LIMIT THEOREMS AND APPROXIMATIONS WITH APPLICATIONS TO INSURANCE RISK AND QUEUEING THEORY

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I certify that I have read this dissertation and that, in my opinion, it is fully adequate in scope and quality as a dissertation for the degree of Doctor of Philosophy.

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Contents

Chapter 1

Introduction

This dissertation focuses on the development of limit theorems and approximations for several performance measures that play an important role in a great variety of applied disciplines including: Insurance Risk Theory, Queueing Theory, Statistical Sequential Analysis, and Time Series Analysis, among others. To be more precise, let us utilize the insurance setting as a vehicle to provide a unified overview of the types of results that are developed in the subsequent chapters of this dissertation.

When dealing with the contingent nature of the insurance business, risk managers take advantage of stochastic models and tools that are used to effectively assess the risk of insurance portfolios (see Bowers et al (1997)). A popular model, widely used in the insurance community to analyze collective risk models is the so-called renewal model (see Bowers et al (1997) p. 432 and Asmussen (2001) Ch. 5). The renewal model assumes that the claims arrive according to a renewal process, with independent and identically distributed (iid) inter-arrival times. It is also assumed that the claims sizes are represented by a sequence of iid non-negative random variables (rv's), independent of the arrival process. Finally, the model specifies a constant (aggregated) premium rate, which is received by the insurance company. A fundamental quantity in the risk analysis of insurance portfolios is the so-called ruin probability or probability of bankruptcy. Of course, if the premium rate charged is less than or equal to the equilibrium pay-out rate, then the LLN implies that the company will go bankrupt eventually with probability one. Consequently, insurance companies would typically charge a positive "safety loading" in addition to the equilibrium pay-out rate. Note, however, that in competitive environments, one would typically expect insurance companies to charge small safety loadings to their customers.

The first portion of this dissertation addresses the problem of understanding the probability of eventual ruin, parametrically in the premium rate, under low safety loading environments. This problem, in turn, involves studying the mathematical structure of random walk with small negative drift. Indeed, the time to bankruptcy in the renewal model can be represented as the first hitting time to a certain level (which is just the initial reserve level of the company) for a random walk that has a negative drift proportional to the safety loading. As a result, the ruin occurs in finite time if the maximum of a random walk with negative drift ever hits a certain level, or, equivalently, if the corresponding first hitting time to this level is finite. Consequently, the aforementioned insurance problem motivates the parametric analysis of the distribution of the maximum of a random walk with small negative drift.

Incidentally, the distribution of the all time maximum of random walk with negative drift corresponds to the steady-state waiting time distribution (excluding service) of the single server queue (which is one of the most fundamental models in the theory of queues). As in the insurance setting discussed previously, the underlying random walk would often have close to zero drift, which translates into the so-called heavy traffic regime that is widely used in the modern analysis of queueing systems. Heavy traffic analysis is often done through diffusion approximations. In fact, as we shall see in Chapter 2, our parametric analysis of the distribution of the maximum of random walk, with close to zero drift, corrects the natural diffusion approximation based on Brownian motion (which provides a crude "first order" approximation to the distribution of the all time maximum of random walk). Corrected diffusion approximations (CDA's) for the distribution of the maximum of random walk were introduced by Siegmund (1979). Siegmund's second order correction to the standard Brownian approximation was motivated by applications in Statistical Sequential Analysis. Specifically, applications related to proper design of statistical tests that run up to a suitably defined first hitting time of an underlying random walk. The theory presented in Chapter 2 extends the development initiated by Siegmund (1979) and subsequent results in Statistical Sequential Analysis (see, for example, Chang (1992) and Chang and Peres (1997)).

The previous discussion presents some examples of applied disciplines that can potentially benefit from the results in the second chapter of this dissertation. Of course, in some of these disciplines, stylized features arising from modeling considerations, and statistical analysis of the data may give rise to additional technical complications that must be addressed. For example, in the insurance setting described before, it turns out that, in several branches of the insurance business (such as property insurance), heavy tailed structure (in particular, claims sizes that do not have exponential moments) seems to be an appropriate modeling feature to consider. (Other examples are discussed in Chapter 3 below.) Unfortunately, techniques (such as exponential changes of measure) that are extremely useful in the analysis of light tailed systems (i.e. assuming the existence of exponential moments) do not extend to the heavy tailed case. For instance, again coming back to the insurance arena, the corrected diffusion approximation by Siegmund (1979), and the extension provided in Chapter 2 of this dissertation, rely on light tailed techniques. Also, another approximation for the ruin probability, which is typically very powerful in light tailed settings, is the celebrated Cramer-Lundberg approximation. It turns out that, in the light tailed case, both the CDA presented in Chapter 2 and the Cramer-Lundberg approximation are intimately connected. Due to its success in applications involving light tailed characteristics, analogous forms of the Cramer-Lundberg approximation have been developed to cover a large class of heavy tailed claims (more precisely, subexponential claims, see Embrechts, Klüppelberg and Mikosch (1997)). These extensions to heavy tailed contexts are developed for large values of the initial reserve and fixed safety loading and typically provide a poor performance for typical values of the initial reserve in practical applications (see Embrechts, Klüppelberg and Mikosch (1997) p. 54). In Chapter 3, we introduce a new interpretation of the Cramer-Lundberg approximation for heavy tailed claims under the low safety loading asymptotic regime. In this dissertation (specifically in Chapter 3) we only focus on the proposed Cramer-Lundberg type of approximation in diffusion scale, which is related to the CDA presented in Chapter 2. Thus, in simple terms, Chapter 3 provides a new Cramer-Lundberg type of approximation for heavy tailed claims, interpreted in a low safety loading asymptotic regime, that seems to perform well in practical applications. (See Asmussen and Binswanger (1997), who analyzed a related approximation provided by Hogan (1986), which is discussed in Chapter 3 of this dissertation.) The approximation provided in Chapter 3 blends accurate approximations in diffusion scale with standard Cramer-Lundberg asymptotics for large values of the reserve in a coherent way; this parallels the relationship between the CDA of Chapter 2 and the Cramer-Lundberg asymptotic in the light tailed case.

As was mentioned before, the analysis of stochastic systems with heavy tailed characteristics gives rise to technical complications due to the fact that standard light tailed techniques are infeasible. In order to deal with the problem of providing accurate approximations for the probability of bankruptcy in heavy tailed contexts, we developed new techniques that, in particular, are applied to obtain the results described in the previous paragraph. These new techniques are presented in Chapter 4 of this dissertation. More precisely, Chapter 4 develops asymptotic expansions of so-called random geometric sums (or geometric convolutions) when the success parameter of the geometric random variable is close to zero. The direct connection to the ruin problem and the distribution of the maximum of random walk comes from a well known representation of the all time maximum of random walk as a geometric number of iid positive random variables. The techniques developed in Chapter 4 have implications beyond the ruin problem previously discussed. In particular, as we shall see, asymptotic expansions of geometric sums are closely related to so-called defective renewal equations. As we discuss in Chapter 4, these types of integral equations arise naturally in many areas of applied probability (including queueing theory and insurance risk theory). The asymptotics developed for geometric sums are then used to obtain expansion for defective renewal equations that are close to being proper. Again, this asymptotic regime arises repeatedly in queueing and insurance. For instance, as we shall see in Chapter 4, these results are useful in the development of corrected heavy traffic approximations for M/G/c queueing models and in the analysis of generalizations of classical renewal risk models.

Finally, it should be recognized that investments may play an important role in the bankruptcy of insurance companies. Indeed, it follows that if one introduces investment effects in the risk reserve, the probability of bankruptcy can be expressed in terms of the distribution of a so-called perpetuity or infinite horizon discounted reward (see Asmussen (2001) Ch. 7). This motivates the theme of the last chapter of this dissertation, namely Chapter 5. Specifically, in Chapter 5 we develop approximation for the distribution of infinite horizon discounted rewards. The theory provided in Chapter 5 is developed, just as in the previous chapters in a "low profit environment" which again is natural in many applications settings (such as the insurance context that we have been emphasizing). In particular, we develop central limit theorems, laws of large numbers, Edgeworth expansions and large deviation principles (rough and exact) for the distribution of perpetuities under low interest rates. As we shall also discuss in Chapter 5, these approximations are relevant not only to the insurance ruin problem, but also for other applied disciplines (including time series analysis and finance).

Chapter 2

Corrected Diffusion Approximation for the Maximum of Random Walk

Let $(X_n : n \geq 1)$ be a sequence of independent and identically distributed (iid) random variables (rv's), and let $S = (S_n : n \geq 0)$ be its associated random walk (so that $S_0 = 0$ and $S_n = X_1 + ... + X_n$ for $n \geq 1$). In this chapter, we focus on the development of high accuracy approximations to the distribution of the maximum r.v.

$$M = \max\{S_n : n \geq 0\}.$$

Clearly, $-\mu \triangleq EX_1$ must typically be negative in order that M be finite-valued. The distribution of M is of importance in a number of different disciplines.

For $x > 0$, $\{M > x\} = \{\tau(x) < \infty\}$, where $\tau(x) = \inf\{n \geq 1 : S_n > x\}$, so that computing the tail of M is equivalent to computing a level crossing probability for the random walk S. Because of this level crossing interpretation, the tail of M is of great interest to both the sequential analysis and risk theory communities. In particular, in the setting of insurance risk, $P(\tau(x) < \infty)$ is the probability that an insurer will face ruin in finite time (when the insurer starts with initial reserve x and is subjected to iid claims over time); see, for example, Asmussen (2000).

The distribution of M also arises in the analysis of the single most important model in queueing theory, namely the single-server queue. If the inter-arrival and service times for successive customers are iid with a mean arrival rate less than the mean service rate, then $W = (W_n : n \geq 0)$ is a positive recurrent Markov chain on $[0, \infty)$, where W_n is the waiting time (exclusive of service) for customer n. If W_∞ is a random variable having the stationary distribution of W, then Kiefer and Wolfowitz (1956) showed that W_∞ has the distribution of M for an appropriately defined random walk. As a consequence, computing the distribution of M is of fundamental importance to queueing theorists.

Since W is a positive recurrent Markov chain, the distribution of M can be computed as the solution to the equation describing the stationary distribution of W. This linear integral equation is known as Lindley's equation (see Lindley (1952)) and is of Wiener-Hopf type; it is challenging to solve, both analytically and numerically. As a result, approximations are frequently employed instead. One important such approximation holds as $\mu \searrow 0$. This asymptotic regime corresponds in risk theory to the setting in which the "safety loading" is small (i.e. the premium charged is close to the typical pay-out for claims) and in queueing theory to the "heavy traffic" setting in which the server is utilized close to 100% of the time. Thus, this asymptotic regime is of great interest from an applications standpoint. Kingman (1963) showed that the approximation

$$P\left(M > x\right) \approx \exp\left(-2\mu x/\sigma^2\right) \tag{1}$$

is valid as $\mu \searrow 0$, where $\sigma^2 = Var\left(X_1\right)$. (A more precise statement of this result will be given in Section 2.) Because the right hand side of (1) is the exact value of the level crossing probability for the natural Brownian approximation to the random walk S, (1) is often called the diffusion approximation to the distribution of M.

As with any such approximation, there are applications for which (1) delivers poor results. Siegmund (1979) therefore proposed a so-called "corrected diffusion approximation" that reflects information in the increment distribution beyond the mean and variance. This corrected diffusion approximation computes the next term in the asymptotic (as $\mu \searrow 0$) beyond that given by the right hand side of (1). The main result in this chapter (Theorem 1) is a development of the full asymptotic expansion initiated by Siegmund. We compute all the terms in the asymptotic expansion for general random walks with increments having exponential moments; see Section 6 for details on the calculation of the relevant coefficients in the expansion. Our theorem can be viewed as a non-Gaussian counterpart to the corresponding expansion provided recently by Chang and Peres (1997) for Gaussian random walks. As perhaps expected, the mathematical approach followed here is quite different from that used by Chang and Peres.

As is well known in the literature, there is a close connection between such corrections and asymptotic expansions for the moments of the ascending ladder height random variables associated with the random walk. Theorem 2 establishes an asymptotic expansion for the mean of the first strict ascending ladder height for random walks with light-tailed symmetric and continuous increments. As indicated in Section 6, this permits one to develop asymptotic expansions for all the moments of the ascending ladder heights (and for the limiting overshoot induced by the associated renewal process); see also Theorem 4.

This chapter is organized as follows. The main results are described in Section 2. A key connection to asymptotic expansions for the "short-time" behavior of the Cauchy process is made in Section 3. Section 4 shows how all the integrals required for our asymptotic expansion can be reduced to the short-time asymptotics of Section 3. Finally, Section 5 provides rigorous support for the remaining details in the argument used to compute the coefficients in the expansion. Section 6 summarizes the computation of the coefficients, and discusses an expansion related to the moments of the strict ascending ladder height. Any proof that does not follows the statement of the result can be found in our final section, namely Section 7.

2.1 The Main Results

To state our main results, we adopt the parameterization utilized by Siegmund (1979). We assume throughout this chapter that the X_i's have exponential moments, so that $E\exp(\theta X_1) < \infty$ for θ in a neighborhood containing the origin. For such θ, define

$$\psi(\theta) = \log E(\exp(\theta X_1)).$$

Then, for each such θ, we can define the probability measure P_θ having the property that for $n \geq 0$,

$$P_\theta(A) = E\left(\exp\left(\theta S_n - n\psi(\theta)\right) 1_A\right)$$

for $A \in \sigma(S_j : 0 \leq j \leq n)$. As is well known, S is again a random walk with iid increments under P_θ, having common increment distribution

$$P_\theta(X_1 \in dx) = \exp(\theta x - \psi(\theta)) P(X_1 \in dx)$$

for $x \in \mathbb{R}$ (with mean $E_\theta X_1 = \psi'(\theta)$ and variance $Var_\theta(X_1) = \psi''(\theta)$). Without any loss of generality, assume that $EX_1 = 0$ and $Var(X_1) = 1$. Since $\psi(\cdot)$ is strictly convex on its domain of finiteness, $E_\theta X_1 < 0$ for $\theta < 0$. Thus, P_θ induces a random walk with negative drift when $\theta < 0$. We therefore focus on corrected approximations to $P_\theta(M > x)$ as $\theta \nearrow 0$.

A key step to the analysis of $P_\theta(M > x)$ is the judicious application of Wald's likelihood ratio identity; see, for example Siegmund (1985), p. 13. For θ_0 in some interval of the form $(-\eta, 0)$, there exists a positive θ_1 such that $\psi(\theta_0) = \psi(\theta_1)$. Set $\Delta = \theta_1 - \theta_0$. Note that parameterizing in terms of Δ is essentially equivalent to parameterization in terms of θ_0 (or parameterization in terms of the drift $\mu = -\psi'(\theta_0)$). The likelihood ratio identity then asserts that

$$\begin{aligned} P_{\theta_0}(\tau(x) < \infty) &= E_{\theta_1} \exp\left(-(\theta_1 - \theta_0) S_{\tau(x)}\right) \\ &= \exp(-(\theta_1 - \theta_0) x) E_{\theta_1} \exp(-(\theta_1 - \theta_0) R(x)), \end{aligned} \qquad (2)$$

where $R(x) = S_{\tau(x)} - x$ is the so-called "overshoot" at level x.

Suppose now that X_1 is strongly non-lattice, in the sense that for each $\delta > 0$,

$$\inf_{|\lambda| > \delta} |1 - g(\lambda)| > 0, \qquad (3)$$

where $g(\lambda) = E\exp(i\lambda X_1)$ is the characteristic function of X_1 (under P_0). Applying renewal theory to the random walk at strictly increasing ladder epochs establishes then

$$E_{\theta_1} \exp(-(\theta_1 - \theta_0) R(x)) \to E_{\theta_1} \exp(-(\theta_1 - \theta_0) R(\infty)) \qquad (4)$$

as $x \to \infty$.

Siegmund (1979) showed that the renewal theorem can be applied uniformly for $\Delta < \eta$ (see also Chang (1992)). Hence, (2) yields

$$P_{\theta_0}(M > x) = \exp(-\Delta x) E_{\theta_1} \exp(-\Delta R(\infty)) + o(\exp(-(\Delta + r) x)) \qquad (5)$$

for some $r > 0$ (uniformly in $\theta_0 > -\eta/2$). In insurance risk theory, Δ is called the "adjustment coefficient" and the quantity $E_{\theta_1} \exp(-\Delta R(\infty))$ is known as the Cramer-Lundberg constant (c.f. Asmussen (2001)).

Relation (5) may alternatively be written as

$$P_{\theta_0}(\Delta M > x) = \exp(-x) E_{\theta_1} \exp(-\Delta R(\infty)) + o(\exp(-rx/\Delta)) \qquad (6)$$

where $o(\exp(-rx/\Delta))$ is uniform in $\theta_0 > -\eta/2$. Note that $\exp(-x)$ is precisely the level crossing probability of level x/Δ for a Brownian motion with drift $-\Delta/2$ and unit variance. Since $E_{\theta_1} X_1 \sim -\Delta/2$ as $\theta_0 \nearrow 0$, (6) provides rigorous support for the diffusion approximation (1). Furthermore, a correction to the diffusion approximation described at the beginning of this chapter can be obtained by developing an asymptotic expansion for $E_{\theta_1} \exp(-\Delta R(\infty))$.

Siegmund (1979) obtained his corrected diffusion approximation by showing that

$$E_{\theta_1} \exp(-\Delta R(\infty)) = \exp(-\Delta \beta_1) + o(\Delta^2) \qquad (7)$$

as $\Delta \downarrow 0$, where β_1 can be computed explicitly as

$$\beta_1 = \frac{1}{6} E X_1^3 - \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{\lambda^2} \operatorname{Re} \log\{2(1 - g(\lambda)) / \lambda^2\} d\lambda. \qquad (8)$$

Note that by computing the single integral (8), Siegmund's corrected diffusion approximation to the distribution of M provides a parametric approximation that is valid for all random walks having negative drift sufficiently close to zero. Such parametric approximations are convenient in many applications settings (i.e. in studying the behavior of a queue when utilization is close to 100%).

Our main theorem shows that there is a full asymptotic expansion for

$$r(\Delta) \triangleq \log E_{\theta_1} \exp(-\Delta R(\infty)).$$

Theorem 1 *Suppose that* X_1 *has exponential moments and is strongly non-lattice. Then,* $r(\cdot)$ *(initially defined on* $[0, \upsilon)$ *for* $\upsilon > 0$*) admits an analytic extension on a neighborhood of the origin in the complex plane.*

Remark An immediate consequence of Theorem 1 and the implicit function theorem is that the Cramer-Lundberg constant, namely $\exp(r(\Delta(\theta_0)))$, initially defined for all $\theta_0 < 0$ sufficiently close to zero, admits an analytic extension on a disc containing the origin in the complex plane.

According to Theorem 1,

$$E_{\theta_1} \exp(-\Delta R(\infty)) = \exp\left(\sum_{n=1}^{\infty} \beta_n \Delta^n\right), \tag{9}$$

where β_1 is given by (8) and $\beta_2 = 0$. (This latter equality follows from the fact that the error term in (7) is $o(\Delta^2)$.) Obviously, in order for (9) to be useful from an applied standpoint, we need a means of numerically computing the β_n's. This issue is discussed in Section 6. We establish there that the β_n's can be successively computed via a finite number of one-dimensional integrations reminiscent of the integral appearing in (8). Thus, the β_n's can easily be computed, thereby yielding cheaply computable high-order parametric corrections to the diffusion approximation (1).

The argument above also permits us to establish asymptotic expansions for certain ladder height quantities. As noted earlier, renewal theory applies to the random walk when sampled at strictly increasing ladder epochs. The renewal theorem invoked above actually establishes that

$$E_{\theta_1} \exp(-\Delta R(\infty)) = \frac{1 - E_{\theta_1} \exp\left(-\Delta S_{\tau_+}\right)}{\Delta E_{\theta_1} S_{\tau_+}}, \tag{10}$$

where $\tau_+ = \inf\{n \geq 1 : S_n > 0\}$ is the first (strict) increasing ladder epoch (see Asmussen (1987)). In view of (2), it follows that

$$1 - E_{\theta_1} \exp\left(-\Delta S_{\tau_+}\right) = P_{\theta_0}(\tau_+ = \infty). \tag{11}$$

Random walk duality (see, for example, p. 173 of Siegmund (1985)) implies that

$$P_{\theta_0}(\tau_+ = \infty) = 1/E_{\theta_0}\tau_-, \tag{12}$$

where $\tau_- = \inf\{n \geq 1 : S_n \leq 0\}$. If the X_i's are symmetric rv's with common continuous distribution function, $\Delta = 2\theta_1$ and $E_{\theta_0}\tau_- = E_{\theta_1}\tau_+$. Furthermore, (10) to (12) then imply that

$$E_{\theta_1} \exp(-\Delta R(\infty)) = \frac{1}{2\theta_1 \left(E_{\theta_1} S_{\tau_+}\right)\left(E_{\theta_1}\tau_+\right)}.$$

In view of Wald's identity, we then obtain the relation

$$E_{\theta_1} \exp(-\Delta R(\infty)) = \frac{\psi'(\theta_1)}{2\theta_1 \left(E_{\theta_1} S_{\tau_+}\right)^2}.$$

As a consequence, Theorem 1 then yields a full asymptotic expansion for the expected ladder height $E_{\theta_1} S_{\tau_+}$. We record this result as our Theorem 2.

Theorem 2 *Assume that X_1 has exponential moments and is symmetric with a continuous distribution function. Then,*

$$E_{\theta_1} S_{\tau_+} = \sqrt{\frac{\psi'(\theta_1)}{2\theta_1}} \exp\left(-\frac{1}{2}\sum_{m=0}^{\infty} \beta_{2m+1} (2\theta_1)^{2m+1}\right).$$

Given our above argument, the only remaining issue in proving Theorem 2 is establishing that $\beta_{2n} = 0$ for $n \geq 1$ in the presence of symmetry. This fact is proven in Section 2.6.

The most important device that we use to prove Theorems 1 and 2 is a convenient representation for $r(\Delta)$. This representation is a key idea in our mathematical development. To introduce our representation put $\phi(\theta) = E\exp(\theta X_1)$ for $\theta \in \mathbb{R}$ and, for $z \in \mathbb{C}$, set $\gamma(z) = E\exp(zX_1)$. Note that ϕ is finite-valued on a neighborhood $\mathcal{N}$ of the origin and γ is analytic on the strip $\{x + iy : x \in \mathcal{N}, y \in \mathbb{R}\}$. For non-negative $\theta \in \mathcal{N}$ and $b \in \mathbb{R}$, put

$$\rho(\theta, b) = \log E_\theta \exp(-bR(\infty)).$$

Note $r(\Delta) = \rho(\theta_1, \Delta)$, where $\theta_1 = \theta_1(\Delta) > \theta_0(\Delta) = \theta_0$ is such that $\psi(\theta_1(\Delta)) = \psi(\theta_0(\Delta))$. Woodroofe (1979) showed that

$$\rho(\theta, b) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{-b}{(b + i\lambda)\, i\lambda} \log\left(\frac{\gamma(\theta) - \gamma(\theta + i\lambda)}{-i\phi'(\theta)\,\lambda} \right) d\lambda; \tag{13}$$

see also Corollary 8.45 and Theorem 8.51 of Siegmund (1985). While (13) is convenient for many purposes, it presents difficulties in the current circumstances because of the singularity (in the logarithm) that arises when $\theta \searrow 0$. The following representation for $\rho(\theta, b)$ is free of such singularities.

Theorem 3 *Suppose X_1 has exponential moments and is strongly non-lattice. Then, for non-negative $\theta \in \mathcal{N}$ and $b > 0$,*

$$\rho(\theta, b) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{-b}{(b + i\lambda)\, i\lambda} \log\left(\frac{2(\gamma(\theta) - \gamma(\theta + i\lambda))}{\lambda(\lambda - 2i\phi'(\theta))} \right) d\lambda. \tag{14}$$

Siegmund's computation of β_1 takes advantage of the fact that the first order behavior of $r(\Delta)$ should match that of

$$s(\Delta) = \log E_0 \exp(\Delta R(\infty)). \tag{15}$$

Since $s(\Delta) = \rho(0, \Delta)$, Theorem 3 implies that

$$s(\Delta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{-\Delta}{(\Delta + i\lambda)\, i\lambda} \log\left(2(1 - g(\lambda))\lambda^{-2}\right) d\lambda; \tag{16}$$

see also p. 226 of Siegmund (1985). We proceed to analyze $\rho(\theta, b)$ by writing $\rho(\theta, b) = s(b) + I(\theta, b)$. In view of both Theorem 3 and (16),

$$I(\theta, b) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{-b}{(b + i\lambda)\, i\lambda} \log\left(\frac{\lambda(\gamma(\theta) - \gamma(\theta + i\lambda))}{(\lambda - 2i\phi'(\theta))(1 - g(\lambda))} \right) d\lambda. \tag{17}$$

In the next sections, we develop asymptotics, as $b \searrow 0$, appropriate to the integrals arising in (16) and (17). Such asymptotics can be used to provide asymptotic expansions for the moments (or, equivalently, the cumulants) of the limiting expected overshoot r.v.$R(\infty)$ under P_θ as $\theta \searrow 0$. Specifically, for $n \geq 1$, let

$$\kappa_n(\theta) = (-1)^n \left.\frac{\partial^n}{\partial b} \rho(\theta, b)\right|_{b=0}.$$

Theorem 4 *Assume that X_1 has exponential moments and is strongly non-lattice. Then (for all $n \geq 1$) $\kappa_n(\cdot)$, initially defined on $[0, \upsilon)$ for $\upsilon > 0$, can be extended to be an analytic function throughout a disc in the complex plane containing the origin.*

An important implication of Theorem 4 is that it can be directly applied to obtain complete asymptotics for the steady-state mean of the waiting time sequence, namely $E_{\theta_0} M$ $(= E_{\theta_0} W_\infty)$. In particular, Siegmund (1979) shows (see also Theorem 6.7, p. 275, of Asmussen (1987)) that

$$\begin{aligned} E_{\theta_0} M &= \frac{E_{\theta_0}\left(S_{\tau_+} \middle| \tau_+ < \infty\right)}{P_{\theta_0}(\tau_+ = \infty)} \\ &= \frac{E_{\theta_1} S_{\tau_+} \exp\left(-\Delta S_{\tau_+}\right)}{1 - E_{\theta_1} \exp\left(-\Delta S_{\tau_+}\right)} \\ &= \frac{E_{\theta_1}(1 - R(\infty)) \exp(-\Delta R(\infty))}{\Delta E_{\theta_1} \exp(-\Delta R(\infty))} \\ &= \frac{1}{\Delta} + \frac{1}{\Delta} \frac{\partial}{\partial b} \rho(\theta_1, \Delta). \end{aligned} \tag{18}$$

Thus, since

$$\frac{\partial}{\partial b} \rho(\theta, b) = \sum_{m=0}^{\infty} (-1)^m \kappa_{m+1}(\theta) \frac{b^m}{m!},$$

it follows that Theorem 4 can be applied directly to provide the full asymptotic expansion for $E_{\theta_0} M$. Indeed, our analysis in Sections 3 to 5 yield an asymptotic expansion for $\kappa_n(\cdot)$ around zero which in turn implies the expansion

$$E_{\theta_0} M = \frac{1}{\Delta} + \sum_{m=0}^{n} \sum_{j=0}^{n-m} (-1)^m \kappa_{m+1}^{(j)}(0) \frac{\theta_1(\Delta)^j}{j!} \frac{\Delta^m}{m!} + O\left(\Delta^{n+1}\right)$$

valid for all $n \geq 0$. The explicit computation of the derivatives $\kappa_{m+1}^{(j)}(0)$, for $j, m \geq 0$, is discussed in Section 2.5.2.

Finally, the analytic extension of $\kappa_n(\cdot)$ and $r(\cdot)$ is a consequence of the following result.

Proposition 1 *If X_1 has exponential moments and strongly non-lattice distribution, then, $I(\cdot)$ (defined as in (17) on a domain containing $[0, \upsilon) \times [0, \upsilon)$ with $\upsilon > 0$) can be analytically extended throughout a disc containing the origin in $\mathbb{C} \times \mathbb{C}$.*

Moreover, with the aid of Theorem 1 it follows easily (from (18) and the implicit function theorem) that $\Delta E_{\theta_0} M$ (initially defined for $\theta_0 < 0$) can be analytically extended (as a function of $\Delta(\theta_0)$) in a neighborhood of the origin in the complex plane.

2.2 Short-time Asymptotics for the Cauchy Process

The approach described in Section 2 suggests computing an asymptotic expansion for $r(\Delta)$ by developing appropriate expansions for $s(\Delta)$ and $I(\theta_1, \Delta)$. In this section, we will show how asymptotics for $s(\Delta)$ can be obtained. Section 4 shows how asymptotics for $I(\theta, \Delta)$ (and, as a result, also for $I(\theta_1(\Delta), \Delta)$) can be reduced to the types of integrals considered here.

Since $s(b)$ is real for b positive, it follows that the integral of the imaginary part of (16) must vanish. Hence, $s(b)$ equals the integral of the real part of (16), so that

$$\begin{aligned} s(b) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{b}{b^2+\lambda^2} \operatorname{Re}\log\left(2\left(1-g(\lambda)\right)\lambda^{-2}\right) d\lambda \\ &\quad -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{b}{\left(b^2+\lambda^2\right)\lambda} \operatorname{Im}\log\left(1-g(\lambda)\right) d\lambda. \end{aligned} \tag{19}$$

Both of the above integrals take the form

$$\begin{aligned} K(b,f) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{b}{b^2+\lambda^2} f(\lambda)\, d\lambda \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{1+\lambda^2} f(\lambda b)\, d\lambda. \end{aligned} \tag{20}$$

for suitably defined f. Note that if $Y = (Y(t) : t \geq 0)$ is a standard Cauchy process (so that $Y(1)$ is distributed as a standard Cauchy r.v.), $K(t, f)$ can then be represented as

$$K(t,f) = \frac{1}{2} E\left(f(Y(t)) \middle| X = 0\right).$$

Hence, representing $K(t, f)$ as a power series in t is equivalent to the development of short-time asymptotics of the Cauchy process. Such asymptotics are also of general analytical interest, because of their relevance to Fourier analysis. Integrals of the type (20) are closely related to "approximate identities of the Fejer type"; see p. 31 of Butzer (1971).

Let $\mathfrak{L}$ be the space of functions $f : \mathbb{R} \to \mathbb{C}$ for which $E\left|f\left(Y\left(1\right)\right)\right|$ is finite and for which f is infinitely differentiable at zero. For $f : \mathbb{R} \to \mathbb{C}$ let $\underline{f}$ be the symmetrization of f defined via $\underline{f}(x) = \left(f\left(x\right) + f\left(-x\right)\right)/2$. The following result provides our short-time asymptotic expansion for $K\left(t, f\right)$.

Proposition 2 *Suppose f belongs to $\mathfrak{L}$. Then, $K\left(\cdot, f\right)$ is infinitely differentiable at the origin and*

$$K^{(n)}\left(0, f\right) = \begin{cases} \left(-1\right)^{n/2} f^{(n)}\left(0\right) & n \text{ even} \\ \left(-1\right)^{(n-1)/2} n! \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(T_{(n-1)/2}\underline{f}\right)\left(\lambda\right) d\lambda & n \text{ odd} \end{cases},$$

where, for $j \geq 0$, T_j acts on even functions in $\mathfrak{L}$ as

$$T_j f\left(\lambda\right) = \frac{f\left(\lambda\right) - \sum_{k=0}^{2j-1} f^{(2k)}\left(0\right)\lambda^{2k}/\left(2k!\right)}{\lambda^{2j}}.$$

Furthermore, the family of linear operators $\left(T_n : n \geq 0\right)$ is a commutative semigroup, so that $T_{n+m} = T_n T_m$ $m, n \geq 0$.

Remark Note that the even derivatives of f match those of $\underline{f}(\cdot)$. One might therefore be tempted to write the derivatives of $K\left(\cdot, f\right)$ in terms of integrals of $T_j f$ rather than $T_j\underline{f}$. The problem is that $T_j f$ typically has a singularity at the origin, unless the odd derivatives of f at zero vanish. As a consequence, the integrals defining the derivative of $K\left(\cdot, f\right)$ may diverge if they were defined directly in terms of f. To avoid this, we use the symmetrization $\underline{f}$.

Proof of Proposition 2. The fact that T_n is a linear operator, and forms a commutative semigroup is straightforward. To obtain the formula for the derivatives of $K\left(\cdot, f\right)$ at the origin, note that $K\left(\cdot, f\right) = K\left(\cdot, \underline{f}\right)$ where $\underline{f}$ is the symmetrization of f given by $\underline{f}(\cdot) = \left(f\left(\cdot\right) + f\left(-\cdot\right)\right)/2$. Furthermore, if $f \in \mathfrak{L}$ then $\underline{f}$ is also in $\mathfrak{L}$. Observe that the Dominated Convergence Theorem implies that

$$K\left(t, \underline{f}\right) \to \underline{f}\left(0\right)/2$$

as $t \searrow 0$. This motivates writing

$$K\left(t, \underline{f}\right) = \underline{f}\left(0\right)/2 + \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{t}{t^2+\lambda^2}\left(\underline{f}\left(\lambda\right) - \underline{f}\left(0\right)\right) d\lambda.$$

Since $E\left|\underline{f}\left(Y\left(1\right)\right)\right|$ is finite, it follows that the above integrand is uniformly dominated by an integrable function for $|\lambda|$ bounded away from zero. On the other hand, $\underline{f}(\lambda) - \underline{f}(0) = O\left(\lambda^2\right)$ as $\lambda \to 0$, so the integrand is also uniformly (in t) dominated for $|\lambda|$ small. Hence, the Dominated Convergence Theorem yields the conclusion that

$$K\left(t, f\right) = \underline{f}\left(0\right)/2 + \frac{t}{2\pi}\int_{-\infty}^{\infty} \frac{1}{t^2}\left(\underline{f}\left(\lambda\right) - \underline{f}\left(0\right)\right) d\lambda + o\left(t\right)$$

as $t \to 0$. In fact,

$$\begin{aligned} K\left(t, f\right) &= \underline{f}\left(0\right)/2 + \frac{t}{2\pi}\int_{-\infty}^{\infty} \frac{1}{\lambda^2}\left(\underline{f}\left(\lambda\right) - \underline{f}\left(0\right)\right) d\lambda \\ &\quad - \frac{t^2}{2\pi}\int_{-\infty}^{\infty} \frac{t}{t^2+\lambda^2}\frac{\left(\underline{f}\left(\lambda\right) - \underline{f}\left(0\right)\right)}{\lambda^2} d\lambda \\ &= \underline{f}\left(0\right)/2 + \frac{t}{2\pi}\int_{-\infty}^{\infty}\left(T_1\underline{f}\right)\left(\lambda\right) d\lambda - t^2 K\left(t, T_1\underline{f}\right). \end{aligned} \tag{21}$$

If we apply (21) recursively to $K\left(\cdot, T_1\underline{f}\right)$, $K\left(\cdot, T_2\underline{f}\right)$,... we find that $K\left(t, f\right)$ satisfies

$$\begin{aligned} K\left(t, f\right) = \sum_{j=0}^{n}\left(-1\right)^j\left(\frac{t^{2j}\left(T_j\underline{f}\right)\left(0\right)}{2} + \frac{t^{2j+1}}{2\pi}\int_{-\infty}^{\infty}\left(T_{j+1}\underline{f}\right)\left(\lambda\right) d\lambda\right) \\ + \left(-1\right)^{n+1} t^{2(n+1)} K\left(t, T_{2(n+1)}\underline{f}\right), \end{aligned}$$

yielding the result.

With Proposition 2 in hand, our asymptotic expansion for $s\left(\Delta\right)$ follows immediately.

2.3 Reducing the Analysis to Cauchy Process Short-time Asymptotics

As we discussed earlier in Section 2, the backbone of our asymptotic analysis for $r\left(\Delta\right)$ is given by the relation $\rho\left(\theta, b\right) = s\left(b\right) + I\left(\theta, b\right)$. In Section 3, we studied how to develop asymptotics for $s(b)$. In this section, we will study how to reduce the analysis of the remaining term $I(\theta,b)$ to that already studied in Section 3. Recall that

$$I(\theta,b)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{-b}{(b+i\lambda)\,i\lambda}\log\left(1-v(\theta,\lambda)\right)d\lambda,$$

where

$$v(\theta,\lambda)=\frac{\lambda}{\lambda-2i\phi'(\theta)}\left(1-\frac{2i\phi'(\theta)}{\lambda}-\frac{\gamma(\theta)-\gamma(\theta+i\lambda)}{1-g(\lambda)}\right).$$

A natural strategy to now follow is to express the logarithm as a power series in $v(\theta,\lambda)$, followed by an expansion for v as

$$v(\theta,\lambda)=\sum_{n=0}^{\infty}v_n(i\lambda)\frac{\theta^n}{n!}. \tag{22}$$

One could then apply Proposition 2 (as for (19)) to the real and imaginary parts in each of the resulting integrals that would appear as coefficients for θ^n. However, the expansion (22) requires that the function v be expressible as a joint power series in non-negative powers of θ and λ. Unfortunately, the presence of the term $(\lambda-2i\phi'(\theta))$ in the denominator of v precludes the existence of such a joint power series.

To avoid this difficulty we write v as

$$v(\theta,\lambda)=\frac{\lambda H(\theta,\lambda)}{\lambda-2i\phi'(\theta)},$$

so that

$$H(\theta,\lambda)=1-\frac{2i\phi'(\theta)}{\lambda}-\frac{\gamma(\theta)-\gamma(\theta+i\lambda)}{1-g(\lambda)}.$$

The function $H(\cdot)$ is well behaved because the term $2i\phi'(\theta)/\lambda$ controls the behavior of $(\gamma(\theta)-\gamma(\theta+i\lambda))(1-g(\lambda))^{-1}$ as $\lambda\searrow 0$. As a consequence, $H(\cdot)$ can be smoothly defined at $\lambda=0$ via the relation $H(\theta,0)=1-\phi''(\theta)$. Our next result describes the analytic structure of $H(\cdot)$.

Proposition 3 *Let* $D_{\eta/2}\triangleq\{z\in\mathbb{C}:|z|<\eta/2\}$ *and, for* $(z_1,z_2)\in D_{\eta/2}\times\left(D_{\eta/2}\bigcup\mathbb{R}\right)$, *put* H

$$H(z_1,z_2)=1-2\frac{i\gamma'(z_1)}{z_2}-\frac{\gamma(z_1)-\gamma(z_1+iz_2)}{1-\gamma(iz_2)}.$$

Then, for every $z_1 \in D_{\eta/2}$*, the function* $H(z_1, \cdot)$ *is analytic on* $D_{\eta/2}$*. Similarly, for every* $z_2 \in D_{\eta/2} \bigcup \mathbb{R}$*, the function* $H(\cdot, z_2)$ *is analytic on* $D_{\eta/2}$*. Finally,* $H(z_1, \lambda)$ *can be represented as an absolutely and uniformly convergent series, for* $\lambda \in \mathbb{R}$ *and* $z_1 \in D_{\eta/2}$*, namely*

$$H(z_1, \lambda) = \sum_{k=1}^{\infty} h_k(i\lambda) \frac{z_1^k}{k!}, \tag{23}$$

where $h_k(i\lambda) \triangleq \left(\gamma^{(k)}(i\lambda) - \mu_k\right) / (1 - g(\lambda)) - \left(2i\mu_{k+1}/\lambda\right)$*. In particular, this implies that*

$$\sup_{\lambda \in \mathbb{R}} |H(z_1, \lambda)| \to 0$$

as $z_1 \to 0$*.*

Remark Note that the function $\widetilde{H}(z_1, z_2) \triangleq H(z_1, z_2) - H(\theta, 0) = H(z_1, z_2) - 1 + \gamma''(z_1)$, satisfies the same properties stated for $H(\cdot)$ in Proposition 3 with $\widetilde{h}_k(i\lambda) \triangleq h_k(i\lambda) + \mu_{k+2}$, this follows from the analyticity of $\gamma(\cdot)$ and the fact that $\gamma''(0) = 1$. Moreover, observe that completely analogous analytic properties apply to the function $\widetilde{G}(z_1, z_2) = (\gamma''(z_1))^{-1} \widetilde{H}(z_1, z_2)$ defined on $D_{\eta/2} \times \left(D_{\eta/2} \bigcup \mathbb{R}\right)$.

Note that $|\lambda / (\lambda - 2i\phi'(\theta))| = |\lambda| \left(\lambda^2 + (2\phi'(\theta))^2\right)^{-1/2} \leq 1$. It follows from Proposition 3 that for $r > 0$ small enough,

$$\sup_{\theta \in (0,r)} \sup_{\lambda \in \mathbb{R}} |v(\theta, \lambda)| < 1.$$

Therefore, for all $0 < \theta < r$, we can proceed to expand $\log(1 - v)$ in powers of v and formally integrate each term in the obtained expansion to express $I(\theta, b)$ in terms of integrals of the form

$$J_k(a, b, f) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{-b}{(b + i\lambda) i\lambda} \left(\frac{i\lambda}{a + i\lambda}\right)^k f(i\lambda) \, d\lambda, \tag{24}$$

where $a, b > 0$, $f(i\cdot) \in \mathcal{L}$ and $k \geq 0$. Because $J_0(a, b, f) \triangleq J_0(b, f)$ can be written as

$$\begin{aligned} J_0(b, f) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{b}{b^2 + \lambda^2} \left(\operatorname{Re} f(i\lambda) - \lambda^{-1} \operatorname{Im} f(i\lambda)\right) d\lambda \\ &\quad + \frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{b}{b^2 + \lambda^2} \left(\operatorname{Im} f(i\lambda) + \lambda^{-1} \operatorname{Re} f(i\lambda)\right) d\lambda, \end{aligned} \tag{25}$$

it follows that asymptotics for J_0 can be computed in terms of asymptotics for the K-type integrals that are subject of Proposition 2. In view of the development leading to (24), a key to our asymptotic expansion for $I(\theta, b)$ is therefore the reduction of integrals $J_k(a, b, f)$ for $k \geq 1$ to integrals of the form $J_0(b, f)$. A key identity in establishing this reduction step is the following.

