Asymptotic Expansions of Defective Renewal Equations with Applications to Perturbed Risk Models and Processor Sharing Queues

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Abstract

We consider asymptotic expansions for defective and excessive renewal equations that are close to being proper. These expansions are applied to the analysis of Processor Sharing queues and perturbed risk models, and yield approximations that can be useful in applications where moments are computable, but the distribution is not.

1 Introduction

A defective renewal equation for a function $a_p(\cdot)$ takes the form

$$a_p(t) = b_p(t) + (1-p)\int_{[0,t)} a_p(t-s)\,P(X_1(p) \in ds), \tag{1}$$

where $b_p(\cdot)$ is a given locally integrable function, which is allowed – together with the distribution of $X_1 := X_1(p)$ – to depend on the parameter $p \in (0,1)$. We are interested in the analysis of the unique bounded solution to equation (1).

It turns out that defective renewal equations such as (1) play an important role in a number of applied probability settings. A prominent example is insurance risk theory; in particular, the so-called "expected discounted penalty" at ruin (from which many quantities of interest, including the ruin probability, can be recovered by judicious choices of the discount rate and the penalty) can be expressed in terms of a defective renewal equation (see Lin and Willmot (2000) p. 162). Many other examples in which defective renewal equations play an important role are also described in Feller (1968) p. 188, 216, Resnick (1992) p. 158, and Lin and Willmot (2000) Ch. 9) these examples include Geiger counters, generalized terminating renewal processes, and age dependent branching processes. The present paper is partly motivated by a different application, namely the $M/G/1$ Processor Sharing queue. The setting in which q is close to one in (1) (or, equivalently, p is close to zero) is common in the application settings described before. For instance, in the insurance setting it arises in environments of low net profits (which occur in competitive conditions). For generalized terminating renewal processes, q close to one corresponds to settings in which the process continues for long periods, and in age dependent branching processes, q close to one reflects a case in which the population is less likely to die. In a queueing environment, q is the traffic intensity, meaning that the system is critically loaded. This, consequently, motivates developing asymptotics for the solution $a_p(\cdot)$ of (1) as $p \searrow 0$.

The results that we develop in this paper apply to perturbed renewal equations of defective or excessive type. We have discussed the case of defective renewal equations as the defective parameter is close to one, this is what we called "defectively perturbed renewal equations". Our analysis is also applicable to "excessively perturbed renewal equations" as the "excessive" parameter is also close to one. More precisely, excessively perturbed renewal equations take the form

$$a_p(t) = b_p(t) + q^{-1} \int_{[0,t)} a_p(t-s) P(X_1(p) \in ds),$$

where $0 < q = 1 - p < 1$, and $b_p(\cdot)$ is an appropriate function. Again, the asymptotic regime that we study is that in which p is close to zero. Excessive renewal equations arise for example in the analysis of branching processes. Specifically, the mean number of living organisms at time t satisfies an excessive renewal equation where q^{-1} is equal to the mean number of offsprings in each generation.

The remainder of the paper is organized as follows. In Section 2 we develop the necessary general theory, building on results in Blanchet, J., and Glynn, P. (2007). In Section 3 we give an application to perturbed risk models, and Section 4 indicates how our results may be applied to obtain corrected diffusion approximations for sojourn time distributions in Processor Sharing queues.

2 Asymptotics of Perturbed Renewal Equations

As we mentioned previously, defective renewal equations take the form

$$a_p(t) = b_p(t) + q \int_{[0,t)} a_p(t-s) P(X_1(p) \in ds), \tag{2}$$

where $q = 1 - p \in (0,1)$ and b is an appropriate function. The following assumptions will be in force throughout the rest of the paper.

A) The non-negative random variables $X_1(p)$ have uniformly strongly non-lattice distribution as $p \searrow 0$. That is, for each $\varepsilon > 0$,

$$\overline{\lim}_{p \longrightarrow 0} \sup_{|\theta| \geq \varepsilon} |E \exp(i\theta X_1(p))| < 1.$$

B) $X_1(p)$ has uniformly bounded moments. That is,

$$\overline{\lim}_{p \longrightarrow 0} E X_1(p)^{2+\alpha_*} < 0$$

for some $\alpha_* \geq 0$.

