot)$ does not hit the reflecting boundary anywhere in the dyadic interval containing T and behaves just like Brownian motion locally in that interval.

In Section 2.1.2, we had mentioned of finding an interval $(T_{left}, T_{right}]$ such that the RBM stays in the interior of positive orthant without hitting the reflecting boundary anywhere in that interval. If the 1-dimensional RBM $Y(\cdot)$ is above the level ε_n at time T, then from Observation B), the whole process $Y(\cdot)$ stays positive in the dyadic interval $((j-1)2^{-n}, j2^{-n}]$ containing T. Since we do not know $Y(T)$, the immediate objective is to figure out how to guarantee that $Y(T)$ is indeed larger than ε_n. From the Lipschitz continuity in observation A), if we have that $S(B_n)(T) > (K+1)\varepsilon_n$

for some n, then $Y(T) > \varepsilon_n$. Since $Y(t)$ is positive almost everywhere, we will indeed have that $S(B_n)(T) > (K+1)\varepsilon_n$ for all large n. Now define,

$$N := \inf\{n \geq 1 : S(B_n)(T) > (K+1)\varepsilon_n\}, \text{ and } \delta := \varepsilon_N.$$

Recall that $\varepsilon_n := \sup_{t\in[0,1]} |B_n^{\uparrow}(t) - B_n^{\downarrow}(t)| = \max\{U_{j,n}^{\uparrow} - L_{j,n}^{\downarrow} : j = 1, \ldots, 2^n\}$. The procedure for generating piecewise linear approximation $Y_n(\cdot)$ for RBM, along with the interval $(T_{left}, T_{right}]$ and δ is summarized here in Algorithm 1:

Algorithm 1 To generate piecewise linear approximation for 1-dimensional RBM $Y(\cdot)$ in the interval [0,1]

procedure StrongApprox(T)
  Initialize $\mathcal{I} = \emptyset, n = 0, \delta = 1, Y_0(t) = 0, t \in [0,1]$
  while $Y_n(T) < (K+1)\delta$ **do**
    Increment $n \longleftarrow n+1$
    Simulate the intersection layers $\mathcal{I}_{j,n}$ for $j = 1, \ldots, 2^n$ conditional on $\mathcal{I}$
    Form $B_n(\cdot)$ as in (10), which serves as piecewise linear approximation to $B(\cdot)$
    Set $\delta = \max\{U_{j,n}^{\uparrow} - L_{j,n}^{\downarrow} : j = 1, \ldots, 2^n\}$ and $\mathcal{I} = \{\mathcal{I}_{j,n} : j = 1, \ldots, 2^n\}$
    Letting $X(\cdot) = B_n(\cdot)$ in (1), solve for the reflected process $Y(\cdot)$; call the solution as $Y_n(\cdot)$
  Find $J = \{1 \leq j \leq 2^n : (j-1)2^{-n} < T \leq j2^{-n}\}$
  Set $N = n, T_{left} = (J-1)2^{-n}$ and $T_{right} = J2^{-n}$
  Return $\mathcal{I}, N, T_{left}, T_{right}, \delta$ and $Y^{\delta}(\cdot) := Y_n(\cdot)$

Similarly for d-dimensional Brownian motion $\mathbf{B}(\cdot) = (B_1(\cdot), \ldots, B_d(\cdot))$, generate approximating processes $B_{i,n}(\cdot), B_{i,n}^{\uparrow}(\cdot)$, and $B_{i,n}^{\downarrow}(\cdot)$ independently for each 1-dimensional Brownian motion $B_i(\cdot)$ and use

$$\mathbf{B}_n(t) = (B_{1,n}(t), \ldots, B_{d,n}(t)) \text{ and } \mathbf{Y}_n(t) = S(\mathbf{B}_n)(t), t \in [0,1]$$

as the piecewise linear approximations for $\mathbf{B}(\cdot)$ and $\mathbf{Y}(\cdot)$ respectively. Setting $\varepsilon_n = \sup\{B_{i,n}^{\uparrow}(t) - B_{i,n}^{\downarrow}(t) : t \in [0,1], i = 1, \ldots, d\}$, we find N, T_{left}, T_{right} and δ as in Algorithm 1. The next step is to suitably propose for $\mathbf{Y}(T)$ based on the simulated information.