Lemma 1 *Suppose that $a, b \geq 0$. Then, for $m, n \geq 0$,*

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{-b}{(b+i\lambda)\,i\lambda}\frac{(i\lambda)^{m+1}}{(a+i\lambda)^{m+n+1}}d\lambda = 0.$$

Furthermore,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{-1}{(1+i\lambda)\,i\lambda}\log(1+ai\lambda)\,d\lambda = 0.$$

Proof. For $a, b > 0$, let the function of a complex variable $f(\cdot)$ be defined as

$$f(z) = \frac{-b}{(b+iz)\,iz}\frac{(iz)^{m+1}}{(a+iz)^{m+n+1}}.$$

Consider the contour (in the clockwise direction) $C(r) = C_1(r) + C_2(r)$, where $C_1(r) = \{re^{i\tau} : -\pi \leq \tau \leq 0\}$ and $C_2(r) = \{\lambda : \lambda \in [-r, r]\}$. Since f is (complex) analytic on $\operatorname{Im}(z) \leq 0$, Cauchy's theorem yields

$$\frac{1}{2\pi}\int_{C(r)}\frac{-b}{(b+iz)\,iz}\frac{(iz)^{m+1}}{(a+iz)^{m+n+1}}dz = 0.$$

This, in turn, implies that

$$\begin{aligned}\frac{1}{2\pi}\int_{-r}^{r}\frac{-b}{(b+i\lambda)\,i\lambda}\frac{(i\lambda)^{m+1}}{(a+i\lambda)^{m+n+1}}dz &= \frac{-1}{2\pi}\int_{C_1(r)}\frac{-b}{(b+iz)\,iz}\frac{(iz)^{m+1}}{(a+iz)^{m+n+1}}dz \\ &= \frac{-1}{2\pi}\int_{-\pi}^{0}\frac{b\,(ir)^{m+1}\,e^{(m+1)\tau i}}{(b+ire^{\tau i})\,(a+ire^{\tau i})^{m+n+1}}d\tau.\end{aligned}$$

Letting $r \to \infty$, we obtain (by virtue of dominated convergence) the first part of the lemma. For the second part, let us define

$$f_1(a) = \frac{1}{2\pi}\int_{-\infty}^{\infty} -((1+i\lambda)\,i\lambda)^{-1}\log(1+ai\lambda)\,d\lambda.$$

A routine dominated convergence argument, combined with our previous analysis, shows that

$$f_1'(a) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{-1}{(1+i\lambda)(1+ai\lambda)} d\lambda = 0.$$

The proof of the lemma is completed by observing that $f_1(a) \to 0$ as $a \searrow 0$.

Let $\mathfrak{L}_0$ be the subspace of $\mathfrak{L}$ (recall the definition of $\mathfrak{L}$ preceding Proposition 2) for which $f(0) = 0$. Also, for $f \in \mathfrak{L}$, let $\widetilde{f}(\cdot) = f(\cdot) - f(0) \,(\in \mathfrak{L}_0)$. We are now ready to offer a proposition that reduces the evaluation of the integrals $J_k(a,b,f)$ for $k \geq 1$ to that of integrals such as $J_0(b,f)$, thereby permitting the application of the short-time asymptotics of Section 3.

Proposition 4 *Suppose that $f \in \mathfrak{L}_0$. Then, for $k \geq 1$ and $n \geq 0$,*

$$J_k(a,b,f) = J_0\left(b, \sum_{j=0}^{n} \binom{k+j-1}{j} (-a)^j \widetilde{T}_j \widetilde{f}\right) + bo(a^n), \tag{26}$$

where the linear operator $\widetilde{T}_j$ $(j \geq 0)$ acts on functions $\widetilde{f}(i\cdot) \in \mathfrak{L}_0$ as

$$\left(\widetilde{T}_j \widetilde{f}\right)(i\lambda) = \frac{\widetilde{f}(i\lambda) - \sum_{m=1}^{j} \widetilde{f}^{(m)}(0)(i\lambda)^m/m!}{(i\lambda)^j}.$$

Moreover, the family of operators $\left(\widetilde{T}_j : j \geq 0\right)$ constitutes a commutative semigroup, so that $\widetilde{T}_m \widetilde{T}_n = \widetilde{T}_{m+n}$.

Remark As for Proposition 2, one might be tempted to express the right-hand side of (26) in terms of f rather that $\widetilde{f}$. However, $\widetilde{T}_j f$ is generally non-integrable with respect to the kernel that defines J_0. Finally, note that, if all integrals are interpreted in terms of Cauchy principal value, one can apply Proposition 4 directly to functions that do not vanish at the origin by defining $J_0(b,f) = J_0(b, f(\cdot) - f(0)) + f(0)/2$.

Proof of Proposition 4. That $\left(\widetilde{T}_j : j \geq 0\right)$ is a family of linear operators forming a commutative semigroup is immediate. By virtue of Lemma 1, it follows that

$$J_m(a,b,f) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{-b}{(b+i\lambda)\, i\lambda} \left(\frac{i\lambda}{a+i\lambda}\right)^m \widetilde{f}(i\lambda)\, d\lambda.$$

Observe that $\widetilde{f}(i\cdot)$ is now in the domain of the operators $\widetilde{T}_n$, $n \geq 1$. On the other hand, we can write

$$J_m\left(a,b,\widetilde{f}\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{-b}{(b+i\lambda)\,i\lambda}\widetilde{f}(i\lambda)\,d\lambda$$
$$+\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{-b}{(b+i\lambda)\,i\lambda}\widetilde{f}(i\lambda)\left(\left(\frac{i\lambda}{a+i\lambda}\right)^m-1\right)d\lambda. \tag{27}$$

Note that

$$\left(\frac{i\lambda}{a+i\lambda}\right)^m - 1 = -\sum_{k=1}^{m}\binom{m}{k}\frac{a^k\,(i\lambda)^{m-k}}{(a+i\lambda)^m}.$$

Once again, by appealing to Lemma 1 and to the definition of $\widetilde{T}_k\widetilde{f}$, it follows that, for $m \geq k \geq 1$,

$$a^k J_m\left(a,b,\widetilde{T}_k\widetilde{f}\right) \triangleq \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{-b}{(b+i\lambda)\,i\lambda}\frac{a^k\,(i\lambda)^{m-k}}{(a+i\lambda)^m}\widetilde{f}(i\lambda)\,d\lambda.$$

Combining this observation with (27), we obtain

$$J_m(a,b,f) = J_m\left(a,b,\widetilde{f}\right) = J_0\left(b,\widetilde{f}\right) - \sum_{k=1}^{m}\binom{m}{k}a^k J_m\left(a,b,\widetilde{T}_k\widetilde{f}\right). \tag{28}$$

The recursive relation (28) can now be expressed in operator form as

$$J_m\left(a,b,\widetilde{f}\right) = J_0\left(b,\widetilde{f}\right) + J_m\left(a,b,\left(1-\left(1+a\widetilde{T}\right)^m\right)\widetilde{f}\right).$$

(Here, we have used the semigroup property of the family of operators $\widetilde{T}_m$). Iterating the previous expression, we arrive at

$$J_m(a,b,f) = J_m\left(a,b,\widetilde{f}\right) = \sum_{k=0}^{n} J_0\left(b,\left(1-\left(1+a\widetilde{T}\right)^m\right)^k\widetilde{f}\right)$$
$$+J_m\left(a,b,\left(1-\left(1+a\widetilde{T}\right)^m\right)^{n+1}\widetilde{f}\right)$$
$$= J\left(b,\sum_{j=0}^{n}\binom{m+j-1}{j}(-a)^j\,\widetilde{T}_j\widetilde{f}\right) + bo\left(a^n\right), \tag{29}$$

where the last equality in (29) has been obtained by using the semigroup property of the operators $\widetilde{T}_m$ and by noting that the coefficient of $a^j \widetilde{T}_j$ in (29) (for $j \leq n$) must match that of x^j in the formal expansion of

$$p(x) = \frac{1 - (1 - (1+x)^m)^{n+1}}{1 - (1 - (1+x)^m)} = \frac{1}{(1+x)^m} + O\left(x^{n+1}\right).$$

That the error term in (29) is $bo(a^n)$ comes from the fact that $aJ_m\left(a, b, \widetilde{f}\right) = bo(1)$, as $a \searrow 0$, as it can be seen as follows,

$$\begin{aligned} \left|aJ_m\left(a, b, \widetilde{f}\right)\right| &= \left|\frac{a}{2\pi}\int_{-\infty}^{\infty} \frac{-b(i\lambda)^{m-1}\widetilde{f}(i\lambda a)}{(b + i\lambda a)(1+i\lambda)^m} d\lambda\right| \\ &\leq \frac{b}{2\pi}\int_{-\infty}^{\infty}\left|\frac{\widetilde{f}(i\lambda a)}{\lambda(1+i\lambda)}\right| d\lambda = bo(1), \end{aligned}$$

where the last step follows by a dominated convergence argument. This concludes the proof of the proposition.

Proposition 4, combined with our development for $K(t, \cdot)$ in Section 3, provides all the elements required to develop asymptotic expansions for integrals of the form $J_m(a, b, f)$. Since, as discussed earlier at a formal level, $I(\theta, b)$ can be expressed as a sum of terms such as $J_m(a, b, f)$, it follows that the whole asymptotic analysis of $r(\Delta)$ and $\rho(\theta, b)$ can be reduced to that of Section 3. A complete rigorous justification for this representation for $I(\theta, b)$ is one of the main issues discussed in Section 5.

2.4 An Asymptotic Expansion for $I(\theta, b)$

In Sections 3 and 4, we have developed the tools required to obtain asymptotic expansions, in powers of b, for $s(b)$ and $I(\theta, b)$. We have done this by showing that the problem can be reduced to short-time asymptotics for the Cauchy process. The purpose of this section is to make rigorous the expansion for $I(\theta, b)$, in powers of θ, that was outlined in Section 4.

Noting the important role that functions vanishing at the origin plays in Proposition 4, it seems appropriate to define

$$\begin{aligned}\widetilde{H}(\theta,\lambda) &\triangleq H(\theta,\lambda)-H(\theta,0)=H(\theta,\lambda)-1+\phi''(\theta)\\ &=\sum_{k=1}^{\infty}\widetilde{h}_k(i\lambda)\frac{\theta^k}{k!},\end{aligned}\tag{30}$$

where $\widetilde{h}_k(i\lambda)\triangleq(\gamma^{(k)}(i\lambda)-\mu_k)/(1-g(\lambda))-\left(2i\mu_{k+1}/\lambda\right)+\mu_{k+2}$ is such that $\widetilde{h}_k(0)=0$. The next proposition shows how a simplified expression for $I(\theta,b)$ in terms of $\widetilde{H}$ can be obtained.

Proposition 5 *Define* $\Psi(\theta)=2\phi'(\theta)/\phi''(\theta)$. *Then,*

$$I(\theta,b)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{-b}{(b+i\lambda)\,i\lambda}\log\left(1-\frac{\phi''(\theta)^{-1}\lambda\widetilde{H}(\theta,\lambda)}{(\lambda-i\Psi(\theta))}\right)d\lambda.\tag{31}$$

Proof. Just note that

$$\begin{aligned}\log(1-v(\theta,\lambda))&=\log\left(1-\frac{\lambda\widetilde{H}(\theta,\lambda)}{(\lambda-i\Psi(\theta)\phi''(\theta))}-\frac{\lambda(1-\phi''(\theta))}{(\lambda-i\Psi(\theta)\phi''(\theta))}\right)\\ &=\log\left(\frac{i\lambda/\Psi(\theta)+1}{i\lambda\phi''(\theta)/\Psi(\theta)+1}\right)+\log\left(1-\frac{\phi''(\theta)^{-1}\lambda\widetilde{H}(\theta,\lambda)}{(\lambda-i\Psi(\theta))}\right).\end{aligned}$$

Thus, (31) follows from Lemma 1 by noting that

$$\begin{aligned}&\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{-b}{(b+i\lambda)\,i\lambda}\log\left(\frac{i\lambda/\Psi(\theta)+1}{i\lambda\phi''(\theta)/\Psi(\theta)+1}\right)d\lambda\\ &=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{-1}{(1+i\lambda)\,i\lambda}\log\left(\frac{i\lambda b/\Psi(\theta)+1}{i\lambda b\phi''(\theta)/\Psi(\theta)+1}\right)d\lambda=0.\end{aligned}$$

Additional simplifications reduce the complexity of the expansion for $I(\theta,b)$. In particular, the expression for the integral $J_0(b,f)$ simplifies when it is known that $J_0(b,f)$ is real; see (25). Fortunately, our analysis of $I(\theta,b)$ gives rise to such real-valued $J_0(b,f)$'s. To establish this result, we introduce the following family of functions.

Definition *A function* $f:\mathbb{R}\to\mathbb{C}$ *is said to have the "parity property" if* $\operatorname{Re}f(i\cdot)$ *and* $\operatorname{Im}f(i\cdot)$ *are even and odd functions respectively. The class of functions possessing the parity property will be denoted by* $\mathcal{P}$.

Note that if $f(i\cdot)$ is in the domain of $J_0(b,\cdot)$ and f possesses the parity property, then we must have that $\operatorname{Im} J_0(b,f) = 0$ (since it corresponds to an integral on the real line of an odd integrable function). The family of functions enjoying the parity property has certain closure characteristics that will be useful for the rest of our development. These closure properties are discussed in the next proposition.

Proposition 6 *The class $\mathcal{P}$ of functions forms an algebra on $\mathbb{R}$ (i.e. a vector space on $\mathbb{R}$ that is closed under product of functions). In addition, if $f \in \mathcal{P}$, then $1/f(\cdot)$ (defined on its domain of finiteness) also possesses the parity property. Finally, if f is in the domain of $\widetilde{T}$ and has the parity property, then $\widetilde{T}f \in \mathcal{P}$.*

Proof. Certainly $\mathcal{P}$ constitutes a vector space on $\mathbb{R}$ and it is almost immediate that $\widetilde{T}$ preserves the parity property. Now, if $f_1,f_2 \in \mathcal{P}$, then $\operatorname{Re}(f_1 f_2) = \operatorname{Re}(f_1)\operatorname{Re}(f_2) - \operatorname{Im}(f_1)\operatorname{Im}(f_2)$ must clearly be even. Similarly, $\operatorname{Im}(f_1 f_2)$ must be odd , which implies that $f_1 f_2 \in \mathcal{P}$. Finally, note that

$$\frac{1}{f} = \frac{\operatorname{Re}(f)}{\operatorname{Re}(f)^2 + \operatorname{Im}(f)^2} - i\frac{\operatorname{Im}(f)}{\operatorname{Re}(f)^2 + \operatorname{Im}(f)^2},$$

which immediately implies that $\operatorname{Re} 1/f$ and $\operatorname{Im} 1/f$ are even and odd functions respectively and thus $1/f \in \mathcal{P}$.

We now present the main result of this section, which yields an expansion for $I(\theta,b)$ in powers of θ and coefficients involving only integrals of the form $J_0(b,f)$ with f satisfying the parity property.

Proposition 7 *For $k,m \geq 1$, let the coefficient multiplying θ^k in the power series representation of $\widetilde{G}(\theta,\lambda)^m \triangleq \left(\phi''(\theta)^{-1}\widetilde{H}(\theta,\lambda)\right)^m$ be defined as $\widetilde{g}_{k,m}(i\lambda)$. Then, $\widetilde{g}_{k,m}(\cdot) \in \mathcal{P}$ can be recursively computed via*

$$\widetilde{g}_{k,m+1}(i\lambda) = \sum_{n=0}^{k} \widetilde{g}_{n+1,m}(i\lambda)\,\widetilde{g}_{k-n,1}(i\lambda).$$

Consider $b > 0$ and let $\chi(\theta) = -\Psi(\theta)/\theta$. Then,

$$I(\theta,b) = \sum_{m=1}^{n} \theta^m \sum_{j=0}^{m-1} \chi(\theta)^j J_0(b,E_{j,m}) + bo(\theta^n), \tag{32}$$

where $E_{j,m}(i\lambda)$, *defined for* $0 \leq j \leq m-1$ *and* $m \geq 1$ *as*

$$E_{j,m} = -\sum_{k=0}^{m-j-1} \frac{1}{m-j-k} \binom{m-k-1}{j} \widetilde{T}_j \widetilde{g}_{k,m-j-k},$$

satisfies the parity property.

Proof. Since $\gamma(i\lambda) = E_0 \cos(\lambda X_1) + iE_0 \sin(\lambda X_1)$, it follows that $\gamma^{(k)}(i\lambda) - \mu_k \in \mathcal{P}$, as does the function $1 - \gamma(i\lambda)$. By the closure properties described in Proposition 6, we may easily conclude that $\widetilde{g}_{k,1} \in \mathcal{P}$. A second application of Proposition 6 shows that $\widetilde{g}_{k,m} \in \mathcal{P}$ and $E_{j,m} \in \mathcal{P}$. The recursive expression provided for $\widetilde{g}_{k,m}$ follows from standard convolution operations of power series. For $n \geq 1$, define

$$\widetilde{G}_n(\theta, \lambda) \triangleq \sum_{k=1}^{n} \widetilde{g}_{k,1}(i\lambda) \frac{\theta^k}{k!}$$

and

$$I_n(\theta, b) \triangleq \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{-b}{(b+i\lambda) i\lambda} \log\left(1 - \frac{\widetilde{G}_n(\theta,\lambda)\lambda}{\lambda - i2\phi'(\theta)}\right) d\lambda.$$

Note that

$$\log\left(1 - \frac{\widetilde{G}(\theta,\lambda)\lambda}{\lambda - i2\phi'(\theta)}\right) - \log\left(1 - \frac{\widetilde{G}_n(\theta,\lambda)\lambda}{\lambda - i2\phi'(\theta)}\right)$$

$$= \log\left(1 - \frac{\lambda}{\lambda - i2\phi'(\theta)} \frac{\left(\widetilde{G}_n(\theta,\lambda) - \widetilde{G}(\theta,\lambda)\right)}{\left(1 - \widetilde{G}_n(\theta,\lambda)\lambda(\lambda - i2\phi'(\theta))^{-1}\right)}\right).$$

On the other hand, from the remark following Proposition 3 and because $\log(1+z) = z(1+\varepsilon(z))$ for $z \in \mathbb{C}$, where $|\varepsilon(z)| \leq |z|$ for $|z| \leq 1/2$ (see Proposition 8.46, Breiman (1992)), we can see that there exists a constant $B > 0$ such that

$$|I(b,\theta) - I_n(b,\theta)|$$

$$\leq \frac{B}{2\pi} \int_{-\infty}^{\infty} \frac{b\left|\widetilde{G}_n(\theta,\lambda) - \widetilde{G}(\theta,\lambda)\right|}{(b^2+\lambda^2)^{1/2}\left(\lambda^2 + (2\phi'(\theta))^2\right)^{1/2}} d\lambda.$$

Essentially by making the change of variables $u = \lambda\theta$ we then see that for all $\theta \in (0, \delta)$ for some $\delta > 0$ we have

$$|I(b,\theta) - I_n(b,\theta)| \leq \frac{Bb\theta^n}{2\pi} \int_{-\infty}^{\infty} \frac{\left|\widetilde{G}_n(\theta,\lambda\theta) - \widetilde{G}(\theta,\lambda\theta)\right|}{\theta^{n+1}|\lambda|\left(\lambda^2+1\right)^{1/2}} d\lambda.$$

It follows easily from the previous inequality and the Dominated Convergence Theorem that

$$I(b,\theta) - I_n(b,\theta) = bo(\theta^n).$$

Using the expansion of $\log(1+z)$ at $z = 0$ and a similar dominated convergence argument, we can write

$$\begin{aligned} I_n(b,\theta) &= \frac{-1}{2\pi}\int_{-\infty}^{\infty} \frac{-b}{(b+i\lambda)\,i\lambda}\sum_{m=1}^{n}\frac{1}{m}\left(\frac{i\lambda}{i\lambda+\Psi(\theta)}\right)^m \widetilde{G}_n(\theta,\lambda)^m\, d\theta + bo(\theta^n) \\ &= \frac{-1}{2\pi}\int_{-\infty}^{\infty} \frac{-b}{(b+i\lambda)\,i\lambda}\sum_{m=1}^{n}\frac{1}{m}\left(\frac{i\lambda}{i\lambda+\Psi(\theta)}\right)^m \sum_{k=0}^{n-m}\theta^{k+m}\widetilde{g}_{k,m}(i\lambda)\, d\lambda \\ &\quad + bo(\theta^n). \end{aligned} \tag{33}$$

Using Proposition 4 and (33), we obtain that

$$\begin{aligned} & I(\theta,b) \\ &= -\sum_{m=1}^{n}\theta^m\sum_{k=0}^{m-1}J_{m-k}\left(\Psi(\theta), b, \frac{\widetilde{g}_{k,m}}{m-k}\right) + bo(\theta^n) \\ &= -\sum_{m=1}^{n}\theta^m\sum_{k=0}^{m-1}J_0\left(b, \sum_{j=0}^{n}\binom{m-k+j-1}{j}(-\Psi(\theta))^j\,\widetilde{T}_j\frac{\widetilde{g}_{k,m}}{m-k}\right) + bo(\theta^n) \\ &= -\sum_{m=1}^{n}\theta^m\sum_{j=0}^{m-1}\theta^j\chi(\theta)^j J_0\left(b, \sum_{k=0}^{m-1}\binom{m-k+j-1}{j}\widetilde{T}_j\frac{\widetilde{g}_{k,m}}{m-k}\right) + bo(\theta^n) \\ &= \sum_{m=1}^{n}\theta^m\sum_{j=0}^{m-1}\chi(\theta)^j J_0\left(b, -\sum_{k=0}^{m-j-1}\binom{m-k-1}{j}\widetilde{T}_j\frac{\widetilde{g}_{k,m}(i\lambda)}{m-j-k}\right) + bo(\theta^n), \end{aligned} \tag{34}$$

which yields the desired conclusion.

In view of the previous result, an explicit expression for the coefficients in the expansion for $J_0(\cdot, f)$, when f satisfies the parity property, deserves special attention. Providing such explicit expressions is the aim of the next proposition.

Proposition 8 *Suppose that $f(i\cdot) \in \mathcal{L}_0$ has the parity property. Then, $J_0(\cdot, f)$ is infinitely differentiable at zero and*

$$J_0^{(n)}(0,f) = \begin{cases} (-1)^{n/2}\left(f_{RE}^{(n)}(0) - \frac{n!}{2\pi}\int_{-\infty}^{\infty}\left(T_{n/2+1}f_{IM}\right)(\lambda)\,d\lambda\right) & n \text{ even} \\ (-1)^{(n+1)/2}\left(\frac{f_{IM}^{(n+1)}(0)}{(n+1)} - \frac{n!}{2\pi}\int_{-\infty}^{\infty}\left(T_{(n+1)/2}f_{RE}\right)(\lambda)\,d\lambda\right) & n \text{ odd} \end{cases}, \tag{35}$$

where $f_{IM}(i\lambda) = \operatorname{Im} f(i\lambda)\lambda^{-1}$ and $f_{RE}(i\lambda) = \operatorname{Re} f(i\lambda)$.

Proof. The proof follows by a direct application of Proposition 2 combined with the fact that $\operatorname{Re} J_0(b,f) = 0$.

We close this section with some remarks that clarify how the expansion just derived for $I(\theta,b)$ can alternatively be viewed through the prism of a formal operator expansion. The analytic properties stated in Proposition 1 provide rigorous justification for the expansions outlined next. First, we note that if $\theta > 0$ is small enough and $b > 0$, we can formally write

$$I(\theta,b) = -\sum_{k=1}^{\infty}\frac{1}{k}J_k\left(\Psi(\theta), b, \phi''(\theta)^{-k}\widetilde{H}^k(\theta,\cdot)\right). \tag{36}$$

Formally interpreting $\left(1+a\widetilde{T}\right)^{-m}$ as

$$\left(1+a\widetilde{T}\right)^{-m} = \sum_{k=0}^{\infty}\binom{m+k-1}{k}(-a)^k\widetilde{T}^k,$$

in combination with the expansion (26) developed for $J_k(a,b,f)$ and equality (36), allows us to write

$$I(\theta,b) = -\sum_{k=1}^{\infty}\frac{1}{k}J_0\left(b, \phi''(\theta)^{-k}\left(1+\Psi(\theta)\widetilde{T}\right)^{-k}\widetilde{H}^k(\theta,\cdot)\right).$$

If we introduce the convention that for commutative operators $B_1(\theta)$, $B_2(\theta)$ and functions $F_1(\theta,\cdot)$, $F_2(\theta,\cdot)$, expressions of the form $B_1(\theta)F_1(\theta,\cdot)B_2(\theta)F_2(\theta,\cdot)$ (or any permutation of this form) are always interpreted as

$$(B_1(\theta)B_2(\theta))(F_1(\theta,\cdot)F_2(\theta,\cdot)),$$

then we can write

$$I(\theta, b) = J_0\left(b, \log\left(1 - \phi''(\theta)^{-1}\left(1 + \Psi(\theta)\widetilde{T}\right)^{-1}\widetilde{H}(\theta, \cdot)\right)\right). \tag{37}$$

Expression (37) provides a convenient shorthand notation for the expansion of $I(\theta, b)$, in powers of θ and with coefficients in terms of integrals of the form $J_0(b, \cdot)$. In addition, note that, in order to recover the coefficients in the expansion for $I(\theta_1(\cdot), \cdot)$ one can apply formal differentiation to (37) in both arguments θ and b (always having in mind that (37) is just a formalism representing a certain asymptotic expansion). Hence, for example, one can obtain the first term in the expansion for $I(\theta_1(\cdot), \cdot)$ as

$$\partial_\Delta I(\theta_1(\Delta), \Delta)|_{\Delta=0} = \partial_\theta I(0,0)\,\partial_\Delta \theta_1(0) + \partial_b I(0,0),$$

where the formal derivatives applied to (37) must be interpreted using the formal operator convention introduced earlier. Thus, for example, if $B(\theta)$ is an operator of the form

$$B(\theta) = \sum_{k=0}^{\infty} b_k \theta^k \frac{\widetilde{T}^k}{k},$$

applied to a function $F(\theta, \lambda) = \sum f_k(i\lambda)\,\theta^k/k!$, we interpret the formal derivative $\partial_\theta \log(1 - B(\theta)F(\theta, \cdot))$ as

$$\begin{aligned}\partial_\theta \log(1 - B(\theta)F(\theta,\cdot)) &= -\partial_\theta B(\theta)(1 - B(\theta)F(\theta,\cdot))^{-1}F(\theta,\cdot) \\ &\quad - B(\theta)(1 - B(\theta)F(\theta,\cdot))^{-1}\partial_\theta F(\theta,\cdot).\end{aligned}$$

where

$$\begin{aligned}&\partial_\theta B(\theta)(1 - B(\theta)F(\theta,\cdot))^{-1}F(\theta,\cdot) \\ =\ &\sum_{k=0}^{\infty}\left(\partial_\theta B(\theta)\,B(\theta)^k\right)F(\theta,\cdot)^{k+1},\end{aligned}$$

and, similarly,

$$\begin{aligned}&B(\theta)(1 - B(\theta)F(\theta,\cdot))^{-1}\partial_\theta F(\theta,\cdot) \\ =\ &\sum_{k=0}^{\infty} B(\theta)^{k+1}\left(F(\theta,\cdot)^k\,\partial_\theta F(\theta,\cdot)\right).\end{aligned}$$

Thus, it is possible to combine this formalism with the expansion

$$J_0(b,f) = \sum_{n=1}^{m} J^{(n)}(0,f)\, b^n/n! + O\left(b^{m+1}\right)$$

to recover the coefficients in the expansion for $I(\theta_1(\cdot),\cdot)$ in powers of Δ.

2.5 Expansions for $r(\Delta)$ and $E_\theta R^k(\infty)$

In previous sections, we developed all the elements required to rigorously compute a full asymptotic expansion for $r(\cdot)$ in powers of Δ. In the first part of this section, as a summary, we indicate how the developments obtained in the previous three sections can be applied to provide an asymptotic expansion for $r(\cdot)$ in powers of Δ. In view of the level of complexity in the computation of the constants β_n, the description in this section is intended to provide guidance for an easy-to-design practical implementation in a computational package such as Mathematica or Matlab. An efficient implementation of the procedure will appear elsewhere. In the second part of this section, also as a direct consequence of the analysis in the previous sections, we will develop a rigorous asymptotic expansion for the cumulants of $R(\infty)$ under P_θ in powers of θ.

2.5.1 The Expansion for $r(\Delta)$

An algorithm for computing β_k for $k \leq n$ proceeds as follows:

1. Expand $s(\Delta)$ up to terms of order $O(\Delta^{n+1})$ using Proposition 8.

2. Similarly, expand the functions $J_0(\cdot, E_{j,m})$ up to terms $O(\Delta^{n-m})$ with $0 \leq j \leq m-1$ and $1 \leq m \leq n$. This also can be done by applying Proposition 8, since $E_{j,m}$ has the parity property.

3. Finally, the terms obtained can be combined with an expansion for $\theta_1(\Delta)$ up to terms of order $O(\Delta^{n+1})$. Such an expansion can be easily obtained using the implicit function theorem and therefore is omitted.

Observe that the previous algorithm provides an asymptotic expansion for $r(\cdot)$ in powers of Δ. However, because of Theorem 1, we actually have that this asymptotic expansion converges absolutely in a neighborhood of the origin.

As a simple application of the previous expansion, we show that $\beta_2 = 0$.

Proposition 9 *Suppose that X_1 has exponential moments and is strongly non-lattice. Then*

$$r(\Delta) = -\Delta\beta_1 + O\left(\Delta^3\right)$$

Proof. We only need to show that $\beta_2 = 0$. Note that by virtue of Proposition 8, the coefficient multiplying Δ^2 in the expansion of $s(\Delta)$ equals

$$s_2 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{\lambda^2}\left(\frac{\operatorname{Im}\log(1-g(\lambda))}{\lambda} - \mu_3\right) - \left(\mu_4/12 - \mu_3^2/18\right).$$

In order to show that $\beta_2 = 0$ it suffices to show that $\theta_1 J(\Delta, E_{0,1}) \sim -\Delta^2 s_2$ or (since $J(\Delta, E_{j,m}) = O(\Delta)$, $\theta_1/2 \sim \Delta$ and $\phi''(\theta_1) \sim 1$), that $\Delta J(\Delta, E_{0,1}) \sim -2\Delta^2 s_2$, where

$$\begin{aligned}\Delta J(\Delta, E_{0,1}) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{-\Delta}{(\Delta+i\lambda)\,i\lambda}\left(\frac{\gamma'(i\lambda)}{1-g(\lambda)} - \frac{2i}{\lambda} + \mu_3\right)d\lambda \\ &= \frac{1}{\pi}\int_0^{\infty}\frac{\Delta^2}{\left(\Delta^2+\lambda^2\right)}\operatorname{Re}\left(\frac{\gamma'(i\lambda)}{1-g(\lambda)} - \frac{2i}{\lambda} + \mu_3\right)d\lambda \qquad (38)\\ &-\frac{\Delta}{\pi}\int_0^{\infty}\frac{\Delta^2}{\left(\Delta^2+\lambda^2\right)\lambda}\operatorname{Im}\left(\frac{\gamma'(i\lambda)}{1-g(\lambda)} - \frac{2i}{\lambda} + \mu_3\right)d\lambda. \qquad (39)\end{aligned}$$

Note that $g'(\lambda) = i\gamma'(i\lambda)$ and that $\operatorname{Im}\log\left(2\lambda^{-2}\right) = 0$; hence, we can write

$$\operatorname{Re}\left(\frac{\gamma'(i\lambda)}{1-g(\lambda)} - \frac{2i}{\lambda} + \mu_3\right) = -\operatorname{Im}\frac{d}{d\lambda}\left(\log\left(2(1-g(\lambda))\lambda^{-2}\right) - \mu_3 i\lambda\right),$$

which implies, using integration by parts, that the integral in (38) equals

$$\begin{aligned}&\frac{-1}{\pi}\int_0^{\infty}\frac{2\lambda\Delta^2}{\left(\Delta^2+\lambda^2\right)^2}\operatorname{Im}\left(\log\left(2(1-g(\lambda))\lambda^{-2}\right) - \mu_3 i\lambda\right)d\lambda \\ &\sim -\frac{\Delta^2}{\pi}\int_0^{\infty}\frac{2}{\lambda^2}\left(\frac{\operatorname{Im}\log(1-g(\lambda))}{\lambda} - \mu_3\right)d\lambda, \qquad (40)\end{aligned}$$

where (40) has been obtained using dominated convergence and simple manipulations. It follows from Proposition 2 and a first order asymptotic expansion of $E_{0,1}(i\lambda)$ that (39) equals $-\Delta^2(\mu_3^2/9 - \mu_4/6)$. Combining this last estimate together with (40) into (38) and (39) yields $\Delta J(\Delta, E_{0,1}) \sim -2\Delta^2 s_2$ which is exactly what we wanted to show to conclude that $\beta_2 = 0$.

2.5.2 The Expansion for $E_\theta R(\infty)^k$ as $\theta \searrow 0$

We shall provide asymptotics for $E_\theta R(\infty)^k = E_\theta\left(S_{\tau_+}^k\right)/\left(k! E_\theta\left(S_{\tau_+}\right)\right)$ via the cumulants $(\kappa_j(\theta) : j \geq k)$ of $R(\infty)$ under P_θ. In particular, these estimates yield the proof of Theorem 4 stated in Section 2. The idea is to develop an asymptotic expansion, in powers of b, for $s(b)$ and $I(\theta, b)$ respectively and to match coefficients in the expression

$$\begin{aligned}\rho(\theta, b) &= -\kappa_1(\theta) b + \kappa_2(\theta) b^2/2 - \kappa_3(\theta) b^3/3! + \ldots \\ &= s(b) + I(\theta, b).\end{aligned} \tag{41}$$

In order to perform this task, we will take advantage of Proposition 7 as follows; first let us define, for $k \geq 1$, $\alpha_{k,j,m} = J_0^{(k)}(0, E_{j,m})/k!$ (which can be explicitly computed via Proposition 8). With this notation, we can write, for $l, n \geq 1$,

$$\begin{aligned}I(\theta, b) &= \sum_{m=1}^{n} \theta^m \sum_{j=0}^{m-1} \chi(\theta)^j \left(\sum_{k=1}^{l} \alpha_{k,j,m} b^k + O\left(b^{l+1}\right)\right) + bo(\theta^n) \\ &= \sum_{k=1}^{l} b^k \sum_{m=1}^{n} \sum_{j=0}^{m-1} \theta^m \chi(\theta)^j \alpha_{k,j,m} + \theta O\left(b^{l+1}\right) + bo(\theta^n)\end{aligned}$$

Therefore, we obtain that, for all $s, n \geq 1$, $\kappa_s(\theta)$ satisfies

$$\kappa_s(\theta) = (-1)^s \left(\kappa_s(0) + s! \sum_{m=1}^{n} \sum_{j=1}^{m-1} \theta^m \chi(\theta)^j \alpha_{s,j,m}\right) + O\left(\theta^{n+1}\right).$$

Consequently, $\kappa_n(\cdot)$ is an infinitely differentiable function at $\theta = 0$ and for $m \geq 0$ and $n \geq 1$ we have

$$\frac{\kappa_n^{(m)}(0)}{n!} = (-1)^n \frac{\kappa_n(0)}{n!} + \sum_{s=0}^{m-1} \sum_{j=0}^{m-1-s} \chi_{s,j} \alpha_{n,j,m-s},$$

where, for $n, j \geq 1$, $\chi_{n,j}$ is the coefficient multiplying θ^n in the expansion for $\chi(\theta)^j$. In particular, the $\chi_{n,j}$ can be computed recursively as

$$\chi_{n,j+1} = \sum_{n=0}^{k} \chi_{n,j} \chi^{(k-n)}(0) / (k-n)!,$$

with $\chi_{n,1} = \chi^{(n)}(0) / n!$.

2.6 Technical Proofs

Proof of Theorem 3. Using Lemma 1, we can add

$$0 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{b}{(b+i\lambda)\, i\lambda} \log\left(1 + i\lambda/2\phi'(\theta)\right) d\lambda$$

to expression (13) for $\rho(\theta, b)$ to obtain

$$\begin{aligned} \rho(\theta, b) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{-b}{(b+i\lambda)\, i\lambda} \log\left(\frac{\gamma(\theta) - \gamma(\theta + i\lambda)}{-i\phi'(\theta)\, \lambda\, (1 + i\lambda/2\phi'(\theta))} \right) d\lambda \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{-b}{(b+i\lambda)\, i\lambda} \log\left(\frac{2\left(\gamma(\theta) - \gamma(\theta + i\lambda)\right)}{\lambda\, (\lambda - 2i\phi'(\theta))} \right) d\lambda, \end{aligned}$$

yielding the conclusion of the theorem.

Proof of Proposition 3. It follows immediately, by a Taylor series expansion of $\gamma(\cdot)$, that a series representation for H can be written (for fixed λ and θ such that $0 < |\lambda| + |\theta| < \eta$) as

$$\begin{aligned} H(\theta, \lambda) &= 1 - \frac{2i\phi'(\theta)}{\lambda} - \frac{\gamma(\theta) - \gamma(\theta + i\lambda)}{1 - g(\lambda)} \\ &= 1 - \frac{2i}{\lambda} \sum_{k=1}^{\infty} \mu_{k+1} \frac{\theta^k}{k!} - \frac{1}{1 - g(\lambda)} \sum_{k=0}^{\infty} \left(\mu_k - \gamma^{(k)}(i\lambda)\right) \frac{\theta^k}{k!} \\ &= \sum_{k=1}^{\infty} h_k(i\lambda) \frac{\theta^k}{k!}. \end{aligned}$$

In fact, the functions $h_k(i\cdot)$ can be analytically extended throughout the disc $D_{\eta/2} = \{z \in \mathbb{C} : |z| < \eta/2\}$. This is easily seen as follows, recall that $\gamma(\cdot)$ (and therefore

$\gamma^{(k)}(\cdot))$ are analytic on $\mathcal{N}$ (defined in Section 2). Also, observe that $1-\gamma(iz) \sim z^2/2$ and $\gamma^{(k)}(iz) - \mu_k \sim iz\mu_{k+1}$ as $z \to 0$. Thus, $\left(\gamma^{(k)}(iz) - \mu_k\right)/(1-\gamma(iz))$ possesses a simple pole at 0 with residue equal to $2i\mu_{k+1}$, which implies that the natural extension of h_k defined as

$$h_k(iz) = \frac{\gamma^{(k)}(iz) - \mu_k}{1-\gamma(iz)} - \frac{2i\mu_{k+1}}{z} = \frac{\left(\gamma^{(k)}(iz) - \mu_k\right)z - 2i\mu_{k+1}(1-\gamma(iz))}{(1-\gamma(iz))z}$$

is analytic on $D_{\eta/2}$. Now, by virtue of the maximum principle (see, for example, Rudin (1987), p. 253) we have that if $\delta > 0$ is suitably small,

$$\sup_{|z|\leq\delta} |h_k(iz)| \leq \sup_{|z|=\delta} |h_k(iz)|.$$

Since $\gamma(z)$ is a non-constant analytic function defined on $D_{\eta/2}$ (which is an open set and thus has an accumulation point), then $1-\gamma(z)$ has an isolated zero at $z=0$. Thus, it is possible to choose $\delta > 0$ in such a way that

$$\inf_{|z|=\delta} |1-\gamma(iz)| > \varepsilon > 0,$$

for some $\varepsilon > 0$. Consequently,

$$\sup_{|z|\leq\delta} |h_k(iz)| \leq \sup_{|z|=\delta} |h_k(iz)| \leq \frac{1}{\varepsilon\delta} \sup_{|z|=\delta} \left|\left(\gamma^{(k)}(iz) - \mu_k\right)z + 2\mu_{k+1}(1-\gamma(iz))\right|.$$

Observe that, for $|z| < \eta/2$, $\gamma^{(k)}(z) = E_0\left(X^k \exp(zX)\right)$. Therefore, if $z = x+iy$, with $|z| = \delta$,

$$\left|\gamma^{(k)}(iz)\right| \leq E_0\left(|X|^k |\exp(izX)|\right) = E_0\left(|X|^k |\exp(yX)|\right) \leq E_0\left(|X|^k \exp(\delta|X|)\right).$$

A similar bound can be obtained for $\gamma(z)$ and we can conclude that $\exists\ B > 0$ such that

$$\sup_{|z|\leq\delta'} |h_k(iz)| \leq B\left(E_0\left(|X|^k(\exp(\delta|X|)+1)\right) + E_0\left(|X|^{k+1}\right)(1+E_0\exp(\delta|X|))\right).$$

Now, suppose that $\delta < \eta/2$. Then, if $z_1 \in D_{\eta/2}$, we can define

$$BE_0\left(|X|^{k+1}\right)(1 + E_0 \exp(\delta |X|)) \frac{z_1^k}{k!} \triangleq N_1(z_1)$$

in such a way that the previous series converges absolutely and uniformly on $D_{\eta/2}$. Similarly, we can define

$$\begin{aligned} B \sum_{k=1}^{\infty} E_0\left(|X|^k (\exp(\delta |X|) + 1)\right) \frac{z_1^k}{k!} &= BE_0\left(\sum_{k=1}^{\infty} |X|^k \frac{z_1^k}{k!} (\exp(\delta |X|) + 1)\right) \\ &= BE_0((\exp(z_1 |X|) - 1)(\exp(\delta |X|) + 1)) \\ &\triangleq N_2(z_1). \end{aligned}$$

Note that, for $j = 1, 2$, $N_j(z_1) \to 0$ as $z_1 \to 0$. On the other hand, since $g(\lambda)$ is strongly non-lattice, we have that

$$\begin{aligned} \sup_{|\lambda| \geq \delta} |h_k(i\lambda)| &= \sup_{|\lambda| \geq \delta} \left| \frac{\gamma^{(k)}(i\lambda) - \mu_k}{1 - g(\lambda)} - \frac{2i\mu_{k+1}}{\lambda} \right| \\ &\leq B\left(E_0\left(|X|^k\right) + E_0\left(|X|^{k+1}\right)\right), \end{aligned}$$

if $B < \infty$ is big enough. The previous estimates imply that there exist constants $0 < M_k \leq B\left(E_0\left(|X|^k (\exp(\delta |X|) + 1)\right) + E_0\left(|X|^{k+1}\right)(1 + E_0 \exp(\delta |X|))\right)$ such that

$$\sup_{z_2 \in \mathbb{R} \bigcup D_{\eta/2}} |h_k(iz_2)| \leq M_k$$

and $\left|\sum_{k=1}^{\infty} M_k \frac{z_1^k}{k!}\right| \leq \sum_{k=1}^{\infty} \left|M_k \frac{z_1^k}{k!}\right| < \infty$ for $z_1 \in D_{\eta/2}$. Thus, using the Weierstrass M test, we obtain the validity of (23). Finally, the invoked Weierstrass M test combined with the analytic functions convergence theorem (see Theorem 10.28, p. 214, of Rudin (1987)) yields the analyticity of $H(z_1, \cdot)$ on $\mathbb{R} \bigcup D_{\eta/2}$ (for $z_1 \in D_{\eta/2}$) and similarly for $H(\cdot, z_2)$ on $D_{\eta/2}$ (for $z_2 \in \mathbb{R} \bigcup D_{\eta/2}$).