C) Let $|b_p|(s)$ be the total variation of the function $b_p(\cdot)$ on $[s, \infty)$. Assume that there exists a bounded monotone function $g : \mathbb{R}_+ \longrightarrow \mathbb{R}$ such that $|b_p|(t) \leq g(t)$ with $g(t) = o(t^{\alpha_*})$ as $t \nearrow \infty$.

Assumption A is a technical condition that is satisfied if the $X_1(p)$'s have (uniformly over p) an absolutely continuous component, which applies in most of the examples that we discussed in the introduction. Assumption B requires the existence of finite variance. As we shall see, the more finite moments are available, the better our asymptotic approximations. Finally, Assumption C) implies direct Riemann integrability, which is required for the application of renewal theory, see Resnick (1992), pag. 237.

In order to state our main results we need a couple of definitions. Let $\overline{X}_1(p) = X_1(p) I(X_1(p) \leq 1/p)$ and define $\theta(p)$ such that

$$E \exp\left(\theta(p) \overline{X}_1(p)\right) = \frac{1}{1-p}.$$

Note that a solution $\theta(p)$ exists for all p sufficiently small. Moreover, θ_p admits an asymptotic expansion of the form (see Proposition 3 in Blanchet and Glynn (2007))

$$\theta(p) = \gamma_1(p) p + \gamma_2(p) p^2 + ... + \gamma_{\alpha_*}(p) p^{\alpha_*} + O\left(p^{\alpha_*+1}\right). \tag{3}$$

The next proposition provides a procedure for recursively computing the γ_k's. The proof is a straightforward calculus exercise and therefore is ommitted.

Proposition 1 *For $k \leq \alpha_* + 1$, the constants $(\gamma_k(p) : 1 \leq k \leq \alpha_*)$ can be computed by solving recursively the following set of equations (note that the kth equation is linear in the γ_k and it only depends on the γ_j's for $j \leq k$).*

$$\sum_{m=1}^{k} \frac{EX_1(p)^m}{m!} \sum_{\{n_1+...+n_m=k-m,\ n_1,..,n_m\geq 0\}} \prod_{j=1}^{m} \gamma_{n_j+1}(p) = 1, \text{ for } 1 \leq k \leq n.$$

In particular, we have that $\gamma_1(p) = 1/EX_1(p)$, $2E^3X_1(p)\gamma_2(p) = 2E^2X_1(p) - EX_1(p)^2$ and $6E^5X_1(p)\gamma_3(p) = 3EX_1(p)^2 - 6EX_1(p)^2 E^2X_1(p) + 6E^4X_1(p) - EX_1(p) EX_1(p)^3$.

The expression given in (3) motivates defining

$$\theta_{\alpha_*}(p) = \gamma_1(p) p + \gamma_2(p) p^2 + ... + \gamma_{\alpha_*}(p) p^{\alpha_*}.$$

Now, let us define, for $j \leq \alpha_* + 1$,

$$b_j(p) = \int_0^\infty t^j b_p(t)\, dt.$$

Our main result in this section takes the following form.

Theorem 1

$$a_p(t/p) = \exp\left(-t\theta_{\alpha^*}(p)/p\right) d(p) + o\left(p^{\alpha_*}\right),$$

where

$$d(p) = \frac{b_0(p) + \sum_{k\leq\alpha_*} b_k(p)\,\theta_{\alpha_*}(p)^k/k!}{q\left(EX_1(p) + \sum_{k\leq\alpha_*}\theta_{\alpha_*}(p)^k\,EX_1(p)^{k+1}/k!\right)}.$$

The rest of the section is devoted to the proof of the previous result. However, before we provide the details of the proof let us comment on its implications in practical settings. In future sections we shall discuss a couple of examples in risk and queueing settings. The first example involves a situation in which $X_i(p)$ and the $b_j(p)$'s do not depend on p. In such case, the approximation given in the previous theorem is an asymptotic expansion in powers of b. In our second example, which involves the sojourn time of a processor sharing queue, the moments of $X_i(p)$ and the $b_j(p)$'s have to be expanded in powers of p in order to provide the required asymptotic expansion.