2.3. **The Acceptance / Rejection Scheme.** Let $\{\mathcal{F}_t : t \in [0,1]\}$ denote the standard Brownian filtration on $C[0,1]^d$. In Algorithm 1, $\mathcal{I}$ contains all the simulated information about the underlying Brownian motion $\mathbf{B}(\cdot)$. Since the RBM $\mathbf{Y}(\cdot)$ does not hit the reflecting boundary anywhere in the interval $(T_{left}, T_{right}]$, the dynamics of $\mathbf{B}(\cdot)$ and $\mathbf{Y}(\cdot)$ match in $(T_{left}, T_{right}]$: to say precisely,

$$\mathbf{Y}(t) - \mathbf{Y}(T_{left}) = \mathbf{B}(t) - \mathbf{B}(T_{left}), \text{ for all } t \in (T_{left}, T_{right}]. \tag{12}$$

Therefore $\mathbf{Y}(T)$ can be expressed as

$$\mathbf{Y}(T) = \mathbf{Y}(T_{left}) + \mathbf{\Delta},$$

where the increment $\mathbf{\Delta} := \mathbf{B}(T) - \mathbf{B}(T_{left}) = \mathbf{Y}(T) - \mathbf{Y}(T_{left})$ is independent of $\mathcal{F}_{T_{left}}$ conditional on the intersection layers $\mathcal{I}$ returned by Algorithm 1. Therefore if we know $\mathbf{Y}(T_{left})$, we can simulate $\mathbf{\Delta}$ independently and add to $\mathbf{Y}(T_{left})$ to arrive at a sample of $\mathbf{Y}(T)$. This necessitates us to know the law of $\mathbf{\Delta}$ conditional on all the simulated collection of random variables $\mathcal{I}$. For ease of exposition, we consider the 1-dimensional case; the law of $\mathbf{\Delta}$ in d-dimensions is given simply by the product form of 1-dimensional densities.

2.3.1. *Probability Density of Δ conditional on $\mathcal{I}$.* At this stage of simulation, all we know about the underlying Brownian motion $B(\cdot)$ is available in the simulated collection $\mathcal{I}$. From Algorithm 1, recall that J is the index corresponding to the dyadic interval $(T_{left}, T_{right}]$; that is $(J-1)2^{-N} = T_{left} < T \le T_{right} = J2^{-N}$. For ease of notation, let

$$L^{\downarrow} := L^{\downarrow}_{J,N} - B(T_{left}),\ L^{\uparrow} := L^{\uparrow}_{J,N} - B(T_{left}),$$
$$U^{\downarrow} := U^{\downarrow}_{J,N} - B(T_{left}),\ U^{\uparrow} := U^{\uparrow}_{J,N} - B(T_{left}),$$
$$l := T_{right} - T_{left},\ s := T - T_{left} \text{ and } v := B(T_{right}) - B(T_{left}).$$

Further let $W(\cdot)$ denote an independent standard Brownian motion on $C[0,1]$ under measure $\mathbb{P}(\cdot)$. Then due to Markov property of $B(\cdot)$, the increment Δ conditional on $\mathcal{I}$ has the following density:

$$f_\Delta(x)dx = \mathbb{P}\left\{ W(s) \in dx \;\middle|\; W(l) = v, \inf_{0\le t\le l} W(t) \in (L^{\downarrow}, L^{\uparrow}), \sup_{0\le t\le l} W(t) \in (U^{\downarrow}, U^{\uparrow}) \right\}. \tag{13}$$