Proof of Proposition 1. We start by writing

$$I(\theta, b) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{-b}{(b + i\lambda) i\lambda} \log\left(1 - \frac{H(\theta, \lambda)\lambda}{\lambda - 2\phi'(\theta) i}\right) d\lambda.$$

The strategy will be to study this integral on $\{|\lambda| < \delta\}$ and $\{|\lambda| \geq \delta\}$ separately (where $\delta > 0$ is some convenient small number to be characterized later).

$$I(\theta, b) = -\frac{1}{2\pi}\int_{-\delta}^{\delta} \frac{b}{(b+i\lambda)}\frac{1}{i\lambda}\log\left(1 - \frac{H(\theta,\lambda)\lambda}{\lambda - 2\phi'(\theta)i}\right)d\lambda \tag{42}$$

$$-\frac{1}{2\pi}\int_{|\lambda|\geq\delta} \frac{b}{(b+i\lambda)}\frac{1}{i\lambda}\log\left(1 - \frac{H(\theta,\lambda)\lambda}{\lambda - 2\phi'(\theta)i}\right)d\lambda. \tag{43}$$

Let us define $I_A(\theta, b)$ and $I_B(\theta, b)$ as (42) and (43) respectively. Suppose that $0 < b < \delta < \eta/2$. By making $u = b\lambda$, we can write

$$I_A(\theta, b) = -\frac{1}{2\pi}\int_{-\delta}^{\delta} \frac{b}{(b+i\lambda)}\frac{1}{i\lambda}\log\left(1 - \frac{H(\theta,\lambda)\lambda}{\lambda - 2\phi'(\theta)i}\right)d\lambda.$$

Let $C = \{w \in \mathbb{C} : |w| \leq \delta\} \cap \{\mathrm{Im}(w) \leq 0\}$, and observe that by virtue of Proposition 3, we can pick $\delta_1 > 0$ in such a way that for all $0 < \theta < \delta_1$ the function

$$f_1(w) = \frac{b}{(b+iw)}\frac{1}{iw}\log\left(1 - \frac{H(\theta,w)w}{w - 2\phi'(\theta)i}\right)$$

is analytic on C. Thus, applying Cauchy's theorem to the contour enclosing C we obtain

$$\begin{aligned}
&I_A(\theta, b) \\
&= -\frac{1}{2\pi}\int_{-\pi}^{0} \frac{b}{\left(b+i\delta e^{i\lambda}\right)}\frac{i\delta e^{i\lambda}}{i\delta e^{i\lambda}}\log\left(1 - \frac{H\left(\theta,\delta e^{i\lambda}\right)\delta e^{i\lambda}}{\delta e^{i\lambda} - i}\right)d\lambda \\
&= \frac{1}{2\pi}\int_{-\pi}^{0} \frac{ib\delta^{-1}e^{-i\lambda}}{\left(1 - ib\delta^{-1}e^{-i\lambda}\right)}\log\left(1 - \frac{H\left(\theta,\delta e^{i\lambda}\right)}{1 - i2\phi'(\theta)\delta^{-1}e^{-i\lambda}}\right)d\lambda. \qquad (44)
\end{aligned}$$

The equality (44) has been obtained by simple algebraic manipulations. Observe that the previous expression in combination with Proposition 3 and the analyticity of the functions $\phi'(\theta)$ (~ 0) at zero immediately gives that $I_A(\theta, b)$ can be represented as an absolutely convergent double power series in θ and b on the set $0 < |\theta| + |b| < \delta_2$ for some $\delta_2 > 0$. Indeed, if we pick δ_2 small enough, it is possible to provide an explicit power series representation for $I_A(\theta, b)$ by using the expansion of $\log(1-w)$ at $w = 0$ in combination with the series representation (23) for the function $H(\theta, \lambda)$ derived in Proposition 3 and a taylor expansion of $(1-w)^{-1}$ around $w = 0$.

The analysis of $I_B(\theta.b)$ is easier,

$$I_B(\theta,b) = \frac{1}{2\pi}\int_{|\lambda|\geq\delta}\frac{b}{(1-b\lambda^{-1})}\frac{1}{\lambda^2}\log\left(1-\frac{H(\theta,\lambda)}{1-i2\phi'(\theta)\lambda^{-1}}\right)d\lambda.$$

Hence, in order to show that $I_B(\cdot)$ can be written as an absolutely convergence double power series in a neighborhood of the origin, it suffices to show (by Fubini's theorem) that

$$\int_{|\lambda|\geq\delta}\sum_{k,j,m\geq 0}^{\infty}\binom{m+k}{k}\frac{b^{j+1}\left(2\left(E\left(\exp\left(\theta|X|\right)-1-|X|\right)\right)\right)^m}{(k+1)|\lambda|^{j+2+m}}\left(\sum_{s=1}^{\infty}|h_s(i\lambda)|\frac{\theta^s}{s!}\right)^{k+1}d\lambda$$

is finite for all non-negative θ and b such that $\theta+b<\delta_3$ for some $\delta_3>0$. But this fact follows easily from Proposition 3, first note, by the change of variables $\lambda = u\delta$, that the previous expression equals

$$\int_{|u|\geq 1}\sum_{k,j,m\geq 0}^{\infty}\binom{m+k}{k}\frac{b^{j+1}\left(2\left(E\left(\exp\left(\theta|X|\right)-1-|X|\right)\right)\right)^m}{\delta^{j+m+1}(k+1)|u|^{j+2+m}}\left(\sum_{s=1}^{\infty}|h_s(i\lambda\delta)|\frac{\theta^s}{s!}\right)^{k+1}du,$$

now pick δ_3 small enough so that $0<\max\left(b,2\left(E\left(\exp\left(\theta|X|\right)-1-|X|\right)\right)\right)<\delta_3<\delta$ (if $\theta+b<\delta_3$), and use Proposition 3 to conclude that one δ_3 can be chosen so that $\sum_{s=1}^{\infty}|h_s(i\lambda\delta)|\frac{\theta^s}{s!}<c<1-\delta_3/\delta$. Therefore, we can bound the previous sum by

$$\begin{aligned}&\int_{|u|\geq 1}\sum_{k,j,m\geq 0}^{\infty}\binom{m+k}{k}\frac{(\delta_3/\delta)^{j+m+1}}{(k+1)|u|^2}c^{k+1}du\\ \leq\ &\frac{2}{3}\frac{1}{1-\delta_3/\delta}\left|\log\left(1-\frac{c}{1-\delta_3/\delta}\right)\right|<\infty.\end{aligned}$$

The conclusions obtained for both $I_A(\cdot)$ and $I_B(\cdot)$, indicate that for all $0\leq\theta,b\leq\upsilon$ (for some $\upsilon>0$) $I(\theta,b)$ can be written as

$$I(\theta,b) = \sum_{j,k\geq 1}\theta^j b^k I_{jk},$$

where the previous series converges absolutely on the specified region on θ and b. The previous expression provides the natural analytic extension of $I(\cdot)$ on $D_\upsilon^2 = \{(z_1,z_2)\in\mathbb{C}\times\mathbb{C} : |z_1|+|z_2|<\upsilon\}$.

Proof of Theorem 1. Since

$$\exp(s(\Delta)) = \frac{1 - E_0\left(\exp\left(-\Delta S_{\tau_+}\right)\right)}{\Delta E_0\left(S_{\tau_+}\right)} = E_0\left(\exp\left(-\Delta R\left(\infty\right)\right)\right),$$

the analytic extension of the term $s(\Delta)$ follows from that of the right hand side, which comes from the fact that S_{τ_+} has exponential moments (see Asmussen (1987)). Thus, since $r(\Delta) = s(\Delta) + I(\theta_1(\Delta), \Delta)$, we just have to analyze $I(\theta_1(\Delta), \Delta)$. However, from the implicit function theorem, we know that $\theta_1(\cdot)$ is analytic in neighborhood of the origin, thus, the analytic functions convergence theorem (see Theorem 10.28, p. 214, of Rudin (1987)) combined with Theorem 1 yields the desired conclusion.

Proof of Theorem 4. From Theorem 1, we know that for $0 \leq \theta, b \leq \upsilon$ (for some $\upsilon > 0$)

$$I(\theta, b) = \sum_{j=1}^{\infty} b^j I_{\cdot,j}(\theta),$$

where each function $I_{\cdot,j}(\theta)$ can be expanded in absolutely convergent power series for $0 \leq \theta \leq \upsilon$, and thus can be analytically extended throughout a neighborhood of the origin in the complex plane. But,

$$\begin{aligned}\rho(\theta, b) &= -\kappa_1(\theta)\, b + \kappa_2(\theta)\, b^2/2 - \kappa_3(\theta)\, b^3/3! + ... \\ &= s(b) + I(\theta, b),\end{aligned}$$

where $s(\cdot)$ is (real) analytic at zero. Hence, the conclusion of the Theorem follows immediately by matching coefficients.

Next, we show that if the distribution of X_1 is symmetric then for $n \geq 1$, $\beta_{2n} = 0$.

Proof of Theorem 2. As we discussed before, all that we need to show is that $\beta_{2n} = 0$. We have shown that an absolutely convergent power series representation is possible for $r(\Delta)$ when Δ is small, thus it suffices to show that if $0 < \Delta < \delta$ (where $\delta > 0$ is suitably small), then an asymptotic expansion for $r(\Delta)$ is given in odd powers of Δ only. Using the integral expression (14), integrating on $|\lambda| \leq \delta$ and

$|\lambda| > \delta$ we can write

$$r(\Delta) = \frac{1}{2\pi}\int_{|\lambda|<\delta} \frac{-\Delta}{(\Delta + i\lambda)\, i\lambda} \log\left(\frac{2(\gamma(\theta_1) - \gamma(\theta_1 + i\lambda))}{\lambda(\lambda - 2i\phi'(\theta_1))}\right) d\lambda \tag{45}$$

$$+ \frac{1}{2\pi}\int_{|\lambda|\geq\delta} \frac{-\Delta}{(\Delta + i\lambda)\, i\lambda} \log\left(\frac{2(\gamma(\theta_1) - \gamma(\theta_1 + i\lambda))}{\lambda(\lambda - 2i\phi'(\theta_1))}\right) d\lambda. \tag{46}$$

Define by $A(\Delta)$ and $B(\Delta)$ the integrals appearing in expressions (45) and (46) respectively. We first analyze $A(\Delta)$. Using a similar argument as in the proof of Theorem 1, we see that

$$A(\Delta) = \frac{1}{2\pi}\int_{C_1} \frac{\Delta}{(\Delta + iz)\, iz} \log\left(\frac{2(\gamma(\theta_1) - \gamma(\theta_1 + iz))}{z(z - 2i\phi'(\theta_1))}\right) dz,$$

where the trajectory C_1 is defined as $C_1 = \{\delta e^{i\lambda} : \lambda \in [0, -\pi)\}$. Also, define the trajectory $C_2 = \{\delta e^{i\lambda} : \lambda \in [-\pi, 0)\}$. The proof of the theorem will be complete if we show that $A(\Delta)$ is an odd function. That is, we must show that $A(\Delta) = -A(-\Delta)$. Note that

$$-A(-\Delta) = \frac{-1}{2\pi}\int_{C_1} \frac{-\Delta}{(-\Delta + iz)\, iz} \log\left(\frac{2(\gamma(-\theta_1) - \gamma(-\theta_1 + iz))}{z(z - 2i\phi'(-\theta_1))}\right) dz$$

$$= \frac{1}{2\pi}\int_{C_2} \frac{-\Delta}{(\Delta + iw)\, iw} \log\left(\frac{2(\gamma(\theta_1) + \gamma(\theta_1 + iw))}{w(w - 2i\phi'(\theta_1))}\right) dz. \tag{47}$$

Equality (47) was obtained by making the change of variables $-w = z$ and using that $\gamma(\theta_1)$ and $\phi'(\theta_1)$ are even and odd functions of θ_1 respectively. In view of (47), in order to show that $A(\Delta) = -A(-\Delta)$, it suffices to show that

$$0 = \frac{1}{2\pi}\int_{C} \frac{\Delta}{(\Delta + iw)\, iw} \log\left(\frac{2(\gamma(\theta_1) - \gamma(\theta_1 + iw))}{w(w - 2i\phi'(\theta_1))}\right) dw,$$

where $C = C_1 + C_2$ is the contour corresponding to the circle with radius δ. Now,

$$\frac{1}{2\pi}\int_{C} \frac{\Delta}{(\Delta + iw)\, iw} \log\left(\frac{2(\gamma(\theta_1) - \gamma(\theta_1 + iw))}{w(w - 2i\phi'(\theta_1))}\right) dw$$

$$= \frac{1}{2\pi}\int_{-C} \frac{\Delta}{w(w - i\Delta)} \log\left(\frac{2(\gamma(\theta_1) - \gamma(\theta_1 + iw))}{w(w - i\Delta)}\right) dw \tag{48}$$

$$+ \frac{1}{2\pi}\int_{-C} \frac{\Delta}{w(w - i\Delta)} \log\left(\frac{w - i\Delta}{w - i2\phi'(\theta_1)}\right) dw. \tag{49}$$

We will show that both terms (48) and (49) vanish. We first consider (49). For $\gamma \in [0,1]$ and $a \in [-\delta, \delta]$, define $f(\gamma)$ as

$$f(\gamma) = \frac{1}{2\pi} \int_{-C} \frac{\Delta}{w(w - i\Delta)} \log(\gamma w - ia)\, dw.$$

Using residue calculus (see Rudin (1987), p. 224) it is easy to see that $f(0) = 0$. A standard dominated convergence argument yields

$$f'(\gamma) = \frac{1}{2\pi} \int_{-C} \frac{\Delta}{(w - i\Delta)(\gamma w - ia)} dw = 0,$$

where the previous integral has again been evaluated using residue calculus. As a result, we obtain that $f(1) = 0$. Applying these considerations with $a = \Delta$ and $a = 2\phi'(\theta_1)$ shows that the integral in (49) equals zero. We also can apply residue calculus to evaluate (48) directly as follows. Consider

$$f_1(w) = \frac{\Delta}{w(w - i\Delta)} \log\left(\frac{2(\gamma(\theta_1) - \gamma(\theta_1 + iw))}{w(w - i\Delta)}\right).$$

Using the change of variables $w = h + i\Delta$ and the definition of $\Delta = \theta_1 - \theta_0$ with $\gamma(\theta_1) = \gamma(\theta_0)$ we can evaluate the residue of f_1 at $w = i\Delta$ as $\mathit{Residue}(f_1; i\Delta) = -i\log(-2\gamma'(\theta_0)/\Delta)$. We also can obtain $\mathit{Residue}(f_1; 0) = i\log(2\gamma'(\theta_1)/\Delta)$. Therefore, using residue calculus we obtain that the integral in (48) equals

$$-i\log(-2\gamma'(\theta_0)/(2\gamma'(\theta_1))) = -i\log(\gamma'(\theta_1)/\gamma'(\theta_1)) = 0,$$

since in the case of symmetric distributions $\gamma'(\lambda)$ is odd and $\theta_1 = -\theta_0$.

Finally, we analyze $B(\Delta)$. Note that

$$B(\Delta) = \frac{1}{2\pi} \int_{|\lambda| \geq \delta} \frac{-\Delta}{(\Delta + i\lambda) i\lambda} \log\left(2(1 - g(\lambda))\lambda^{-2}\right) d\lambda \tag{50}$$

$$+ \frac{1}{2\pi} \int_{|\lambda| \geq \delta} \frac{-\Delta}{(\Delta + i\lambda) i\lambda} \log\left(1 - \frac{\lambda H(\theta_1, \lambda)}{\lambda - 2i\phi'(\theta_1)}\right) d\lambda. \tag{51}$$

Let $B_1(\Delta)$ and $B_2(\Delta)$ be defined as (50) and (51) respectively. Since X_1 is symmetric, it follows that $\log\left(2(1 - g(\lambda))\lambda^{-2}\right)$ is real. As a result, we obtain, just by integrating the real and imaginary parts of the integrand in $B_1(\Delta)$,

$$B_1(\Delta) = \frac{1}{2\pi} \int_{|\lambda| \geq \delta} \frac{\Delta}{\Delta^2 + \lambda^2} \log\left(2(1 - g(\lambda))\lambda^{-2}\right) d\lambda. \tag{52}$$

Expression (52) yields an asymptotic expansion in odd powers of Δ for $B_1(\Delta)$. Again, integrating the real and imaginary parts in $B_2(\Delta)$ we obtain

$$B_2(\Delta) = \frac{1}{2\pi}\int_{|\lambda|\geq\delta} \frac{\Delta}{(\Delta^2+\lambda^2)} \operatorname{Re}\log\left(1-\frac{\lambda H(\theta_1,\lambda)}{\lambda-2i\phi'(\theta_1)}\right)d\lambda \tag{53}$$

$$-\frac{1}{2\pi}\int_{|\lambda|\geq\delta} \frac{\Delta^2}{(\Delta^2+\lambda^2)\lambda} \operatorname{Im}\log\left(1-\frac{\lambda H(\theta_1,\lambda)}{\lambda-2i\phi'(\theta_1)}\right)d\lambda. \tag{54}$$

The previous identity for $B_2(\Delta)$ is obtained by observing that the integral of the imaginary part must vanish. This occurs because for all θ_1 small the function $\log(1-i\lambda H(\theta_1,\lambda)/(i\lambda+2\phi'(\theta_1)))$ satisfies the parity property, which can be verified by observing that, since $\gamma(i\lambda)=E_0\cos(\lambda X)+iE_0\sin(\lambda X)$, it follows that $h_k(i\lambda)\in\mathcal{P}$; also, using Proposition 6, we obtain that $i\lambda/(i\lambda+2\phi'(\theta_1))$ satisfies the parity property. Therefore, the closure properties proved in Proposition 6 together with an expansion of the logarithm yield that $\log(1-i\lambda H(\theta_1,\lambda)/(i\lambda+2\phi'(\theta_1)))\in\mathcal{P}$. which justifies (53) and (54). For notational convenience let us define

$$C(\theta_1,\lambda)=\sum_{k=1}^{\infty} h_{2k}(i\lambda)\,\theta_1^{2k}/2k! \tag{55}$$

and

$$D(\theta_1,\lambda)=-i\sum_{k=1}^{\infty} h_{2k-1}(i\lambda)\,\theta_1^{2k-1}/(2k-1)!, \tag{56}$$

where $h_k(i\lambda)=(\gamma^{(k)}(i\lambda)-\mu_k)/(1-\gamma(i\lambda))-2i\mu_{k+1}/\lambda$. Since the distribution of X_1 is symmetric we have that $\gamma(i\lambda)$ is even and real. Moreover, we also have that $h_k(i\lambda)$ is even if and only if k is even. We also can see that $\operatorname{Re}(H(\theta,\lambda))\triangleq C(\theta,\lambda)$ and $\operatorname{Im}(H(\theta,\lambda))\triangleq D(\theta,\lambda)$ are even and odd functions of both θ and λ (meaning that for every $\theta\in(-\eta/2,\eta/2)$ fixed, $C(\theta,\cdot)$ is even and, similarly, for each $\lambda\in\mathbb{R}$, $C(\cdot,\lambda)$ is also even on $(-\eta/2,\eta/2)$, say). Using this notation, we can write

$$\frac{\lambda H(\theta_1,\lambda)}{\lambda-2\phi'(\theta_1)\,i} = \frac{\lambda^2 C(\theta_1,\lambda)-2\phi'(\theta_1)\,\lambda D(\theta_1,\lambda)}{\lambda^2+(2\phi'(\theta_1))^2} \tag{57}$$

$$+\,i\frac{2\phi'(\theta_1)\,\lambda C(\theta_1,\lambda)+\lambda^2 D(\theta_1,\lambda)}{\lambda^2+(2\phi'(\theta_1))^2}. \tag{58}$$

Let us define $\overline{C}(\theta_1, \lambda)$ and $\overline{D}(\theta_1, \lambda)$ as the real and imaginary parts of $\lambda H(\theta_1, \lambda) / (\lambda - 2\phi'(\theta_1) i)$, respectively, as indicated in the corresponding expressions (57) and (58). Since $\lambda H(\theta_1, \lambda) / (\lambda - 2\phi'(\theta_1) i)$ holds the parity property, $\overline{C}(\theta_1, \lambda)$ and $\overline{D}(\theta_1, \lambda)$, are even and odd function in both arguments θ_1 and λ. By symmetry of the distribution of X_1 we have that $\Delta = 2\theta_1$, also as a consequence of symmetry, $2\phi'(\theta_1)$ is an odd (real) analytic function of θ_1 at the origin, which implies that $(2\phi'(\theta_1))^2$ is even. Hence, using the expansion of $\log(1 - z)$ at $z = 0$ in expressions (53) and (54) (justified by virtue of Proposition 3), we see that an asymptotic expansion for the integral (53) involves expanding expressions of the form

$$K(\theta_1) \frac{1}{2\pi} \int_{|\lambda| \geq \delta} \frac{\Delta}{(\Delta^2 + \lambda^2)} \overline{C}(\theta_1, \lambda)^k \, \overline{D}(\theta_1, \lambda)^{2m} \, d\lambda \tag{59}$$

where $K(\theta_1)$ is an even function of θ_1 which is also (real) analytic at the origin. This implies in view of (55) to (58) and the properties of $2\phi'(\theta_1)$ discussed before, that an asymptotic expansion for (59) must be given in odd powers of Δ only, which must be also the case for the integral in (53). The treatment for the integral (54) is completely analogous and also yields an asymptotic expansion in odd powers of Δ. This yields the conclusion of the theorem.

Chapter 3

The Cramer-Lundberg Theorem in the Presence of Heavy Tails

Let $S = (S_n : n \geq 0)$ be the random walk generated by the sequence $X = (X_n : n \geq 1)$ of independent and identically distributed random variables (iid rv's) with $EX_1 = 0$ and $EX_1^2 = 1$ (so that $S_0 = 0$ and $S_n = X_1 + ... + X_n$ for $n \geq 1$). Assume that the X_i's are strongly non-lattice, in the sense that $g(\lambda) \triangleq E\exp(i\lambda X_1)$ satisfies, for each $\varepsilon > 0$,

$$\inf_{|\lambda|>\varepsilon} |1 - g(\lambda)| > 0.$$

Or, in other words, that $\overline{\lim}_{|\lambda|\to\infty} |g(\lambda)| < 1$ (see Siegmund (1985), p. 176).

Let us introduce a small location parameter $\delta > 0$ representing the drift of the random walk. More precisely, let us consider a parametric family of random walks, $S^\delta = \left(S_n^\delta : n \geq 0\right)$, generated by the sequence $X^\delta = (X_n - \delta : n \geq 1)$. So that

$$S_n^\delta = S_n - n\delta.$$

We shall focus on developing highly accurate approximations, of Cramer-Lundberg type, for the distribution of

$$M_\delta = \max_{n\geq 0} S_n^\delta$$

in the presence of heavy tailed increments. For our purposes here, we say that X has "heavy tails" if for all $\theta \neq 0$, $E \exp(\theta |X|) = \infty$.

Driven by a number of important applications in several disciplines, a great deal of effort has been put into understanding the distributional properties of M_δ. The book by Asmussen (2003) provides a detailed account of several important applications settings in which the distribution of M_δ plays a major role. Most notably we mention: insurance risk theory, in which $P(M_\delta > x)$ is the probability of eventual ruin of an insurer that faces iid claims and possesses initial reserve x; queueing theory, in which the waiting time sequence (excluding service) in the single-server queue, under iid inter-arrival and processing times, and first-come first-served service discipline, turns out to converge in distribution to M_δ (see Kiefer and Wolfowitz (1956)), and sequential analysis, in which the tail probability $P(M_\delta > x)$ can be interpreted as the power of a one-sided sequential probability ratio test (see Siegmund (1985)).

Many problems in applied probability motivate study of models with heavy tails. For instance, in certain lines of the insurance business, such as fire insurance, statistical evidence suggest that claims sizes generally exhibit heavy tailed behavior (see, for example, p. 436 of Bowers et al (1997) and Embrechts, Klüppelberg and Mikosch (1997)). Queueing theory also gives rise to heavy-tails. For example, when modeling data traffic in communication networks, evidence has been found suggesting that exponential tail features (present in traditional models of data traffic) are not compatible with empirical observations (see Adler, Feldman and Taqqu (1998), and Willinger et al (1995)). Therefore, developing asymptotic analysis for systems with heavy tail characteristics is an important applied problem.

Computing the exact distribution of M_δ (either numerically or analytically) under general increment distributions is well known to be a challenging problem. Essentially, it entails solving a Wiener-Hopf type equation known as Lindley's equation (see Lindley (1952)). This integral equation corresponds to the equation describing the stationary distribution of the positive recurrent Markov chain $W_{n+1} = (W_n + X_n - \delta)^+$. Consequently, most of the literature has been focused on developing approximations and numerical algorithms for computing the distribution of M_δ. One of the most popular approximations is based on the so-called Cramer-Lundberg asymptotic formula (see equation (1) below). This formula was initially developed for light tailed increment distributions (i.e. $E\exp(\eta|X_1|) < \infty$ for η in a neighborhood of the origin). The Cramer-Lundberg approximation is a celebrated result in insurance risk and queueing theory (see Asmussen (2001, 2003) and Grandell (1991)), and it is widely accepted that it tends to perform very well in practice (see the discussion in Asmussen (2001) and Grandell (1991)). This performance can be explained via the exponential rate of convergence that actually holds in many practical applications, see equation (2) below.

As we shall see in Section 2, the Cramer-Lundberg representation for $P(M_\delta > x)$ in the case of light tailed increments can be interpreted in a "scaled" form as a function of $\delta > 0$ only (i.e. we allow $x = y(\delta) = O(\delta^{-b})$ for $b \geq 1$ as $\delta \searrow 0$, see (2) and (3) below). The case of $y(\delta) = O(\delta^{-1})$ (as $\delta \searrow 0$) is of great interest, since it corresponds to the so-called diffusion scale (see equation (4)). With this scaled interpretation we can see that the Cramer-Lundberg approximation has an error that is exponentially small as $\delta \searrow 0$ (or, equivalently, $y(\delta) \nearrow \infty$). Note that the case δ close to zero is encountered often in practice. For instance, in the queueing setting described before, $\delta \approx 0$ corresponds to the so-called heavy traffic regime in which the server is busy close to 100% of the time (this terminology actually motivated the title of this chapter). In insurance risk theory, δ close to zero implies that the premium charged is close to the typical pay-out for claims (in the language of risk theorists, the "safety loading" is small). Furthermore, our scaled form of the Cramer-Lundberg representation allows us to obtain a corresponding heavy tailed version (assuming $E|X_1|^{3+\alpha}$ for $\alpha > 0$) of the standard Cramer-Lundberg approximation that provides a good fit (as $\delta \searrow 0$) at essentially every region of the quantile space (see (5)). In particular, the error obtained is of polynomial form in δ, at a rate that depends on the number of moments available. Although we state the complete form of our scaled Cramer-Lundberg representation (see (5)), we focus only on the diffusion region of the space (i.e. $y(\delta) = O(\delta^{-1})$), which yields the most important result of this chapter, namely, Theorem 1. The details for the case $y(\delta) = O(\delta^{-b})$ for $b > 1$ are given in Blanchet, Olvera-Cravioto and Glynn (2004).

Initial forms of heavy tailed Cramer-Lundberg asymptotics for $P(M_\delta > x)$ were given by Bahr (1975) and Borovkov (1976) in the context of the so-called classical risk model or (equivalently) the single-server queue with Poisson arrivals. Further generalizations were developed by Embrechts and Veraverbeeke (1982). These heavy tailed versions of the Cramer-Lundberg approximation tend to perform well only at very large quantile values (see Asmussen and Binswanger (1997), and also the discussion in Embrechts, Klüppelberg and Mikosch (1997) p. 54). The approximations provided in this chapter (in particular, see Theorem 1) are intended to yield good fit in more "typical" values of the distribution (i.e. on the region $x = y(\delta) = O\left(\delta^{-1}\right)$, which corresponds to the diffusion scale). For large quantiles (i.e. $x = y(\delta) = O\left(\delta^{-b}\right)$ for $b \geq 1$) our approximations match earlier results mentioned above.

A closely related approximation, of the type of so-called "corrected diffusion approximations" (CDA's), has been tested in practical applications by Asmussen and Binswanger (1997) and shows satisfactory performance. This first order CDA was developed by Hogan (1986). As we shall see, Theorem 1 not only allows one to strengthen and recover Hogan's CDA but it also significantly reduces the error of the diffusion approximation (see (4) below) as $\delta \searrow 0$.

Section 2 introduces our "scaled" Cramer-Lundberg representation and discusses our main results (see Theorem 1) using ideas from the light tailed case. Section 3 studies the connection between our proposed representation and corrected diffusion approximations. The technical development is given in Section 4.

3.1 A Cramer-Lundberg Representation

As we mentioned previously, the so-called Cramer-Lundberg asymptotic formula was initially developed for light tailed random walks. In particular, suppose that there exists a positive solution θ^δ to the equation

$$\phi\left(\theta^\delta\right) = \exp\left(\theta^\delta \delta\right),$$

where $\phi(\theta) \triangleq E\exp(\theta X_1)$. For $x > 0$ define $\tau(x) = \inf\{n \geq 1 : S_n^\delta > x\}$. Since $\{\tau(x) < \infty\} = \{M_\delta > x\}$, the fundamental identity of sequential analysis establishes that

$$P(M_\delta > x) = P(\tau(x) < \infty) = \exp\left(-\theta^\delta x\right) E_{\theta^\delta} \exp\left(-\theta^\delta \left(S^\delta_{\tau(x)} - x\right)\right),$$

where

$$P_{\theta^\delta}(A) = E\left(\exp\left(\theta^\delta (S_n - n\delta)\right) 1_A\right)$$

for every set $A \in \sigma(X_1, ..., X_n)$ (where $\sigma(X_1, .., X_n)$ is the sigma-field generated by $X_1, ..., X_n$). The "overshoot" $R(x) \triangleq S^\delta_{\tau(x)} - x$ can be interpreted as the residual life time of the embedded renewal process generated by the strictly ascending ladder heights of S^δ. The standard Cramer-Lundberg asymptotic is then obtained by applying renewal theory at strictly ascending ladder heights yielding

$$P(M_\delta > x) \sim \exp\left(-\theta^\delta x + r(\delta)\right) \tag{1}$$

as $x \to \infty$, where $r(\delta) = \log E \exp\left(-\theta^\delta R(\infty)\right)$.

Moreover, since we are assuming strongly non-lattice increment distributions, a result by Stone (1965) on rates of convergence in renewal theory guarantees an exponential rate of convergence in (1). In particular, the Cramer-Lundberg representation

$$P(M_\delta > x) = \exp\left(-\theta^\delta x + r(\delta)\right) + O\left(e^{-ax}\right) \tag{2}$$

holds for some $a > 0$ (see Asmussen (2003) p. 196). It turns out that the exponential rate of convergence in (2) is uniform in $\delta > 0$ (see Lemma 5 of Siegmund (1979) or Lemma 1 below), allowing us to write (see Chang (1992)) the following scaled Cramer-Lundberg representation for $P(M_\delta > y(\delta))$

$$P(M_\delta > y(\delta)) = \exp\left(-\theta^\delta y(\delta) + r(\delta)\right) + O\left(e^{-ay(\delta)}\right), \tag{3}$$

which is valid for some $a > 0$ (uniformly on $\delta \in [0, \delta_1]$ for some $\delta_1 > 0$) and $y(\delta) = O\left(\delta^{-b}\right)$ for $b > 0$. Of special importance is the case in which $b = 1$ (i.e. $y(\delta) = O\left(\delta^{-1}\right)$). Using the implicit function theorem it is easy to see that $\theta^\delta = 2\delta + O\left(\delta^2\right)$, we therefore can recover, from (3), Kingman's (1963) diffusion approximation

$$P(M_\delta > x/\delta) \approx \exp(-2x) + o(1), \tag{4}$$

valid as $\delta \searrow 0$, for $x > 0$.

In this chapter, we introduce a heavy tailed version of the scaled Cramer-Lundberg representation (3). In particular, if $E|X_1|^{3+\alpha} < \infty$ for $\alpha > 0$, then, for each $\varepsilon > 0$ sufficiently small, our proposed representation takes the form

$$P(M_\delta > y(\delta)) = \begin{cases} \exp\left(-\theta_\alpha^\delta y(\delta) + r_\alpha(\delta)\right) + o\left(\delta^{\alpha-\varepsilon}\right) & \text{if } y(\delta) = o\left(\delta^{-b}\right) \text{ for } b = 1 \\ \frac{1}{\delta}\int_{y(\delta)}^{\infty} P(X_1 > u)\,du + o\left(y(\delta)^{\alpha+1}\right) & \text{if } y(\delta) = o\left(\delta^{-b}\right) \text{ for } b > 1 \end{cases}. \quad (5)$$

The constants θ_α^δ and $r_\alpha(\delta)$ correspond to natural approximations for θ^δ and $r(\delta)$ respectively $-$ their form is discussed in detail below. The case $y(\delta) = O\left(\delta^{-b}\right)$ for $b > 1$ is derived under the additional assumption that the increments possess regularly varying tails, although the technical details are not discussed in this dissertation (see Blanchet, Olvera-Cravioto, and Glynn (2004)) for additional detail on this case). It suffices to remark in the present discussion that representation (5) generalizes the scaled Cramer-Lundberg representation (3) and reconciles our proposed representation with previous Cramer-Lundberg type asymptotics developed for fixed values of δ (see Embrechts, Klüppelberg and Mikosch (1997) p. 39). In our development here, we will focus only on the "diffusion" region of the space, namely, $y(\delta) = O\left(\delta^{-1}\right)$.

In order to understand the nature of the constants θ_α^δ and $r_\alpha(\delta)$ let us analyze the elements describing (3). Using the implicit function theorem, it is possible to develop an approximation for θ^δ in terms of

$$\theta_\alpha^\delta \triangleq 2\delta + \sum_{0\le j\le\alpha} \xi_{j+2}\frac{\delta^{j+2}}{(j+2)!} = \theta^\delta + o\left(\delta^{\lfloor\alpha\rfloor}\right), \quad (6)$$

where $\xi_2 = 8EX_1^3/3$, and ξ_j depends on the first $j+1$ moments of X_1. Also, it turns out that $r(\delta)$ can be computed explicitly in terms of Woodroofe's (1979) integral form

$$r(\delta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{-\theta^\delta}{\left(\theta^\delta + i\lambda\right)i\lambda} \log\left(\frac{\phi\left(\theta^\delta\right) - e^{-\delta i\lambda}\phi\left(\theta^\delta + i\lambda\right)}{-i\left(\phi'\left(\theta^\delta\right) - \delta\phi\left(\theta^\delta\right)\right)\lambda}\right) d\lambda, \quad (7)$$

see also Siegmund (1985) p. 176. It is not hard to verify, using a dominated convergence argument and Proposition 8.44 of Breiman (1992), that if

$$r_\alpha(\delta) \triangleq \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{-\theta_\alpha^\delta}{\left(\theta_\alpha^\delta + i\lambda\right) i\lambda} \log\left(\frac{\gamma_\alpha\left(\theta_\alpha^\delta\right) - e^{-\delta i\lambda}\sum_{k\leq\alpha+1} g^{(k)}(\lambda)\left(\theta_\alpha^\delta\right)^k/k!}{-i\left(\gamma_\alpha'\left(\theta_\alpha^\delta\right) - \delta\gamma_\alpha\left(\theta_\alpha^\delta\right)\right)}\right) d\lambda \tag{8}$$

with $\gamma_\alpha(\cdot)$ defined as

$$\gamma_\alpha(\theta) = 1 + \sum_{0\leq j\leq\alpha} \frac{\theta^{j+2} E X^{j+2}}{(j+2)!},$$

then,

$$r(\delta) = r_\alpha(\delta) + o(\delta^\alpha).$$

In view of the fact that for θ_α^δ and $r_\alpha(\delta)$ to be meaningful only finitely many moments of X_1 are required to exist, the previous estimates together with (3) suggest the natural scaled Cramer-Lundberg representation provided. Summarizing, the main result of this chapter is the following.

Theorem 1 *Suppose that $E|X_1|^{\alpha+3} < \infty$ for $\alpha > 0$, and that the distribution of X_1 is strongly non-lattice. Then,*

$$P(M_\delta > x/\delta) = \exp\left(-\theta_\alpha^\delta x/\delta + r_\alpha(\delta)\right) + o\left(\delta^{\alpha-\varepsilon}\right) \tag{9}$$

as $\delta \searrow 0$ for $\varepsilon > 0$ sufficiently small and $x > 0$ fixed.

Remark 1 As we shall, the slack term $\varepsilon > 0$ comes from an estimate involving Spitzer identities and the Wiener-Hopf factorization (see Proposition 3). In other words, if we could set $\varepsilon = 0$ in Proposition 3, then Theorem 1 would hold assuming only $E|X_1|^{\alpha+3} < \infty$ for $\alpha \geq 0$ with an error of order $o(\delta^\alpha)$.

Remark 2 We also will see, that Theorem 1 could also have been formulated in a more robust form as follows.

Theorem 2 (Robust form) *Let G be the class of random variables Y such that*

i) $\sup_{Y\in\mathcal{G}} E\left|Y\right|^{\alpha+3}<\infty$ *for all* $\alpha>0$.
ii) The distribution of Y equals the distribution of X_1 on $[-1/\delta, 1/\delta]$.
iii) X_1 is strongly non-lattice and $EY = o\left(\delta^{\alpha}\right)$.
Then, for $\varepsilon>0$ sufficently small and each $x>0$ fixed,

$$P\left(M_{\delta}^{Y}>x/\delta\right)=\exp\left(-\theta_{\alpha}^{\delta}x/\delta+r_{\alpha}\left(\delta\right)\right)+o\left(\delta^{\alpha-\varepsilon}\right),$$

as $\delta\searrow 0$ (uniformly in $Y\in G$) where M_{δ}^{Y} is the all time maximum of the random walk $S_n = Y_1+...+Y_n-n\delta$, and the Y_i's are iid rv's members of class G.

3.2 Connection to Corrected Diffusion Approximations

The approximation suggested by Theorem 1 is closely related to so-called "corrected diffusion approximations" (CDA's). These approximations are developed in the form of asymptotic expansions in powers of $\delta>0$. These asymptotic expansions follow the spirit of Edgeworth expansions for the central limit theorem and provide parametric information (in $\delta>0$) about the distribution of the whole time maximum of random walk. CDA's for the distribution of M_δ were introduced by Siegmund (1979). Assuming light tailed increments, Siegmund (1979) developed an expansion that corrects the diffusion approximation (4) up to an error of order $o\left(\delta^{2}\right)$. Chang and Peres (1997) obtained a complete asymptotic expansion for Gaussian random walks and, as we have seen, a complete asymptotic expansion for general strongly non-lattice increments with exponential moments was developed in the second chapter of this dissertation.

A first order CDA (corrected diffusion approximation) to (4) in the case of heavy tailed increments was proposed by Hogan (1986). In particular, assuming that $E\left|X_1\right|^5<\infty$, and under some integrability conditions on the characteristic function of X_1 (which in particular imply the continuity of X_1) Hogan showed that

$$P(M_\delta > x/\delta) = \exp(-2x)\left(1 + \delta\frac{4xEX_1^3}{3} - 2\delta\beta\right) + o(\delta).$$

The constant β was computed by Siegmund (1979) as

$$\beta = \frac{1}{6}EX_1^3 - \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{\theta^2}\operatorname{Re}\log\{2(1-g(\theta))/\theta^2\}d\theta. \tag{10}$$

Hogan's strategy consists, essentially, in applying direct Fourier inversion to the characteristic function of M_δ. His method of proof does not seem to extend directly to higher order correction terms.