The proof of Theorem 1 is given in several steps. First we estimate the difference between $a_p(t)$ and $\overline{a}_p(t)$, where $\overline{a}_p(\cdot)$ is the solution to

$$\overline{a}_p(t) = b_p(t) + q\int_{[0,t)} \overline{a}_p(t-s)\,P\left(\overline{X}_1(p)\in ds\right). \tag{4}$$

The function $\overline{a}_p(\cdot)$ is more convenient to work with because one can transform (as we shall discuss momentarily) the previous defective equation into a proper renewal equation.

Lemma 1 *Under assumptions B) and C) we have that*

$$\left|a_p(t/p) - \overline{a}_p(t/p)\right| = o\left(p^{\alpha_*+1}\right)$$

as $p\searrow 0$ *uniformly over* t *in compact sets.*

Proof. Let M be a geometric random variable with parameter p. In other words, $P(M=k) = pq^{k-1}$ for $k\geq 1$. Suppose that M is independent of the $X_i(p)$'s and define

$$\overline{S}_M = \overline{X}_1(p) + \ldots + \overline{X}_M(p).$$

Iterating the renewal equation for $\overline{a}(\cdot)$ (we concentrate on the unique bounded solution to the renewal equation) we obtain that

$$\overline{a}_p(t) = \frac{1}{p}\int_{[0,t)} b_p(t-s)\, P\left(\overline{S}_M \in ds\right).$$

Similarly, we also have that

$$a_p(t) = \frac{1}{p}\int_{[0,t)} b_p(t-s)\, P\left(S_M \in ds\right),$$

where

$$S_M = X_1(p) + ... + X_M(p).$$

Therefore

$$\begin{aligned}
&\overline{a}_p(t/p) - a_p(t/p) \\
&= \frac{1}{p}\int_{[0,t/p)} b_p(t/p-s)\left(P\left(\overline{S}_M \in ds\right) - P\left(S_M \in ds\right)\right) \\
&= \frac{1}{p}\int_{[0,t/2p)} b_p(t/p-s)\left(P\left(\overline{S}_M \in ds\right) - P\left(S_M \in ds\right)\right) \qquad (5)\\
&+ \frac{1}{p}\int_{[t/2p,t/p)} b_p(t/p-s)\left(P\left(\overline{S}_M \in ds\right) - P\left(S_M \in ds\right)\right). \qquad (6)
\end{aligned}$$

Let J_1 and J_2 be the integrals in (5) and (6) respectively. Now, since $|b_p(t)| \leq |b_p|(t) \leq g(t) = o(t^{\alpha+2})$ and $g(\cdot)$ is monotone we have (thanks to assumption C) that

$$\max_{1/2\leq u\leq 1} |b_p(ut/p)| \leq g(t/(2p)) = o\left(p^{\alpha_*+2}\right).$$

Thus, it follows that

$$\frac{1}{p}\left|\int_{[0,t/2p)} b_p(t/p-s)\, P\left(\overline{S}_M \in ds\right)\right| \leq \frac{1}{p}\max_{1/2\leq u\leq 1}|b_p(ut/p)| = o\left(p^{\alpha_*+1}\right).$$

Which implies that $J_1 = o\left(p^{\alpha_*+1}\right)$. For J_2 we note that

$$
\begin{aligned}
J_2 &= E\left(b_p\left(t/p - \overline{S}_M\right) 1\left(p\overline{S}_M \in [1/2,1)\right)\right) \\
&\quad - E\left(b_p\left(t/p - S_M\right) 1\left(pS_M \in [1/2,1)\right)\right) \\
&\quad = E\left(b_p\left(t/p - S_M\right) 1\left(pS_M \in [1/2,1)\right) 1\left(\max_{k=1}^{M} X_k \le 1/p\right)\right) \\
&\quad + E\left(b_p\left(t/p - \overline{S}_M\right) 1\left(p\overline{S}_M \in [1/2,1)\right)\left(1 - 1\left(\max_{k=1}^{M} X_k \le 1/p\right)\right)\right) \\
&\quad - E\left(b_p\left(t/p - S_M\right) 1\left(pS_M \in [1/2,1)\right)\right) \\
&\quad \le 2g\left(0\right)\left(1 - P\left(\max_{k=1}^{M} X_k\left(p\right) \le 1/p\right)\right) \le 2g\left(0\right) P\left(X_1 > 1/p\right)/p,
\end{aligned}
$$