Note that the support of $f_\Delta(\cdot)$ is $(L^{\downarrow}, U^{\uparrow})$. A closed form expression for $f_\Delta(\cdot)$ follows from Proposition 5.1 of [3], and is given here (we thus verify that 1) and 2) from Assumption 1 indeed hold):

$$f_\Delta(x) \propto \rho(x) \times \pi(x), \qquad x \in (L^{\downarrow}, U^{\uparrow})$$

where for any fixed $L^{\downarrow}, L^{\uparrow}, U^{\downarrow}, U^{\uparrow}, v, s$ and l,

$$\pi(x) := \exp\left(-\frac{1}{2}\left(x - \frac{s}{l}v\right)^2 \Big/ \left(\frac{s(l-s)}{l}\right)\right), \text{ and} \tag{14}$$

$$\rho(x) := \mathbb{P}\left\{ \inf_{0\le t\le l} W(t) \in (L^{\downarrow}, L^{\uparrow}), \sup_{0\le t\le l} W(t) \in (U^{\downarrow}, U^{\uparrow}) \;\middle|\; W(s) = x, W(l) = v \right\}$$
$$= \gamma_1(x)\gamma_2(x) - \gamma_3(x)\gamma_4(x) - \gamma_5(x)\gamma_6(x) + \gamma_7(x)\gamma_8(x). \tag{15}$$

To define $\gamma_1, \ldots, \gamma_8$, first consider the probability that the Brownian bridge from a to b in the time interval $[0, r]$ stays within (L, U) :

$$\gamma(L, U; r, a, b) := \mathbb{P}\left\{ L < \inf_{0\le t\le l} W(t) \le \sup_{0\le t\le l} W(t) < U \;\middle|\; W(0) = a, W(r) = b \right\}$$
$$= \left(1 - \sum_{j=1}^{\infty} (\sigma_j - \tau_j)\right) \mathbf{1}(a, b \in (L, U)), \tag{16}$$

where, $\sigma_j := \exp\left(-\frac{2}{r}((U-L)j + L - a)((U-L)j + L - b)\right)$
$$+ \exp\left(-\frac{2}{r}((U-L)j - U + a)((U-L)j - U + b)\right), \text{ and}$$
$$\tau_j := \exp\left(-\frac{2(U-L)j}{r}((U-L)j + a - b)\right) + \exp\left(-\frac{2(U-L)j}{r}((U-L)j + b - a)\right).$$

The expression (16) for $\gamma(L, U; l, a, b)$ is originally from [10]. Now we are ready to define $\gamma_1, \ldots, \gamma_8$ mentioned in (15):

$$\gamma_1(x) = \gamma(L^{\downarrow}, U^{\uparrow}; s, 0, x), \quad \gamma_2(x) = \gamma(L^{\downarrow}, U^{\uparrow}; l-s, x, v), \quad \gamma_3(x) = \gamma(L^{\uparrow}, U^{\uparrow}; s, 0, x),$$
$$\gamma_4(x) = \gamma(L^{\uparrow}, U^{\uparrow}; l-s, x, v), \quad \gamma_5(x) = \gamma(L^{\downarrow}, U^{\downarrow}; s, 0, x), \quad \gamma_6(x) = \gamma(L^{\downarrow}, U^{\downarrow}; l-s, x, v),$$
$$\gamma_7(x) = \gamma(L^{\uparrow}, U^{\downarrow}; s, 0, x), \quad \gamma_8(x) = \gamma(L^{\uparrow}, U^{\downarrow}; l-s, x, v).$$

2.3.2. *The Proposal and its Likelihood.* Define $\tilde{\Delta} := Y(T) - Y^\delta(T_{left})$. Note that $\tilde{\Delta}$ cannot take values outside the interval $(L^\downarrow - \delta, U^\uparrow + \delta)$ because

$$|\tilde{\Delta} - \Delta| = |Y(T_{left}) - Y^\delta(T_{left})| < \delta,$$

and the support of $f_\Delta(\cdot)$ is $(L^\downarrow, U^\uparrow)$. Since the support of $\tilde{\Delta}$ is bounded, we propose samples of $\tilde{\Delta}$ simply from the uniform distribution on $(L^\downarrow - \delta, U^\uparrow + \delta)$. The proposal Z for $Y(T)$ is made as follows:

1) Sample $\tilde{Z}$ uniformly from $(L^\downarrow - \delta, U^\uparrow + \delta)$
2) Use $Y^\delta(T_{left})$ returned by Algorithm 1 to propose for $Y(T)$ as below:

$$Z = Y^\delta(T_{left}) + \tilde{Z}. \tag{17}$$

As noted before, $\tilde{\Delta} = \Delta + Y(T_{left}) - Y^\delta(T_{left})$. Therefore,

$$\mathbb{P}\{\tilde{\Delta} \in dx | \mathcal{I}, \mathcal{F}_{T_{left}}\} = f_\Delta\left(x + Y^\delta(T_{left}) - Y(T_{left})\right) dx.$$

Since the proposal density is uniform on $(L^\downarrow - \delta, U^\uparrow + \delta)$, the likelihood ratio of Z under $\mathbb{P}(\cdot)$ with respect to that of the proposal measure (conditional on $\mathcal{I}, \mathcal{F}_{T_{left}}$) is then simply,

$$\begin{aligned} L(Z) &= \frac{f_\Delta\left(Z + Y^\delta(T_{left}) - Y(T_{left})\right)}{(U^\uparrow - L^\downarrow + 2\delta)^{-1}} \\ &\propto \rho\left(Z + Y^\delta(T_{left}) - Y(T_{left})\right) \times \pi\left(Z + Y^\delta(T_{left}) - Y(T_{left})\right). \end{aligned} \tag{18}$$

In the following section, we explain how to appropriately reject the proposed sample Z so that we obtain an exact sample of $Y(T)$.

2.3.3. *The Rejection Sampler.* To perform acceptance / rejection sampling, we require the likelihood ratio $L(\cdot)$ to be bounded. Further, as mentioned in Section 2.1.3, it is desirable to have the likelihood ratio to be Lipschitz continuous. The following result guarantees that the likelihood ratio $L(\cdot)$ is indeed bounded and Lipschitz continuous for our choice of proposal density.

Lemma 1. *There exists positive constants c_π, K_π, c_ρ and K_ρ such that for any fixed $L^\downarrow, L^\uparrow, U^\downarrow, U^\uparrow$, v, s and l,*

$$\begin{aligned} \pi(x) < c_\pi, \quad |\pi(x) - \pi(y)| < K_\pi |x - y| \text{ and} \\ \rho(x) < c_\rho, \quad |\rho(x) - \rho(y)| < K_\rho |x - y|, \end{aligned}$$

for all $x, y \in (L^\downarrow, U^\uparrow)$.

The previous result implies part 1) in Assumption 2. Explicit closed-form expressions for the constants K_π, K_ρ, c_π and c_ρ are presented in the Appendix. Let V_1 and V_2 be two independent random variables distributed uniformly in $[0, 1]$. Because of (18), if we accept the proposal Z only when

$$V_1 < \frac{\pi\left(Z + Y^\delta(T_{left}) - Y(T_{left})\right)}{c_\pi} \text{ and } V_2 < \frac{\rho\left(Z + Y^\delta(T_{left}) - Y(T_{left})\right)}{c_\rho}, \tag{19}$$

then the resulting sample has distribution same as that of $Y(T)$. Note that Algorithm 1 does not return $Y(T_{left})$ as a part of its output, and this leads to the following question: How to algorithmically perform the comparisons (19) without knowing the value of $Y(T_{left})$? As mentioned

in the Introduction, to perform these comparisons, it is not necessary to know the values of $Y(T_{left})$. If we have access to $Y^{\varepsilon}(\cdot)$ satisfying

$$\sup_{t\in[0,T_{left}]} |Y(t) - Y^{\varepsilon}(t)| < \varepsilon,$$

then $Y^{\varepsilon}(T_{left}) \in (Y(T_{left}) - \varepsilon, Y(T_{left}) + \varepsilon)$ can serve as a proxy for $Y(T_{left})$. Due to the Lipschitz continuity of $\pi(\cdot)$ and $\rho(\cdot)$, the following hold:

$$\left|\pi\left(Z + Y^{\delta}(T_{left}) - Y(T_{left})\right) - \pi\left(Z + Y^{\delta}(T_{left}) - Y^{\varepsilon}(T_{left})\right)\right| < K_{\pi}\varepsilon, \text{ and}$$
$$\left|\rho\left(Z + Y^{\delta}(T_{left}) - Y(T_{left})\right) - \rho\left(Z + Y^{\delta}(T_{left}) - Y^{\varepsilon}(T_{left})\right)\right| < K_{\rho}\varepsilon.$$

Additionally if ε is smaller than

$$\frac{\left|c_{\pi}V_1 - \pi\left(Z + Y^{\delta}(T_{left}) - Y^{\varepsilon}(T_{left})\right)\right|}{K_{\pi}} \wedge \frac{\left|c_{\rho}V_2 - \rho\left(Z + Y^{\delta}(T_{left}) - Y^{\varepsilon}(T_{left})\right)\right|}{K_{\rho}},$$

then

$$V_1 < \frac{\pi\left(Z + Y^{\delta}(T_{left}) - Y(T_{left})\right)}{c_{\pi}} \text{ if and only if } V_1 < \frac{\pi\left(Z + Y^{\delta}(T_{left}) - Y^{\varepsilon}(T_{left})\right)}{c_{\pi}}, \text{ and} \tag{20}$$

$$V_2 < \frac{\rho\left(Z + Y^{\delta}(T_{left}) - Y(T_{left})\right)}{c_{\rho}} \text{ if and only if } V_2 < \frac{\rho\left(Z + Y^{\delta}(T_{left}) - Y^{\varepsilon}(T_{left})\right)}{c_{\rho}}. \tag{21}$$

The rejection sampler in Algorithm 2 takes the intersection layers $\mathcal{I}$ returned by Algorithm 1 as input, and generates further refined approximations $(Y^{\varepsilon}(t) : t \in [0, T_{left}])$ of the RBM which can then be used to perform the equivalent comparisons in (20) and (21). This verifies part 2) of Assumption 2.

Algorithm 2 To accept/reject the proposal $Z := Y^{\delta}(T_{left}) + \tilde{Z}$. If both the comparisons in (19) hold true, the algorithm returns Z; otherwise it rejects Z and returns nothing

procedure RejectionSampler($\mathcal{I}, T_{left}, N, Y^{\delta}(T_{left}), Z$)
  Initialize $n = N, \varepsilon = \max\{U^{\uparrow}_{j,n} - L^{\downarrow}_{j,n} : j = 1, \ldots, 2^n\}$ and $\varepsilon_{thr} = 0$.
  Draw V_1, V_2 uniformly from $[0, 1]$
  while $\varepsilon > \varepsilon_{thr}$ **do**
    Increment $n \longleftarrow n + 1$
    Simulate the intersection layers $\mathcal{I}_{j,n}$ for $j = 1, \ldots, 2^n$ conditional on $\mathcal{I}$
    Form $B_n(\cdot)$ as in (10), which serves as piecewise linear approximation to $B(\cdot)$
    Set $\varepsilon = \max\{U^{\uparrow}_{j,n} - L^{\downarrow}_{j,n} : j = 1, \ldots, 2^n\}$ and $\mathcal{I} = \{\mathcal{I}_{j,n} : j = 1, \ldots, 2^n\}$
    Letting $X(\cdot) = B_n(\cdot)$ in (1), solve for the reflected process $Y(\cdot)$; call the solution as $Y_n(\cdot)$
    Update ε_{thr} as below:

$$\varepsilon_{thr} = \frac{\left|c_{\pi}V_1 - \pi\left(Z + Y^{\delta}(T_{left}) - Y_n(T_{left})\right)\right|}{K_{\pi}} \wedge \frac{\left|c_{\rho}V_2 - \rho\left(Z + Y^{\delta}(T_{left}) - Y_n(T_{left})\right)\right|}{K_{\rho}}$$

  if $c_{\pi}V_1 < \pi\left(Z + Y^{\delta}(T_{left}) - Y_n(T_{left})\right)$ and $c_{\rho}V_2 < \rho\left(Z + Y^{\delta}(T_{left}) - Y_n(T_{left})\right)$ **then**
    Return Z
  else
    Return $\emptyset$