A more convenient representation for Hogan's approximation (which is guaranteed to give only non-negative values) can be written as

$$P(M_\delta > x/\delta) \approx \exp\left(-2x\left(1 - 2\delta EX_1^3/3\right) - 2\delta\beta\right). \tag{11}$$

In order to recover Hogan's approximation (11) from (9) note that (using the same technique as in the proof of Theorem 3 in Chapter 2 of this dissertation),

$$\begin{aligned} r_\alpha(\delta) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{-\theta_\alpha^\delta}{\left(\theta_\alpha^\delta + i\lambda\right)i\lambda} \\ &\quad \log\left(\frac{2\left(\gamma_\alpha\left(\theta_\alpha^\delta\right) - e^{-\delta i\lambda}\sum_{k\le\alpha+1} g^{(k)}(\lambda)\left(\theta_\alpha^\delta\right)^k/k!\right)}{\lambda\left(\lambda - 2i\left(\phi'\left(\theta^\delta\right) - \delta\phi\left(\theta^\delta\right)\right)\right)}\right)d\lambda \\ &\sim \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{-\theta_\alpha^\delta}{\left(\theta_\alpha^\delta + i\lambda\right)i\lambda}\log\left(2(1-\phi(i\lambda))\lambda^{-2}\right)d\lambda \sim 2\delta\beta. \end{aligned} \tag{12}$$

The estimate (12) was obtained from the expansion $\theta_\alpha^\delta = 2\delta + \delta^2 8EX_1^3/3 + o\left(\delta^2\right)$, which is valid, as we show in Corollary 1 below, as long as $E|X_1|^{4+\alpha} < \infty$ for $\alpha \ge 0$. Approximation (11) can therefore be recovered by combining (12) and the expansion for θ_α^δ into (9). We stress that (9) does not provide a CDA in the parametric sense introduced by Siegmund (1979). Furthermore, the techniques introduced in Chapter 2 do not apply directly to provide an asymptotic expansion of $r_\alpha(\delta)$ in powers of the drift δ under the parameterization utilized here.

3.3 Technical Development

Throughout the rest of the chapter we will suppose, in addition to the assumptions discussed at the beginning of this chapter, that $E|X_1|^{\alpha+3} < \infty$ for $\alpha > 0$. The strategy that we will pursue follows a truncation argument. Consider the sequence $\overline{X}^\delta$ of rv's $\overline{X}_k^\delta = X_k 1(|X_k| \leq 1/\delta) - \delta$, for $k \geq 1$, and its associated random walk $\overline{S}^\delta = \left(\overline{S}_n^\delta : n \geq 0\right)$ (i.e. $\overline{S}_0^\delta = 0$ and $\overline{S}_n^\delta = \overline{X}_1^\delta + ... + \overline{X}_n^\delta$). The idea is first to develop approximation (9) for the distribution of

$$\overline{M}_\delta = \max_{n\geq 0} \overline{S}_n^\delta.$$

Later, we will show that $P(M_\delta > x/\delta)$ and $P\left(\overline{M}_\delta > x/\delta\right)$ are suitably close.

Put $\overline{\phi}_\delta(\theta) = E\exp\left(\theta\overline{X}_1\right)$ and set $\overline{\psi}_\delta(\theta) = \log\overline{\phi}_\delta(\theta)$. Note that $\overline{\psi}_\delta'(0) = -\delta + o\left(\delta^{\alpha+2}\right)$; therefore, if δ is small enough, we can guarantee that there is a strictly positive solution to the equation $\overline{\psi}_\delta\left(\theta_*^\delta\right) = 0$. A similar argument to that given previously to obtain (1) yields

$$P\left(\overline{M}_\delta > x\right) = \exp\left(-\theta_*^\delta x\right) E_\delta^* \exp\left(-\theta_*^\delta \overline{R}_\delta(x)\right), \tag{13}$$

where $\overline{\tau}(x) = \inf\{n \geq 1 : \overline{S}_n^\delta > x\}$, $\overline{R}_\delta(x) \triangleq \overline{S}_{\overline{\tau}(x)}^\delta - x$ is the overshoot at level x, and

$$P_\delta^*(A) = E\left(\exp\left(\theta_*^\delta \overline{S}_n^\delta\right) 1_A\right)$$

for every set $A \in \sigma\left(\overline{X}_1^\delta, ..., \overline{X}_n^\delta\right)$ (where $\sigma\left(\overline{X}_1^\delta, ..., \overline{X}_n^\delta\right)$ is the sigma-field generated by $\overline{X}_1^\delta, ..., \overline{X}_n^\delta$). Renewal theory applied at the strictly increasing ladder heights of the random walk $\overline{S}$ implies that

$$E_\delta^* \exp\left(-\theta_*^\delta \overline{R}_\delta(x)\right) \to E_\delta^* \exp\left(-\theta_*^\delta \overline{R}_\delta(\infty)\right)$$

as $x \to \infty$, for fixed $\delta > 0$. Here, we are interested in applying renewal theory uniformly on $\delta \in (0, \delta_1)$. The next proposition (which is analogous to Lemma 5 of Siegmund (1979)) provides the means for doing so.

Lemma 1 *Let $\mathcal{F}$ be a family of distribution functions supported on $[0, \infty)$. For each $F \in \mathcal{F}$, let $E_F(\cdot)$ be the expectation operator associated to $F \in \mathcal{F}$, and define*

$E_F g(\tau) \triangleq \int_{[0,\infty)} g(t) F(dt)$ *for each continuous and bounded function* $g : [0,\infty) \to \mathbb{C}$. *Suppose that the family* $\mathcal{F}$ *is uniformly strongly non-lattice, (i.e. the corresponding characteristic functions* $\chi_F(\lambda) = E_F \exp(i\lambda\tau)$ *satisfy*

$$\inf_{F\in\mathcal{F}} \inf_{|\lambda|>\varepsilon} |1 - \chi_F(\lambda)| > 0. \tag{14}$$

Then, $U_F(t) \triangleq \sum_{n=0}^{\infty} F^{*n}(t)$ *satisfies the following.*

1. *If* $\sup_{F\in\mathcal{F}} E_F \exp(\eta X_1) < \infty$ *for some* $\eta > 0$*, then*

$$\sup_{F\in\mathcal{F}} \left| U_F(t) - \frac{t}{E_F\tau} - \frac{E_F\tau^2}{2E_F^2\tau} \right| = O\left(e^{-at}\right)$$

as $t \to \infty$ *for some* $a > 0$.

2. *Moreover, if* $\sup_{F\in\mathcal{F}} E_F \tau^{\varepsilon+2} < \infty$ *for* $\varepsilon \geq 0$*, then,*

$$\sup_{F\in\mathcal{F}} \left| U_F(t) - \frac{t}{E_F\tau} - \frac{E_F\tau^2}{2E_F^2\tau} - \frac{H_2^F(t)}{E_F\tau_\delta^2} - H_1^F * H_1^F(t) \right| = o\left(t^{\alpha+2}\log(t)\right)$$

as $t \to \infty$*, where* $H_1^F(t) = \int_t^\infty (1 - F(s))\, ds \,/ E_F\tau$ *and* $H_2^F(t) = \int_t^\infty H_1^F(s)\, ds$.

Proof. See Theorem 1 in Chapter 4 of this dissertation.

A crucial assumption that must be verified when applying the previous lemma is the strongly non-lattice condition (14). A key result that we shall use to verify this assumption repeatedly throughout the rest of this chapter is the so-called Wiener-Hopf factorization, which we now state without proof (see Theorem 8.3.1 of Asmussen (2003) for a proof of this classical result).

Lemma 2 (Wiener-Hopf) *Suppose that* $Y = (Y_j : j \geq 1)$ *is a sequence of iid rv's with characteristic function* $g(\lambda) \triangleq E\exp(i\lambda Y_1)$. *Define* $S_n \triangleq Y_1 + ... + Y_n$ *and* $S_0 \triangleq 0$. *Put* $\tau_+ = \inf\{n \geq 0 : S_n > 0\}$ *and set* $\tau_- = \inf\{n \geq 1 : S_n \leq 0\}$. *Finally, let* $g_+(\lambda) = E(\exp(i\lambda S_{\tau+}); \tau_+ < \infty)$ *and put* $g_-(\lambda) = E(\exp(i\lambda S_{\tau-}); \tau_- < \infty)$. *Then,*

$$1 - g(\lambda) = (1 - g_+(\lambda))(1 - g_-(\lambda)).$$

With Lemma 1 in hand, we now can provide a detailed asymptotic analysis of $E_\delta^* \exp\left(-\theta_*^\delta \overline{R}_\delta\left(x/\delta\right)\right)$ as $\delta \searrow 0$ (for fixed $x > 0$), as the following proposition shows.

Proposition 1 *There exists $\delta^* > 0$ and a function $f_1 : (0,\infty) \to (0,\infty)$, such that $f_1(z) = o\left(z^{-(1+\alpha)}\right)$ as $z \nearrow \infty$ for which*

$$\sup_{0\leq\delta\leq\delta^*}\left|P_\delta^*\left(\overline{R}_\delta(x) > y\right) - P_\delta^*\left(\overline{R}_\delta(\infty) > y\right)\right| \leq y^{-1} f_1(y)\, x f_1(x) + f_1(x+y)$$

for $x, y > 0$. Also, if $x = O(1/\Delta)$ we have

$$\left|E_\delta^* \exp\left(-\Delta \overline{R}_\delta(x)\right) - E_\delta^* \exp\left(-\Delta \overline{R}_\delta(\infty)\right)\right| \leq o\left(\Delta^{1+\alpha}\right)$$

as $\Delta \searrow 0$ uniformly in $\delta \in (0, \delta^*)$.

Proof. An analogous result was obtained by Chang (1992) when exponential moments exist. Our argument here follows Chang's argument, we provide the details for completeness. Applying renewal theory at strictly increasing ladder heights we have that

$$P_\delta^*\left(\overline{R}_\delta(x) > y\right) = \int_{[0,x)} P_\delta^*\left(\overline{S}_{\overline{\tau}_+}^\delta > x + y - t\right) U_\delta^*(dt),$$

where $\overline{\tau}_+ = \inf\{n \geq 0 : \overline{S}_n^\delta > 0\}$ is the first strictly increasing ladder epoch, $\overline{S}_{\overline{\tau}_+}^\delta$ is the first strictly increasing ladder height, and U_δ^* is the corresponding renewal measure generated by the strictly increasing ladder heights under the probability measure P_δ^*. We also know from renewal theory (for fixed $\delta > 0$) that

$$\begin{aligned} P_\delta^*\left(\overline{R}(\infty) > y\right) &= \frac{1}{E_\delta^* \overline{S}_{\overline{\tau}_+}^\delta} \int_y^\infty P_\delta^*\left(\overline{S}_{\overline{\tau}_+}^\delta > t\right) dt \\ &= \frac{1}{E_\delta^* \overline{S}_{\overline{\tau}_+}^\delta} \int_{-\infty}^x P_\delta^*\left(\overline{S}_{\overline{\tau}_+}^\delta > x + y - t\right) dt. \end{aligned}$$

Thus,

$$\begin{aligned} & P_\delta^*\left(\overline{R}_\delta(x) > y\right) - P_\delta^*\left(\overline{R}_\delta(\infty) > y\right) \\ = & \int_0^x P_\delta^*\left(\overline{S}_{\overline{\tau}_+}^\delta > x + y - t\right)\left(U_\delta^*(dt) - \frac{dt}{E_\delta^* \overline{S}_{\overline{\tau}_+}^\delta}\right) \qquad (15) \\ & + \frac{1}{E_\delta^* \overline{S}_{\overline{\tau}_+}^\delta} \int_{-\infty}^0 P_\delta^*\left(\overline{S}_{\overline{\tau}_+}^\delta > x + y - t\right) dt. \qquad (16) \end{aligned}$$

Note that (15) can be written as

$$\int_0^x P_\delta^*\left(\overline{S}_{\overline{\tau}_+}^\delta > x+y-t\right)\varepsilon_\delta^*(dt),$$

where

$$\varepsilon_\delta^*(t) = U_\delta^*(t) - \frac{t}{E_\delta^*\overline{S}_{\overline{\tau}_+}^\delta} - \frac{E_\delta^*\left(\overline{S}_{\overline{\tau}_+}^\delta\right)^2}{2E_\delta^*\overline{S}_{\overline{\tau}_+}^\delta}.$$

Using properties of the convolution and the change of variable $u = x/t$ we obtain

$$\begin{aligned} &\int_0^x P_\delta^*\left(\overline{S}_{\overline{\tau}_+}^\delta > x+y-t\right)\varepsilon_\delta^*(dt) \\ =\; & -\int_0^x \varepsilon_\delta^*(x-t)\,P_\delta^*\left(\overline{S}_{\overline{\tau}_+}^\delta \in y+dt\right) \\ =\; & -\int_0^1 \varepsilon_\delta^*(x-xt)\,P_\delta^*\left(\overline{S}_{\overline{\tau}_+}^\delta \in y+xdt\right). \end{aligned} \tag{17}$$

Now, since $E\left(|X_1|^{3+\alpha}\right) < \infty$, $0 \le \overline{S}_{\tau+}^\delta \le 1/\delta$ and $\theta_*^\delta \sim 2\delta$, we can guarantee that there exists $\delta_1 > 0$ such that

$$\begin{aligned} \sup_{0\le\delta\le\delta_1} E_\delta^*\left(\left(\overline{S}_{\overline{\tau}_+}^\delta\right)^{2+\alpha}\right) &= \sup_{0\le\delta\le\delta_1} E\left(\left(\overline{S}_{\overline{\tau}_+}^\delta\right)^{2+\alpha}\exp\left(\theta_*^\delta\overline{S}_{\overline{\tau}_+}^\delta\right)\right) \\ &\le M\sup_{0\le\delta\le\delta_1} E\left(\left(\overline{S}_{\overline{\tau}_+}^\delta\right)^{2+\alpha}\right) < \infty. \end{aligned}$$

Let us verify that the laws $P_\delta^*\left(\overline{S}_{\overline{\tau}+}^\delta \in ds\right)$ are uniformly strongly non-lattice. First, it is almost immediate to see that the laws $P_\delta^*\left(\overline{X}_1 \in ds\right)$ are uniformly strongly non-lattice. From Lemma 2 we have that

$$\frac{1}{2}\left|1 - e^{-\delta i\lambda}E_\delta^*\exp\left(i\lambda\overline{X}_1\right)\right| \le \left|1 - e^{-\delta i\lambda}E_\delta^*\exp\left(i\lambda\overline{S}_{\overline{\tau}+}^\delta\right)\right|.$$

The uniform strongly non-lattice assumption can be easily verified from the previous inequality. Consequently, we can apply Lemma 1 to conclude, from (17), that

$$\begin{aligned}\left|\int_0^x P_\delta^*\left(\overline{S}_{\overline{\tau}_+}^\delta > x+y-t\right)\varepsilon_\delta^*(dt)\right| &\leq \int_0^1 |\varepsilon_\delta^*(x-xt)|\, P_\delta^*\left(\overline{S}_{\overline{\tau}_+}^\delta \in y+xdt\right) \\ &\leq \int_0^{1/2} |\varepsilon_\delta^*(x-xt)|\, P_\delta^*\left(\overline{S}_{\overline{\tau}_+}^\delta \in y+xdt\right) + \\ &\quad \int_{1/2}^1 |\varepsilon_\delta^*(x-xt)|\, P_\delta^*\left(\overline{S}_{\overline{\tau}_+}^\delta \in y+xdt\right) \\ &\leq o\left(x^{-\alpha}\right) P_\delta^*\left(\overline{S}_{\overline{\tau}_+}^\delta \geq y\right) + P_\delta^*\left(\overline{S}_{\overline{\tau}_+}^\delta \geq y+x/2\right) \\ &= o\left(x^{-\alpha}\right) o\left(y^{-2-\alpha}\right) + o\left((y+x)^{-2-\alpha}\right).\end{aligned}$$

On the other hand, the term (16) equals

$$\frac{1}{E_\delta^*\overline{S}_{\overline{\tau}_+}^\delta}\int_{x+y}^\infty P_\delta^*\left(\overline{S}_{\overline{\tau}_+}^\delta > t\right)dt = o\left((x+y)^{-1-\alpha}\right).$$

This yields the first part of this proposition. For the second part, note that

$$E_\delta^*\exp\left(-\Delta\overline{R}_\delta(x)\right) = \int_0^\infty e^{-u}P_\delta^*\left(\overline{R}_\delta(x)\leq u/\Delta\right)du,$$

thus

$$\begin{aligned}&\left|E_\delta^*\exp\left(-\Delta\overline{R}_\delta(x)\right) - E_\delta^*\exp\left(-\Delta\overline{R}_\delta(\infty)\right)\right| \\ \leq& \int_0^\infty e^{-u}\left(o\left(u^{-2-\alpha}\Delta^{2+\alpha}\right)o\left(x^{-\alpha}\right) + o\left(\Delta^{1+\alpha}(x\Delta+u)^{-1-\alpha}\right)\right)du = o\left(\Delta^{1+\alpha}\right)\end{aligned}$$

as long as $x = O(1/\Delta)$, this provides the second part of the statement.

We are almost ready to show that our stated approximation (9) is valid for the truncated random walk $\overline{S}^\delta$. Let us just provide, a couple of elementary results describing the asymptotic behavior of θ_*^δ and $\overline{\phi}_\delta\left(\theta_*^\delta\right)$ as $\delta \searrow 0$.

Proposition 2 *Let*

$$\widetilde{\phi}_\delta(\theta) \triangleq E\left(\exp\left(\theta X_1 1\left(|X_1|\leq 1/\delta\right)\right)\right).$$

(Observe that $\overline{\phi}_\delta(\theta) = \exp(-\theta\delta)\widetilde{\phi}_\delta(\theta)$*.) Then, for all* $\theta \in [-M\delta, M\delta]$ *with* $M > 0$

$$\left|\widetilde{\phi}_\delta(\theta) - \sum_{1\leq j\leq\alpha+3} EX_1^j 1\left(|X_1|\leq 1/\delta\right)\frac{\theta^j}{j!}\right| \leq o\left(\delta^{\alpha+2}\right).$$

Furthermore,

$$\left|\widetilde{\phi}_{\delta}\left(\theta\right)-\sum_{1\leq j\leq\alpha+3}EX_{1}^{j}\frac{\theta^{j}}{j!}\right|\leq o\left(\delta^{\alpha+2}\right),$$

for $\theta\in[-M\delta,M\delta]$.

Proof. The proof proceeds by expanding for fixed δ the function

$$\begin{aligned}&\widetilde{\phi}_{\delta}\left(\theta\right)\\=&\sum_{1\leq j\leq\alpha+2}EX_{1}^{j}1\left(\left|X_{1}\right|\leq1/\delta\right)\frac{\theta^{j}}{j!}\\&+E\left(X_{1}^{\lfloor\alpha+3\rfloor}\exp\left(\eta X_{1}1\left(\left|X_{1}\right|\leq1/\delta\right)\right)\right)\frac{\theta^{\lfloor\alpha+3\rfloor}}{\left(\lfloor\alpha+3\rfloor\right)!},\end{aligned}$$

where $|\eta|\leq|\theta|\leq M\delta$. Hence,

$$\begin{aligned}&\left|E\left(X_{1}^{\lfloor\alpha+3\rfloor}\exp\left(\eta X_{1}1\left(\left|X_{1}\right|\leq1/\delta\right)\right)\right)\frac{\theta^{\lfloor\alpha+3\rfloor}}{\left(\lfloor\alpha+3\rfloor\right)!}\right|\\\leq&\frac{M\delta^{\lfloor\alpha+3\rfloor}}{\left(\lfloor\alpha+3\rfloor\right)!}\exp\left(M\right)E\left|X_{1}^{\alpha+3}\right|=o\left(\delta^{\alpha+2}\right).\end{aligned}$$

The fact that $EX_{1}^{j}1\left(\left|X_{1}\right|\leq1/\delta\right)-EX_{1}^{j}=o\left(\delta^{\alpha+3-j}\right)$ can be easily checked, this yields the conclusion of the proposition.

As a consequence of the previous proposition we obtain the next corollary.

Corollary 1

$$\left|\theta_{*}^{\delta}-\sum_{j\leq\alpha+2}\frac{\xi_{j}}{j!}\delta^{j}\right|\leq o\left(\delta^{\alpha+1}\right).$$

The constants ξ_j *are computed via the system of linear equations:*

$$\sum_{m=0}^{n}\binom{n}{m}\frac{\kappa_{m+2}}{m+2}\xi_{n-m+1}=0;\quad 0\leq n\leq\alpha+1,$$

where κ_j *is the jth cumulant of* X_1.

Proof. The interesting case arises when exponential moments fail to exists, in that case $\log \widetilde{\phi}_\delta(1) > \delta$ for all δ small. Therefore, by strict convexity of $\log \widetilde{\phi}_\delta(\cdot)$, we must have $\theta_*^\delta \leq \delta$. This implies that θ_*^δ is in the domain in which the expansion of Proposition 2 is valid. The rest of the conclusion follows from the implicit function theorem.

Proposition 3 *If $y(\delta) = O(\delta^{-b})$ for $b \leq 1$, and $x > 0$, then*

$$P\left(\overline{M}_\delta > x/\delta\right) = \exp\left(-\theta_\alpha^\delta x/\delta + r_\alpha(\delta)\right) + o(\delta^\alpha).$$

Proof. By Corollary 1 we have

$$P\left(\overline{M}_\delta > x/\delta\right) = \exp\left(-\theta_*^\delta x/\delta\right) E_\delta^* \exp\left(-\theta_*^\delta \overline{R}_\delta(x/\delta)\right).$$

On the other hand, Proposition 1 asserts that

$$\left|E_\delta^* \exp\left(-\theta_*^\delta \overline{R}_\delta(x/\delta)\right) - E_\delta^* \exp\left(-\theta_*^\delta \overline{R}_\delta(\infty)\right)\right| \leq o\left(\delta^{\alpha+1}\right),$$

as long as $\theta_* = O(\delta)$ which holds by virtue of Corollary 1. Now, observe that

$$\begin{aligned} & \log E_\delta^* \exp\left(-\theta_*^\delta \overline{R}_\delta(\infty)\right) \\ = \; & \frac{-1}{2\pi} \int_{-\infty}^{\infty} \frac{\theta_*^\delta}{\left(\theta_*^\delta + i\lambda\right) i\lambda} \log\left(\frac{\widetilde{\phi}_\delta\left(\theta_*^\delta\right) - e^{-\delta i\theta} \widetilde{\phi}_\delta\left(\theta_*^\delta + i\theta\right)}{-i\left(\widetilde{\phi}_\delta'\left(\theta_*^\delta\right) - \delta \widetilde{\phi}_\delta\left(\theta_*^\delta\right)\right)\theta}\right) d\theta. \end{aligned}$$

Since

$$\left|g^{(m)}(i\theta) - \widetilde{\phi}_\delta^{(m)}(i\theta)\right| \leq 2E\left(|X^m| \, 1\left(|X| > 1/\delta\right)\right) = o\left(\delta^{\alpha+3-m}\right),$$

a routine dominated convergence argument (obtained with the aid of Proposition 8.44 of Breiman (1992)) yields

$$\log E_\delta^* \exp\left(-\theta_*^\delta \overline{R}_\delta(\infty)\right) - r_\alpha(\delta) = o(\delta^\alpha).$$

The proposition is proved by combining these estimates.

The next step is to show that $P\left(\overline{M}_\delta > x/\delta\right) - P\left(M_\delta > x/\delta\right) = o(\delta^\alpha)$. We well do this by taking advantage of a geometric sum representation of the maximum of random walks with negative drift. Specifically, let us write $\tau_+ = \tau(0)$ and $S^\delta_{\tau_+}$ (resp. $\overline{\tau}_+ = \overline{\tau}(0)$ and $\overline{S}^\delta_{\overline{\tau}_+}$) to denote the first strictly ascending ladder epoch and first strictly ascending ladder height of the random walk S^δ (resp. $\overline{S}^\delta$). It is well known that

$$M_\delta \overset{D}{=} Z \triangleq \sum_{j=1}^{G(p_\delta)} T_{j,\delta} \;, \tag{18}$$

where $T^\delta = (T_{j,\delta} : j \geq 1)$ is a sequence of iidrv's with distribution function given by $P(T_{1,\delta} \leq t) = P\left(S^\delta_{\tau_+} \leq t \middle| \tau_+ < \infty\right)$ and $G(p_\delta)$ is geometrically distributed with parameter $p_\delta = P(\tau_+ = \infty)$ (i.e. $P(G(p_\delta) = k) = p_\delta (1 - p_\delta)^k$ for $k \geq 0$). A completely analogous representation is also valid for $\overline{M}_\delta$, namely

$$\overline{M}_\delta \overset{D}{=} \overline{Z} \triangleq \sum_{j=1}^{\overline{G}(\overline{p}_\delta)} \overline{T}_{j,\delta} \;, \tag{19}$$

with an iid sequence $\overline{T} = \left(\overline{T}_{j,\delta} : j \geq 1\right)$ such that $P\left(\overline{T}_{1,\delta} \leq t\right) = P\left(\overline{S}^\delta_{\overline{\tau}_+} \leq t \middle| \overline{\tau}^\delta_+ < \infty\right)$ and a parameter $\overline{p}_\delta = P(\overline{\tau}_+ = \infty)$ for the geometric rv $\overline{G}$.

It is natural to expect that if the moments of M_δ and $\overline{M}_\delta$ are close, then their corresponding distributions do not differ significantly. The next result (whose proof is given at the end of the section) shows that the moments of $T_{1,\delta}$ and $\overline{T}_{1,\delta}$ are close as $\delta \searrow 0$ (this implies, in view of representations (18) and (19), that the moments of $\overline{M}_\delta$ provide good approximations, in some sense, for those of M_δ).

Theorem 3 *For each $\varepsilon > 0$ small enough*

$$\begin{aligned} p_\delta &= \overline{p}_\delta + o\left(\delta^{\alpha+1-\varepsilon}\right), \\ E\left(\overline{T}_{1,\delta}\right) &= E\left(T_{1,\delta}\right) + o\left(\delta^{\alpha-\varepsilon}\right). \end{aligned}$$

Moreover, for $2 \leq j \leq \alpha + 2$

$$E\left(\overline{T}^j_{1,\delta}\right) = E\left(T^j_{1,\delta}\right) + o\left(\delta^{\alpha+2-j-\varepsilon}\right).$$

Proof. Given at the end of the section.

Theorem 1 is just an immediate consequence of the next final proposition.

Proposition 4

$$P\left(\overline{M}_\delta > x/\delta\right) - P\left(M_\delta > x/\delta\right) = o\left(\delta^\alpha\right)$$

Proof. Applying Theorem 1.1 of Kalashnikov (1997) (see also Proposition 1 in the fourth chapter of this dissertation for a somewhat shorter argument) we obtain

$$P\left(M > x/p\right) = qEq^{N(x/p)}. \tag{20}$$

A similar argument as that as the one given in the proof of Proposition 1 (by means of the Wiener-Hopf factorization) can be used to easily verify the uniform strong non-latticity (for $\delta > 0$ sufficiently small) of the distributions of both $T_{1,\delta}$ and $\overline{T}_{1,\delta}$. In addition, note that both $T_{1,\delta}$ and $\overline{T}_{1,\delta}$ have uniformly (in $\delta > 0$) bounded moments of order $\lfloor \alpha + 2 \rfloor$. Using renewal theory (in its uniform version, as in Lemma 1) we shall obtain, in Theorem 3 of this dissertation's fourth chapter, asymptotic expansions (as $p \searrow 0$) for $P\left(Z > x/p\right)$, which, combined with (20), allows writing

$$P\left(M_\delta > x/\delta\right) = \exp\left(a_\delta\left(p_\delta\right) x/\delta + b_\delta\left(p_\delta\right)\right) + o\left(\delta^\alpha\right),$$

where $a\left(p_\delta\right)$ and $b\left(p_\delta\right)$ satisfy

$$a\left(p_\delta\right) = \sum_{k \leq \alpha} \frac{a_\delta^{(k+1)}\left(0\right)}{(k+1)!} p^{k+1}, \tag{21}$$

$$b\left(p_\delta\right) = \sum_{k \leq \alpha} \frac{b_\delta^{(k+1)}\left(0\right)}{(k+1)!} p^{k+1}, \tag{22}$$

with $a_\delta^{(m)}\left(0\right)$ and $b_\delta^{(m)}\left(0\right)$ depending algebraically on the first m and $m+1$ moments respectively of $T_{1,\delta}$ (see Theorem 4 in the fourth chapter). Similarly,

$$P\left(\overline{M}_\delta > x/\delta\right) = \exp\left(\overline{a}_\delta\left(\overline{p}_\delta\right) x/\delta + \overline{b}_\delta\left(\overline{p}_\delta\right)\right) + o\left(\delta^\alpha\right),$$

where $\overline{a}_\delta\left(\overline{p}_\delta\right)$ and $\overline{b}_\delta\left(\overline{p}_\delta\right)$ have analogous representations as (21) and (22) above. This implies, by virtue of Theorem 3, that

$$\delta^{-1}\left(a_\delta\left(p_\delta\right) - \overline{a}_\delta\left(\overline{p}_\delta\right)\right) = o\left(\delta^{\alpha-\varepsilon}\right) = b_\delta\left(p_\delta\right) - \overline{b}_\delta\left(\overline{p}_\delta\right),$$

which in turn implies the statement of the proposition.

Proof of Proposition 3. We first estimate $|p_\delta - \overline{p}_\delta|$. Recall that

$$p_\delta = P\left(\overline{\tau}_+^\delta = \infty\right) = \sum_{n=1}^{\infty} \frac{1}{n} P\left(S_n^\delta > 0\right) = \sum_{n=1}^{\infty} \frac{1}{n} P\left(\frac{S_n}{n} > \delta\right),$$

similarly

$$\overline{p}_\delta = \sum_{n=1}^{\infty} \frac{1}{n} P\left(\frac{\overline{S}_n}{n} > \delta\right).$$

Thus,

$$\begin{aligned} |p_\delta - \overline{p}_\delta| &\leq \sum_{n=1}^{\infty} \frac{1}{n} E\left|1\left(\frac{S_n}{n} > \delta\right) - 1\left(\frac{\overline{S}_n}{n} > \delta\right)\right| \\ &= \sum_{n=1}^{\infty} \frac{1}{n} P\left(\frac{S_n}{n} > \delta; \frac{\overline{S}_n}{n} \leq \delta\right) \\ &\quad + \sum_{n=1}^{\infty} \frac{1}{n} P\left(\frac{\overline{S}_n}{n} > \delta; \frac{S_n}{n} \leq \delta\right). \end{aligned}$$

Now, fix $\varepsilon > 0$ small and write

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n} P\left(\frac{\overline{S}_n}{n} > \delta; \frac{S_n}{n} \leq \delta\right) &= \sum_{n \leq 1/\delta^{2+\varepsilon_0}} \frac{1}{n} P\left(\frac{\overline{S}_n}{n} > \delta; \frac{S_n}{n} \leq \delta\right) \\ &\quad + \sum_{n > 1/\delta^{2+\varepsilon}} \frac{1}{n} P\left(\frac{\overline{S}_n}{n} > \delta; \frac{S_n}{n} \leq \delta\right). \end{aligned}$$

Observe that

$$\begin{aligned} P\left(\frac{\overline{S}_n}{n} > \delta; \frac{S_n}{n} \leq \delta\right) &\leq P\left(\max_{k=1}^{n} |X_k| > 1/\delta\right) \\ &= 1 - \left(1 - \overline{F}(1/\delta)\right)^n, \end{aligned}$$

where $\overline{F}(x) = P(X > x)$. Since $E\left(|X|^{3+\alpha}\right) < \infty$ we have that $\overline{F}(1/\delta) = o\left(\delta^{3+\alpha}\right)$. Thus, we can write

$$P\left(\frac{S_n}{n} > \delta; \frac{\overline{S}_n}{n} \leq \delta\right) \leq 1 - \left(1 - o\left(\delta^{3+\alpha}\right)\right)^n.$$

However,

$$\begin{aligned}\sum_{n\leq 1/\delta^{2+\varepsilon}}\frac{1}{n}P\left(\frac{S_n}{n}>\delta;\frac{\overline{S}_n}{n}\leq\delta\right) &\leq M\log\left(1/\delta^{2+\varepsilon}\right)\left(1-\left(1-o\left(\delta^{3+\alpha}\right)\right)^{1/\delta^{2+\varepsilon}}\right)\\ &= M\log\left(1/\delta^{2+\beta}\right)\left(1-\left(1-\delta^{2+\varepsilon}o\left(\delta^{1+a-\varepsilon}\right)\right)^{1/\delta^{2+\varepsilon}}\right)\\ &= o\left(\delta^{1+\alpha-\varepsilon_0}\right)\end{aligned}$$

for $\varepsilon_0>\varepsilon>0$ small enough. Now, put $\widetilde{\psi}_\delta(\theta)=\log\widetilde{\phi}_\delta(\theta)$ (recall that $\widetilde{\phi}_\delta(\theta)$ was defined in Proposition 2) and use Chernoff's bound to obtain

$$P\left(\frac{\overline{S}_n}{n}>\delta;\frac{S_n}{n}\leq\delta\right)\leq P\left(\frac{\overline{S}_n}{n}>\delta\right)\leq\exp\left(-n\left(\delta\overline{\theta}_\delta-\widetilde{\psi}_\delta\left(\overline{\theta}_\delta\right)\right)\right),$$

where $\overline{\theta}_\delta$ satisfies the equation $\widetilde{\psi}'_\delta\left(\overline{\theta}_\delta\right)=\delta$ (which, can be easily seen to have a solution for $\delta>0$ small enough). Hence,

$$\begin{aligned}\sum_{n>1/\delta^{2+\varepsilon}}\frac{1}{n}P\left(\frac{\overline{S}_n}{n}>\delta;\frac{S_n}{n}\leq\delta\right) &\leq \sum_{n>1/\delta^{2+\varepsilon}}\frac{1}{n}\exp\left(-n\left(\delta\overline{\theta}_\delta-\widetilde{\psi}_\delta\left(\overline{\theta}_\delta\right)\right)\right)\\ &\leq \frac{\exp\left(-\left\lfloor 1/\delta^{2+\varepsilon}\right\rfloor\left(\delta\overline{\theta}_\delta-\widetilde{\psi}_\delta\left(\overline{\theta}_\delta\right)\right)\right)}{1-\exp\left(-\left(\delta\overline{\theta}_\delta-\widetilde{\psi}_\delta\left(\overline{\theta}_\delta\right)\right)\right)}\\ &= o\left(\exp\left(-r/\delta^{\varepsilon}\right)\right)\end{aligned}$$

for $r>0$ (since $\left(\delta\overline{\theta}_\delta-\widetilde{\psi}_\delta\left(\overline{\theta}_\delta\right)\right)\sim\delta^2/2$), the previous term is obviously of order $o\left(\delta^{1+\alpha+\varepsilon_0}\right)$. For the term

$$\sum_{n>1/\delta^{2+\varepsilon}}\frac{1}{n}P\left(\frac{S_n}{n}>\delta;\frac{\overline{S}_n}{n}\leq\delta\right)\leq\sum_{n>1/\delta^{2+\varepsilon}}\frac{1}{n}P\left(\frac{S_n}{n}>\delta\right),$$

we first note that (since $E\left|X_1\right|^{3+\alpha}<\infty$)

$$P\left(X_1>x\right)\leq P\left(\left|X_1\right|>x\right)\leq C\left(1+x\right)^{-(\alpha+3)}\triangleq V\left(x\right)$$

for some constant $C>0$. Corollary 4.2 of Borovkov (2000) implies that

$$\sup_{x\geq t\sqrt{(\alpha+1)n\log n}}\frac{P\left(\max_{k\leq n}S_n>x\right)}{nV\left(x\right)}\leq 1+h\left(t\right),$$

where $h(t) \to 0$ as $t \nearrow \infty$. In our case $n > 1/\delta^{2+\varepsilon}$, therefore

$$\begin{aligned} x &= n\delta \geq \delta^{-(1+\varepsilon)} \geq t_1 \delta^{-(1+5\varepsilon/6)} \\ &\geq t\sqrt{(\alpha+1)\left(1/\delta^{2+\varepsilon}\right)\log\left(1/\delta^{2+\varepsilon}\right)} \geq t\sqrt{(\alpha+1)\, n \log n}, \end{aligned}$$

for large enough (but fixed) constants $t_1, t > 0$. We therefore conclude that there exists $t_2 > 0$ such that

$$\sum_{n>1/\delta^{2+\varepsilon}} \frac{1}{n} P\left(\frac{S_n}{n} > \delta\right) \leq t_2 \sum_{n>1/\delta^{2+\varepsilon}} V(n\delta) = o\left(\delta^{\alpha+1}\right)$$

(is analogous. This gives the estimate $p_\delta = \overline{p}_\delta + o\left(\delta^{\alpha+1-\varepsilon}\right)$.

Now let us write $\mu_+(j,\delta) = ET_{1,\delta}^j$ and $\overline{\mu}_+(j,\delta) = E\overline{T}_{1,\delta}^j$ for $j \leq \lfloor \alpha + 2 \rfloor$. Similarly, we use the symbol $\mu_-(j,\delta)$ (resp. $\overline{\mu}_-(j,\delta)$) to denote the jth moment of the first weakly descending ladder height of the random walk S^δ (resp. the jth moment of the first weakly descending ladder height of $\overline{S}^\delta$). Finally, let $\mu_j = E(X_1 - \delta)^j$ and $\overline{\mu}_j = E\left(\overline{X}_1^\delta\right)^j$. The Wiener-Hopf factorization (Lemma 2) then asserts that $\mu_1 = p\mu_{-,1}$ (and that $\overline{\mu}_1 = \overline{p\mu}_{-,1}$), that is

$$\begin{aligned} \overline{\mu}_-(j,\delta) - \mu_-(j,\delta) &= \frac{\overline{\mu}_1}{p + o\left(\delta^{\alpha+1-\varepsilon}\right)} - \frac{\mu_1}{p} \\ &= \frac{\overline{\mu}_1 - \mu_1}{p + o\left(\delta^{\alpha+1-\varepsilon}\right)} - \mu_1\left(\frac{1}{p + o\left(\delta^{\alpha+1-\varepsilon}\right)} - \frac{1}{p}\right) \\ &= o\left(\delta^{\alpha+1}\right) - \delta\left(\frac{o\left(\delta^{\alpha+1-\varepsilon}\right)}{\left(p + o\left(\delta^{\alpha+1-\varepsilon}\right)\right)p}\right) = o\left(\delta^{\alpha-\varepsilon}\right). \end{aligned} \tag{23}$$

Also from the Wiener-Hopf factorization we obtain

$$\mu_+(1,\delta) = \frac{p\mu_-(2,\delta) - \mu_2}{2\mu_-(1,\delta)}. \tag{24}$$

Therefore, in order to continue, we need to estimate the difference between $\overline{\mu}_-(j,\delta)$ and $\mu_-(j,\delta)$. This differences will be estimated via Fourier methods.

Since we are assuming strongly non-lattice we can use the following identity

$$\begin{aligned} \log\left(\frac{1 - E\left(\exp\left(\Delta S_{\tau-}^\delta\right)\right)}{-E\left(\Delta S_{\tau-}^\delta\right)}\right) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\Delta}{\left(\Delta^2 + \lambda^2\right)} \operatorname{Re}\log\left(\frac{1 - e^{\delta i\theta} g(-\lambda)}{-i\delta\lambda}\right) d\lambda \\ &\quad - \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\Delta^2}{\left(\Delta^2 + \lambda^2\right)\lambda} \operatorname{Im}\log\left(\frac{1 - e^{\delta i\theta} g(-\lambda)}{-i\delta\lambda}\right) d\lambda, \end{aligned} \tag{25}$$

for $\Delta > 0$. This identity is almost the same as the one derived via Corollary 8.45 and Theorem 8.51 of Siegmund (1985) which is obtained for the strictly ascending ladder height, however a straightforward adaptation of Siegmund's argument shows that the result also holds for the descending ladder height as displayed in (25). An expansion of the left hand side of (25) in powers of Δ (up to order $\lfloor \alpha + 2 \rfloor$) generates a sequence of coefficients $c_j(\delta)$. Note that the ratios $\mu_-(k,\delta)/\mu_-(1,\delta)$ (for $k \leq \lfloor \alpha + 2 \rfloor$) can be recovered from the coefficients $c_j(\delta)$, for $j \leq k$ by solving a system of equations (in fact, $(-1)^j c_j(\delta)$ is the jth order cumulant of the limiting overshoot of the random walk $-S^\delta$, and $\mu_-(j+1,\delta)/\mu_-(1,\delta)$ is proportional to its jth moment). Hence, we can compute the magnitude of the error between $\mu_-(j+1,\delta)/\mu_-(1,\delta)$ and $\overline{\mu}_-(j+1,\delta)/\overline{\mu}_-(1,\delta)$ by estimating $c_j(\delta) - \overline{c}_j(\delta)$. Consequently, it suffices to study the coefficients in the asymptotic expansion (in powers of $\Delta > 0$) of

$$\begin{aligned}
&\log\left(\frac{E\overline{S}^\delta_{\tau-}\left(1 - E\exp\left(\Delta S^\delta_{\tau-}\right)\right)}{ES^\delta_{\tau-}\left(1 - E\exp\left(\Delta \overline{S}^\delta_{\tau-}\right)\right)}\right) \\
= \; & \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\Delta}{\left(\Delta^2 + \lambda^2\right)} \operatorname{Re}\log\left(\frac{\left(\delta + o\left(\delta^{\alpha+2}\right)\right)\left(1 - e^{i\delta\lambda} g(-\lambda)\right)}{\delta\left(1 - e^{i\delta\theta}\widetilde{g}_\delta(-\lambda)\right)}\right) d\lambda \qquad (26) \\
& -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\Delta^2}{\left(\Delta^2 + \lambda^2\right)\lambda} \operatorname{Im}\log\left(\frac{\left(\delta + o\left(\delta^{\alpha+2}\right)\right)\left(1 - e^{i\delta\lambda} g(-\lambda)\right)}{\delta\left(1 - e^{i\delta\theta}\widetilde{g}_\delta(-\lambda)\right)}\right) d\lambda, \qquad (27)
\end{aligned}$$

where $\widetilde{g}_\delta(\lambda) = E\exp\left(i\lambda X_1 1\left(|X_1| \leq 1/\delta\right)\right)$. The expansion of the integrals (26) and (27) can be easily obtained using Proposition 2 in the second chapter of this dissertation. For instance,

$$\begin{aligned}
c_1 - \overline{c}_1 &= \frac{1}{2}\left(\frac{\mu_-(2,\delta)}{\mu_-(1,\delta)} - \frac{\overline{\mu}_-(2,\delta)}{\overline{\mu}_-(1,\delta) + o\left(\delta^{\alpha-\varepsilon}\right)}\right) \text{ (using the LHS of (25))} \\
&= \frac{1}{4}\left(\frac{\mu_2 + o\left(\delta^{\alpha+2}\right)}{\delta + o\left(\delta^{\alpha+3}\right)} - \frac{\mu_2}{\delta}\right) \text{ (expanding (26) and (27))} \qquad (28) \\
& \quad \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{\lambda^2} \operatorname{Re}\log\left(\frac{\left(\delta + o\left(\delta^{k-1}\right)\right)\left(1 - e^{i\delta\lambda} g(-\lambda)\right)}{\delta\left(1 - e^{i\delta\theta}\widetilde{g}_\delta(-\lambda)\right)}\right) d\theta. \qquad (29)
\end{aligned}$$

Since $|g(-\lambda) - \widetilde{g}_\delta(-\lambda)| \leq o\left(\delta^{\alpha+3}\right)$, the term (29) is smaller than the term (28), and it is straightforward to verify that

$$\frac{\mu_2 + o\left(\delta^{\alpha+1}\right)}{\delta + o\left(\delta^{\alpha+2}\right)} - \frac{\mu_2}{\delta} = o\left(\delta^{\alpha}\right);$$

which implies that

$$\overline{\mu}_-\left(2,\delta\right) - \mu_-\left(2,\delta\right) = o\left(\delta^{\alpha-\varepsilon}\right).$$

We can continue in this fashion; for example, we observe that the error term in the difference between $c_2\left(\delta\right)$ and $\overline{c}_2\left(\delta\right)$ is determined by that of the difference between μ_3/μ_1 and $\overline{\mu}_3/\overline{\mu}_1$ (because now the coefficient of Δ^2 in the expansion of (26) involves $\mu_3/\delta - \overline{\mu}_3/\left(\delta + o\left(\delta^{\alpha+2}\right)\right)$ and that of (27) is an integral involving $\mu_2/\delta - \overline{\mu}_2/\left(\delta + o\left(\delta^{\alpha+2}\right)\right)$). So, we obtain that $\mu_-\left(3,\delta\right) - \overline{\mu}_-\left(3,\delta\right) = o\left(\delta^{\alpha-1}\right)$. Similarly, for the difference between $\overline{\mu}_-\left(n,\delta\right)/\overline{\mu}_-\left(1,\delta\right)$ and $\mu_-\left(n,\delta\right)/\mu_-\left(1,\delta\right)$ for $n > 3$, we observe that we must look at the difference between μ_n/δ and $\overline{\mu}_n/\left(\delta + o\left(\delta^{\alpha+2}\right)\right)$ which yields that

$$\mu_-\left(n,\delta\right) - \overline{\mu}_-\left(n,\delta\right) = o\left(\delta^{\alpha+2-n}\right).$$

Furthermore, using the Wiener-Hopf factorization we see that the error in $\overline{\mu}_+\left(n,\delta\right) - \mu_+\left(n,\delta\right)$ is determined by that of $\overline{\mu}_-\left(n,\delta\right) - \mu_-\left(n,\delta\right)$ which, in particular, implies the statement of the proposition.