which implies $J_2 = o\left(p^{\alpha_*+1}\right)$. ■

A standard technique for handling defective renewal equations consists in transforming them into proper equations. We shall apply this technique to $\overline{a}_p\left(\cdot\right)$. Note that the definition of $\overline{\theta}_p$ implies that we can define a probability distribution $P_{\overline{\theta}_p}\left(\cdot\right)$ via

$$
P_{\overline{\theta}_p}\left(\overline{X}_1\left(p\right) \in dx\right) = q\exp\left(\overline{\theta}_p x\right) P\left(\overline{X}_1\left(p\right) \in dx\right).
$$

Therefore, multiplying through the left and right hand sides of equation (4) we obtain the proper renewal equation

$$
\widetilde{a}_p\left(t\right) = \widetilde{b}_p\left(t\right) + \int_{[0,t)} \widetilde{a}_p\left(t-s\right) P_{\overline{\theta}_p}\left(\overline{X}_1\left(p\right) \in dx\right),
$$

where $\widetilde{a}_p\left(t\right) = \overline{a}_p\left(t\right)\exp\left(\overline{\theta}_p t\right)$ and $\widetilde{b}_p\left(t\right) = b_p\left(t\right)\exp\left(\overline{\theta}_p t\right)$.

The next step is to analyze $\widetilde{a}_p\left(\cdot\right)$ using uniform renewal theory. In order to invoke such theory let us introduce some notation. Given a non-negative random variable τ with distribution function τ let us write

$$
E_F g\left(\tau\right) = \int_{[0,\infty)} g\left(t\right) dF\left(t\right)
$$

for all integrable functions $g\left(\cdot\right)$. In addition, let $\left(U_F\left(t\right) : t \ge 0\right)$ be the renewal function associated to F. In other words

$$
U_F\left(t\right) = I\left(0 \le t\right) + \sum_{n=1}^{\infty} P\left(\tau_1 + \ldots + \tau_n \le t\right),
$$

where the τ_i's are iid copies of τ. A family $\mathcal{F}$ of distributions is said to be strongly non-lattice if for each $\varepsilon > 0$

$$\sup_{F\in\mathcal{F},|\theta|\geq\varepsilon} |E_F \exp(i\theta\tau)| < 1.$$

The next result, which follows from Theorem 1 in Blanchet and Glynn (2007); see also Borovkov and Foss (2000) and Fuh (2004), will be useful in the analysis of $\overline{a}_p(\cdot)$.

Theorem 2 *Suppose that $\mathcal{F}$ is a strongly non-lattice family of distributions. If $\sup_{F\in\mathcal{F}} E_F\tau^{\alpha_*+2} < \infty$, then*

$$\left| U_F(t) - \frac{t}{E_F\tau} - \frac{E_F\tau^2}{2E_F^2\tau} \right| = o(t^{\alpha_*})$$

uniformly over $F \in \mathcal{F}$ as $t \nearrow \infty$.

We shall apply the previous result to the family of distributions $(P_{\overline{\theta}_p} : 0 \leq p \leq p_0)$ for some $p_0 \in (0,1)$. Let us write

$$\overline{U}_p(t) = I(0 \leq t) + \sum_{n=1}^{\infty} P_{\overline{\theta}_p}\left(\overline{X}_1(p) + ... + \overline{X}_1(p) \leq t\right).$$

The next lemma verifies the conditions in Theorem 2.

Lemma 2 *Under assumptions A) to B), there exists $p_0 \in (0,1)$ such that*

$$\left| \overline{U}_p(t) - \frac{t}{E_{\overline{\theta}_p}\overline{X}_1(p)} - \frac{E_{\overline{\theta}_p}\overline{X}_1(p)^2}{2E^2_{\overline{\theta}_p}\overline{X}_1(p)} \right| = o(t^{\alpha_*})$$

uniformly over $p \in [0, p_0]$ as $t \nearrow \infty$.