For d-dimensional processes, the probability density of the increment $\mathbf{\Delta}$ and proposal density $g(\cdot)$ are both given by product of 1-dimensional densities. This results in a likelihood ratio which is also of product form, leading to a straightforward generalization of the acceptance / rejection procedure given in Algorithm 2.

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Appendix

Here we provide the proof of Lemma 1, and present explicit expressions for the constants K_π, K_ρ, c_π and c_ρ. For proving Lemma 1, we need the following result.

Lemma 2. *For any given $U > L, r > 0$, the function $\gamma(L,U;r,a,b)$ defined in (16) is Lipschitz continuous with respect to the variables a and b; that is,*

$$|\gamma(L,U;r,a_1,\cdot) - \gamma(L,U;r,a_2,\cdot)| < K(L,U,r)|a_1 - a_2|$$
$$|\gamma(L,U;r,\cdot,b_1) - \gamma(L,U;r,\cdot,b_2)| < K(L,U,r)|b_1 - b_2|,$$

for all $a_1, a_2, b_1, b_2 \in (L,U)$. The Lipschitz constant $K(L,U,r)$ is given by

$$K(L,U,r) := \sum_{j\geq 1} K_j = \frac{8(U-L)}{r} \sum_{j\geq 1} j \exp\left(-\frac{2}{r}(U-L)^2(j-1)^2\right).$$

Proof. Let $\gamma_n(a,b) = 1 - \sum_{j=1}^n (\sigma_j - \tau_j)$. Since a, b take values in (L,U), it is easily checked that for all $j \geq 1$ both

$$\left|\frac{d}{da}(\sigma_j - \tau_j)\right| < K_j, \text{ and } \left|\frac{d}{db}(\sigma_j - \tau_j)\right| < K_j,$$

where

$$K_j := \frac{8(U-L)j}{r} \exp\left(-\frac{2}{r}(U-L)^2(j-1)^2\right).$$

Then it is immediate that for all n,

$$\left|\frac{d\gamma_n(a,b)}{da}\right| < \sum_{j=1}^{\infty} K_j \text{ and } \left|\frac{d\gamma_n(a,b)}{db}\right| < \sum_{j=1}^{\infty} K_j.$$

As a consequence, we use the following elementary properties of Lipschitz continuity to establish the Lipschitz continuity of $\gamma(L,U;r,a,b)$ with respect to variables a and b:

1) If a differentiable function $f(\cdot)$ on a convex domain is such that its first derivative $|f'(x)| < K$ for some constant K, then the function $f(\cdot)$ is Lipschitz continuous with Lipschitz constant at most K.
2) If a sequence of Lipschitz functions $f_n(\cdot)$ all having Lipschitz constant bounded by K converge uniformly to $f(\cdot)$, then $f(\cdot)$ is also Lipschitz continuous with Lipschitz constant at most K.

Since $\gamma_n(a,b)$ converge uniformly to $\gamma(a,b)$ for $a,b \in (L,U)$, it follows immediately from the above two facts that $\gamma(L,U;r,a,b)$ is Lipschitz continuous with Lipschitz constant at most

$$K(L,U,r) := \sum_{j\geq 1} K_j = \frac{8(U-L)}{r} \sum_{j\geq 1} j \exp\left(-\frac{2}{r}(U-L)^2(j-1)^2\right).$$

□

Proof of Lemma 1. For all $x \in (L^{\downarrow}, U^{\uparrow})$,

$$\pi(x) = \exp\left(-\frac{1}{2}\left(x - \frac{s}{l}v\right)^2 \Big/ \left(\frac{s(l-s)}{l}\right)\right) \leq 1 =: c_\pi. \tag{22}$$