Chapter 4

Asymptotic Expansions for Geometric Sums with Applications to Defective Renewal Equations

Consider a sequence $X = (X_k : k \geq 1)$ of non-negative independent and identically distributed (iid) random variables (rv's). Suppose that X_1 is strongly non-lattice in the sense that its characteristic function, $g(\lambda) = E\exp(i\lambda X_1)$, satisfies that for every $\varepsilon > 0$

$$\inf_{|\lambda|>\varepsilon} |1 - g(\lambda)| > 0 \tag{1}$$

or, equivalently, that $\overline{\lim}_{|\lambda|\to\infty} |g(\lambda)| < 1$ (see Siegmund (1985) p. 176).

Let M be a geometrically distributed random variable independent of X. That is,

$$P(M = k) = p(1-p)^k = pq^k; \quad k \geq 0.$$

Our focus here is on the distribution of

$$S_M \triangleq \sum_{k=1}^{M} X_k,$$

($S_M \triangleq 0$ on $M = 0$) when the success probability p of the geometric random variable M is small. The rv S_M is called a geometric sum. Renyi's theorem for geometric sums of random variables establishes that if $EX_1 < \infty$, then

$$P\left(pS_M > x\right) = \exp\left(-x/E\left(X_1\right)\right) + o\left(1\right) \tag{2}$$

as $p \searrow 0$. In this chapter, under the assumption of strongly non-lattice increments, we develop additional order correction terms (in powers of p) to approximation (2) (see equation (18) below). These types of expansions are similar in spirit to the Edgeworth expansions for the central limit theorem. As in Edgeworth expansions, the existence of certain order moments has to be imposed in order to provide the *nth* order correction term. See, for example, Theorem 3.

The rv S_M is utilized in many applied probability settings. For example, in queueing theory, it is well known (by appealing to the ascending ladder heights representation for the maximum of random walk) that the steady-state waiting time distribution of the standard single server queue can be represented as a geometric sum with non-negative increments (c.f. Asmussen (1987) or Kalashnikov (1997), Section 1.3.3). In insurance risk theory, the ruin probability in the renewal model can also be expressed as a tail probability of a geometric sum with non-negative increments (see Asmussen (2001) or Kalashnikov (1997), Section 1.3.4). Finally, in the context of reliability models, the first break-down time of a system that consists of an operating element, $N - 1$ unloaded redundant elements and M identical repair units, can also be expressed as a geometric sum such as S_M (refer also to Kalashnikov (1997), Section 1.3.5). Other applications include program debugging and the total reward until visiting a rare set in a Markov setting. (See the book on geometric sums by Kalashnikov (1997) for additional details.)

The setting in which the success probability p is close to zero arises often in applications. For instance, in the queueing example mentioned in the previous paragraph, this setting corresponds to the so-called heavy traffic regime in which the server utilization is close to 100%; in the risk insurance context, p close to zero describes the setting in which the security margin, included in the risk premium received by the insurance company, is close to zero. Finally, in the reliability example, p close to zero reflects a setting with a low break-down rate. In several of the examples above, the distribution of the increments X_k depends on p as well. We must therefore develop a theory that can handle this dependence. As an important application of the results developed in this chapter (in particular Theorem 4) recall our results in Chapter 3 on high accuracy approximations for the maximum of random walk with heavy-tailed increments. These approximations, as we have pointed out repeatedly, are very useful in some of the applied settings mentioned at the beginning of our discussion (e.g. the steady-state distribution of the single server queue and the ruin probability in the insurance context).

In addition to the applications described above, there exists a close connection between so-called defective renewal equations and geometric sums. Indeed, if $a(\cdot)$ satisfies the defective renewal equation

$$a(t) = b(t) + q \int_{[0,t)} a(t-s) P(X_1 \in ds), \tag{3}$$

Then, in great generality, (see Lin and Willmot (2000) p. 152) it follows that

$$a(t) = \frac{1}{p} \int_{[0,t)} b(t-s) P(S_M \in ds). \tag{4}$$

Equation (4) makes the connection clear between solutions of defective renewal equations (such as (3)) and the distribution of geometric sums. It turns out that defective renewal equations such as (3) play an important role in a number of applied probability settings. A prominent example is insurance risk theory; in particular, the so-called "expected discounted penalty" at ruin (from which many quantities of interest, including the ruin probability, can be recovered by judicious choices of the discount rate and the penalty) can be expressed in terms of a defective renewal equation (see Lin and Willmot (2000) p. 162). Many other examples in which defective renewal equations play an important role are also described in Feller (1968) p. 188, 216, Resnick (1992) p. 158, and Lin and Willmot (2000) Ch. 9) these examples include Geiger counters, generalized terminating renewal processes, and age dependent branching processes. The setting in which q is close to one in (3) (or, equivalently, p is close to zero) is common in the application settings described before. For instance, in the insurance setting it arises in environments of low net profits (which occur in competitive conditions). For generalized terminating renewal processes, q close to one corresponds to settings in which the process continues for long periods, and in age dependent branching processes, q close to one reflects a case in which the population is less likely to die. This, consequently, motivates developing asymptotics for the solution $a(\cdot)$ of (3) as $p \searrow 0$.

In Section 2, we develop the asymptotic expansion in powers of p for $P(pS_M > x)$ (see Theorems 2, 3, and 4). The implications for asymptotic expansions of defective renewal equations are studied in Section 3 (see Theorem 5).

4.1 Asymptotics for Geometric Sums

We first start with a useful representation for the tail probability of a geometric sum. Set $S_n = X_1 + ... + X_n$ (with $S_0 = 0$) and put $N(t) = \sup\{n \geq 0 : S_n \leq t\}$. Observe that, for each non-negative integer m, $\{S_m > x\} = \{N(x) < m\}$. Thus, combining the independence between X and M with the fact that $P(M > m) = q^{m+1}$, we can write

$$P(S_M > x) = P(N(x) > M) = E(P(N(x) > M|\, X)) = qE\left(q^{N(x)}\right).$$

We therefore have shown the next proposition.

Proposition 1

$$P(S_M > x) = qEq^{N(x)}.$$

The previous proposition implies that in order to study $P(pS_M > x)$ it suffices to study the behavior of $Eq^{N(x/p)}$ for $x > 0$ and small $p > 0$. A renewal theoretic argument yields the following (defective) renewal equation

$$Eq^{N(t)} = P(X_1 > t) + q \int_{[0,t)} Eq^{N(t-s)} P(X_1 \in ds). \tag{5}$$

Note that, if $p > 0$ small enough and $E\exp(\eta X_1) < \infty$ for some $\eta > 0$, the equation

$$E\exp\left(\widehat{\theta} X_1\right) = 1/q \tag{6}$$

has a unique solution $\widehat{\theta} > 0$. Therefore, (5) can be transformed into the non-defective renewal equation

$$e^{\widehat{\theta}t} E q^{N(t)} = e^{\widehat{\theta}t} P\left(X_1 > t\right) + \int_{[0,t)} e^{\widehat{\theta}(t-s)} E q^{N(t-s)} F_{\widehat{\theta}}\left(ds\right),$$

where $F_{\widehat{\theta}}\left(ds\right) = q e^{\widehat{\theta}s} P\left(X_1 \in ds\right)$. Renewal theory then implies that

$$e^{\widehat{\theta}t} E q^{N(t)} = \int_{[0,t)} e^{\widehat{\theta}(t-s)} P\left(X_1 > t-s\right) U_{\widehat{\theta}}\left(ds\right), \tag{7}$$

where $U_{\widehat{\theta}}\left(t\right) = E_{\widehat{\theta}}\left(N\left(t\right)+1\right)$, and under $P_{\widehat{\theta}}$ the X_i's are iid with distribution $F_{\widehat{\theta}}$. Using results by Stone (1965) it is not hard to verify that, for fixed but small $p > 0$

$$\left| e^{\widehat{\theta}t} E q^{N(t)} - \frac{1}{E_{\widehat{\theta}} X_1} \int_{[0,t)} e^{\widehat{\theta}s} P\left(X_1 > s\right) ds \right| \leq K\left(p\right) \exp\left(-a\left(p\right)t\right). \tag{8}$$

Since our situation involves sending simultaneously $t \nearrow \infty$ and $p \searrow 0$ we would like the bound on the right hand side of (8) to hold uniformly in $p \in [0, \delta_1]$ for some $\delta_1 > 0$. That is, we would like to show that we can find $a, K \in (0, \infty)$ such that $\sup_{p\in[0,\delta_1]} K\left(p\right) \leq K < \infty$ and $\sup_{p\in[0,\delta_1]} a\left(p\right) \geq a$. The following theorem provides means to obtain these uniform estimates.

Theorem 1 *Let $\mathcal{F}$ be a family of distribution functions supported on $[0,\infty)$. For each $F \in \mathcal{F}$, let $E_F\left(\cdot\right)$ be the expectation operator associated to $F \in \mathcal{F}$, and define $E_F g\left(\tau\right) \triangleq \int_{[0,\infty)} g\left(t\right) F\left(dt\right)$ for each continuous and bounded function $g : [0,\infty) \to \mathbb{C}$. Suppose that the family $\mathcal{F}$ is uniformly strongly non-lattice, (i.e. the corresponding characteristic functions $\chi_F\left(\lambda\right) = E_F \exp\left(i\lambda\tau\right)$ satisfy*

$$\inf_{F\in\mathcal{F}} \inf_{|\lambda|>\varepsilon} \left|1 - \chi_F\left(\lambda\right)\right| > 0. \tag{9}$$

Then, $U_F\left(t\right) \triangleq \sum_{n=0}^{\infty} F^{*n}\left(t\right)$ satisfies the following.

1. *If $\sup_{F\in\mathcal{F}} E_F \exp\left(\eta X_1\right) < \infty$ for some $\eta > 0$, then*

$$\sup_{F\in\mathcal{F}} \left| U_F\left(t\right) - \frac{t}{E_F\tau} - \frac{E_F\tau^2}{2E_F^2\tau} \right| = O\left(e^{-at}\right)$$

as $t \to \infty$ for some $a > 0$.

2. *Moreover, if* $\sup_{F\in\mathcal{F}} E_F\tau^{\varepsilon+2} < \infty$ *for* $\varepsilon \geq 0$*, then,*

$$\sup_{F\in\mathcal{F}}\left|U_F(t) - \frac{t}{E_F\tau} - \frac{E_F\tau^2}{2E_F^2\tau} - \frac{H_2^F(t)}{E_F\tau_\delta^2} - H_1^F * H_1^F(t)\right| = o\left(t^{\alpha+2}\log(t)\right)$$

as $t\to\infty$*, where* $H_1^F(t) = \int_t^\infty (1-F(s))\,ds\,/E_F\tau$ *and* $H_2^F(t) = \int_t^\infty H_1^F(s)\,ds$.

Proof. Part 1 is essentially Siegmund's (1979) lemma. Part 2 follows the same steps as in Carlsson (1983), the key assumption is the uniform strongly non-lattice condition (9). The Fourier inversion expressions provided by Carlsson (1983) are the same for each fixed F. At the end, Carlsson's estimates of the error rate depend on the application of a uniform version of the Riemann-Lebesgue lemma to his equation (11) which can be obtain following his same argument in the presence of the strongly nonlattice assumption imposed.

We now are ready to provide our asymptotic expansion for $P(pS_M > x)$ in the presence of exponential moments.

Theorem 2 *Suppose that* X_1 *has strongly non-lattice distribution and that* $\phi(\eta) \triangleq E\exp(\eta X) < \infty$ *for some* $\eta > 0$*. Then, for some* $a > 0$*, and as* $x/p\to\infty$*,*

$$P(pS_M > x) = \exp\left(-x\widehat{\theta}/p + r(p)\right) + O\left(\exp(-ax/p)\right), \tag{10}$$

where $\widehat{\theta}$ *solves (6) and*

$$\exp(r(p)) = \frac{p}{q\widehat{\theta}\phi'\left(\widehat{\theta}\right)} \triangleq c(p). \tag{11}$$

Moreover, both $\widehat{\theta}$ *and* r *are real analytic functions of* p *at the origin.*

Proof. The argument preceding Theorem 1 led us to equation (7). We now verify that the assumptions in Theorem 1 are satisfied. Let us define $g_{\widehat{\theta}}(\lambda) \triangleq E_{\widehat{\theta}}\exp(i\lambda X_1) = qE\exp\left(\left(i\lambda + \widehat{\theta}\right)X_1\right)$. Using the implicit function theorem on (6) it follows easily that $\widehat{\theta} = p/EX_1 + O(p^2)$. As a consequence, the following inequality can be easily derived for all $p > 0$ sufficiently small and some $M_1 \in (0,\infty)$

$$\left|g_{\widehat{\theta}}(\lambda) - g(\lambda)\right| \leq M_1 p.$$

Hence, we conclude that for each $\varepsilon > 0$ it is possible to pick $\delta > 0$ sufficiently small so that

$$\inf_{p\in[0,\delta]} \inf_{|\lambda|>\varepsilon} \left|1 - g_{\widehat{\theta}}(\lambda)\right| \geq \inf_{p\in[0,\delta]} \inf_{|\lambda|>\varepsilon} |1 - g(\lambda)| - M_1\delta > 0.$$

Finally, also because $\widehat{\theta} = O(p)$, it is possible to pick $p > 0$ small enough so that $E_{\widehat{\theta}} \exp(\eta X_1) = qE_{\widehat{\theta}} \exp\left(\left(\eta + \widehat{\theta}\right) X_1\right) < \infty$ for some $\eta > 0$. We now can apply Theorem 1 to equation (7) and obtain

$$\begin{aligned}
& \left| e^{\widehat{\theta}x/p} Eq^{N(x/p)} - \frac{1}{E_{\widehat{\theta}}X_1} \int_0^\infty e^{\widehat{\theta}s} P(X_1 > s)\, ds \right| \\
\leq\ & \frac{1}{E_{\widehat{\theta}}X_1} \int_{x/p}^\infty e^{\widehat{\theta}s} P(X_1 > s)\, ds \qquad (12)\\
& + \left| \frac{1}{E_{\widehat{\theta}}X_1} \int_{[0,x/p)} e^{\widehat{\theta}(x/p-s)} P(X_1 > x/p - s)\, V(ds) \right|, \qquad (13)
\end{aligned}$$

where, $V(t)$ is a function that we are introducing here and it corresponds to the left hand side of 1 in Theorem 1), therefore $|V(t)| = O(e^{-at})$ for some $a > 0$. The integral in (12) is easily seen to be bounded by $Ke^{-ax/p}$ for some finite constants $K, a > 0$ (assuming that $p > 0$ is sufficiently small). We just need to analyze the integral in (13). Integration by parts yields

$$\begin{aligned}
& \int_{[0,x/p)} e^{\widehat{\theta}(x/p-s)} P(X_1 > x/p - s)\, V(ds) \\
=\ & V(x/p)\, P(X_1 > 0) - e^{\widehat{\theta}x/p} P(X_1 > x/p)\, V(0) \qquad (14)\\
& + \widehat{\theta} e^{\widehat{\theta}x/p} \int_{[0,x/p)} V(s)\, e^{-\widehat{\theta}s} P(X_1 > x/p - s)\, ds \qquad (15)\\
& + e^{\widehat{\theta}x/p} \int_{[0,x/p)} V(s)\, e^{-\widehat{\theta}s} P(X_1 > x/p - ds). \qquad (16)
\end{aligned}$$

The absolute value of (14) is also bounded by $Ke^{-ax/p}$ for some finite constants $K, a > 0$. For the integral (15) observe that

$$\begin{aligned}
& \widehat{\theta} e^{\widehat{\theta} x/p} \left| \int_{[0,x/p)} V(s) e^{-\widehat{\theta} s} P(X_1 > x/p - s)\, ds \right| \\
\leq & \ \widehat{\theta} e^{\widehat{\theta} x/p} \int_{[0,x/2p)} |V(s)| e^{-\widehat{\theta} s} P(X_1 > x/p - s)\, ds \\
& + \widehat{\theta} e^{\widehat{\theta} x/p} \int_{[x/2p,x/p)} |V(s)| e^{-\widehat{\theta} s} P(X_1 > x/p - s)\, ds \\
\leq & \ \widehat{\theta} e^{\widehat{\theta} x/p} P(X_1 > x/(2p)) M + \widehat{\theta} e^{\widehat{\theta} x/p} \int_{[x/2p,\infty)} |V(s)|\, ds.
\end{aligned}$$

Since $\widehat{\theta} = O(p)$ and X_1 has exponential moments, we conclude that the previous expression is bounded by $Ke^{-ax/p}$ (for appropriate positive constants K and a). The treatment for integral (16) is very similar to that of (15). Thus, we conclude that

$$Eq^{N(x/p)} = \frac{\int_0^\infty e^{\widehat{\theta} s} P(X_1 > s)\, ds}{E_{\widehat{\theta}} X_1} + O(\exp(-rx/p)). \tag{17}$$

In order to recover the required expression for $c(p)$, note that

$$E_{\widehat{\theta}} X_1 = q \int_{[0,\infty)} s e^{\widehat{\theta} s} P(X_1 \in ds) = q\phi'\left(\widehat{\theta}\right).$$

On the other hand, using integration by parts and the definition of $\widehat{\theta}$, we see that

$$\int_0^\infty e^{\widehat{\theta} s} P(X_1 > s)\, ds = \frac{\left(\phi\left(\widehat{\theta}\right) - 1\right)}{\widehat{\theta}} = \frac{p}{q\widehat{\theta}}.$$

Combining the previous last two identities together into (17) yields equation (10). The analytic properties of $\widehat{\theta}$ follow directly from the implicit function theorem. It is easy to see that $r(\cdot)$ is well defined at zero (i.e. that the right hand side of (11) is strictly positive when p is close to zero). However, it is almost immediate to verify that $c(p)$ is real analytic at the origin with $c(0) = 1$. This implies the real analyticity of r and the conclusion of the theorem.

Theorem 2 indicates that

$$\widehat{\theta}(p) = \sum_{k=1}^{\infty} \frac{\widehat{\theta}^{(k)}(0)}{k!} p^k, \text{ and } r(p) = \sum_{k=0}^{\infty} \frac{r^{(k)}(0)}{k!} p^k.$$

For notational convenience let us write $\widehat{\theta}^{(k)}(0)/k! = \gamma_k$ and $r^{(k)}(0)/k! = \xi_k$. We know that $\widehat{\theta}^{(1)}(0) = 1/EX_1$ and $r(0) = 0$, the rest of the γ_k's and ξ_k's can be easily computed via the implicit function theorem. For instance, $2\gamma_2 = 1 - EX_1^2/(2E^2X_1)$ and $\xi_1 = 1 - \gamma_2 EX_1 - EX_1^2/(2E^2X_1)$. For completeness we provide a set of recursive equations to compute the γ_k's and ξ_k's.

Proposition 2 *For $n \geq 1$ and each $k \leq n$, the constants $(\gamma_k : 1 \leq k \leq n)$ can be computed by solving recursively the following set of equations (note that the kth equation is linear in the γ_k and it only depends on the γ_j's for $j \leq k$).*

$$\sum_{m=1}^{k} \frac{EX_1^m}{m!} \sum_{\{n_1+\ldots+n_m=k-m,\ n_1,..,n_m \geq 0\}} \prod_{j=1}^{m} \gamma_{n_j+1} = 1, \text{ for } 1 \leq k \leq n.$$

Consequently, the constants $(\xi_k : 0 \leq k \leq n-1)$ can be obtained through a Taylor expansion up to order n of the function

$$\widetilde{r}_n(p) = \log\left(\frac{1}{q\sum_{k=1}^{n}\gamma_k p^{k-1}\sum_{m=0}^{n-1}\left(\sum_{k=1}^{n}\gamma_k p^k\right)^m EX_1^{m+1}/m!}\right)$$

around $p = 0$. In particular, for $k \leq n-1$, $\xi_k = \widetilde{r}_n^{(k)}(0)/k!$.

Proof. The proof follows directly by applying the implicit function theorem. The details are omitted

Consequently, the previous theorem provides the means to develop an algorithm, that can be implemented easily, for computing an asymptotic expansion for the tail probability $P(S_M > x/p)$ in powers of p.

Theorem 2 corrects Renyi's approximation (2) by providing a full asymptotic expansion in powers of p with an exponential error term. In other words, the last theorem provides rigorous support for the parametric (in $p > 0$) approximation

$$P(S_M > x/p) \approx \exp\left(-x/EX_1 + \sum_{k=1}^{\infty} p^k\left(\xi_k - \gamma_{k+1}x\right)\right), \tag{18}$$

valid up to an error exponentially small as $p \searrow 0$. It is easy to see that γ_k and ξ_k depend on the first k and $(k+1)$ order moments of X_1 respectively. This suggests that, if $EX_1^{\alpha+2} < \infty$, say, the approximation

$$P\left(S_M > x/p\right) \approx \exp\left(-x/EX_1 + \sum_{k=1}^{\alpha} p^k \left(\xi_k - \gamma_{k+1} x\right)\right) \tag{19}$$

should be more accurate than (2). Providing rigorous support for approximation (19) in the presence of heavy tails (we say here that a non-negative random variable X_1 is heavy tailed if for every $\eta > 0$, $E\exp\left(\eta X_1\right) = \infty$) presents an additional mathematical complication. Note that a crucial ingredient in the proof of Theorem 2 is the existence of a root for equation (6). This indicates that the strategy followed in the proof of Theorem 2 is infeasible in the heavy tailed case. Our idea is then to proceed via truncation. Define the sequence $\overline{X} = \left(\overline{X}_k : k \geq 1\right)$ as $\overline{X}_k = X_k 1\left(X_k \leq x/p\right)$ and consider its associated random walk $\overline{S} = \left(\overline{S}_n : n \geq 0\right)$ (i.e. $\overline{S}_n = \overline{X}_1 + ... + \overline{X}_n$ with $\overline{S}_0 = 0$). We first argue that the distribution of S_M is suitably close to that of $\overline{S}_M$.

Lemma 1 *Suppose that $EX_1^{\beta} < \infty$ for $\beta \geq 1$, then*

$$\left|P\left(pS_M > x\right) - P\left(p\overline{S}_M > x\right)\right| = o\left(\frac{p^{\beta-1}}{x^{\beta}}\right)$$

Proof. Note that

$$\begin{aligned}
&\left|P\left(pS_M > x\right) - P\left(p\overline{S}_M > x\right)\right| \\
\leq\ & p\sum_{k=0}^{\infty} q^k P\left(S_n > x/p; \overline{S}_n \leq x/p\right) + p\sum_{k=0}^{\infty} q^k P\left(\overline{S}_n > x/p; S_n \leq x/p\right) \\
\leq\ & 2p\sum_{k=0}^{\infty} q^k P\left(\max_{k\leq n} X_k > x/p\right) = 2p\sum_{k=0}^{\infty} q^k \left(1 - \left(1 - o\left(\left(p/x\right)^{\beta}\right)\right)^k\right).
\end{aligned}$$

On the other hand,

$$\left(1 - o\left(\left(p/x\right)^{\beta}\right)\right)^k = 1 - ko\left(\left(p/x\right)^{\beta}\right) + \frac{k\left(k-1\right)}{2}\left(1-\eta_k\right)^{k-2} o\left(\left(p/x\right)^{2\beta}\right),$$

where $|\eta_k| \leq o\left((p/x)^{\beta}\right)$. Hence, we can write

$$\begin{aligned}\left|P\left(pS_M > x\right) - P\left(p\overline{S}_M > x\right)\right| &\leq 2p\sum_{k=0}^{\infty} q^k k o\left((p/x)^{\beta}\right) + 2p\sum_{k=0}^{\infty} q^k k^2 o\left((p/x)^{2\beta}\right) \\ &= o\left(\frac{p^{\beta-1}}{x^{\beta}}\right) + o\left(\frac{p^{2\beta-2}}{x^{2\beta}}\right) = o\left(\frac{p^{\beta-1}}{x^{\beta}}\right).\end{aligned}$$

We now would like to study $P\left(p\overline{S}_M > x\right)$ just as we did in Theorem 2. Theorem 1 can also be applied here to obtain a suitable approximation for $P\left(p\overline{S}_M > x\right)$, as our next result shows.

Theorem 3 *Assume that the distribution of X_1 is strongly non-lattice. Also, suppose that*

$$EX_1^{2+\alpha} < \infty$$

for $\alpha \geq 0$. Then,

$$P\left(pS_M > x\right) = \exp\left(-x\widehat{\theta}_{\alpha}/p + r_{\alpha}\left(p\right)\right) + o\left(p^{\alpha}\right)$$

as $p \searrow 0$, where

$$\widehat{\theta}_{\alpha} = p/EX_1 + \sum_{k\leq\alpha} \gamma_k p^k, \text{ and } r_{\alpha}\left(p\right) = \sum_{k\leq\alpha} \xi_k p^k$$

and the γ_k's and ξ_k's are defined recursively via Proposition 2.

Proof. Let $\overline{N}\left(t\right) = \sup\{n \geq 0 : \overline{S}_n \leq t\}$, then, by virtue of Proposition 1 and Lemma 1 it suffices to compute $Eq^{\overline{N}(x/p)}$. Following similar steps as in the proof of Theorem 2 we obtain

$$Eq^{\overline{N}(x/p)} = \int_{[0,x/p)} e^{\overline{\theta}(x/p-s)} P\left(X_1 > x/p - s; X_1 \leq x/p\right) \overline{U}_{\overline{\theta}}\left(ds\right). \quad (20)$$

The elements in equation (20) are indicated next. First, $\overline{\theta}$ is the solution to the equation

$$\overline{\phi}\left(\overline{\theta}\right) \triangleq E\exp\left(\overline{\theta X}_1\right) = \frac{1}{q},$$

which clearly exists if p is small enough. In order to describe $\overline{U}_\theta(s)$ define $P_{\overline{\theta}}(\cdot)$ via

$$P_{\overline{\theta}}(B) = q^n E\left(\exp\left(\overline{\theta}S_n\right); 1(B)\right),$$

for every B in the sigma-field $\sigma\left(\overline{X}_1, .., \overline{X}_n\right)$. Next, we will show that

$$\overline{V}(s) \triangleq \overline{U}_\theta(s) - \frac{s}{E_{\overline{\theta}}\overline{X}_1} - \frac{E_{\overline{\theta}}\overline{X}_1^2}{2E_{\overline{\theta}}^2\overline{X}_1} - \frac{\int_t^\infty \int_s^\infty P_{\overline{\theta}}\left(\overline{X}_1 > u\right) du ds}{EX_1^2}, \tag{21}$$

where, $\left|\overline{V}(s)\right| = o\left(s^{-(\alpha+1)}\right)$ as $s \nearrow \infty$ uniformly in $p > 0$ small enough. The previous expression follows from Theorem 1, as we now illustrate. (Note that the term $\overline{V}$ in (21) includes the last two terms in the right hand side of the equation in the part 2 of Theorem 1.) Observe that $\overline{g}_p(\lambda) \triangleq E_{\overline{\theta}} \exp\left(i\lambda\overline{X}_1\right) = qE_{\overline{\theta}}\exp((i\lambda + \overline{\theta})\overline{X}_1)$ satisfies

$$\begin{aligned}\left|\overline{g}_p(\lambda) - E\exp(i\lambda X_1)\right| &\leq \left|\overline{g}_p(\lambda) - E\exp\left(i\lambda\overline{X}_1\right)\right| + o\left(p^{\alpha+2}\right)\\ &\leq p\left|E\exp\left(i\lambda\overline{X}_1\right)\right| + \overline{\theta}E\overline{X}_1 + o\left(p^{\alpha+2}\right) = O(p).\end{aligned}$$

This implies that $\overline{g}_p(\cdot)$ satisfies the uniform strongly non-lattice condition (14). On the other hand, since $\overline{\theta} = O(p)$, we have that for all $p > 0$ small enough

$$E_{\overline{\theta}}\overline{X}_1^{\alpha+2} = qE\exp\left(\overline{\theta}X_1\right)\overline{X}_1^{\alpha+2} \leq ME\overline{X}_1^{\alpha+2} < MEX_1^{\alpha+2} < \infty.$$

This justifies the validity of representation (21). Furthermore, (21) implies that

$$\begin{aligned}&Eq^{\overline{N}(x/p)}\\ =\;& \int_{[0,x/p)} \frac{e^{\overline{\theta}(x/p-s)}P\left(X_1 > x/p - s; X_1 \leq x/p\right)}{E_{\overline{\theta}}\overline{X}_1} ds \qquad (22)\\ &+\int_{[0,x/p)} \frac{e^{\overline{\theta}(x/p-s)}P\left(X_1 > x/p - s; X_1 \leq x/p\right)}{E_{\overline{\theta}}\overline{X}_1^2} \int_s^\infty P_{\overline{\theta}}\left(\overline{X}_1 > u\right) du ds \qquad (23)\\ &+\int_{[0,x/p)} e^{\overline{\theta}(x/p-s)}P\left(X_1 > x/p - s; X_1 \leq x/p\right)\overline{V}(ds). \qquad (24)\end{aligned}$$

Let us denote by I_1, I_2, and I_3 the expressions (22), (23), and (24) respectively. We first show that $I_3 = o\left(p^{\alpha+1}\right)$. To see this, we use integration by parts, the triangle inequality and the fact that $\overline{\theta} = O(p)$ to obtain

$$\begin{aligned}
|I_3| &\leq \left|\overline{V}(x/p)\right| + M_1 \left|\int_{[0,x/p)} \overline{V}(s)\, de^{-\overline{\theta}s} P(X_1 > x/p - s; X_1 \leq x/p)\right| \\
&\leq \left|\overline{V}(x/p)\right| + M_1 \left|\int_{[0,x/p)} \overline{V}(s)\, e^{-\overline{\theta}s} P(X_1 > x/p - ds; X_1 \leq x/p)\right| \\
&\quad + M_1 \left|\int_{[0,x/p)} \overline{V}(s)\, e^{-\overline{\theta}s} P(X_1 > x/p - s; X_1 \leq x/p)\, ds\right|. \qquad (25)
\end{aligned}$$

The term $\left|\overline{V}(x/p)\right| = o(p^{\alpha+1})$. Now, observe that

$$\begin{aligned}
&\left|\int_{[0,x/p)} \overline{V}(s)\, e^{-\overline{\theta}s} P(X_1 > x/p - ds; X_1 \leq x/p)\right| \\
&\leq \left|\int_{[0,x/2p)} \overline{V}(s)\, e^{-\overline{\theta}s} P(X_1 > x/p - ds; X_1 \leq x/p)\right| \\
&\quad + \left|\int_{[x/2p,x/p)} \overline{V}(s)\, e^{-\overline{\theta}s} P(X_1 > x/p - ds; X_1 \leq x/p)\right| \\
&\leq K_2 P(X_1 > x/(2p)) + K_1 \max_{1/2 \leq u \leq 1} \left|\overline{V}(ux/p)\right| \qquad (26) \\
&= o\left(p^{\alpha+2}\right) + o\left(p^{\alpha+1}\right) = o\left(p^{\alpha+1}\right),
\end{aligned}$$

for some constants K_1 and K_2. The integral in (25) follows the same lines as (26). For I_2 we have

$$I_2 = \frac{1}{E_{\overline{\theta}} \overline{X}_1^2} \int_{[0,x/p)} e^{\overline{\theta}(x/p-s)} P(X_1 > x/p - s) \int_s^\infty P_{\overline{\theta}}\left(\overline{X}_1 > u\right) du ds + o\left(p^{\alpha+1}\right). \quad (27)$$

A parallel argument to that given for I_3 shows that

$$\frac{1}{E_{\overline{\theta}} \overline{X}_1^2} \int_{[0,x/p)} e^{\overline{\theta}(x/p-s)} P(X_1 > x/p - s) \int_s^\infty P_{\overline{\theta}}\left(\overline{X}_1 > u\right) du ds = o(p^\alpha),$$

which yields

$$I_2 = o(p^\alpha).$$

Finally, we analyze I_1

$$
\begin{aligned}
& I_1 + o\left(p^{\alpha+1}\right) \\
= \; & \frac{1}{E_{\overline{\theta}}\overline{X}_1}\int_0^{x/p} e^{\overline{\theta}u} P\left(X_1 > u\right) du = \frac{1}{\overline{\theta}E_{\overline{\theta}}\overline{X}_1}\int_0^{x/p} P\left(X_1 \geq u\right) de^{\overline{\theta}u} \\
= \; & o\left(p^{\alpha+1}\right) - \frac{1}{\overline{\theta}E_{\overline{\theta}}\overline{X}_1} + \frac{1}{\overline{\theta}E_{\overline{\theta}}\overline{X}_1}\int_0^{x/p} e^{\overline{\theta}u} P\left(X_1 \in du\right) \\
= \; & \frac{\left(1 - E\exp\left(\overline{\theta}X_1\right)\right)}{\overline{\theta}E\left(\exp\left(\overline{\theta}X_1\right)\overline{X}_1\right)} + o\left(p^{\alpha+1}\right).
\end{aligned}
$$

Lastly, the implicit function theorem yields that

$$
\frac{p}{q^2\overline{\theta}\overline{\phi}'\left(\overline{\theta}\right)} = \frac{\left(1 - E\exp\left(\overline{\theta}X_1\right)\right)}{q\overline{\theta}E\left(\exp\left(\overline{\theta}X_1\right)\overline{X}_1\right)} = \exp\left(\sum_{k\leq\alpha} p^k\xi_k\right) + o\left(p^{\alpha}\right),
$$

and

$$
\overline{\theta} = \sum_{k\leq\alpha} p^k\gamma_k + o\left(p^{\alpha}\right).
$$

This concludes the proof of the theorem.

Remark Note that (26) and 27) combined with Lemma 1 indicate that, in principle, it is possible to develop an approximation for $P\left(pS_M > x\right)$ up to an error of order $o\left(p^{\alpha+1}\right)$ given by

$$
P\left(pS_M > x\right) \approx \exp\left(-x\overline{\theta}/p\right)\left(\frac{p}{q^2\overline{\theta}\overline{\phi}'\left(\overline{\theta}\right)} + I_2\right).
$$

This approximation, however, involves computing explicitly I_2 and $\overline{\theta}$ which may be cumbersome in practice.

As we indicated at the beginning of this chapter, in many applications settings the increment distributions are actually changing with p. In this context, it is desirable to develop approximations similar to those provided in the previous theorems. Fortunately, Theorem 1 also provides a means to deal with the typical situations that arise in practice. To fix ideas, consider a family of probability measures $\mathcal{P} = \{P_p, \, p \in [0, \overline{\delta}]$

for some $\overline{\delta} > 0\}$. Suppose that, under each P_p, the random variables $(X_k : k \geq 1)$ form an iid sequence. Also, assume that the distribution of X_1 is uniformly strongly non-lattice with respect to $\mathcal{P}$ (i.e. the characteristic functions $g_p(\lambda) = E_p \exp(i\theta X_1)$ satisfy condition (9)). In addition, suppose that one of the following conditions hold:

A) for some $\eta > 0$, $\sup_{0 \leq p \leq \overline{\delta}} E_p \exp(\eta X_1) < \infty$ or

B) $\sup_{0 \leq p \leq \overline{\delta}} E_p X_1^{2+\alpha} < \infty$, for some $\alpha \geq 0$.

Under this set of assumptions, we have the following analogue to Theorems 2 and 3.

Theorem 4 *Assume that the family P_p, $p \in [0, \overline{\delta}]$ is uniformly strongly non-lattice (see equation (9)). If condition A) above holds, then, there exists constants $K_1, K_2 > 0$ such that for $p > 0$ small*

$$|P_p(pS_M > x) - \exp(-\theta^* x/p + r_p(p))| \leq K_1 \exp(-K_2 x/p),$$

where $\theta^* = \theta^*(p)$ solves $\phi_p(\theta^*) \triangleq E_p \exp(\theta^* X_1) = 1/q$ and

$$\exp(r_p(p)) = \frac{p}{q^2 \theta^* \phi_p'(\theta^*)}.$$

Moreover, $\theta^*(p) = \sum_{k=1}^{\infty} p^k \gamma_k(p)$ and $r_p(p) = \sum_{k=1}^{\infty} p^k \xi_k(p)$ (where the $\gamma_k(p)$'s and $\xi_k(p)$'s depend on the first k and $(k+1)$ moments of X_1 under P_p respectively). Finally, if condition B) is in force, then

$$\left| P_p(pS_M > x) - \exp\left(-x/E_p X_1 + \sum_{k \leq \alpha} p^k \left(\xi_k - \gamma_{k+1} x\right)\right) \right| = o(p^\alpha).$$

Proof. The proof parallels the arguments given in Theorems 2 and 3 using Theorem 1. The details are omitted.

Remark Note that the γ_k's and the ξ_k's also depend on p. The previous result would yield the desired asymptotic expansion assuming that the problem at hand has enough structure, so that an asymptotic expansion of ξ_k's and γ_k's can be obtained. The expansion for the distribution of the all time maximum of a random walk with small negative drift given in Chapter 1 of this dissertation, provides an example in which the previous result would have been applicable.

4.2 Asymptotics of Defective Renewal Equations

As we discussed at the beginning of the chapter, in many applied probability settings one often deals with defective renewal equations, which are integral equations that can be written as

$$a(t) = b(t) + q\int_{[0,t)} a(t-s)\, P(X_1 \in ds),$$

where $q = 1-p \in (0,1)$ and b is a given function for which we shall assume certain regularity properties (see Theorem 5). As an application of our developments in Section 2, we provide means to obtain asymptotic expansion for $a(\cdot)$ as $p \searrow 0$.