Proof. Note that $E_{\overline{\theta}_p}\exp\left(i\lambda\overline{X}_1(p)\right) = qE\exp((i\lambda + \overline{\theta}_p)\overline{X}_1(p))$ satisfies

$$\begin{aligned}
&\left| E_{\overline{\theta}_p}\exp\left(i\lambda\overline{X}_1(p)\right) - E\exp\left(i\lambda X_1(p)\right) \right| \\
&\leq \left| E_{\overline{\theta}_p}\exp\left(i\lambda\overline{X}_1(p)\right) - E\exp\left(i\lambda\overline{X}_1(p)\right) \right| + o\left(p^{\alpha+2}\right) \\
&\leq p\left| E\exp\left(i\lambda\overline{X}_1(p)\right) \right| + \overline{\theta}_p E\overline{X}_1(p) + O(p) = O(p).
\end{aligned}$$

The strongly non-lattice condition for the family $\left(P_{\overline{\theta}_p} : 0 \leq p \leq p_0\right)$ follows immediately from assumption A. Since $\overline{\theta} = O(p)$, we have that for all $p > 0$ small enough there exists $m \in (0, \infty)$ such that

$$
\begin{aligned}
& E_{\overline{\theta}} \overline{X}_1(p)^{\alpha_*+2} \\
& = qE \exp\left(\overline{\theta}_p \overline{X}_1(p)\right) \overline{X}_1(p)^{\alpha_*+2} \leq mE\overline{X}_1(p)^{\alpha_*+2} < mEX_1(p)^{\alpha_*+2}.
\end{aligned}
$$

The result then follows as an immediate application of Theorem 2 after invoking assumption B. ■

We now are ready to provide the proof Theorem 1.

Proof of Theorem 1. By iterating the proper renewal equation for $\widetilde{a}_p(\cdot)$ we have that

$$
\widetilde{a}_p(t/p) = \int_{[0,t/p)} \widetilde{b}_p(t/p - s)\, d\overline{U}_p(s).
$$

Define

$$
\overline{V}_p(t) = \overline{U}_p(t) - \frac{t}{E_{\overline{\theta}_p} \overline{X}_1(p)} - \frac{E_{\overline{\theta}_p} \overline{X}_1(p)^2}{2E^2_{\overline{\theta}_p} \overline{X}_1(p)}.
$$

We then obtain that

$$
\widetilde{a}_p(t/p) = \frac{1}{E_{\overline{\theta}_p} \overline{X}_1(p)} \int_{[0,t/p)} \widetilde{b}_p(s)\, ds + R_p(t/p),
$$

where

$$
\begin{aligned}
R_p(t/p) &= \int_{[0,t/p)} \widetilde{b}_p(t/p - s)\, d\overline{V}_p(s) \\
&= \overline{V}_p(t/p)\, \widetilde{b}_p(0_+) - \widetilde{b}_p(t/p_+)\, \overline{V}_p(0) \\
&\quad - \int_{[0,t/p)} \overline{V}_p(s)\, \widetilde{b}_p(t/p - ds).
\end{aligned}
$$

Define

$$
\begin{aligned}
R_{1,p}(t/p) &= \int_{[0,t/(2p))} \overline{V}_p(s)\, \widetilde{b}_p(t/p - ds), \\
R_{2,p}(t/p) &= \int_{[t/(2p),t/p)} \overline{V}_p(s)\, \widetilde{b}_p(t/p - ds).
\end{aligned}
$$

Because of assumption C) we have that there exists a constant $c \in (0, \infty)$, independent of p, such that

$$|R_{1,p}(t/p)| \leq \exp(ct) \sup_{0 \leq s < \infty} \overline{V}_p(s) \, g(t/2p) = o\left(p^{\alpha_* + 1}\right),$$

similarly

$$|R_{2,p}(t/p)| \leq \exp(ct) \, g(0) \sup_{t/(2p) \leq s \leq t/p} \left|\overline{V}_p(s)\right| = o\left(p^{\alpha_*}\right).$$

The contribution of the terms $\overline{V}_p(t/p) \, \widetilde{b}_p(0_+)$ and $\widetilde{b}_p(t/p_+) \, \overline{V}_p(0)$ is handled similarly so that $R_p(t) = o(p^{\alpha_*})$ as $p \searrow 0$ and therefore

$$\widetilde{a}_p(t/p) = \frac{1}{E_{\overline{\theta}_p} \overline{X}_1(p)} \int_{[0,t/p)} \widetilde{b}_p(s) \, ds + o\left(p^{\alpha_*}\right),$$

which implies the statement of the theorem.