The Lipschitz continuity of $\pi(\cdot)$ follows from the boundedness of its first derivative $\pi'(\cdot)$ on the convex domain (L,U) : for all $x \in (L,U)$,

$$\left|\frac{d\pi(x)}{dx}\right| \leq \frac{|xl - sv|}{s(l-s)} < \frac{\max\{|Ul - sv|, |Ll - sv|\}}{s(l-s)} =: K_\pi. \tag{23}$$

To prove the Lipschitz continuity of $\rho(\cdot)$, we first note the boundedness of $\gamma(L,U;r,\cdot,\cdot)$: Simple substitution will yield that $\gamma(L,U;r,a,b) = 0$ whenever either a or b equals one of L, U. Then due to the Lipschitz continuity of $\gamma(L,U;r,\cdot,\cdot)$ from Lemma 2, we have that

$$|\gamma(L,U;r,a,b)| \leq K(L,U,r)(U-L). \tag{24}$$

Now consider the first term $\gamma_1(x)\gamma_2(x)$ in (15):

1) Because of (24), $|\gamma_1(\cdot)|$ and $|\gamma_2(\cdot)|$ are bounded by $K(L^{\downarrow}, U^{\uparrow}, s)(U^{\uparrow} - L^{\downarrow})$ and $K(L^{\downarrow}, U^{\uparrow}, l-s)(U^{\uparrow} - L^{\downarrow})$, respectively, in the interval $x \in (L^{\downarrow}, U^{\uparrow})$.
2) From Lemma 2, we have that $\gamma_1(\cdot)$ and $\gamma_2(\cdot)$ are Lipschitz continuous (with respect to the variable x) with Lipschitz constants at most $K(L^{\downarrow}, U^{\uparrow}, s)$ and $K(L^{\downarrow}, U^{\uparrow}, l-s)$ respectively.

From the above two observations, we conclude that $\gamma_1(\cdot)\gamma_2(\cdot)$ is Lipschitz continuous with respect to x with Lipschitz constant at most

$$K_{1,2} := 2K(L^{\downarrow}, U^{\uparrow}, s)K(L^{\downarrow}, U^{\uparrow}, l-s)(U^{\uparrow} - L^{\downarrow}).$$

This is because if f, g are Lipschitz continuous with respective Lipschitz constants K_f and K_g and absolute bounds C_f and C_g, then fg is Lipschitz continuous with Lipschitz constant at most $C_f K_g + C_g K_f$. Using the same reasoning, the Lipschitz constants of other terms in (15), namely $\gamma_3(\cdot)\gamma_4(\cdot), \gamma_5(\cdot)\gamma_6(\cdot)$ and $\gamma_7(\cdot)\gamma_8(\cdot)$ are at most

$$\begin{aligned}
K_{3,4} &:= 2K(L^{\uparrow}, U^{\uparrow}, s)K(L^{\uparrow}, U^{\uparrow}, l-s)(U^{\uparrow} - L^{\uparrow}),\\
K_{5,6} &:= 2K(L^{\downarrow}, U^{\downarrow}, s)K(L^{\downarrow}, U^{\downarrow}, l-s)(U^{\downarrow} - L^{\downarrow}), \text{ and}\\
K_{7,8} &:= 2K(L^{\uparrow}, U^{\downarrow}, s)K(L^{\uparrow}, U^{\downarrow}, l-s)(U^{\downarrow} - L^{\uparrow})
\end{aligned}$$

respectively. Therefore, $\rho(x)$ is Lipschitz continuous with Lipschitz constant K_ρ given by,

$$K_\rho := K_{1,2} + K_{3,4} + K_{5,6} + K_{7,8}.$$

Since $\rho(x) = 0$ whenever x takes either $L^{\downarrow}$ or $U^{\uparrow}$, using Lipschitz continuity of ρ we reason that,

$$|\rho(x)| \le K_\rho(U^{\uparrow} - L^{\downarrow}) =: c_\rho.$$

This along with (22) and (23) proves the claim. □

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