Theorem 5 *Suppose that the distribution of X_1 is strongly non-lattice and that $EX_1^{2+\alpha} < \infty$. In addition, suppose that b is right-continuous with left limits, has finite variation and $|b|(t) \le g(t)$ with $\int_0^\infty t^{\alpha+1} g(t) < \infty$. Finally, let us write, for $j \le \alpha+1$, $b_j = \int_0^\infty t^j b(t)\, dt$. Then, as $p \searrow 0$*

$$a(t/p) = \exp\left(-t\widehat{\theta}_\alpha/p\right) d(p) + o(p^\alpha),$$

where

$$d(p) = \frac{b_0 + \sum_{k\le\alpha} b_k \widehat{\theta}_\alpha^k/k!}{q\left(EX_1 + \sum_{k\le\alpha} \widehat{\theta}_\alpha^k EX_1^{k+1}/k!\right)}.$$

Proof. First we note that if $\overline{a}(\cdot)$ satisfies

$$\overline{a}(t) = b(t) + q\int_{[0,t)} \overline{a}(t-s)\, P\left(\overline{X}_1 \in ds\right), \tag{28}$$

where $\overline{X}_1 = X_1 1(X_1 < 1/p)$, then, by applying Laplace transforms we can verify that $\overline{a}$ satisfies (see Theorem 9.1.1 of Lin and Willmot (2000))

$$\overline{a}(t) = \frac{1}{p}\int_{[0,t)} b(t-s)\, P\left(\overline{S}_M \in ds\right).$$

Therefore

$$\begin{aligned}
&\overline{a}\left(t/p\right)-a\left(t/p\right)\\
=\ &\frac{1}{p}\int_{[0,t/p)} b\left(t/p-s\right)\left(P\left(\overline{S}_M\in ds\right)-P\left(S_M\in ds\right)\right)\\
=\ &\frac{1}{p}\int_{[0,t/2p)} b\left(t/p-s\right)\left(P\left(\overline{S}_M\in ds\right)-P\left(S_M\in ds\right)\right) \qquad (29)\\
&+\frac{1}{p}\int_{[t/2p,t/p)} b\left(t/p-s\right)\left(P\left(\overline{S}_M\in ds\right)-P\left(S_M\in ds\right)\right). \qquad (30)
\end{aligned}$$

Let J_1 and J_2 be the integrals in (29) and (30) respectively. Now, since $b\left(t\right)=o\left(t^{\alpha+2}\right)$ and is right continuous with left limits, it is not hard to see that

$$\max_{1/2\leq u\leq 1}\left|b\left(ut/p\right)\right|=o\left(p^{\alpha+2}\right).$$

Thus, it follows that

$$\frac{1}{p}\left|\int_{[0,t/2p)} b\left(t/p-s\right)P\left(\overline{S}_M\in ds\right)\right|\leq\frac{1}{p}\max_{1/2\leq u\leq 1}\left|b\left(ut/p\right)\right|=o\left(p^{\alpha+1}\right).$$

Which implies that $J_1=o\left(p^{\alpha}\right)$. For J_2 we can use integration by parts to obtain

$$\begin{aligned}
\left|J_2\right| =\ &\frac{b\left(0\right)}{p}\left(P\left(p\overline{S}_M\leq t\right)-P\left(pS_M\leq t\right)\right)\\
&+\left|\frac{b\left(t/p\right)}{p}\right|\left(P\left(p\overline{S}_M\leq t/2\right)-P\left(pS_M\leq t/2\right)\right)\\
&+\int_{[1/2,1)}\frac{1}{p}\left|P\left(\overline{S}_M\leq st/p\right)-P\left(S_M\leq st/p\right)\right|\left|b\right|\left(t/p-ds\right).
\end{aligned}$$

From Lemma 1 and the fact that $\int_{[0,\infty)}\left|b\right|\left(ds\right)<\infty$ we can easily obtain that $J_2=o\left(p^{\alpha+1}\right)$. The rest of the argument follows just as in the proof of Theorem 3, by finding a root for the equation $E\exp\left(\overline{\theta X}_1\right)=1/q$, transforming (28) into a non-defective renewal equation and applying Theorem 1.

As a final remark we note that a straightforward generalization of the previous theorem can be obtained in a completely analogous setting as the one described in Theorem 4. As an application of the previous results we consider a couple of examples in insurance risk theory and queueing theory.

Example 1 (Perturbed ruin model) *Consider the case of the classical ruin model perturbed by a diffusion introduced by Dufresne and Gerber (1991). That is, suppose that the risk process is a Levy process of the form*

$$R(t) = x + ct - S(t) + \sigma B(t); \quad t \geq 0,$$

where $S(\cdot)$ represents the aggregate claim process, which follows a compound Poisson process with Poisson parameter λ and increments (claims) $Y = (Y_k : k \geq 1)$; x represents the initial reserve, c is a constant premium rate satisfaying $c > \lambda EY$, and $\sigma B(\cdot)$ is a Brownian motion independent of S with diffusion coefficient equal to σ (i.e. instantaneous variance equal to σ^2). The term involving the Brownian motion B, represents noise that may incorporate non-systematic fluctuations in the composition of the insurance portfolio, measurement errors, etc. We are interested in computing the probability of eventual ruin in this model. Note that this model cannot be reduced directly to the standard renewal model discussed at the beginning of the chapter because, in this case, ruin can occur between claim arrivals. In order to apply Theorem 5, let us introduce some additional notation. Let Z be a rv having the equilibrium distribution generated by Y, that is

$$P(Z \leq z) = \frac{1}{EY} \int_0^z P(Y > y)\, dy.$$

Also, define $p = 1 - \lambda EY/c$ and $q = 1 - p$, and $V = Z + \sigma^2 W/(2c)$, where W is distributed exponential with mean one and Z and W are independent. Finally, let $\tau(x) = \inf(t \geq 0 : R(t) < x)$, and note that the ruin occurs if and only if $\{\tau(x) < \infty\}$. Dufresne and Gerber (1991) proved that if $P(\tau(x) < \infty) = a(x)$, then

$$\begin{aligned} a(x) &= qP(V > x) + pP\left(W > 2cx/\sigma^2\right) \\ &\quad + q \int_0^x a(x - y)\, P(V \in dy). \end{aligned}$$

In this context, p close to zero and x large are reasonable assumptions, hence we can use Theorem 5 can be directly applied here to provide asymptotics for $a(x/p)$ as $p \to 0$. In particular, for $j \geq 0$, it is easy to verify that

$$b_j = \frac{1}{j+1} \left(qEV^{j+1} + q\left(\sigma^2/(2c)\right)^{j+1} (j+1)! \right),$$

and that

$$EZ^j = \frac{EZ^j}{(j+1)\,EZ}, \quad EW^j = j!.$$

These expressions, combined with Proposition 2 and Theorem 5 provide all the necessary means to compute the desired asymptotic expansion. For instance, assuming that $EZ^4 < \infty$ *(which implies that* $EY^3 < \infty$*), we obtain that*

$$a\,(x/p) = \exp\left(-x/EV + 1/2\left(1 - EV^2/\left(2E^2V\right)\right)p\right)d\,(p) + o\,(p)\,,$$

where

$$d\,(p) = \frac{(qEV + p\sigma^2/\,(2c)) + (qEV^2 + \sigma^4/\,(2c^2))\,p/EV}{q\,(EV + pEV^2/EV)},$$

and

$$\begin{aligned} EV &= EY^2/\,(2EY) + \sigma^2/\,(2c) \\ EV^2 &= EY^3/\,(3EY) + \sigma^4/\left(2c^2\right) + \sigma^2 EY^2/\,(2cEY)\,. \end{aligned}$$

Note that these asymptotics correspond to corrected diffusion approximations for the present model.

Example 2 (M/G/c waiting time) *A standard (first-come first-served) M/G/c queue can be described as follows. Customers arrive according to a Poisson process with rate* λ*. The system is composed by* c *servers and a buffer of infinite size. The amount of time required by service of each customer is described by a sequence* $V = (V_j : j \geq 1)$ *of non-negative iid random variables independent of the arrival process. The stability condition of this queue requires less than 100% utilization, which is expressed via* $\rho = \lambda EV/c < 1$*. Just as in the standard M/G/1 case, the so-called equilibrium distribution of the service time sequence, namely* $H\,(t) = \int_0^t P\,(V > s)\,ds/EV$*, plays an important role in discribing the steady-state waiting time distribution, say* $W = (W_n : n \geq 1)$*, (excluding service) of this queueing system. In particular (see Van Hoorn (1984)), it turns out that if* $a\,(t) = P\,(W_\infty > t|\,W_\infty > 0)$*, then*

$$\begin{aligned} a\,(t) &= (1-\rho)\,(1 - H\,(t))^c \\ &\quad + \rho\,(1 - H\,(tc)) + \rho\int_0^t a\,(t-s)\,dH\,(sc)\,, \end{aligned}$$

and

$$P\left(W_{\infty}>0\right)=\frac{\left(\lambda EV\right)^{c-1}}{(c-1)!}\frac{\rho}{\left(1-\rho\right)\sum_{j=0}^{c-1}\left(\lambda EV\right)^{j}/j!+\left(\lambda EV\right)^{c}/\left(c!\right)}.$$

Therefore, as a straightforward application of Theorem 5, we can develop corrected diffusion approximations (in the spirit of Chapter 2 of this dissertation) for the steady-state waiting time of the M/G/c queue.

Chapter 5

Approximations for the Distribution of Infinite Horizon Discounted Rewards

For $t \geq 0$, let $\Lambda(t)$ be a real-valued random variable representing the cumulative reward associated with running a system over $[0, t]$. In the presence of a stochastic inflation rate, the infinite horizon discounted reward takes the form

$$D = \int_{[0,\infty)} \exp\left(-\Gamma\left(t_{-}\right)\right) d\Lambda\left(t\right),$$

where $\Gamma = (\Gamma(t) : t \geq 0)$ is a real-valued process representing the cumulative inflation to time t. An enormous literature exists within the performance modeling and stochastic control communities that focuses on computing and/or optimizing the expected infinite-horizon discounted reward, namely $E(D)$. Our focus, in this paper, is on the development of approximations for the distribution of the random variable (r.v.) D (and not just its expected value).

As we shall see in Section 2, the distribution of the random variable D plays a key role in a number of different applications contexts. Since, clearly, computing the exact distribution of D is, in general, very difficult, the emphasis in this paper is on the development of approximations. All of our approximations are rigorously supported by limit theorems that are valid in the asymptotic regime in which the "inflation rate" is small.

Study of approximations for the distribution of D can be traced back to the early seventies. Gerber (1971) established a Central Limit Theorem (CLT), as well as its Berry-Esséen companion, for

$$D = \sum_{k=0}^{\infty} \exp\left(-\alpha k\right) X_k,$$

in the case of a (small) deterministic discount rate α and iid rewards $\left(X_k\right)_{k\geq 0}$. Whitt (1972) obtained more general central limit theorems for D, also under the assumption of deterministic interest rates. The aim of Whitt's paper was to establish discounted stochastic limit theorems based on postulating a functional limit theorem for the (undiscounted) reward process (in our notation, Λ).

The stochastic discount rate has also been widely studied. Pollack and Siegmund (1985) computed the distribution of D in the case in which Γ follows a Brownian motion with negative drift and $\Lambda\left(t\right) = t$; see also Dufresne (1990). The distribution of D has also been computed explicitly by Gjessing and Paulsen (1997) in some other particular cases in which both Γ and Λ follow particular types of Levy processes) . Computing the distribution of D in complete generality is clearly unfeasible. And even in Markovian settings, such as those previously described, the type of integro-differential equations that arise (see Gjessing and Paulsen (1997) and Yor (2001)) are challenging to solve both analytically and numerically. Hence, our goal is to provide approximations to D that hold in great generality and require relatively "easy-to-obtain" information for their implementation.

It is important to recognize that D arises as the stationary distribution of certain processes that have been well studied in the context of time series analysis (such as AR and ARCH processes). By properly scaling certain types of auto-regressive processes, Nelson (1990) obtained sample-path weak convergence results a Gaussian Ornstein-Uhlenbech process as the sample frequency increases. More recently Forniari and Mele (1997) extended Nelson's results to cover more general type of non-linear ARCH and GARCH time series models. From the time series analysis perspective, the central limit theorem (CLT) derived here in Section 4 is related to the convergence of the stationary distributions of auto-regressive type models to that of Ornstein-Uhlenbeck (namely a Gaussian law). One of the contributions of Whitt (1972) is to show that weak convergence of properly scaled processes Γ and Λ in the standard Skorohod topology is not enough to guarantee weak convergence of a suitably scaled and normalized version of D. Thus, the general weak convergence analysis at the level of stationary distributions in auto-regressive processes does not follow directly from previous results in the literature. Our laws of large numbers (LLNs) and CLTs derived in Section 4 hold in great generality (in particular, in the cases considered by Nelson (1990) and Forniari and Mele (1997)), hence our results complement previous analysis on the structure of auto-regressive processes.

Some results similar in spirit to our results in Section 4 have been obtained by Kushner (1984) and Benveniste, Metiver, and Priouret, (1990) in the context of stochastic approximation algorithms, more precisely the so-called least mean squares (LMS) algorithm, which gives rise to a linear stochastic recursive equation of order one. Although these results hold in the vector valued case, the dependence assumptions imposed are stronger than ours and are only given in discrete time, which is not convenient for some of the applications discussed in the next section (e.g. finance and insurance). Also in the context of stochastic approximation algorithms, Bucklew, Kurtz, and Sethares (1993) analyzed weak convergence (on compact sets) of processes following certain stochastic recursive equations that give rise to stationary distributions related to D. As in the previous discussion regarding the time series setting, this type of analysis does not directly imply weak convergence of stationary distributions.

In this paper, we not only complement previous results in the literature (such as those discussed for in the context of time series analysis) by providing rigorous general statements that support LLNs and CLTs at the level of stationary distributions, but also provide new approximations and further refinements for the LLN's and CLT's previously indicated. The approximations proposed take the form of Edgeworth expansions and large deviation principles (LDP's), and can typically be implemented at a modest computational cost (see, for example, (7) and (12)). The assumptions under which these results are derived are stated at the beginning of the corresponding sections.

The rest of the paper is organized as follows. As we indicated before our first approximation takes the form of a law of large numbers (LLN) and is derived in Section 3. In Section 4, we provide a central limit theorem (CLT) correction to the LLN derived in Section 3. The approximations developed in Sections 3 and 4 are shown to be valid under very general settings. Under additional assumptions, refinements to the CLT are introduced in Section 5. These refined approximations take the form of Edgeworth expansions and are provided in both the discrete and the continuous time settings. Finally, under exponential tail conditions on Λ and Γ, general large deviation principles (LDP's) are given in Section 6. In this section also, sharp large deviation asymptotics are discussed as well.

5.1 Motivating Examples

The distribution of D plays a key role in a number of different applications contexts. In the world of finance and pension funds, D is called a "perpetuity" (see Embrechts, Klüppelberg, and Mikosch (1997)). As an example of how D arises in pension funds we mention Dufresne (1990), who proposed a model, based on perpetuities, for computing the value of a pension fund. He argued that the value at time t can be expressed as

$$V(t) = \int_{-\infty}^{t} \exp\left(-\int_{s}^{t} \gamma(u)\,du\right) \lambda(s)\,ds,$$

where $(\gamma(t), \lambda(t) : -\infty < t < \infty)$ is stationary and ergodic, with $0 < E\gamma(t) < \infty$ and $E\log(1 + |\lambda(t)|) < \infty$. The processes $\gamma(t)$ and $\lambda(t)$ depend on the parameters that serve to characterize the pension fund (i.e. benefit payments, actuarial liability, net premium, and rate of return). As explained in Dufresne (1990), the distribution of the value process plays an important role in risk management, as it serves to compute critical rates ensuring that the fund is being managed in a balanced manner with respect to its actuarial liabilities; see Dufresne (1990) and Bédard and Dufresne (2001) for additional detail on pension funding. The random variable $V(t)$ can be recast as a special case of D, so that our results apply directly.

The random variable D also arises in non-pension fund insurance settings. Consider a company that receives premiums at a rate of p dollars per unit time, and pays out claims according to the random process $A(\cdot)$. If $\gamma(t)$ represents the rate of return on the invested risk reserve at time t, the risk reserve $R(t)$ evolves according to the equation

$$dR(t) = \gamma(t) R(t) dt + (pdt - dA(t)),$$

subject to the initial condition $R(0) = r_0$. Harrison (1977) shows that the ruin probability $P(\inf_{t\geq 0} R_t < 0)$ can be computed in terms of D (for Λ and Γ suitably defined) when γ is deterministic. Paulsen (1998) extends this result to the case of stochastic $\gamma(\cdot)$; see also Nyrhinen (2001). Thus, the key to calculating such ruin probabilities is computing the distribution of D.

As we mentioned before, it turns out that D also plays a major role in the theory of ARCH processes. This class of time series is widely used within the statistics and econometrics communities, and has been employed to model log-returns, exchange rates, inflation, and many other financial and economic time series; see Campbell, Lo and Mackinlay (1999), Shephard (1996), Mills (1993) and Wilkie (1986). An ARCH(1) model satisfies the stochastic recursion

$$Y_{n+1} = A_n + B_{n+1} Y_n,$$

where the sequence $((A_i, B_i) : i \geq 1)$ is iid (independent and identically distributed.) Under mild stability conditions (see, for example, Kesten (1973), Verbaat (1979), Goldie (1991), Embrechts and Goldie (1994)), this Markov chain has a stationary distribution. This stationary distribution is a special case of D.

We also note several other applications settings in which the distribution of D arises as a central object. Goldie and Grübel (1996) describe its relevance to complexity theory (in the context of sorting algorithms related to "Quicksort") and analytic number theory. Carmona, Petit, and Yor (2001) describe several other applications arising in mathematical physics and finance.

Apart from the financial Whitt (1972) also reports two application contexts in which our limit theorems may have potential important implications. and, second, the dynamic programming context, in which the approximations derived may be used in developing stochastic criteria and sensitivity analysis for small interest rates, see Whitt (1972) for additional detail.

As stated earlier, our work is intended to provide approximations to the distribution of D. Of particular importance (in view of the above applications) is the setting in which the "interest rate" corresponding to Γ is small. Our theorems establish that our approximations become asymptotically exact as the "interest rate" goes to zero.

5.2 Law of Large Numbers

We assume throughout this chapter (except in some cases explicitly indicated) that for each $t \geq 0$, $(\Lambda(s) : 0 \leq s \leq t)$ is of bounded variation. We also require Λ and $(\Gamma(s) : s \geq 0)$ to be right continuous functions with left limits (RCLL). (Note that we do not require bounded variation for Γ.) Let $|\Lambda|(t)$ be the total variation of Λ over $[0, t]$, and suppose that $|\Lambda|$ satisfies:

$$\overline{\lim}_{t\to\infty} \frac{1}{t} |\Lambda|(t) < \infty \quad a.s. \tag{1}$$

We further assume that:

A1 There exist deterministic constants $\lambda \in R$ and $\gamma \in (0, \infty)$ such that:

$$\begin{aligned} \Gamma(t) &= \gamma t + o_p(t) \\ \Lambda(t) &= \lambda t + o_p(t), \end{aligned}$$

where $o_p(t)$ means that for every $c > 0$,

$$\sup_{0 \leq t \leq c} \left| \frac{o_p(t\beta)}{\beta} \right| = o(1) \text{ as } \beta \to \infty.$$

Our first proposed approximation for D takes the form

$$D \overset{D}{\approx} \lambda/\gamma. \tag{2}$$

Here, $\overset{D}{\approx}$ means "has approximately the same distribution as", and is intended to hold no rigorous mathematical meaning. The relation (2) should be merely interpreted as a statement of a proposed approximation.

Of course, given such an approximation, it is important to identify conditions under which the approximation can be guaranteed to be good. We shall argue that the approximation (2) tends to be good when the discount rate γ is small. To make this statement mathematically rigorous, we shall introduce a parameter α that will control the magnitude of the discount rate. We will show that as $\alpha \searrow 0$, the approximation (2) becomes asymptotically valid. More precisely, let

$$D(\alpha) = \int_{[0,\infty)} \exp(-\alpha\Gamma(t_-))\,\Lambda(dt)\,.$$

For $D(\alpha)$, the approximation (2) takes the form

$$D(\alpha) \overset{D}{\approx} \lambda/\alpha\gamma. \tag{3}$$

The following theorem shows that the approximation (3) becomes accurate as $\alpha \searrow 0$.

Theorem 1 *Under A*1*,*

$$\alpha D(\alpha) \to \frac{\lambda}{\gamma} \quad a.s. \ \ as\ \alpha \searrow 0.$$

Note that the "law of large numbers" (LLN) offered by Theorem 1 does not assume that the instantaneous discount rate is non-negative (i.e. that Γ is non-decreasing). The lack of such an assumption introduces some technical complications in our proofs. We prove (3) by replacing Γ with the non-decreasing function

$$\overline{\Gamma}(t) \triangleq \sup\{\Gamma(s) : 0 \le s \le t\}.$$

The hope is that

$$\overline{D}(\alpha) = \int_{[0,\infty)} \exp\left(-\alpha\overline{\Gamma}(t_-)\right)\Lambda(dt)$$

then has a behavior similar to that of $D(\alpha)$ when α is small. Theorem 1 can be established by proving the corresponding law of large numbers for $\overline{D}(\alpha)$. Thus, the proof of Theorem 1 follows from Propositions 1 and 2 below.

Proposition 1 *Assume A*1*. Then,*

$$\alpha\left(D(\alpha) - \overline{D}(\alpha)\right) \to 0 \quad a.s.$$

as $\alpha \searrow 0$.

Proof. Observe that

$$\begin{aligned} D(\alpha) &= \int_0^\infty \exp(-\alpha\Gamma(t_-))\Lambda(dt) \\ &= \int_0^\infty \int_{\alpha\Gamma(t_-)}^\infty \exp(-u)du\Lambda(dt) \\ &= \int_0^\infty \left(\int_{\alpha\overline{\Gamma}(t_-)}^\infty \exp(-u)du + \int_{\alpha\Gamma(t_-)}^{\alpha\Gamma(t_-)} \exp(-u)du \right) \Lambda(dt) \\ &= \int_0^\infty \int_{\alpha\overline{\Gamma}(t_-)}^\infty \exp(-u)du\Lambda(dt) + \int_{\alpha\Gamma(t_-)}^{\alpha\overline{\Gamma}(t_-)} \exp(-u)du\Lambda(dt) \\ &= \overline{D}(\alpha) + \int_0^\infty \int_{\alpha\Gamma(t_-)}^{\alpha\overline{\Gamma}(t_-)} \exp(-u)du\Lambda(dt). \end{aligned}$$

Therefore, we must show that

$$\left| \alpha \int_0^\infty \int_{\alpha\Gamma(t_-)}^{\alpha\overline{\Gamma}(t_-)} \exp(-u)du\Lambda(dt) \right| \to 0 \;\; a.s. \;\; as \;\; \alpha \searrow 0,$$

but

$$\left| \alpha \int_0^\infty \int_{\alpha\Gamma(t_-)}^{\alpha\overline{\Gamma}(t_-)} \exp(-u)du\Lambda(dt) \right| \leq \alpha \int_0^\infty \int_{\alpha\Gamma(t_-)}^{\alpha\overline{\Gamma}(t_-)} \exp(-u)du\,|\Lambda|(dt).$$

Now the right hand side of the last inequality is equal to

$$\begin{aligned} &\int_0^\infty \alpha \int_0^{\alpha\overline{\Gamma}(t/\alpha-)-\alpha\Gamma(t/\alpha-)} \exp(-u-\alpha\Gamma(t/\alpha))du\,|\Lambda|\left(\frac{dt}{\alpha}\right) \\ =& \int_0^\infty \alpha\exp\left(-\gamma t+\alpha o_p(t/\alpha)\right) \int_0^{\alpha\overline{\Gamma}(t/\alpha-)-\alpha\Gamma(t/\alpha-)} \exp(-u)du\,|\Lambda|\left(\frac{dt}{\alpha}\right) \\ =& \int_0^\infty h(t,\alpha)\,\mu_a(dt). \end{aligned}$$

Where,

$$h(t,\alpha) = \exp\left(-\frac{\gamma}{2}t+\alpha o_p(t/\alpha)\right) \int_0^{\alpha\overline{\Gamma}(t/\alpha-)-\alpha\Gamma(t/\alpha-)} \exp(-u)du,$$

and

$$\mu_a(dt) = \alpha\exp\left(-\frac{\gamma}{2}t\right)|\Lambda|\left(\frac{dt}{\alpha}\right).$$

Observe that

$$0 \leq \sup_{\alpha, t \geq 0} h\left(t, \alpha\right) \leq M.$$

Moreover, it is not hard to verify that $h\left(t, \alpha\right) \to 0$ as $\alpha \to 0$ uniformly on compact intervals. Also, notice that

$$\begin{aligned}
\mu_a(0, \infty) &= \int_0^\infty \alpha \exp\left(-\frac{\gamma}{2}t\right) |\Lambda| \left(\frac{dt}{\alpha}\right) \\
&= \int_0^\infty \alpha \exp\left(-\frac{\gamma}{2}t\alpha\right) |\Lambda| \left(dt\right) \\
&= \int_0^\infty e^{-u} \alpha \left|\Lambda\right| \left(\frac{2u}{\alpha\gamma}\right) du \\
&= \int_0^\infty e^{-u/2} e^{-u/2} \alpha \left|\Lambda\right| \left(\frac{2u}{\alpha\gamma}\right) du.
\end{aligned}$$

Since $|\Lambda|$ satisfies (1), we have for some $B > 0$,

$$0 \leq \int_0^\infty e^{-u/2} \alpha \left|\Lambda\right| \left(\frac{2u}{\alpha\gamma}\right) du \leq B.$$

Thus, we have that for all $\alpha > 0$,

$$0 \leq \mu_a(0, \infty) \leq B < \infty.$$

And, if we fix $\varepsilon > 0$, then there exists $C > 0$ such that $\mu_a(C, \infty) \leq \varepsilon$, and such that

$$\sup_{t \in [0, C]} h\left(t, \alpha\right) \leq \varepsilon,$$

for α small enough, this implies that if α is small,

$$\int_0^\infty h\left(t, \alpha\right) \mu_a\left(dt\right) \leq \varepsilon(B + M),$$

since ε was arbitrary, we deduce that

$$\left| \alpha \int_0^\infty \int_{\alpha\Gamma(t_-)}^{\alpha\overline{\Gamma}(t_-)} \exp(-u) du \Lambda\left(dt\right) \right| \leq \int_0^\infty h\left(t, \alpha\right) \mu_a\left(dt\right) \to 0.$$

as claimed.

To prove Proposition 2, we need to recall the following definition of the generalized inverse of a non-decreasing function.

Definition 1 *Let* $\Gamma : \mathbb{R}_+ \to \mathbb{R}$ *non-decreasing, RCLL (right continuous with left limits) then we define* Γ^{-1} *as*

$$\Gamma^{-1}(u) = \inf\{t \geq 0 : \Gamma(t) > u\}.$$

Proposition 2 *Assume* $A1$*. Then,*

$$\alpha \overline{D}(\alpha) \to \lambda/\gamma \quad a.s.$$

as $\alpha \searrow 0$.

Proof.

$$\begin{aligned}
\alpha \overline{D}(\alpha) &= \alpha \int_0^\infty \exp(-\alpha \overline{\Gamma}(t_-)) \Lambda(dt) \\
&= \alpha \int_0^\infty \int_{\overline{\Gamma}(t_-)}^\infty \exp(-u) du \Lambda(dt) \\
&= \alpha \int_0^\infty \exp(-u) \Lambda\left(\overline{\Gamma}^{-1}(u/\alpha)\right) du.
\end{aligned}$$

Now,

$$\frac{t}{\overline{\Gamma}^{-1}(t_-)} = \frac{\overline{\Gamma}\left(\overline{\Gamma}^{-1}(t_-)\right)}{\overline{\Gamma}^{-1}(t_-)} = \gamma + \frac{o_p\left(\overline{\Gamma}^{-1}(t_-)\right)}{\overline{\Gamma}^{-1}(t_-)}.$$

Hence

$$\Gamma^{-1}(t) = \frac{t}{\gamma} + o_p(t) \quad a.s.$$

This implies that

$$\alpha \Lambda\left(\Gamma^{-1}(u/\alpha)\right) = \alpha \Gamma^{-1}(u/\alpha) \frac{\Lambda\left(\Gamma^{-1}(u/\alpha)\right)}{\Gamma^{-1}(u/\alpha)} \to u \frac{\lambda}{\gamma}.$$

Thus, in order to apply the Dominated Convergence Theorem, it suffices to show that for almost every sample path, we have that

$$\left|\alpha \Lambda\left(\Gamma^{-1}(u/\alpha)\right)\right| \leq H(u, \omega) \in L^1\left(e^{-u} du\right),$$

for some measurable function H. However,

$$\left|\alpha \Lambda\left(\Gamma^{-1}(u/\alpha)\right)\right| = O(u)$$

which suffices to apply dominated convergence.

5.3 The Central Limit Theorem

In this section, we assume that Λ and Γ jointly satisfy a "strong approximation principle", namely:

A2 There exists a probability space supporting (Λ, Γ) and a two-dimensional standard Brownian motion

$$(B_1, B_2) = ((B_1(t), B_2(t)) : t \geq 0)$$

for which deterministic constants $\lambda \in (-\infty, \infty)$ and $\gamma \in (0, \infty)$ and a covariance matrix Σ can be found such that

$$\begin{pmatrix} \Gamma(t) \\ \Lambda(t) \end{pmatrix} = \begin{pmatrix} \gamma \\ \lambda \end{pmatrix} t + G \begin{pmatrix} B_1(t) \\ B_2(t) \end{pmatrix} + o_p\left(t^{1/2}\right)$$

$a.s.$ as $t \to \infty$.

The entries of the covariance matrix $C = GG^T$ can typically be identified as follows:

$$\begin{aligned} C_{11} &= \lim_{t\to\infty} \frac{1}{t} E(\Lambda(t) - \lambda t)^2 \\ C_{12} &= \lim_{t\to\infty} \frac{1}{t} E(\Lambda(t) - \lambda t)(\Gamma(t) - \gamma t) = C_{21} \\ C_{22} &= \lim_{t\to\infty} \frac{1}{t} E(\Gamma(t) - \gamma t)^2 . \end{aligned}$$

The strong approximation principle $A2$ holds in great generality, the prototypical example is a sequence of (independent and identically distributed) iid random variables with finite second moment, some other cases, in which dependence is allowed, and under which the validity of this principle has been proved are briefly described (along with references) next. (Some relevant references on this topic are Philipp and Stout (1975) and Csörgo and Révész (1981).)

Case 1 (Philipp and Stout (1975) Thm. 4.1) Suppose $X = (X_n : n \geq 1)$ is strictly stationary sequence of random variables, such that $E\left(|X_1|^{2+\delta}\right) < \infty$ for some $\delta > 0$. Also, assume that X is a $\phi-$mixing with

$$\sum_{k=1}^{\infty} \phi^{1/2}(k) < \infty,$$

then $S(t) = \sum_{k \leq t} X_k$ satisfies a strong approximation principle with rate $o_p\left(t^{1/2-\lambda}\right)$ for some $\lambda > 0$ small.

Case 2 Philipp and Stout's Thm. 8.1, also provides a strong approximation principle with rate $o_p\left(t^{1/2-\lambda}\right)$ (for $\lambda > 0$ small) when $X = (X_n : n \geq 1)$, is not necessarily stationary, but $\phi-$strong mixing with

$$\phi(k) << k^{-300(1+2\delta)},$$

for some $\delta \in (0, 2]$. This result also requires moments of order $2 + \delta$ and other technical assumptions to control the growth of the second moment of the random elements X_k.

Case 3 In the context of positive recurrent irreducible Markov sequence $(\zeta_n)_{n \geq 1}$ with stationary transition probabilities and countable state space, Thm. 10.1 of Philipp and Stout (1975), provides strong approximation principles for the case in which $X_k = f(\zeta_k)$. The results in this case depend on moment conditions of the type described before for the cumulative reward within a cycle of the Markov process.

Case 4 Horvath (1984a, 1984b and 1986) developed strong approximation theorems in the contexts of vector valued cases under higher moment conditions, and also for the cases of renewal processes and extended renewal processes. Also, Philipp and Stout (1975) Ch. 12 deals with various types of continuous parameter stochastic processes (e.g. Gaussian increments and mixing increments, the later case includes as a particular case Levy processes).

Given $A2$, we propose the following (refined) Gaussian approximation for D, namely

$$D \stackrel{D}{\approx} \lambda/\gamma + \sigma/\gamma^{1/2} N(0,1), \tag{4}$$

where

$$\sigma^2 = \frac{1}{2}\left(C_{11} - 2\frac{\lambda}{\gamma}C_{12} + \frac{\lambda^2}{\gamma^2}C_{22}\right).$$

Note that (4) improves upon (2) by providing a normal approximation for the stochastic variability that is present in the $r.v.$ D.

As for the approximation (2), we claim that (4) is accurate when the discount rate is small. In particular, note that (4) suggests the approximation

$$D(\alpha) \overset{D}{\approx} \lambda/\alpha\gamma + \sigma/\sqrt{\alpha\gamma} N(0,1), \tag{5}$$

where

$$\begin{aligned}\sigma &= \sqrt{\frac{1}{2}\left(C_{11} - 2\frac{\lambda}{\gamma\alpha}\alpha C_{12} + \frac{\lambda^2}{(\gamma\alpha)^2}\alpha^2 C_{22}\right)} \\ &= \sqrt{\frac{1}{2}\left(C_{11} - 2\frac{\lambda}{\gamma}C_{12} + \frac{\lambda^2}{\gamma^2}C_{22}\right)}.\end{aligned}$$

The following central limit theorem (CLT) asserts that the approximation (5) is indeed valid as $\alpha \searrow 0$.

Theorem 2 *If A2 is in force, then*

$$\alpha^{-1/2}\left(\alpha D(\alpha) - \frac{\lambda}{\gamma}\right) \Longrightarrow \sigma N(0,1)$$

as $\alpha \searrow 0$, *where*

$$\sigma^2 = \frac{1}{2\gamma}\left(C_{11} - 2\frac{\lambda}{\gamma}C_{12} + \frac{\lambda^2}{\gamma^2}C_{22}\right).$$

Again, just as in the case of the LLN derived previously, the strategy is to show that the behavior of the random variable

$$\overline{D}(\alpha) = \int_{[0,\infty)} \exp\left(-\alpha\overline{\Gamma}(t_-)\right)\Lambda(dt),$$

is comparable to that of $D(\alpha)$ for the purposes of approximation (5). This is the aim of Proposition 3 below, whose proof follows using the same technique as in Proposition 1 together with an application of the next Lemma.

Lemma 1 *Under A2,*

$$\sqrt{\alpha}\left(\overline{\Gamma}(t/\alpha) - \Gamma(t/\alpha)\right) \to 0 \quad as \quad \alpha \to 0$$

uniformly on compact sets.

Proof.

$$\left|\sqrt{\alpha}\left(\overline{\Gamma}(t/\alpha)-\Gamma(t/\alpha)\right)\right| \leq \sqrt{\alpha}\max_{0\leq s\leq t/\alpha}\left(\gamma(s-t/\alpha)+\Sigma_{2\cdot}\left(B(s)-B(t/\alpha)\right)\right)+ \sqrt{\alpha}o_p\left((t/\alpha)^{1/2}\right).$$

Observe that

$$\begin{aligned}
&\sqrt{\alpha}\max_{0\leq s\leq t/\alpha}\left(\gamma(s-t/\alpha)+\Sigma_{2\cdot}\left(B(s)-B(t/\alpha)\right)\right)\\
=\;&\max_{0\leq s\leq t}\left(\gamma\frac{(s-t)}{\sqrt{\alpha}}+\sqrt{\alpha}\Sigma_{2\cdot}\left(B(s/\alpha)-B(t/\alpha)\right)\right)\\
\leq\;&\max_{0\leq s\leq t}\left(C(s-t)^{1/2+\varepsilon}\alpha^{-\varepsilon}-\gamma(s-t)\alpha^{-1/2}\right),\\
\leq\;&\max_{0\leq u\leq t}\left(Cu^{1/2+\varepsilon}\alpha^{-\varepsilon}-\gamma u\alpha^{-1/2}\right)\leq M\alpha^{1/2}\to 0.
\end{aligned}$$

The first inequality holds by virtue of the law of iterated logarithms (LIL) and from the last inequality we can see that the convergence holds uniformly on compact sets.

Proposition 3 *Under A2,*

$$\alpha^{1/2}\left(D(\alpha)-\overline{D}(\alpha)\right)\to 0 \quad a.s.$$

as $\alpha\searrow 0$.

Proof. As in the proof of Proposition 1,

$$D(\alpha)=\overline{D}(\alpha)+\int_0^\infty\int_{\alpha\Gamma(t_-)}^{\alpha\overline{\Gamma}(t_-)}\exp(-u)du\Lambda(dt).$$

Hence, we must show

$$\alpha^{1/2}\left|\int_0^\infty\int_{\alpha\Gamma(t_-)}^{\alpha\overline{\Gamma}(t_-)}\exp(-u)du\Lambda(dt)\right|\to 0 \quad a.s. \quad as \quad \alpha\searrow 0.$$

Now, observe that

$$\begin{aligned}
\left|\alpha\int_0^\infty\int_{\alpha\Gamma(t_-)}^{\alpha\overline{\Gamma}(t_-)}\exp(-u)du\Lambda(dt)\right| &\leq \alpha\int_0^\infty\int_{\alpha\Gamma(t_-)}^{\alpha\overline{\Gamma}(t_-)}\exp(-u)du\left|\Lambda\right|(dt)\\
&= \int_0^\infty h(t,\alpha)\mu_a(dt).
\end{aligned}$$

Where,

$$h(t,\alpha) = \exp\left(-\frac{\gamma}{2}t + \alpha o_p(t/\alpha)\right)\frac{1}{\sqrt{\alpha}}\int_0^{\alpha\overline{\Gamma}(t/\alpha-)-\alpha\Gamma(t/\alpha-)} \exp(-u)du,$$

and

$$\mu_a(dt) = \alpha \exp\left(-\frac{\gamma}{2}t\right)|\Lambda|\left(\frac{dt}{\alpha}\right).$$

Observe that

$$0 \leq h(t,\alpha) \leq M,$$

and virtue of the previous lemma, $h(t,\alpha) \rightarrow 0$ as $\alpha \rightarrow 0$ uniformly on compact intervals. The rest of the proof follows by repeating the same steps used in Proposition 1.

In light of the previous proposition, Theorem 2 follows by combining the last result together with Proposition 4 below. The following lemma will be used in the proof of Proposition 4.

Lemma 2 *Let Σ be a $d-$dimensional vector and $\tau(t)$ such that*

$$\tau(t) = \gamma t + o_p(t)$$

and suppose that $B(t)$ is a $d-$dimensional Brownian motion. Then,

$$\alpha^{1/2}\int_0^\infty e^{-u}\Sigma' B(\tau(u/\alpha))\,du \Rightarrow N\left(0,\sigma^2\right),$$

where $\sigma^2 = \frac{1}{2\gamma}\Sigma'\Sigma'^T$.