THIS MIGHT BE ON THE SHORT SIDE? ■

A completely analogous result to that provided for defective renewal equations holds for excessive renewal equations, which can be written as

$$a_p(t) = b_p(t) + q^{-1} \int_{[0,t)} a_p(t-s) \, P(X_1 \in ds), \qquad (7)$$

where $q = 1 - p \in (0, 1)$. The development behind expansion for solutions to equations such as (7) is completely analogous, we ommit the details here.

Theorem 3 *Suppose that the distribution of the non-negative rv X_1 is strongly non-lattice and that $EX_1^{2+\alpha} < \infty$. In addition, suppose that there exists a bounded monotone function $g : \mathbb{R}_+ \rightarrow \mathbb{R}$ such that $|b(t)| \leq g(t)$ with $\int_0^\infty t^{\alpha+1} g(t) \, dt < \infty$. Finally, let us write, for $j \leq \alpha + 1$, $b_j = \int_0^\infty t^j b(t) \, dt$, and consider the solution $a(\cdot)$ to equation (7). Then, as $p \searrow 0$,*

$$a(t/p) \exp\left(t\widehat{\theta}/p\right) = d(p) + o\left(p^{\alpha+1}\right),$$

where $\widehat{\theta}$ is the minimal negative solution to

$$\phi\left(\widehat{\theta}\right) = q,$$

and

$$d(p) = \frac{b_0 + \sum_{k \leq \alpha+1} b_k \widehat{\theta}^k / k!}{q\left(EX_1 + \sum_{k \leq \alpha+1} \widehat{\theta}^k EX_1^{k+1} / k!\right)}.$$

Remark Straightforward set of equations can be developed here for obtain an asymptotic expansion of $\widehat{\theta}$ up to a term of order $p^{\lfloor \alpha+2 \rfloor}$.

Proof. This setting is actually easier than that of Theorem 1. We can work directly with the excessive renewal equation (7) just as we did before. The details are omitted. ■

3 Application to a Perturbed Risk Model

As an application of our results we consider the following example in insurance risk theory which generalizes the so-called classical ruin model. In the insurance context corrected diffusion approximations are developed in the context in which the premium rate charged by the insurance company is close to the equilibrium pay-out rate of claims. Since, in competitive environments, insurance companies will tend to charge relatively low premium rates, this asymptotic regime is very natural in the insurance context. The model that we consider is a perturbation of the classical risk model in which a Brownian component is added to the risk reserve.

Specifically, we consider the case of the classical ruin model perturbed by a diffusion introduced by Dufresne and Gerber (1991). That is, suppose that the risk process is a Levy process of the form

$$R(t) = x + ct - S(t) + \sigma B(t); \quad t \geq 0,$$

where $S(\cdot)$ represents the aggregate claim process, which follows a compound Poisson process with Poisson parameter λ and increments (claims) $Y = (Y_k : k \geq 1)$; x represents the initial reserve, c is a constant premium rate satisfying $c > \lambda EY$ and $\sigma B(\cdot)$ is a Brownian motion independent of S with diffusion coefficient equal to σ (i.e. instantaneous variance equal to σ^2). The term involving the Brownian motion B, represents noise that may incorporate non-systematic fluctuations in the composition of the insurance portfolio, measurement errors, etc. We are interested in computing the probability of eventual ruin in this model. Note that this model cannot be reduced directly to the classical ruin model because, in this case, ruin can occur between claim arrivals. As we discussed at the beginning of this section, we are interested in finding asymptotics in a low net-premium environment, specifically, when c is close to λEY.

Let us introduce some additional notation. Let Z be a rv having the equilibrium distribution generated by Y, that is

$$P(Z \leq z) = \frac{1}{EY} \int_0^z P(Y > y)\, dy.$$

Also, define $p = 1 - \lambda EY/c$ and $q = 1 - p$, and $V = Z + \sigma^2 W/(2c)$, where W is distributed exponential with mean one and Z and W are independent. Finally, let $\tau(x) = \inf(t \geq 0 : R(t) < 0)$, and note that the ruin occurs if and only if $\{\tau(x) < \infty\}$. Dufresne and Gerber (1991) proved that if $P(\tau(x) < \infty) = a(x)$, then

$$\begin{aligned} a(x) &= qP(V > x) + pP\left(W > 2cx/\sigma^2\right) \\ &+ q \int_0^x a(x - y)\, P(V \in dy). \end{aligned}$$

In this context, p close to zero is a reasonable assumption because is exactly the low net-profit environment discussed before. Hence, Theorem 1 can be directly applied here to provide asymptotics for $a(x/p)$ as $p \to 0$. In particular, for $j \geq 0$, it is easy to verify that

$$b_j = \frac{1}{j+1} \left(qEV^{j+1} + q\left(\sigma^2/(2c)\right)^{j+1} (j+1)! \right),$$

and that

$$EZ^j = \frac{EY^{j+1}}{(j+1)\, EY}, \quad EW^j = j!.$$

These expressions, combined with Theorem 1 provide all the necessary means to compute the desired asymptotic expansion. For instance, assuming that $EY^3 < \infty$ (which implies that $EZ^2 < \infty$), we obtain that

$$a(x/p) = \exp\left(-x/EV + 1/2\left(1 - EV^2/\left(2E^2V\right)\right) p\right) d(p) + o(p),$$

where

$$d(p) = \frac{\left(qEV + p\sigma^2/(2c)\right) + \left(qEV^2 + \sigma^4/(2c^2)\right) p/EV}{q\left(EV + pEV^2/EV\right)},$$

and

$$\begin{aligned} EV &= EY^2/(2EY) + \sigma^2/(2c) \\ EV^2 &= EY^3/(3EY) + \sigma^4/\left(2c^2\right) + \sigma^2 EY^2/(2cEY). \end{aligned}$$

Note that these asymptotics correspond to corrected diffusion approximations for the present model.

4 The $M/G/1$ Processor Sharing system

In the present section we explain how our methodology can be applied to a non-trivial queueing example. We consider the $M/G/1$ queue where the service discipline is Processor Sharing, i.e. each customer receives service at rate $1/n$ if the total number of customers in the system is n. Assuming the system load if $\rho < 1$, the sojourn time $V(\tau)$ of a customer with job size τ can be represented as

$$V(\tau) = V_0(\tau) + \sum_{i=1}^{N} C_i(\tau). \tag{8}$$

Here $C_i(\tau), i \geq 1$ is an i.i.d. sequence of random variables independent of $V_0(\tau)$ and N. We refer to Yashkov (1983), Ott (1984) to keep the presentation in this paper compact. This example obviously fits into the framework of this paper, and was in fact an important motivation for the general setup in Section 2, since $V_0(\tau)$ and $C_1(\tau)$ depend on ρ. Both $V_0(\tau)$ and $C_1(\tau)$ can be interpreted as functionals of a Crump-Mode-Jagers branching process, as noted in Yashkov (1983) and Grischechkin (1992). This branching interpretation allows one to conclude that both $V_0(\tau)$ and $C_1(\tau)$ are stochastically increasing in ρ. In addition, $V_0(\tau)$ and $C_1(\tau)$ have finite exponential moments - even if $\rho = 1$. Also, if B is a generic service time, then the law $C_1(\tau)$ has a density component on $(0, \tau)$ which is bounded from below by $e^{-\lambda\tau} P(B > x)/E[B]$. See the arguments in the proof of Theorem 2.1 in Egorova & Zwart (2007) for more details.

The focus in the remainder of this section is on deriving a corrected diffusion approximation for $V(\tau)$ by utilizing information on $V_0(\tau)$ and $C_1(\tau)$.