Proof.

$$\begin{aligned} & \alpha^{1/2}\int_0^\infty e^{-u}\Sigma' B(\tau(u/\alpha))\,du \\ = \ & \alpha^{1/2}\int_0^{t/\alpha} e^{-u}\Sigma' B(\tau(u/\alpha))\,du + \alpha^{1/2}\int_{t/\alpha}^\infty e^{-u}\Sigma' B(\tau(u/\alpha))\,du \end{aligned}$$

Observe that

$$\begin{aligned}\left|\alpha^{1/2}\int_{t/\alpha}^{\infty}e^{-u}\Sigma' B\left(\tau\left(u/\alpha\right)\right)du\right| &\leq \alpha^{1/2}\int_{t/\alpha}^{\infty}e^{-u}\Sigma'\left|B\left(\tau\left(u/\alpha\right)\right)\right|du\\ &= \alpha^{-\varepsilon}\int_{t/\alpha}^{\infty}Ce^{-u}u^{1/2+\varepsilon}du \quad a.s.,\end{aligned}$$

by virtue of the LIL (for all $\varepsilon > 0$ and for some $C > 0$), and the integral above, goes to zero faster than α^{ε}. Hence, in order to show weak convergence, it suffices to study (by virtue of Slutsky's lemma)

$$\begin{aligned}\alpha^{1/2}\int_{0}^{t/\alpha}e^{-u}\Sigma' B\left(\tau\left(u/\alpha\right)\right)du &= \alpha^{1/2}\int_{0}^{t/\alpha}e^{-u}\Sigma'\left(B\left(\tau\left(u/\alpha\right)\right)-B\left(u/\gamma\alpha\right)\right)du\\ &\quad+\alpha^{1/2}\int_{0}^{t/\alpha}e^{-u}\Sigma' B\left(u/\gamma\alpha\right)du.\end{aligned}$$

We first show that

$$\left|\alpha^{1/2}\int_{0}^{t/\alpha}e^{-u}\Sigma'\left(B\left(\tau\left(u/\alpha\right)\right)-B\left(u/\gamma\alpha\right)\right)du\right| \Longrightarrow 0.$$

Observe that

$$\begin{aligned}\sup_{0\leq u\leq t/\alpha}\alpha e^{-u}\left|\tau\left(u/\alpha\right)-\mu u/\alpha\right| &= \sup_{0\leq u\leq t/\alpha}e^{-u}\left|\alpha o_p\left(u/\alpha\right)\right|\\ &\leq o\left(1\right)\sup_{0\leq u\leq\infty}e^{-u}u\end{aligned}$$

therefore, for every $\delta > 0$ we can choose α_0 such that if $\alpha > \alpha_0$

$$P\left(\sup_{0\leq u\leq t/\alpha}e^{-u}\left|\tau\left(u/\alpha\right)-\mu u/\alpha\right| > \delta/\alpha\right)\leq\delta$$

Let us define

$$A_{\delta}=\{\omega:\sup_{0\leq u\leq t/\alpha}e^{-u}\left|\tau\left(u/\alpha\right)-\mu u/\alpha\right|\leq\delta/\alpha\}$$

and

$$A_{\delta}^{k}=\{\omega:\sup_{k/\alpha\leq u\leq(k+1)/\alpha}e^{-u}\left|\tau\left(u/\alpha\right)-\mu u/\alpha\right|\leq\delta/\alpha\},$$

then

$$
\begin{aligned}
& P\left(\left|\alpha^{1/2}\int_0^{t/\alpha} e^{-u}\Sigma'\left(B\left(\tau\left(u/\alpha\right)\right)-B\left(u/\gamma\alpha\right)\right)du\right|>\varepsilon\right)\\
\leq\ & P\left(\alpha^{1/2}\int_0^{t/\alpha} e^{-u}\Sigma'\left\|\left(B\left(\tau\left(u/\alpha\right)\right)-B\left(u/\gamma\alpha\right)\right)\right\|du>\varepsilon\right)\\
\leq\ & P\left(\alpha^{1/2}\int_0^{t/\alpha} e^{-u}\Sigma'\left\|\left(B\left(\tau\left(u/\alpha\right)\right)-B\left(u/\gamma\alpha\right)\right)\right\|du>\varepsilon; A_\delta\right)+\delta.
\end{aligned}
$$

Notice that

$$
\begin{aligned}
& P\left(\alpha^{1/2}\int_0^{t/\alpha} e^{-u}\Sigma'\left\|\left(B\left(\tau\left(u/\alpha\right)\right)-B\left(u/\gamma\alpha\right)\right)\right\|du>\varepsilon; A_\delta\right)\\
\leq\ & \sum_{k=0}^{\lfloor t\rfloor} P\left(\alpha^{1/2}\int_{k/\alpha}^{(k+1)/\alpha} e^{-u}\Sigma'\left\|\left(B\left(\tau\left(u/\alpha\right)\right)-B\left(u/\gamma\alpha\right)\right)\right\|du>\frac{\varepsilon}{t}; A_\delta\right)\\
\leq\ & \sum_{k=0}^{\lfloor t\rfloor} P\left(\alpha^{1/2}\int_{k/\alpha}^{(k+1)/\alpha} e^{-u}\Sigma'\left\|\left(B\left(\tau\left(u/\alpha\right)\right)-B\left(u/\gamma\alpha\right)\right)\right\|du>\frac{\varepsilon}{t}; A_\delta^k\right).
\end{aligned}
$$

The $kth-$term in the last expression is less or equal than

$$
\begin{aligned}
& P\left(\alpha^{1/2}\int_{k/\alpha}^{(k+1)/\alpha} e^{-u}\Sigma'\sup_{0\leq s\leq\frac{2\delta}{\alpha}e^{k/\alpha}}\left|B\left(s\right)\right|du>\frac{\varepsilon}{2t}\right)\\
=\ & P\left(\int_{k/\alpha}^{(k+1)/\alpha} e^{-u}\sup_{0\leq s\leq 2\delta e^{k/\alpha}}\left|B\left(s\right)\right|du>\frac{\varepsilon}{2t}k\right)\\
\leq\ & \frac{t}{2\varepsilon c}e^{-k/\alpha}\left(1-e^{-1/\alpha}\right)E\left(\sup_{0\leq s\leq 2\delta e^{k/\alpha}}\left|B\left(s\right)\right|\right)\\
=\ & \frac{t}{2\varepsilon c}e^{-k/\alpha}\left(1-e^{-1/\alpha}\right)E\left(\sup_{0\leq u\leq 1}\sqrt{2\delta e^{k/\alpha}}\left|B\left(s\right)\right|\right)\\
=\ & \frac{Mt\sqrt{\delta}}{\varepsilon}e^{-k/2\alpha}\left(1-e^{-1/\alpha}\right).
\end{aligned}
$$

This implies that

$$\begin{aligned} & P\left(\alpha^{1/2}\int_0^{t/\alpha} e^{-u}\Sigma'\left\|\left(B\left(\tau\left(u/\alpha\right)\right)-B\left(u/\gamma\alpha\right)\right)\right\|du>\varepsilon;A_\delta\right) \\ \leq\ & \frac{Mt\sqrt{\delta}}{\varepsilon}\left(1-e^{-1/\alpha}\right)\sum_{k=0}^{\infty}e^{-k/2\alpha} \\ =\ & \frac{Mt\sqrt{\delta}}{\varepsilon}\frac{1-e^{-1/\alpha}}{1-e^{-1/2\alpha}}=\frac{Mt\sqrt{\delta}}{\varepsilon}\frac{1-e^{-1/\alpha}}{1-e^{-1/2\alpha}}. \end{aligned}$$

Therefore,

$$\overline{\lim}_{\alpha\to 0}P\left(\alpha^{1/2}\int_0^{t/\alpha} e^{-u}\Sigma'\left\|\left(B\left(\tau\left(u/\alpha\right)\right)-B\left(u/\gamma\alpha\right)\right)\right\|du>\varepsilon;A_\delta\right)\leq\frac{2Mt\sqrt{\delta}}{\varepsilon}.$$

Since δ was arbitrary, we conclude that

$$\left|\alpha^{1/2}\int_0^{t/\alpha} e^{-u}\Sigma'\left(B\left(\tau\left(u/\alpha\right)\right)-B\left(u/\gamma\alpha\right)\right)du\right|\stackrel{P}{\to}0,$$

in particular the last term goes to zero weakly; finally we observe that

$$\alpha^{1/2}\int_0^{t/\alpha} e^{-u}\Sigma' B\left(u/\gamma\alpha\right)du\stackrel{D}{=}\int_0^{t/\alpha} e^{-u}\Sigma' B\left(u/\gamma\right)du\Longrightarrow\int_0^{\infty} e^{-u}\Sigma' B\left(u/\gamma\right)du,$$

which is Gaussian, with mean zero and variance (which can be computed using integration by parts and the Ito isometry) $\sigma^2=\frac{1}{2\gamma}\Sigma'\Sigma'^T$.

Proposition 4 *If A2 is in force,*

$$\alpha^{-1/2}\left(\alpha\overline{D}\left(\alpha\right)-\frac{\lambda}{\gamma}\right)\Longrightarrow\sigma N\left(0,1\right)\quad as\ \alpha\to 0,$$

where

$$\begin{aligned}\sigma^2 &= \frac{1}{2\gamma}\begin{bmatrix}1 & -\frac{\lambda}{\gamma}\end{bmatrix}C\begin{bmatrix}1\\ -\frac{\lambda}{\gamma}\end{bmatrix} \\ &= \frac{1}{2\gamma}\left(C_{11}-2\frac{\lambda}{\gamma}C_{12}+\frac{\lambda^2}{\gamma^2}C_{22}\right).\end{aligned}$$

Proof. We write

$$\alpha\overline{D}\left(\alpha\right)-\frac{\lambda}{\gamma}=\int_{0}^{\infty}e^{-u}\left(\alpha\Lambda\left(\overline{\Gamma}^{-1}\left(\frac{u_{-}}{\alpha}\right)\right)-\frac{\lambda}{\gamma}\right)du.$$

Let $W\left(\alpha,u\right)=\left(\alpha\Lambda\left(\overline{\Gamma}^{-1}\left(u/\alpha\right)\right)-\frac{\lambda}{\gamma}\right)$, using the strong approximation assumption we can see that:

$$\begin{aligned} W\left(\alpha,u\right) &= \alpha\left(\frac{\lambda}{\gamma}u-\frac{\lambda}{\gamma}\right)-\frac{\alpha\lambda}{\gamma}G_{2\cdot}B\left(\overline{\Gamma}^{-1}\left(\frac{u_{-}}{\alpha}\right)\right)+\alpha G_{1\cdot}B\left(\overline{\Gamma}^{-1}\left(\frac{u_{-}}{\alpha}\right)\right) \\ &\quad +\alpha o_{p}\left(\left(\overline{\Gamma}^{-1}\left(\frac{u_{-}}{\alpha}\right)\right)^{1/2}\right) \\ &= \alpha\left(\frac{\lambda}{\gamma}u-\frac{\lambda}{\gamma}\right)+\alpha e^{-u}\Sigma'B\left(\Gamma^{-1}\left(\frac{u_{-}}{\alpha}\right)\right)du+\alpha o_{p}\left(\Gamma^{-1}\left(\frac{u_{-}}{\alpha}\right)^{1/2}\right), \end{aligned}$$

where

$$\Sigma'=\left[\begin{array}{cc}1 & -\frac{\lambda}{\gamma}\end{array}\right]G.$$

Integrating out $\alpha^{-1/2}W\left(\alpha,u\right)$, we obtain

$$\alpha^{-1/2}\int_{0}^{\infty}e^{-u}W\left(\alpha,u\right)du=I_{1}\left(\alpha\right)+\alpha^{1/2}\int_{0}^{\infty}e^{-u}\Sigma'B\left(\overline{\Gamma}^{-1}\left(u/\alpha\right)\right)du+I_{2}\left(\alpha\right).$$

We analyze one by one each of these terms. First, it is clear that

$$I_{1}=\alpha^{-1/2}\int_{0}^{\infty}e^{-u}\left(\frac{\lambda}{\gamma}u-\frac{\lambda}{\gamma}\right)du=0.$$

Next,

$$I_{2}=\alpha^{-1/2}\int_{0}^{\infty}e^{-u}\alpha o_{p}\left(\left(\overline{\Gamma}^{-1}\left(u/\alpha\right)\right)^{1/2}\right)du=\int_{0}^{\infty}e^{-u}\alpha^{1/2}o_{p}\left(\left(\overline{\Gamma}^{-1}\left(u/\alpha\right)\right)^{1/2}\right)du.$$

Observe that

$$\sqrt{\alpha}o_{p}\left(\left(\overline{\Gamma}^{-1}\left(u/\alpha\right)\right)^{1/2}\right)=u^{1/2}\sqrt{\frac{\alpha}{u}\Gamma^{-1}\left(u/\alpha\right)}\frac{o_{p}\left(\left(\overline{\Gamma}^{-1}\left(u/\alpha\right)\right)^{1/2}\right)}{\left(\overline{\Gamma}^{-1}\left(u/\alpha\right)\right)^{1/2}}\rightarrow 0\quad a.s.\quad \alpha\rightarrow 0.$$

We wish to apply dominated convergence. Notice that

$$\left|\sqrt{\alpha}o_p\left(\left(\overline{\Gamma}^{-1}(u/\alpha)\right)^{1/2}\right)\right| = O\left(u^{1/2}\right) \leq H(u,\omega) \in L\left(e^{-u}du\right)$$

this implies that $I_2(\alpha) \to 0$ $a.s.$ Therefore, we obtain that

$$\alpha^{-1/2}W(\alpha,u) - \alpha^{1/2}\int_0^\infty e^{-u}\Sigma' B\left(\overline{\Gamma}^{-1}(u/\alpha)\right)du \Rightarrow 0,$$

which combined with the previous lemma and standard converging together results yields the conclusion of the proposition.

Finally, Theorem 2 is a direct consequence of Propositions 3 and 4.

5.4 Edgeworth Expansion

In this section, we provide refined versions of the approximations given in the previous sections. The refined approximation takes the form of an Edgeworth expansion for the distribution of D. We shall derive these approximations in the iid setting for the discrete time case and under Markovian assumptions for the continuous time case. More precisely, in the discrete time case, motivated by the applications to ARCH processes described in Section 2, we consider

$$D = \sum_{k=0}^{\infty}\exp\left(-\sum_{j=0}^{k-1}Z_j\right)X_k,$$

where $(X_k, Z_k)_{k\geq 1}$ is a sequence of iid random vectors satisfying certain assumptions to be described later (see assumptions AI1 to AI4 below); while in the continuous time context, we work with

$$D = \int_0^\infty \exp\left(-\int_0^t \gamma(Y(s))\,ds\right)d\Lambda(t),$$

where $Y = (Y_s : s \geq 0)$ is a suitably defined homogeneous Markov process Λ is a stationary independent increment process, this setting is commonly used in the risk theory example discussed in Section 2 (see Ch. 7 of Asmussen (2001)).

5.4.1 The discrete time setting

In this section, we shall consider the following set of assumptions.

ED1 Assume that $Z_1 \geq 0$, $E(Z_1) = \gamma < \infty$, $E(Z_1^2) = \mu_Z^{(2)} < \infty$, and $E\left(|Z_1|^3\right) < \infty$. Let σ_Z^2 be the variance of Z_1 and $\kappa_Z^{(3)}$ its third order cumulant, which can be written as

$$\kappa_Z^{(3)} = \mu_Z^{(3)} - 3\mu_Z^{(2)}\gamma + 2\gamma^3.$$

ED2 Suppose that X_1 has non-lattice distribution with $E(X_1) = \lambda$, $Var(X_1^2) = \sigma_X^2$, and $E\left(|X_1|^3\right) < \infty$. Let $E(X_1^3) = \mu_3^X$ and write $\kappa_X^{(3)}$ to denote the third order cumulant of X_1. In addition, assume that the distribution of X_1 given Z_1 is non-lattice.

ED3 Suppose that $E\left(|X_1|^j |Z_1|^k\right) < \infty$ for $0 < j + k \leq 3$ and for $j, k \geq 1$ denote $\mu_{jk} = E\left(X_1^j Z_1^k\right)$. Moreover, let us define,

$$\delta(\theta, Z_1) = \left|E\left(e^{i\theta X_1} \middle| Z_1\right)\right|$$

and assume that

$$\overline{\lim_{h\to 0}} \sup_{\varepsilon \leq |\theta| \leq 1/\varepsilon} \frac{P(\delta(\theta, Z_1) > 1 - h)}{h} < \infty, \tag{6}$$

for $\varepsilon > 0$.

Condition (6) is technical, and may be seen as a form of strong non-latticity of X_1 given Z_1. Notice that, in the important special case in which the X_k's are independent of the Z_k's, assumption AI3 is an immediate consequence of AI2. Indeed, if X_1 is non-lattice, we have that $\delta(\theta, Z_1) = \delta(\theta) < 1$. Therefore, for all $h > 0$ sufficiently small, $\delta(\theta) < 1 - h$. This implies that the limit in (6) is zero.

As a remark, we also note that, alternatively, the non-negativity of Z_1 required in assumption AI1 can be replaced by the existence of exponential moments, we record this observation as our alternative assumption AI1'.

ED1' Assume that $E\exp(\rho Z_1) < \infty$ for ρ in a vicinity of the origin.

Under these assumptions, we improve the approximation (4) by providing an Edgeworth expansion for the distribution of D when $(X_k, Z_k)_{k\geq 1}$ is a sequence of *i.i.d.* random vectors and the discount rate γ is small. In particular, by defining

$$\sigma^2 = \frac{1}{2}\left(\sigma_X^2 - 2\frac{\lambda}{\gamma}\sigma_{XZ} + \frac{\lambda^2}{\gamma^2}\sigma_Z^2\right),$$

we can write the approximation proposed as

$$\begin{aligned} P\left(D \leq y\right) \;\; \approx \;\; & P\left(N\left(\lambda/\gamma, \sigma^2/\gamma\right) \leq y\right) - \sqrt{\gamma}\beta_1\eta\left(\left(y - \lambda/\gamma\right)\frac{\sqrt{\gamma}}{\sigma}\right) \\ & -\frac{\sqrt{\gamma}}{18}\beta_2 H\left(\left(y-\lambda/\gamma\right)\frac{\sqrt{\gamma}}{\sigma}\right). \end{aligned} \tag{7}$$

The constants β_1 and β_2 satisfy

$$\begin{aligned} \beta_1 &= \frac{\mu_Z^{(2)}\lambda}{2\gamma^2\sigma}, \\ \sigma^3\beta_2 &= \kappa_X^{(3)} - 2\kappa_{21}\frac{\lambda}{\gamma} + 3\kappa_{12}\frac{\lambda^2}{\gamma^2} - 3\frac{\kappa_{11}}{\gamma}\left(\sigma_X^2 - 2\frac{\lambda}{\gamma}\sigma_{XZ} + \frac{\lambda^2}{\gamma^2}\sigma_Z^2\right) \\ & \quad +3\sigma_Z^2\frac{\lambda}{\gamma^2}\left(\sigma_X^2 - 2\frac{\lambda}{\gamma}\sigma_{XZ} + \frac{\lambda^2}{\gamma^2}\sigma_Z^2\right) - \frac{\kappa_Z^{(3)}\lambda^3}{\gamma^3}, \end{aligned}$$

with

$$\begin{aligned} \kappa_{12} &= \mu_{12} + \mu_{11} - \mu_Z^{(2)} - 3\gamma\mu_{11} + 2\gamma^2\lambda, \\ \kappa_{21} &= \mu_{21} + \mu_{11} - \mu_X^{(2)} - 3\lambda\mu_{11} + 2\lambda^2\gamma, \\ \kappa_{11} &= \mu_{11} - \lambda\gamma = \sigma_{XZ} \triangleq cov\left(X, Z\right); \end{aligned}$$

and

$$\begin{aligned} \eta\left(y\right) &= \frac{1}{\sqrt{2\pi}}\exp\left(-y^2/2\right) \\ H\left(y\right) &= \left(y^2 - 1\right)\eta\left(y\right). \end{aligned}$$

The application of the approximation (7), requires estimation of the joint moments μ_{ij}, which can be easily done (even non-parametrically) using standard methods. Also, observe that in the case in which the sequences $(X_k)_{k>0}$ and $(Z_k)_{k>0}$ are independent, the constants σ^2, β_1 and β_2 take the simplified form

$$\sigma^2 = \frac{1}{2}\left(\sigma_X^2 + \frac{\lambda^2}{\gamma^2}\sigma_Z^2\right)$$

and

$$\beta_1 = \frac{\mu_Z^{(2)}\lambda}{2\gamma^2\sigma}, \quad \beta_2 = \frac{1}{\sigma^3}\left(\kappa_X^{(3)} + 3\frac{\sigma_Z^2\lambda\sigma_X^2}{\gamma^2} - \frac{\kappa_Z^{(3)}\lambda^3}{\gamma^3} + 3\frac{\sigma_Z^4\lambda^3}{\gamma^4}\right).$$

In order to understand the nature of approximation (7), we introduce a small scaling parameter $\alpha > 0$ and define

$$D(\alpha) = \sum_{k=0}^{\infty} \exp\left(-\alpha\sum_{j=0}^{k-1} Z_j\right) X_k.$$

approximation (7) becomes (since the quantities σ, β_1 and β_2 are not affected by the scaling)

$$\begin{aligned} P(D(\alpha) \le y) \approx\ & P\left(N\left(\lambda/\alpha\gamma, \sigma^2/\alpha\gamma\right) \le y\right) - \sqrt{\gamma\alpha}\beta_1\eta\left((y-\lambda/\gamma\alpha)\frac{\sqrt{\gamma\alpha}}{\sigma}\right) \\ & -\frac{\sqrt{\gamma\alpha}}{18}\beta_2 H\left((y-\lambda/\gamma\alpha)\frac{\sqrt{\gamma\alpha}}{\sigma}\right). \end{aligned} \quad (8)$$

Or, in other words,

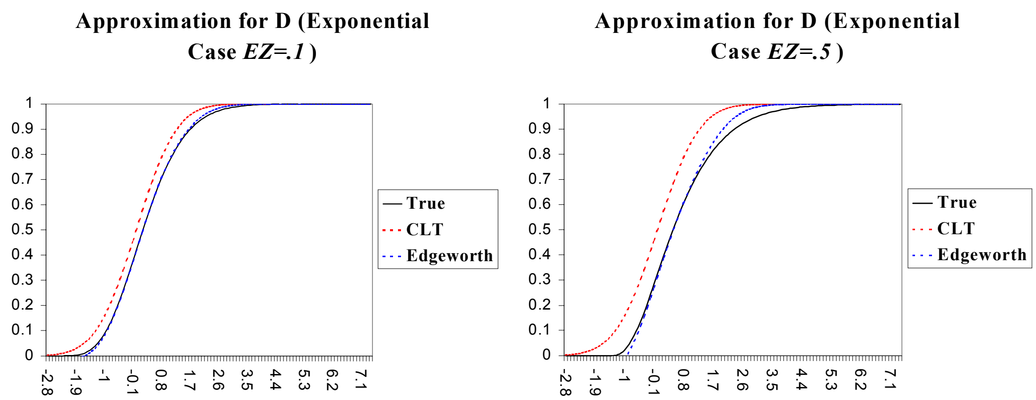
$$\begin{aligned} P\left(\sqrt{\alpha}\left(D(\alpha) - \lambda/\alpha\gamma\right) \le y\right) \approx\ & P\left(N\left(0, \sigma^2/\gamma\right) \le y\right) - \sqrt{\gamma\alpha}\beta_1\eta\left(\frac{\sqrt{\gamma}}{\sigma}y\right) \\ & -\frac{\sqrt{\gamma\alpha}}{18}\beta_2 H\left(\frac{\sqrt{\gamma}}{\sigma}y\right) \end{aligned}$$

with an error of order $o(\sqrt{\alpha})$ (uniformly on y). The precise mathematical statement concerning the previous approximations is the content of Theorem 3 below, which provides the first order correction in the Edgeworth expansion for $D(\alpha)$. However, before moving on to Theorem 3, we present a simple example to illustrate the accuracy of the approximations proposed.

Example 1 *Suppose that $X_1 \sim \lambda \exp(1)$ and $Z_1 \sim \gamma \exp(1)$. Under these assumptions it follows (see Gjessing and Paulsen (1997)) that*

$$D = \sum_{k=0}^{\infty} \exp\left(-\sum_{j=0}^{k-1} Z_j\right) X_k \sim \lambda\Gamma(1/\gamma + 1, 1),$$

where $\Gamma(1/\gamma+1,1)$ *represents a random variable with distribution gamma with the parameters given. In order to illustrate the numerical fit of the approximation provided we consider the case in which* $\lambda = 1$ *and* $\gamma = .1$ *and* $\gamma = .5$ *respectively. The following graphs compare the CLT and Edgeworth approximations developed against the true distribution of* D*:*



CLT and Edgeworth Based Approximations

We now provide the rigorous statement supporting approximation (7).

Theorem 3 *If the set of assumptions* $ED1$ *(or* $ED1'$*) to* $ED4$ *are in force, then*

$$P\left(\sqrt{\alpha}\left(D(\alpha)-\frac{\lambda}{\gamma\alpha}\right)\leq y\right) = P\left(N\left(0,\frac{\sigma^2}{\gamma}\right)\leq y\right)-\sqrt{\alpha}\beta_1 n(y) - \frac{\sqrt{\alpha}}{18}\frac{\beta_2}{\gamma}H(y)+G_\alpha(y); \quad (9)$$

where G_α *represents a signed measure with* $G_\alpha^+(R)+G_\alpha^-(R)\triangleq\|G_\alpha(dy)\| = o(\sqrt{\alpha})$.

In order to prove this theorem, we need some preliminary results. As it is standard in obtaining Edgeworth expansions via Fourier analytic methods (see Feller (1968) p. 512), one first proceeds to obtain an asymptotic expansion for the cumulant moment generating function of interest. Hence, our first result provides an asymptotic expansion for $\psi_\alpha(\theta)\triangleq\log E\exp\left(i\theta\alpha^{-1/2}(\alpha D(\alpha)-\lambda/\gamma)\right)$ in powers of $\sqrt{\alpha}$.

Lemma 3 *Assume $ED1$ (or $ED1'$) to $ED3$. Then, there exists $\delta > 0$ for which we have that*

$$\begin{aligned}\psi_\alpha(\theta) &= \left(\frac{\mu_Z^{(2)}\lambda}{2\gamma^2} + O(\alpha)\right) i\theta\alpha^{1/2} \\ &+ \left(\frac{1}{2\gamma\alpha}\left(\sigma_X^2 - 2\frac{\lambda}{\gamma}\sigma_{XZ} + \frac{\lambda^2}{\gamma^2}\sigma_Z^2\right) + O(1)\right)\frac{(i\theta)^2}{2}\alpha \\ &+ \left(\frac{C_3}{\alpha} + O(1)\right)\frac{(i\theta)^3}{6}\alpha^{3/2} + o\left(\alpha^{1/2}\right),\end{aligned}$$

(uniformly in $\theta \in (-\delta, \delta)$, $\delta > 0$) where

$$\begin{aligned}3\gamma C_3 &= \kappa_X^{(3)} - 2\kappa_{21}\frac{\lambda}{\gamma} + 3\kappa_{12}\frac{\lambda^2}{\gamma^2} \\ &+3\left(\sigma_Z^2\frac{\lambda}{\gamma^2} - \frac{\sigma_{XZ}}{\gamma}\right)\left(\sigma_X^2 - 2\frac{\lambda}{\gamma}\sigma_{XZ} + \frac{\lambda^2}{\gamma^2}\sigma_Z^2\right) - \frac{\kappa_Z^{(3)}\lambda^3}{\gamma^3}.\end{aligned}$$

Proof. The idea is to write

$$\phi_\alpha(\theta) = \exp\left(i\theta\lambda/\gamma\sqrt{\alpha}\right)\phi\left(\theta\sqrt{\alpha}, \alpha\right),$$

where $\phi_\alpha(\theta) \triangleq \exp(\psi_\alpha(\theta))$ and $\phi(\theta, \alpha) \triangleq E\exp(i\theta D(\alpha))$. Notice that $\phi(\theta, \alpha)$ satisfies

$$\phi(\theta, \alpha) = E\left(\exp\left(i\theta\left(X_1 + \exp(-\alpha Z_1) D_1(\alpha)\right)\right)\right),$$

with $D_1(\alpha)$ independent of (X_1, Z_1). Thus, we have,

$$\begin{aligned}\phi(\theta, \alpha) &= E\left(\exp\left(i\theta\left(X_1 + \exp(-\alpha Z_1) D_1(\alpha)\right)\right)\right) \\ &= E\left(E\left(\exp\left(i\theta\left(X_1 + \exp(-\alpha Z_1) D_1(\alpha)\right)\right) \middle| X_1, Z_1\right)\right) \\ &= E\left(E\left(\exp(i\theta X_1)\,\phi\left(\theta\exp(-\alpha Z_1), \alpha\right) \middle| X_1, Z_1\right)\right) \\ &= E\left(\exp(i\theta X_1)\,\phi\left(\theta\exp(-\alpha Z_1), \alpha\right)\right).\end{aligned}$$

Using the Taylor development for characteristic functions (see Feller (1968) App. Sec. XV.5 and Breiman (1992) Prop. 8.44) applied to $\phi(\theta, \alpha)$ and $\phi_\alpha(\theta)$, together with the moment conditions implied by assumptions $ED1$ (or $ED1'$) to $ED3$, we arrive at the expression stated for $\psi_\alpha(\theta)$.

Lemma 4 *Under assumptions* $ED1$ *(or* $ED1'$*) to* $ED4$, $\phi(\theta,\alpha) \triangleq E\exp(i\theta D(\alpha))$ *satisfies*

$$|\phi(\theta,\alpha)| = o\left(\alpha^{1/2}\right)$$

as $\alpha \to 0$ *uniformly in* θ *over compact sets not containing the origin.*

Proof. Let $\phi_X(\theta, Z_1) = E\left(e^{i\theta X_1}\middle| Z_1\right)$, and let $T_\alpha = \inf\{k : S_k > 1/\alpha\}$. Then,

$$\begin{aligned}
|\phi(\theta,\alpha)| &= \left|E\left(E\left(\exp\left(i\theta\sum_{k=1}^{\infty} X_k \exp(-\alpha S_{k-1})\right)\middle| Z\right)\right)\right| \\
&= \left|E\left(\Pi_{k=1}^{\infty}\phi_X\left(\theta e^{-\alpha S_{k-1}}, Z_k\right)\right)\right| \\
&\leq E\left(\Pi_{k=1}^{\infty}\left|\phi_X\left(\theta e^{-\alpha S_{k-1}}, Z_k\right)\right|\right) \\
&\leq E\left(\Pi_{k=1}^{T\alpha-1}\left|\phi_X\left(\theta e^{-\alpha S_{k-1}}, Z_k\right)\right|\right) \\
&\leq E\left(\Pi_{k=1}^{T\alpha-1}\left|\Delta(\theta, Z_k)\right|\right),
\end{aligned}$$

where $\Delta(\theta, Z_1) = \sup\{|\phi_X(\theta^*, Z_1)| : |\theta^*| > |\theta e^{-1}|\}$. Since the distribution of X_1 given Z_1 is non-lattice, we must have that $0 < \Delta(\theta, Z_1) < 1$. So,

$$\begin{aligned}
|\phi(\theta,\alpha)| &\leq E\left(\Pi_{k=1}^{T\alpha-1}|\Delta(\theta, Z_k)|\right) \\
&\leq P\left(\alpha\left|T_\alpha - \frac{1}{\alpha\gamma}\right| > \varepsilon\right) + E\left(\Pi_{k=1}^{T\alpha-1}|\Delta(\theta, Z_k)|; \alpha|T_\alpha - 1/\alpha\gamma| \leq \varepsilon\right) \\
&\leq P\left(\alpha\left|T_\alpha - \frac{1}{\alpha\gamma}\right| > \varepsilon\right) + E\left(|\Delta(\theta, Z_1)|^{1/\alpha(1/\gamma-\varepsilon)-1}\right).
\end{aligned}$$

Since condition $AF1$ ($AF1'$) imply that $0 < EZ_1 < \infty$ and $Var(Z_1) < \infty$, we have that $\left(\alpha^{1/2}\left|T_\alpha - \frac{1}{\alpha\gamma}\right|\right)^2$ is uniformly integrable (see Gut (1988) p. 92.) In particular, this implies, using Chebyshev's inequality, that

$$P\left(\alpha\left|T_\alpha - \frac{1}{\alpha\gamma}\right| > \varepsilon\right) = O(\alpha).$$

Finally, if we choose $\varepsilon > 0$ small enough so that $c \triangleq 1/\gamma - \varepsilon > 0$, we must show (for θ not in a neighborhood of the origin) that

$$E\left(|\Delta(\theta, Z_1)|^{c/\alpha}\right) = o\left(\sqrt{\alpha}\right).$$

Let $W = -\log(|\Delta(\theta, Z_1)|)$ and $\beta = c/\alpha$. Then,

$$E\left(|\Delta(\theta, Z_1)|^\beta\right) = E(\exp(-\beta W)) = \int_0^\infty \exp(-u) P(u/\beta > W)\, du.$$

Thus,

$$\beta E\left(|\Delta(\theta, Z_1)|^\beta\right) = \int_0^\infty \exp(-u) \beta P(u/\beta > W)\, du.$$

Fix $\varepsilon > 0$ and write

$$\begin{aligned} \beta E\left(|\Delta(\theta, Z_1)|^\beta\right) &= \int_0^\varepsilon \exp(-u) \beta P(u/\beta > W)\, du \\ &\quad + \int_\varepsilon^\infty u \exp(-u) \beta/u P(u/\beta > W)\, du \qquad (10) \\ &\leq \beta P(\varepsilon/\beta > W) + \int_\varepsilon^\infty u \exp(-u) \beta/u P(u/\beta > W)\, du. \end{aligned}$$

We want to apply Fatou's Lemma in the form

$$\begin{aligned} &\overline{\lim}_{\beta \longrightarrow \infty} \int_\varepsilon^\infty u \exp(-u) \beta/u P(u/\beta > W)\, du \\ \leq &\int_\varepsilon^\infty \overline{\lim}_{\beta \longrightarrow \infty} u \exp(-u) \beta/u P(u/\beta > W)\, du. \end{aligned}$$

In order to do this, we must show that

$$0 \leq \beta/u P(u/\beta > W) \leq M$$

for some $M > 0$ for $u \in [\varepsilon, \infty]$, and β large. So, by right continuity and the existence of left limits, it suffices to show that

$$\overline{\lim}_{\beta \longrightarrow \infty} \frac{P(h > W)}{h} < \infty.$$

But

$$\begin{aligned} \overline{\lim}_{h \longrightarrow 0} \frac{P(h > W)}{h} &= \overline{\lim}_{h \longrightarrow 0} \frac{P(h > -\log(|\Delta(\theta, Z_1)|))}{h} \\ &= \overline{\lim}_{h \longrightarrow 0} \frac{P(\exp(-h) < |\Delta(\theta, Z_1)|)}{h} \\ &= \overline{\lim}_{h \longrightarrow 0} \frac{P(|\Delta(\theta, Z_1)| > 1 - h)}{h} < \infty, \end{aligned}$$

by virtue of assumption $ED4$. This is what we require in order to apply Fatou's lemma. Consequently, we have

$$\overline{\lim}_{\beta \longrightarrow \infty} \beta E\left(\left|\Delta\left(\theta, Z_1\right)\right|^{\beta}\right) < \infty,$$

which implies

$$\overline{\lim}_{\beta \longrightarrow \infty} \sqrt{\beta} E\left(\left|\Delta\left(\theta, Z_1\right)\right|^{\beta}\right) = 0,$$

and this is what we needed to conclude the proof of the lemma.

We now are ready to proof Theorem 3.

Proof of Theorem 3. . The proof of this theorem follows closely the steps of Feller (1968) p.512. To simplify the notation, let us consider $E\left(X_1\right) = 0$ and $E\left(X_1^2\right) = 2\gamma$ and the X_k's independent of the Z_k's (as we shall see from the proof, these are just simplifying assumptions and the adaptation of the present proof is straightforward using the corresponding local expansion given in Lemma 3)). Let $\gamma\left(\theta\right) = \widehat{G}\left(\theta\right) = e^{-\theta^2/2}\left(1 + \frac{(i\theta)^3 \kappa_X^{(3)}}{18\gamma}\sqrt{\alpha}\right)$. Esséen's lemma applies here since

$$G\left(x\right) = \Phi\left(x\right) - \frac{\kappa_X^{(3)}}{18}\sqrt{a}\left(x^2 - 1\right)\eta\left(x\right)$$

is bounded by some constant C. Also $\gamma\left(0\right) = 1$ and $\gamma'\left(0\right) = 0$. Therefore,

$$\left|F_\alpha\left(x\right) - G\left(x\right)\right| \leq \frac{1}{\pi}\int_{-T}^{T} \frac{1}{|\theta|}\left|\phi\left(\sqrt{\alpha}\theta, \alpha\right) - \gamma\left(\theta\right)\right| d\theta + \frac{24C}{\pi T}.$$

Let $T = M/\sqrt{\alpha}$, for some $M > 0$ big. Then, for any $\delta > 0$ small, we have

$$\left|F_\alpha\left(x\right) - G\left(x\right)\right| \leq I_1 + I_2 + I_3 + \sqrt{\alpha}\frac{24C}{\pi M},$$

where

$$\begin{aligned} I_1 &= \frac{1}{\pi}\int_{-\delta/\sqrt{\alpha}}^{\delta/\sqrt{\alpha}} \frac{1}{|\theta|}\left|\phi\left(\sqrt{\alpha}\theta, \alpha\right) - \gamma\left(\theta\right)\right| d\theta, \\ I_2 &= \frac{1}{\pi}\int_{\delta/\sqrt{\alpha}}^{M/\sqrt{\alpha}} \frac{1}{|\theta|}\left|\phi\left(\sqrt{\alpha}\theta, \alpha\right) - \gamma\left(\theta\right)\right| d\theta, \\ I_3 &= \frac{1}{\pi}\int_{-M/\sqrt{\alpha}}^{\delta/\sqrt{\alpha}} \frac{1}{|\theta|}\left|\phi\left(\sqrt{\alpha}\theta, \alpha\right) - \gamma\left(\theta\right)\right| d\theta. \end{aligned}$$

Observe that

$$\begin{aligned} I_2 &\leq \frac{1}{\pi}\int_{\delta/\sqrt{\alpha}}^{M/\sqrt{\alpha}} \frac{1}{|\theta|}\left|\phi\left(\sqrt{\alpha}\theta,\alpha\right)\right| d\theta + \frac{1}{\pi}\int_{\delta/\sqrt{\alpha}}^{M/\sqrt{\alpha}} \frac{1}{|\theta|}\left|\gamma\left(\theta\right)\right| d\theta \\ &= \frac{1}{\pi}\int_{\delta}^{M} \frac{1}{|\theta|}\left|\phi\left(\theta,\alpha\right)\right| d\theta + \frac{1}{\pi}\int_{\delta/\sqrt{\alpha}}^{M/\sqrt{\alpha}} \frac{1}{|\theta|}\left|\gamma\left(\theta\right)\right| d\theta. \end{aligned}$$

By virtue of our previous lemma, it is clear that I_2 goes to zero faster than $\sqrt{\alpha}$, similarly for I_3. Thus, we just have to study I_1. Let

$$\begin{aligned} \zeta\left(\theta,\alpha\right) &\triangleq \log\left(\phi\left(\theta,\alpha\right)\right) + \frac{\theta^2 2\gamma}{2\left(1-m\left(-2\alpha\right)\right)} \\ &= \log\left(\phi\left(\theta,\alpha\right)\right) + \frac{\theta^2\gamma}{\left(1-m\left(-2\alpha\right)\right)} \end{aligned}$$

where $m\left(-\lambda\right) = E\left(e^{-\lambda Z_1}\right)$. Hence, we can write

$$\begin{aligned} I_1 &= \frac{1}{\pi}\int_{-\delta/\sqrt{\alpha}}^{\delta/\sqrt{\alpha}} \frac{1}{|\theta|}\left|\phi\left(\sqrt{\alpha}\theta,\alpha\right) - \gamma\left(\theta\right)\right| d\theta \\ &= \frac{1}{\pi}\int_{-\delta/\sqrt{\alpha}}^{\delta/\sqrt{\alpha}} \frac{1}{|\theta|}\left|\exp\left(\zeta\left(\sqrt{\alpha}\theta,\alpha\right) - \frac{\theta^2\gamma}{\left(1-m\left(-2\alpha\right)\right)}\right) - \gamma\left(\theta\right)\right| d\theta \\ &= \frac{1}{\pi}\int_{-\delta/\sqrt{\alpha}}^{\delta/\sqrt{\alpha}} \frac{1}{|\theta|} e^{-\theta^2/2}\left| e^{\left(\zeta\left(\sqrt{\alpha}\theta,\alpha\right) - \frac{\theta^2}{2}\left(\frac{\alpha\gamma}{\left(1-m\left(-2\alpha\right)\right)}-1\right)\right)} - 1 - \frac{\left(i\theta\right)^3\mu_3\sqrt{\alpha}}{18}\right| d\theta. \end{aligned}$$

Using Feller (1968), p. 507, we have that for any $\widetilde{\beta}_1$ and $\widetilde{\beta}_2$ complex numbers,

$$\left|e^{\widetilde{\beta}_1} - 1 - \widetilde{\beta}_2\right| \leq \left(\left|\widetilde{\beta}_1 - \widetilde{\beta}_2\right| + \frac{1}{2}\widetilde{\beta}_2^2\right)\exp\left(\upsilon\right), \tag{11}$$

where $\upsilon \geq \max\left(\left|\widetilde{\beta}_1\right|, \left|\widetilde{\beta}_2\right|\right)$. Given $\varepsilon > 0$, we can choose $\delta > 0$ small enough so that $|\theta\sqrt{\alpha}| < \delta$ (as in Feller (1968), p. 507) and

$$\left|\zeta\left(\theta\sqrt{\alpha},\alpha\right) - \frac{\alpha^{3/2}\left(i\theta\right)^3\kappa_X^{(3)}}{3!\left(1-m\left(-3\alpha\right)\right)}\right| \leq \varepsilon\frac{\theta^3\alpha^{3/2}}{\left|\left(1-m\left(-3\alpha\right)\right)\right|} \leq \varepsilon K\theta^3\alpha^{1/2}$$

for α small enough and some constant K_1 independent of α (because $\frac{\alpha^{3/2}\kappa_X^{(3)}}{(1-m(-3\alpha))}$ is the cumulant of order 3 for the random variable $\sqrt{\alpha}D\left(\alpha\right)$). At the same time, δ can also be chosen satisfying

$$\left|\zeta\left(\theta\sqrt{\alpha},\alpha\right)\right| < \frac{1}{2}\frac{\gamma\alpha\theta^2}{2\left(1-m\left(-2\alpha\right)\right)} \leq \frac{K_2}{3}\theta^2$$

for some $K_2 \leq 1$ for α small enough. Now, δ can be chosen also with the property that

$$\left|\frac{\alpha^{3/2}\left(i\theta\right)^3\kappa_X^{(3)}}{3!\left(1-m\left(-3\alpha\right)\right)}\right| < \frac{K_2}{3}\theta^2.$$

Notice that

$$\begin{aligned}
&\left|e^{\left(\zeta\left(\sqrt{\alpha}\theta,\alpha\right)-\frac{\theta^2}{2}\left(\frac{\alpha\gamma}{(1-m(-2\alpha))}-1\right)\right)}-1-\frac{\left(i\theta\right)^3\kappa_X^{(3)}}{18}\right| \\
\leq\ &\left|e^{\left(\zeta\left(\sqrt{\alpha}\theta,\alpha\right)-\frac{\theta^2}{2}\left(\frac{\alpha\gamma}{(1-m(-2\alpha))}-1\right)\right)}-1-\frac{\alpha^{3/2}\left(i\theta\right)^3\kappa_X^{(3)}}{3!\left(1-m\left(-3\alpha\right)\right)}\right|+ \\
&\left|\frac{\alpha^{3/2}\left(i\theta\right)^3\kappa_X^{(3)}}{3!\left(1-m\left(-3\alpha\right)\right)}-\frac{\left(i\theta\right)^3\kappa_X^{(3)}}{18}\sqrt{\alpha}\right|,
\end{aligned}$$

and observe that

$$\left|\frac{\alpha^{3/2}\left(i\theta\right)^3\kappa_X^{(3)}}{3!\left(1-m\left(-3\alpha\right)\right)}-\frac{\left(i\theta\right)^3\kappa_X^{(3)}}{18}\sqrt{\alpha}\right| \leq \sqrt{\alpha}\theta^3 o\left(1\right).$$

Finally, we apply inequality (11) with $\widetilde{\beta}_1 = \zeta\left(\sqrt{\alpha}\theta,\alpha\right) - \frac{\theta^2}{2}\left(\frac{\alpha\gamma}{(1-m(-2\alpha))}-1\right)$ and $\widetilde{\beta}_2 = \frac{\alpha^{3/2}(i\theta)^3\kappa_X^{(3)}}{3!(1-m(-3\alpha))}$ for $\delta > 0$ small enough so that

$$\begin{aligned}
I_1 \ \leq\ &\frac{\varepsilon}{\pi}\kappa_1\sqrt{\alpha}\int_{-\infty}^{\infty}\theta^2 e^{-\theta^2/6}d\theta+\frac{\alpha}{\pi}K_1^2\int_{-\infty}^{\infty}e^{-\theta^2/6}\theta^6 d\theta+ \\
&\frac{\sqrt{\alpha}}{\pi}o\left(1\right)\int_{-\infty}^{\infty}\left|\theta\right|^3 e^{-\theta^2/6}d\theta.
\end{aligned}$$

Hence we conclude that

$$\limsup_{\alpha\to 0}\frac{1}{\sqrt{\alpha}}\sup_x\left|F_\alpha\left(x\right)-G\left(x\right)\right| \leq \varepsilon\kappa,$$

for some constant κ. Since ε was arbitrary, this concludes the proof of the theorem.