Let $\beta(s)$ be the LST of the service time distribution and let $\beta^r(s) = (1 - \beta(s))/sE[B]$. Let $\omega(s)$ be the LST of the steady-state workload distribution. It is well-known that $\omega(s) = \frac{1-\rho}{1-\rho\beta^r(s)}$.

The following results seems new and of independent interest. A related result for $V(\tau)$ itself has been derived in Zwart & Boxma (2000).

Lemma 3

$$\int_0^\infty e^{-v\tau} d\left(1/E[e^{-sV_0(\tau)}]\right) = \frac{1}{1 - \frac{s}{1-\rho}\frac{\omega(v)}{v}}, \tag{9}$$

so that

$$E[e^{-sV_0(\tau)}] = \left(\sum_{n=0}^{\infty} s^n \alpha_{0,n}(\tau)\right)^{-1}, \tag{10}$$

with $\alpha_{0,n}(\tau) = 1$ *and*

$$\alpha_{0,n}(\tau) = \frac{1}{(1-\rho)^n} \frac{1}{n!} \int_0^\tau (\tau - x)^n dW^{n*}(x). \tag{11}$$

In this expression, $W(x)$ *is the* $M/G/1$ *workload distribution.*

Proof. The first result follows in a straightforward manner from (3.16)–(3.19) in Yashkov (1983), see also p. 111 of Egorova & Zwart (2007). The second result follows from the first result by expanding the expression in powers of s and inverting term by term. ■

By straightforward differentiation, we obtain

$$E[V_0(\tau)] = \alpha_{0,1}(\tau), \tag{12}$$

$$E[V_0^2(\tau)] = 2\alpha_{0,1}(\tau)^2 - 2\alpha_{0,2}(\tau). \tag{13}$$

In the setting of Section 2, we have $p = 1 - \rho$, and $b_j(p) = E[V_0^j(\tau)]$, for $j = 1, 2$. Expressions for higher value of j can be derived from the above lemma at the expense of more involved computations. To be able to apply the results in Section 2 we also need expressions for $E[X_1(p)^j]$, which coincide with $E[C_1(\tau)^j]$. For that, we develop a series representation for the LST of $C_1(\tau)$ in the next lemma.

Let $B^r(x)$ be a distribution function with density $P(B > x)/E[B]$.

Lemma 4

$$E[e^{-sC_1(\tau)}] = E[e^{-sV_0(\tau)}] \left(1 + \sum_{s=1}^{\infty} s^n \alpha_{1,n}(\tau)\right), \tag{14}$$

with $\alpha_{1,n}(\tau) = \alpha_{0,n} * B^r(\tau)$*.*

Proof. From the (3.16)–(3.19) in Yashkov (1983) it follows that

$$E[e^{-sC_1(\tau)}] = E[e^{-sV_0(\tau)}] \left(\bar{B}^r(\tau) + \int_0^\tau \frac{1}{E[e^{-sV_0(\tau-x)}]} dB^r(x)\right).$$

The result follows by combining this with the previous lemma. ■

From this expression, it is readily seen that

$$E[C_1(\tau)] = E[V_0(\tau)] - \alpha_{1,1}(\tau) \tag{15}$$

$$E[C_1^2(\tau)] = E[V_0^2(\tau)] - 2\alpha_{0,1}(\tau)\alpha_{1,1}(\tau) + 2\alpha_{1,2}(\tau). \tag{16}$$

These expressions yield the correct formulae for the moments of $V(\tau)$. In particular, we verified the well-known result $E[V(\tau)] = \tau/(1-\rho)$. Combined with Proposition 1 and Theorem 1, the above results yield a two-term corrected diffusion approximation for the distribution of $V(\tau)$. Higher order terms in this expansion can be obtained from higher order moments of $V_0(\tau)$ and $C_1(\tau)$. These, in turn, can be obtained at the expense of more cumbersome computations, leading to explicit expressions in terms of the constants $\alpha_{i,j}(\tau)$. Nevertheless, the resulting formulae are much easier to compute than the original distribution of $V(\tau)$, since many examples of service-time distributions allow fast evaluation of (convolution of the workload distribution W), and the constants $\alpha_{i,j}(\tau)$ are completely determined by W.

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