5.4.2 The continuous time setting

A popular model in the risk theory setting discussed in Section 2 consists of considering the processes Γ as Λ two independent Levy processes (i.e. two stationary independent increment processes, see Gjessing and Paulsen (1992)). The stationary independent increment assumption of the risk process Λ has been argued to hold by several authors in the risk theory community (this setting includes the so-called classical risk model, see Asmussen (2001) and Grandell (1991)). On the other hand, in finance, short rate processes usually are modelled as positive functions of a Markov process (typically with mean reverting characteristics). This motivates the following setting in which we develop the desired Edgeworth expansion.

Suppose that $\Lambda = (\Lambda(t) : t \geq 0)$ is a Levy process. In addition, let $Y = (Y(s) : s \geq 0)$ be a homogeneous Markov process taking values in a Polish space Ξ and let $\mathcal{B}(\Xi)$ be the Borel sigma-field in Ξ. Let $P(t, y, B)$ ($t \in \mathbb{R}_+$, $y \in \Xi$ and $B \in \mathcal{B}(\Xi)$) be the corresponding transition probability function. Assume that Y satisfies the Feller condition (i.e. $P(t, y, B_\delta(x)) \to 1$ as $t \searrow 0$, for all $\delta > 0$) and that the mapping $y \to E_y f(Y_t)$ is continuous for all $f(\cdot) \in C(\Xi)$ (the space of continuous function taking values on Ξ). Let A be the associated infinitesimal generator of the process Y, defined via the relation

$$Af(y) = \lim_{t \downarrow 0} \frac{E_y f(Y(t)) - f(y)}{t},$$

where $f \in C(\Xi)$. The domain $D(A)$ of A is composed by those functions $f \in C(\Xi)$ for which the previous limit exists (uniformly, for all $y \in \Xi$) (See Skorohod, Hoppensteadt and Salehi (2002)). In addition, suppose that $Y(\cdot)$ has right continuous with left limits sample paths and that it is geometrically ergodic (see Kontoyiannis and Meyn (2003), p. 9).

The following set of assumptions are in force throughout this section.

EC1 Λ and Γ are independent and the distribution of $\Lambda(1)$ is non-lattice.

EC2 Suppose Y is geometrically ergodic (see Kontoyiannis and Meyn (2003) p. 9).

Suppose that $\widetilde{\gamma}(\cdot) : \Xi \to \mathbb{R}$ is a continuous mapping such that $\widetilde{\gamma}(x) > 0$ for all $x \in \Xi$ and define Γ as

$$\Gamma(t) = \int_0^t \widetilde{\gamma}(Y(s))\, ds.$$

Under EC1 and EC2, we shall provide rigorous support for the approximation

$$\begin{aligned} P\left(D\leq y\right) &\approx P\left(N\left(\lambda/\gamma,\chi^{(2)}\left(0\right)/2\right)\leq y\right)-\sqrt{\gamma}\frac{\lambda}{\gamma}F\left(y_{0}\right)\eta\left(\left(y-\lambda/\gamma\right)\sqrt{2/\chi^{(2)}\left(0\right)}\right) \\ &\quad -\frac{\sqrt{\gamma}}{18}\chi^{(3)}\left(0\right)H\left(\left(y-\lambda/\gamma\right)\sqrt{2/\chi^{(2)}\left(0\right)}\right), \end{aligned} \tag{12}$$

where (if $\pi\left(dy\right)$ denotes the stationary distribution of Y), F can be characterized as the solution of the Poisson equation

$$AF=E_{\pi}\gamma\left(Y\left(1\right)\right)-\gamma\left(y\right),$$

and $\chi\left(\cdot\right)$ depends on the log-moment generating function of Λ and the Perron-Frobenius eigenvalue associated with cumulative Markov reward Γ. More precisely, for every $\theta\in\mathbb{R}$ consider the (unique) solution pair $\left(u\left(y,\theta\right),\psi_{\Gamma}\left(\theta\right)\right)$ (such that $u\left(y,0\right)=1$) satisfying

$$\left(Au\right)\left(y,\theta\right)=\left(\psi_{\Gamma}\left(\theta\right)-\theta\widetilde{\gamma}\left(y\right)\right)u\left(y,\theta\right). \tag{13}$$

Note that the geometric ergodicity guarantees existence and uniqueness of the solution pair $\left(u,\psi_{\Gamma}\right)$, see Kontoyiannis and Meyn (2003)). Let $\psi_{\Lambda}\left(i\theta\right)=\log E\exp\left(i\theta\Lambda\left(1\right)\right)$ (we work with the branch $\{arg\left(z\right)\in[0,2\pi)\}$ when operating with complex logarithms) then $\chi\left(i\theta\right)=-\psi_{\Gamma}^{-1}\left(-\psi_{\Lambda}\left(i\theta\right)\right)$ (note that $\chi'\left(0\right)=\lambda/\gamma$). Just as in the discrete time case, the approximation (12) will be supported in the context of small interest rates for a suitably parameterized family of discounted rewards. In particular, we shall prove that the approximation

$$\begin{aligned} & P\left(\sqrt{\alpha}\left(D\left(\alpha\right)-\lambda/\left(\gamma\alpha\right)\right)\leq y\right) \\ \approx\ & P\left(N\left(0,\chi^{(2)}\left(0\right)/2\right)\leq y\right)-\sqrt{\gamma\alpha}\frac{\lambda}{\gamma}F\left(y_{0}\right)\eta\left(y\sqrt{2/\chi^{(2)}\left(0\right)}\right) \\ & -\frac{\sqrt{\gamma\alpha}}{18}\chi^{(3)}\left(0\right)H\left(y\sqrt{2/\chi^{(2)}\left(0\right)}\right) \end{aligned}$$

holds with an error of order $o\left(\sqrt{\alpha}\right)$ (uniformly on y), where

$$D\left(\alpha\right)=\int_{0}^{\infty}\exp\left(-\alpha\Gamma\left(t\right)\right)d\Lambda\left(t\right).$$

(Note that the previous integral can be interpreted, via integration by parts, path by path as a Lebesgue-Stieltjes integral.)

Theorem 4 *Suppose that EC1 and EC2 hold. Then,*

$$
\begin{aligned}
& P\left(\sqrt{\alpha}\left(D\left(\alpha\right)-\chi'\left(0\right)/\alpha\right)\leq y\right) \\
= \; & P\left(N\left(0,\chi^{(2)}\left(0\right)/2\right)\leq y\right)-\sqrt{\gamma\alpha}F\left(y_0\right)\eta\left(y\sqrt{2/\chi^{(2)}\left(0\right)}\right) \\
& -\frac{\sqrt{\gamma\alpha}}{18}\chi^{(3)}\left(0\right)H\left(y\sqrt{2/\chi^{(2)}\left(0\right)}\right)+G_\alpha([-\infty,y));
\end{aligned}
$$

where G_α represents a signed measure with $G_\alpha^+\left(\mathbb{R}\right)+G_\alpha^-\left(\mathbb{R}\right)\triangleq\|G_\alpha\|=o\left(\sqrt{\alpha}\right)$.

The proof of the previous theorem parallels its corresponding continuous time analogue described in the previous section. We first obtain a local description of $\psi_\alpha\left(\theta\right)=\log E\exp\left(i\theta\sqrt{\alpha}\left(D\left(\alpha\right)-\lambda/\left(\gamma\alpha\right)\right)\right)$.

Lemma 5 *Under assumptions EC1 and EC2 we have that*

$$
\psi_\alpha\left(\theta\right)=-\frac{\chi^{(2)}\left(0\right)}{2}\theta^2+\sqrt{\alpha}\left(\frac{\chi^{(3)}\left(0\right)}{18}\left(i\theta\right)^3-\frac{\lambda}{\gamma}F\left(y_0\right)i\theta\right)+o\left(\sqrt{\alpha}\right)
$$

(uniformly in $\theta\in\left(-\delta,\delta\right)$, $\delta>0$).

Proof. It is known that for every $u\in D\left(A\right)$ such that $\inf_{x\in\Xi}|u\left(x\right)|>0$ we have that

$$
M_t\left(z\right)=\frac{u\left(Y\left(t\right),\theta\right)}{u\left(Y_0,\theta\right)}\exp\left(-\int_0^t\left(\frac{Au}{u}\right)\left(Y\left(s\right),\theta\right)ds\right) \tag{14}
$$

is a Martingale with respect to the filtration generated by Y (see Lemma 2, p. 82 of Skorohod, Hoppensteadt and Salehi (2001)). Since Y is geometrically ergodic it follows that the generalized eigenvalue problem

$$
\left(Au\right)\left(y,\theta\right)=\left(\psi_\Gamma\left(\theta\right)-\theta\widetilde{\gamma}\left(y\right)\right)u\left(y,\theta\right),\quad u\left(y,0\right)=1 \tag{15}
$$

has a unique solution pair $\left(u\left(y,\theta\right),\psi_\Gamma\left(\theta\right)\right)$ for every $\theta\in\mathbb{R}$. In addition, $\inf_{\theta\in\Xi}u\left(y,\theta\right)>0$ for all $\theta\in\mathbb{R}$ and $\psi_\Gamma\left(\cdot\right)$ is a strictly increasing function (since

$$
\psi_\Gamma\left(\theta\right)=\lim_{t\to\infty}\frac{1}{t}\log E\exp\left(\theta\Gamma\left(t\right)\right)).
$$

Observe that the solution to (15) automatically provides the solution to the problem

$$\begin{aligned}\frac{1}{\widetilde{\gamma}(y)}(Au)(y,\theta) &= \left(\frac{\psi_\Gamma(\theta)}{\widetilde{\gamma}(y)}-\theta\right)u(y,\theta)\\ &= \left(-\psi_\Gamma^{-1}(-\nu)-\frac{\nu}{\widetilde{\gamma}(y)}\right)u\left(y,\psi_\Gamma^{-1}(-\nu)\right),\end{aligned}$$

(where $\nu=-\psi_\Gamma(\theta)$). In addition, Proposition 4.8 of Kontoyiannis and Meyn (2003) states that for each $\theta\in\Xi$, both $u(y,\cdot)$ and $\psi_\Gamma(\cdot)$ are analytic in $\mathcal{N}=\{z\in\mathbb{C}:|z|\le\delta\}$ for some $\delta>0$ (which immediately implies the analyticity of $\zeta(\cdot)=-\psi_\Gamma^{-1}(-\cdot)$) and $\inf_{x\in\Xi,z\in\mathcal{N}}|u(x,z)|>0$. Note that the Markov process $\widetilde{Y}=\left(\widetilde{Y}(t):t\ge0\right)$ defined as $\widetilde{Y}(t)=Y(\Gamma^{-1}(t))$ is also a geometrically ergodic Markov process with generator $\widetilde{A}=\frac{1}{\widetilde{\gamma}}A$ (the reason is that $\widetilde{\gamma}$ being continuous and positive implies $\inf_{x\in\Xi}\widetilde{\gamma}(x)>0$, which yields that the Lyaponuv bound needed in the definition of geometric ergodicity is immediately satisfied after scaling factors (see Kontoyiannis and Meyn (2003) p. 9). Therefore, by considering the Markov generator $\partial_t+\widetilde{A}$ and the function $u(y,\psi_\Lambda(i\theta e^{-\alpha t}))$, (for $\theta\in\mathbb{R}$ with $|\theta|<\delta$) in the relation (14) we can build the Martingales

$$\begin{aligned}M_t(i\theta) &= \frac{u\left(\widetilde{Y}(t),-\chi(i\theta e^{-\alpha t})\right)}{u\left(Y_0,-\chi(i\theta e^{-\alpha t})\right)}\exp\left(\int_0^t\frac{\psi_\Lambda(i\theta e^{-\alpha t})}{\widetilde{\gamma}\left(\widetilde{Y}(t)\right)}dt-\int_0^t\chi\left(i\theta e^{-\alpha t}\right)dt\right)\\ &\quad\exp\left(-\alpha\int_0^t i\theta e^{-\alpha t}\frac{u_\theta\left(\widetilde{Y}(t),-\chi(i\theta e^{-\alpha t})\right)}{u\left(\widetilde{Y}(t),-\chi(i\theta e^{-\alpha t})\right)}\dot{\chi}\left(i\theta e^{-\alpha t}\right)dt\right).\end{aligned}$$

Note that $M_t(i\theta)$ is a bounded martingale (in particular, uniformly integrable). Thus it possesses a last element $M_\infty(i\theta)$, which implies that

$$\exp\left(\int_0^\infty\chi\left(i\theta e^{-\alpha t}\right)dt\right)u(Y_0,i\theta)=E\exp\left(\int_0^\infty\frac{\psi_\Lambda(i\theta e^{-\alpha t})}{\widetilde{\gamma}\left(\widetilde{Y}(t)\right)}dt-\xi(\alpha,i\theta)\right),$$

where

$$\xi(\alpha,i\theta)=\alpha\int_0^t i\theta e^{-\alpha t}\frac{u_\theta\left(\widetilde{Y}(t),-\chi(i\theta e^{-\alpha t})\right)}{u\left(\widetilde{Y}(t),-\chi(i\theta e^{-\alpha t})\right)}\dot{\chi}\left(i\theta e^{-\alpha t}\right)dt.$$

Therefore, we conclude that

$$\begin{aligned} & \exp\left(\int_0^\infty \left(\chi\left(\sqrt{\alpha} i\theta e^{-\alpha t}\right) - \sqrt{\alpha} i\theta e^{-\alpha t}\lambda/\gamma\right) dt\right) u\left(Y_0, -\chi\left(\sqrt{\alpha} i\theta\right)\right) \\ = \; & E \exp\left(\int_0^\infty \frac{\psi_\Lambda\left(\sqrt{\alpha} i\theta e^{-\alpha t}\right)}{\widetilde{\gamma}\left(\widetilde{Y}(t)\right)} dt - i\theta \frac{\lambda}{\gamma\sqrt{\alpha}} - \xi\left(\alpha, \sqrt{\alpha} i\theta\right)\right) \\ = \; & E \exp\left(\int_0^\infty \frac{\psi_\Lambda\left(\sqrt{\alpha} i\theta e^{-\alpha t}\right)}{\widetilde{\gamma}\left(\widetilde{Y}(t)\right)} dt - i\theta \frac{\lambda}{\gamma\sqrt{\alpha}}\right) + o\left(\sqrt{\alpha}\right) \end{aligned} \tag{16}$$

(uniformly in $\theta \in (-\delta, \delta)$). The previous equality follows from the fact that

$$\begin{aligned} E\xi\left(\alpha, \sqrt{\alpha} i\theta\right) &= \sqrt{\alpha} i\theta\alpha \int_0^\infty e^{-\alpha t} \frac{u_\theta\left(\widetilde{Y}(t), -\chi\left(\sqrt{\alpha} i\theta\right)\right)}{u\left(\widetilde{Y}(t), -\chi\left(\sqrt{\alpha} i\theta\right)\right)} \dot{\chi}\left(\sqrt{\alpha} i\theta e^{-\alpha t}\right) dt \\ &= \sqrt{\alpha} i\theta \frac{\lambda}{\gamma} E\alpha \int_0^\infty e^{-\alpha t} u_\theta\left(\widetilde{Y}(t), 0\right) dt + O(\alpha), \end{aligned}$$

and (using Theorem 1) in combination with the bounded convergence theorem) it follows that

$$\alpha E \int_0^\infty e^{-\alpha t} u_\theta\left(\widetilde{Y}(t), 0\right) dt \rightharpoonup E u_\theta\left(Y(\infty), 0\right) = E_\pi F\left(Y(1)\right) = 0$$

(since $u_\theta(y, 0) = F(y)$). On the other hand, notice that

$$\begin{aligned} E\exp\left(i\theta D(\alpha)\right) &= E\left(E\left(\exp\left(i\theta\int_0^\infty \exp\left(-\alpha\Gamma(t)\right) d\Lambda(t)\right) \Big| \Gamma\right)\right) \\ &= E\exp\left(\int_0^\infty \psi_\Lambda\left(i\theta\exp\left(-\alpha\Gamma(t)\right)\right) dt\right) \\ &= E\exp\left(\int_0^\infty \frac{\psi_\Lambda\left(i\theta e^{-\alpha u}\right)}{\widetilde{\gamma}\left(\widetilde{Y}(t)\right)} du\right). \end{aligned} \tag{17}$$

Combining expressions (10) and (17) with a Taylor expansion of $\chi(\cdot)$ and $u(Y_0, \cdot)$ yields the conclusion of the Theorem.

The proof of Theorem 4 can be completed along the same lines as in the discrete time case after showing that $\phi(\theta, \alpha) \triangleq E\exp\left(i\theta D(\alpha)\right)$ goes to zero fast enough for $|\theta| \in (w_0, w_1)$ for any $0 < w_0 < w_1 < \infty$.

Lemma 6 *Suppose that EC1 and EC2 are in force, then* $\phi(\theta,\alpha) \triangleq E\exp(i\theta D(\alpha))$ *satisfies*

$$\sup_{|\theta|\in(\theta_0,\theta_1)} |\phi(\theta,\alpha)| = o\left(\sqrt{\alpha}\right),$$

for all $0 < \theta_0 < \theta_1 < \infty$.

Proof. We proceed as in the discrete time case, first we write

$$\begin{aligned} |\phi(\theta,\alpha)| &= \left| E\exp\left(\int_0^\infty \psi_\Lambda\left(i\theta\exp\left(-\alpha\Gamma(t)\right)\right)dt\right)\right| \\ &\leq E\left|\exp\left(\int_0^\infty \psi_\Lambda\left(i\theta\exp\left(-\alpha\Gamma(t)\right)\right)dt\right)\right| \end{aligned}$$

(note that $\psi_\Lambda(i\cdot)$ is well defined except for at most countably many values, in those cases we can assign the value $-\infty$ and that will not affect the value of the integral above). The proof now follows just as in the discrete time case, by spliting the integral up to $\Gamma^{-1}(1/\alpha)$ and using the non-lattice property of the distribution of $\Lambda(1)$. In fact, since $1/(\alpha\sup_{x\in\Xi}\widetilde{\gamma}(x)) \leq \Gamma^{-1}(1/\alpha)$ we actually can obtain an exponential rate of convergence instead of the rate $o\left(\alpha^{1/2}\right)$.

Remarks

a) The assumption that Ξ is compact does not really play an essential role. It was only used to ensure that the martingale property of $M_t(i\theta)$ in the proof of Lemma 5. A local description for $\psi_\alpha(i\theta)$ could also have been obtained by computing the moments of $D(\alpha)$, which is relatively easy in the present setting.

b) The independence between Γ and Λ can also be relaxed. For example, one could have assumed that both processes are conditionally independent given another Markov process, say Z, provided that Λ remains a possibly non-time homogeneous Levy process with a suitably non-lattice conditional distribution type assumption analogous to condition AI3 in the previous subsection.

c) Following the same ideas as in Lemma 5, a local expansion for $\psi_\alpha(\theta)$ can be obtained for the case in which

$$D(a) = \int_0^\infty \exp\left(-\alpha\int_0^t \widetilde{\gamma}(Y(s))\,ds\right)\widetilde{\lambda}(Y(s)).$$

(where $\widetilde{\lambda}$ is, say, continuous on the compact Polish space Ξ). In this case, the corresponding generalized eigenvalue problem takes the form

$$\frac{1}{\widetilde{\gamma}}\left(Au\right)\left(y,\theta\right)=\left(\chi\left(\theta\right)-\frac{\widetilde{\lambda}\left(y\right)}{\widetilde{\gamma}\left(y\right)}\right)u\left(y,\theta\right),\ u\left(y,0\right)=1,$$

and a formal corrected approximation can be written as

$$\begin{aligned}P\left(D\leq y\right) &\approx P\left(N\left(\lambda/\gamma,\chi^{(2)}\left(0\right)/2\right)\leq y\right)-\sqrt{\gamma}u_{\theta}\left(y_0,0\right)\eta\left(\left(y-\lambda/\gamma\right)\sqrt{2/\chi^{(2)}\left(0\right)}\right)\\ &\quad -\frac{\sqrt{\gamma}}{18}\chi^{(3)}\left(0\right)H\left(\left(y-\lambda/\gamma\right)\sqrt{2/\chi^{(2)}\left(0\right)}\right).\end{aligned}$$

The only step required to make the previous approximation rigorous is to show that for all $0<\theta_0<\theta_1<\infty$, $\sup_{|\theta|\in(\theta_0,\theta_1)}\left|\phi\left(\theta,\alpha\right)\right|=o\left(\sqrt{\alpha}\right)$ as in Lemma 6. This essentially involves assuming enough structure to ensure strongly non-lattice properties of D. We have chosen Levy process in our exposition because they provide a convenient framework to easily verify, from the model primitives, the non-lattice conditions that yield the described Edgeworth expansion.

5.5 Large Deviations

To fix ideas, let us begin by considering the same setting under which we derived our LLN in Section 3. In the previous section, we derived accurate approximations for the distribution of D (in the iid setting) for small interest rates when the D is close to its typical value (according to the LLN this implies looking at D close to λ/γ). In a number of applications (including those discussed in Section 2 regarding time series analysis and risk theory), one is often interested in computing $P\left(D>x\right)$ for x suitably large. In particular, these types of applications motivate interest in the analysis of the tail probability $P\left(D>x\right)$ for $x>>\lambda/\gamma$. As we shall see, under certain exponential moment conditions on $\left(\Gamma,\Lambda\right)$, the approximation proposed here will take the form

$$P\left(D>x\right)\approx\exp\left(-I\left(x\right)/\gamma\right), \tag{18}$$

where $I(x) > 0$ corresponds to the so-called rate function and will typically take the form $I(x) = x\theta^* - \int_0^\infty \chi(\theta^* e^{-s})\, ds$, where θ^* satisfies $\theta^* x = \chi(\theta^*)$ and $\chi(\cdot)$ is a suitably defined convex function. The goal of this section is to provide, under general conditions, rigorous justification (at least in a rough logarithmic sense) for the previous approximation. In addition, we will also explore, under additional structure, exact asymptotics (also known as precise large deviations).

Applications in finance and risk theory motivate study of continuous time processes, including the case in which the processes Γ and Λ take the form

$$\Gamma(t) = \int_0^t \widetilde{\gamma}(s)\, ds \quad \text{and} \quad \Lambda(t) = \int_0^t \widetilde{\lambda}(s)\, ds,$$

where, for all s, $\widetilde{\gamma}(s) > 0$ represents the "short rate" process and $\widetilde{\lambda}(s)$ represent the reward rate. Also, other applied contexts such as the analysis of ARCH processes in time series motivate study of the discrete time setting, in which

$$\Gamma(t) \triangleq \sum_{k=1}^{\lfloor t \rfloor} Z_k \quad \text{and} \quad \Lambda(t) \triangleq \sum_{k=1}^{\lfloor t \rfloor} X_k,$$

and $(X_k, Z_k)_{k\geq 0}$ is a (typically stationary) sequence of two dimensional random vectors with the property that $Z_k > 0$ for all $k \geq 0$.

In order to provide rigorous justification for the approximation (18), we shall consider

$$\begin{aligned} \alpha D(\alpha) &= \int_{[0,\infty)} \exp(-\alpha\Gamma(t_-))\, d\Lambda(t) \\ &= \alpha \int_{[0,\infty)} \exp(-u)\, \Lambda\left(\Gamma^{-1}(u/\alpha)\right) du, \end{aligned} \tag{19}$$

and study $P(\alpha D(\alpha) > x)$ for $x > \lambda$. Note that the previous identity holds in general provided that $\Gamma(\cdot)$ is non-decreasing and $\Lambda(\cdot)$ has RCLL sample paths. In other words, (19) may hold even if Λ does not have bounded variation. Expression (19) suggests a natural strategy to derive a LDP for $\{\alpha D(\alpha)\}_{\alpha>0}$ as $\alpha \searrow 0$; namely, to apply the contraction principle (under appropriate sample path large deviations assumptions on $\alpha(\Gamma(\cdot/\alpha), \Lambda(\cdot/\alpha))$), to the mapping $\Psi : D[0,\infty) \times D[0,\infty) \to \mathbb{R}$

defined as

$$\Psi(x,y)=\int_0^\infty \exp(-t)\,y\left(x^{-1}(t)\right)dt.$$

Actually, we will follow more or less this idea, although with important modifications arising due to the fact that Ψ is not continuous. Indeed, if we consider the map Ψ_1, acting on $D[0,\infty)$ endowed with the Skorohod J_1 topology (see Whitt (2001)) and defined as $\Psi_1(x)=\int_0^\infty \exp(-t)\,x(t)\,dt$, then, we can see, aside from the fact that Ψ_1 is not well defined for every element in $D[0,\infty)$, that Ψ_1 is discontinuous at every single point. In order to see this, just consider the sequence of functions $(x_n : n\geq 1)$, defined as $x_n(t)=e^n I(n\leq t<n+1)$, and note that $x_n\rightarrow 0$ while $\Psi_1(x_n)=1-e^{-1}$. (This example was given by Whitt (1972); that $\Psi_1(\cdot)$ is discontinuous at every element of $D[0,\infty)$ follows by linearity of Ψ_1.)

The idea, then, is to restrict the domain of Ψ_1 to a proper subspace of $D[0,\infty)$, endowed with a finer topology under which $\Psi_1(\cdot)$ is continuous. This idea will be studied in detail in the next subsection, in which we treat the continuous setting. Later, we will return to the discrete setting.

5.5.1 The continuous time setting

We will restrict the domain of Ψ_1 to the subspace

$$L_\beta[0,\infty)\triangleq\{x\in C[0,\infty):\overline{\lim}_{t\rightarrow\infty}\left|\frac{x(t)}{t^\beta}\right|=0\},$$

for some $\beta>0$, with the topology generated by the weighted norm

$$\|x\|_\beta=\sup_{t\geq 0}\frac{|x(t)|}{1+t^\beta}.$$

Whitt (1972) proved that Ψ_1 is continuous on $\left(L_\beta[0,\infty),\|\cdot\|_\beta\right)$, which suggests using the contraction principle on this space. The following proposition constitutes an intermediate step in this direction.

Proposition 5 *Suppose that the family of processes* $\alpha\left(\Gamma(\cdot/\alpha),\Lambda(\cdot/\alpha)\right)_{\alpha>0}$ *satisfies a LDP on* $C[0,\infty)\times C[0,\infty)$ *(endowed with the product topology generated by the* *uniform convergence on compact sets also known as Stone's topology) with a good rate function* $I(x,y)$*. Then,* $R_\alpha(\cdot) = \alpha\Lambda(\Gamma^{-1}(\cdot/\alpha))$ *satisfies a LDP on* $C[0,\infty)$ *(endowed with Stone's topology) with good rate function* $I'(z) = \inf\{I(x,y) : z = y \circ x^{-1}\}$*.*

Proof. This is just a direct consequence of the contraction principle (see Theorem 4.2.1, p. 126 of Dembo and Zeitouni (1999)) and the fact that the mapping $(x,y) \to y \circ x^{-1}$ in the topological spaces described (see Whitt (2001), Theorem 13.2.2., p. 430).

At this point, one may be tempted to invoke, once again, the contraction principle in combination with Proposition 5 to obtain the desired LDP. However, in order to proceed with this program, we must show that the LDP developed in Proposition 5 actually holds on $\left(L_\beta[0,\infty), \|\cdot\|_\beta\right)$ (since, in order to apply the contraction principle, the continuity of Ψ_1 must be compatible with the topology under which the original LDP was derived). In order to show the LDP on $\left(L_\beta[0,\infty), \|\cdot\|_\beta\right)$ we will need to show that the random elements $(\alpha\Lambda(\Gamma^{-1}(\cdot/\alpha)))_{\alpha>0}$ are exponentially tight (see Dembo and Zeitouni (1998)). (This type of reasoning parallels similar arguments in the context of weak convergence theory and the important role that tightness plays in this theory). Recall that a sequence of probability measures P_n is said to be exponentially tight if for every $a > 0$ there exist compact sets K_a, such that

$$\overline{\lim}_{n\to\infty} \frac{1}{n} \log P_n(K_a) \leq -a,$$

or, if the P_n's take values on subsets of a Polish space, then the P_n's are exponentially tight if for $\varepsilon > 0$, there exists a compact set K_ε such that, for all $n \geq 1$,

$$\varepsilon^n > 1 - P_n(K_\varepsilon),$$

(see Zajic (1993) p. 11). In view of these observations, we must characterize exponential tightness in $L_\beta[0,\infty)$. This is the aim of the following theorem.

Lemma 7 *Consider a sequence of probability measures* $(P_n : n \geq 1)$ *on* $L_\beta[0,\infty)$ *(such that* $P_n\{x : x(0) = 0\} = 1$*) and acting on the Borel sigma-field corresponding to the topology generated by the norm* $\|\cdot\|_\beta$*. Then,* $(P_n : n \geq 1)$ *is exponentially tight* *if and only if* $(P_n : n \geq 1)$ *is exponentially tight under the (relative) Stone topology, and that for each* $\delta > 0$

$$\overline{\lim}_{n\to\infty}\frac{1}{n}\log P_n\left(x : \sup_{t\geq t_0}\frac{|x(t)|}{t^{\beta}} > \delta\right) \to -\infty \quad \textit{as} \quad t_0 \nearrow \infty. \tag{20}$$

Proof. Lemma 3.3 of Whitt (1972) establishes that relatively compact sets in $\left(L_\beta[0,\infty), \|\cdot\|_\beta\right)$ are those sets B with compact closure under the relative Stone topology, and satisfying

$$\lim_{t\to\infty}\sup_{x\in B}\frac{|x(t)|}{t^{\beta}} = 0.$$

Also, recall that (if $x(0) = 0$ a.s. with respect to each P_n) for exponential tightness under Stone's topology, it is necessary and sufficient (see Feng and Kurtz (2000), p. 30) that, for each $\varepsilon, T > 0$,

$$\overline{\lim}_{n\to\infty}\frac{1}{n}\log P_n\left(x : \omega(x,\delta,T) > \varepsilon\right) \to -\infty \quad \text{as} \quad \delta \searrow 0, \tag{21}$$

where $\omega(x,\delta,T)$ is the modulus of continuity of x, on the interval $[0,T]$, evaluated at δ. We now show that if conditions (20) and (21) are satisfied, then the sequence $(P_n : n \geq 1)$ is exponentially tight. Pick $\lambda > 0$, choose δ_k so that

$$P_n\left(x : \omega(x,\delta_k,T) > 1/k\right) \leq e^{-n\lambda}/2^{k+1},$$

and let $B_k = \{x : \omega(x,\delta_k,T) \leq 1/k\}$. Also, pick t_k so that

$$P_n\left(x : \sup_{t\geq t_k}\frac{|x(t)|}{t^{\beta}} > 1/k\right) \leq e^{-n\lambda}/2^{k+1},$$

and let $C_k = \{x : \sup_{t>t_k}|x(t)|/t^{\beta} \leq 1/k\}$. Consider the closure, $\bar{A}_\lambda$, of $A_\lambda = \cap_k(B_k \cap C_k)$. Note that

$$1 - P\left(\bar{A}_\lambda\right) \leq 1 - P(A_\lambda) = P\left(\cup_k\left(B_k^c \cap C_k^c\right)\right) \leq e^{-n\lambda}$$

We claim that A_λ is relatively compact (i.e. that $\bar{A}_\lambda$ is compact), to see this, choose $\varepsilon > 0$ and let $k_0 > 1/\varepsilon$. Then, for all $\delta < \delta_{k_0}$ we have that

$$\sup_{x\in A}\omega(x,\delta,T) < \varepsilon.$$

Similarly, for every $T > t_{k_0}$ we have that

$$\varepsilon > \sup_{x\in A}\sup_{t>T}\frac{|x(t)|}{t^{\beta}},$$

which implies that

$$\overline{\lim}_{t\to\infty}\sup_{x\in A}\frac{|x(t)|}{t^{\beta}} \le \varepsilon$$

for all $\varepsilon > 0$. Thus, by virtue of the Arzela-Ascoli theorem (see Billingsley (1999) p.81) and Lemma 3.3 of Whitt (1972), which concludes the argument for sufficiency. The necessity part is easier and follows just as in Feng and Kurtz (2000) p. 30. Therefore, it is omitted.

With the aid of the previous lemma, the exponential tightness of $(\alpha\Lambda(\Gamma^{-1}(\cdot/\alpha)))_{\alpha>0}$ follows easily.

Lemma 8 *Suppose that* $\alpha(\Gamma(\cdot/\alpha), \Lambda(\cdot/\alpha))_{\alpha>0}$ *satisfies a full LDP with rate function* $I(x, y)$ *(under Stone's topology). (Recall that a full LDP means an LDP with convex good rate function). Then,*

a) The family $(\alpha\Gamma(\cdot/\alpha) - \gamma\cdot, \alpha\Lambda(\cdot/\alpha) - \lambda\cdot)_{\alpha>0}$ *is exponentially tight in* $L_1[0,\infty)\times L_1[0,\infty)$ *with the product topology generated by the norm* $\|\cdot\|_1$

b) The class of random elements $(\alpha\Lambda(\Gamma^{-1}(\cdot/\alpha)) - \lambda\cdot/\gamma)_{\alpha>0}$*, is exponentially tight in* $(L_1[0,\infty), \|\cdot\|_1)$ *.*

Remark The convexity of the rate function does not really play a role in this lemma, but only the goodness of the rate function is required.

Proof. For part a), it suffices to show that $\alpha\Gamma(\cdot/\alpha)-\gamma\cdot$ and $\alpha\Lambda(\cdot/\alpha)-\gamma\cdot$ are both exponentially tight in $(L_1[0,\infty), \|\cdot\|_1)$. Since $\alpha\Lambda(\cdot/\alpha)$ satisfies a full LDP in $C[0,\infty)$ (under Stone's topology), which is a topological group (which implies the addition is a continuous operation), it follows from the contraction principle that $\alpha\Lambda(\cdot/\alpha) - \gamma\cdot$ also satisfies a full LDP. Note that $C[0,\infty)$, endowed with Stone's topology, is a Polish space. Thus, the existence of a full LDP guarantees the exponential tightness of $\alpha\Lambda(\cdot/\alpha) - \gamma\cdot$ (see Dembo and Zeitouni (1999), p. 120 (c)). Therefore, we just have to prove condition (20). Note that for any $0 < a < b < \infty$, the mapping $x \to \sup_{t\in[a,b]} |x(t)/t|$ is continuous (under Stone's topology), which implies that the family $V_\alpha = \sup_{t\in\{a,b\}} |\alpha\Lambda(t/\alpha)/t - \gamma|$ satisfies an LDP with good rate function J, say. Hence, we can write

$$\begin{aligned} & P\left(\sup_{t>t_0}\left|\frac{\alpha\Lambda(t/\alpha)-\gamma t}{t}\right| \geq \delta\right) \\ \leq & \sum_{k=1}^{\infty} P\left(\sup_{t>t_0[k,k+1]}\left|\frac{\alpha\Lambda(t/\alpha)-\gamma t}{t}\right| \geq \delta\right) \\ \leq & \sum_{k=1}^{\infty} P\left(\sup_{u=\frac{t}{kt_0}>[1,2]}\left|\frac{\alpha\Lambda(ukt_0/\alpha)-\gamma ukt_0}{ukt_0}\right| \geq \delta\right) \\ = & \sum_{k=1}^{\infty} \exp\left(-\left(J(\delta)+o_{kt_o/\alpha}(1)\right)kt_0/\alpha\right), \end{aligned}$$

where the subindex in $o_{kt_o/\alpha}(1)$ has been used just to emphasize that $o_{kt_o/\alpha}(1) \to 0$ as $kt_0/\alpha \to \infty$. So we can choose k_0 big enough so that for every $k > k_0$ we have $J(\delta) + o_{kt_o/\alpha}(1) > J(\delta)/2 > 0$. From these estimates it is easy to conclude that

$$\overline{\lim}_{\alpha\to 0}\alpha \log P\left(\sup_{t>t_0}\left|\frac{\alpha\Lambda(t/\alpha)-\gamma t}{t}\right| \geq \delta\right) \to -\infty \quad \text{as} \quad t_0 \nearrow \infty,$$

which yields, by virtue of Lemma 7, the corresponding exponential tightness for $\alpha\Lambda(\cdot/\alpha) - \gamma\cdot$. The argument for $\alpha\Gamma(\cdot/\alpha) - \gamma\cdot$ is exactly the same and therefore has been omitted. Part b) also proceeds along the same lines as the previous argument, since it follows from Proposition 5 that $\alpha\Lambda(\Gamma^{-1}(\cdot/\alpha))$ satisfies a full LDP under Stone's topology.

We are ready to derive the LDP for $(\alpha D(\alpha))_{\alpha>0}$ in the continuous setting.

Theorem 5 *Suppose that the family of processes $\alpha(\Gamma(\cdot/\alpha), \Lambda(\cdot/\alpha))_{\alpha>0}$ satisfies a full LDP on $C[0,\infty) \times C[0,\infty)$ (endowed with the corresponding product Stone's topology) with a good rate function $I(x,y)$. Then, $\{\alpha D(\alpha)\}_{\alpha>0}$ satisfies an LDP on $\mathbb{R}$ with good rate function*

$$I(z) = \inf\{I(x,y) : z = \int_0^\infty e^{-t}\left(y \circ x^{-1}\right)(t)\,dt\}.$$

Proof. Proposition 5 combined with the contraction principle tells us that the family of random variables $(\alpha\Lambda(\Gamma^{-1}(\cdot/\alpha)) - \lambda\cdot/\gamma)_{\alpha>0}$ satisfies a full LDP on $C[0,\infty)$. Since the product topology generated by the norm $\|\cdot\|_1$ in the subspace $L_1[0,\infty)$ is finer than Stone's topology, Corollary 4.2.6 of Dembo and Zeitouni (1999) (which is a simple consequence of the inverse contraction principle applied with the identity mapping) applies yielding that $(\alpha\Lambda(\Gamma^{-1}(\cdot/\alpha)) - \lambda\cdot/\gamma)_{\alpha>0}$ satisfies a full LDP on $(L_1[0,\infty), \|\cdot\|_1)$. Since the mapping Ψ_1 is continuous on $(L_1[0,\infty), \|\cdot\|_1)$, we can apply the contraction principle once again here thereby yielding the conclusion of the theorem.

The previous theorem provides rigorous justification for approximation (18) in very general setting (essentially all those in which functional LDPs for (Γ, Λ) exist in the space of continuous functions). This includes, for example, the setting in which Λ and Γ are diffusion processes (see Dembo and Zeitouni (1999) Section 5.6). However, in order for the previous theorem to be useful from an applied standpoint, sufficient conditions must be provided to guarantee the validity of an LDP with good rate function on $C[0,\infty)\times C[0,\infty)$. Fortunately, these types of conditions have been well studied in the literature.

The following set of assumptions taken from Zajic (1993) are useful to guarantee the existence of a full LDP (more than we actually need), and their validity has been shown in many different settings (see Zajic (1993) Ch. 3 and Ch. 4).

ACL1 For all $\theta, \eta \in \mathbb{R}$ suppose that

$$g(\eta,\theta) \triangleq \sup_{s,t} \frac{1}{t} \log E \exp\left(\eta \int_s^{s+t} \gamma(u)\, du + \theta \int_s^{s+t} |\lambda(u)|\, du\right) < \infty.$$

In addition, assume that there exists $\varepsilon > 0$ and a pair of functions (f, h) such that $h(\delta) \to 0$ as $\delta \to 0$ and

$$\underline{\lim}_{\delta\to 0} (\varepsilon \log(\delta) + f(\delta) h(\delta) - \delta g(f(\delta))) = \infty.$$

ACL2 If $0 = t_0 < t_1 < \ldots < t_m < \infty$ then

$$W_{\alpha,m} \triangleq \alpha\left((\Gamma(t_i/\alpha) - \Gamma(t_{i-1}/\alpha)), (\Lambda(t_i/\alpha) - \Lambda(t_{i-1}/\alpha))\right)_{i=1}^m$$

satisfies a Large Deviations Principle (LDP) on $\mathbb{R}^{2m}$ with good rate function

$$I_m(z) = \sum_{i=1}^{m} (t_i - t_{i-1}) I\left(\frac{z_i}{t_i - t_{i-1}}\right),$$

where $I(x_{1,}x_2)$ is the rate function governing the LDP of $n^{-1}(\Gamma(n), \Gamma(n))$.

The following theorem provides a form of the LDP that is well suited for applications. Define (as in Zajic (1993) p. 9)

$$\psi(\eta,\theta) = \overline{\lim}_{n\to\infty} \frac{1}{n} \log E \exp(\eta\Gamma(n) + \theta\Lambda(n)) < \infty.$$

Theorem 6 *Suppose that assumptions ACL1 and ACL2 are in force. Let AC_0 be the set of absolutely continuous functions, defined on $[0,\infty)$, taking values on the real line and vanishing at the origin. Then, if $y > \lambda/\gamma$, we have that*

$$\begin{aligned} & \lim_{\alpha\to\infty} \alpha \log P(\alpha D(\alpha) \geq y) \\ = \quad & -I(y) \triangleq -\inf_{x\in AC_0} \{\int_0^\infty \sup_\theta (\theta\dot{x}(s) - \chi(\theta))\, ds : y = \int_0^\infty e^{-s} x(s)\, ds\}, \end{aligned}$$

where $\chi(\cdot)$ is defined via $\psi(-\chi(\cdot),\cdot) = 0$. In addition, if there exists $\theta^* = \theta^*(y)$ such that $y\theta^* = \chi(\theta^*)$, then we have

$$\begin{aligned} \lim_{\alpha\to\infty} \alpha \log P(\alpha D(\alpha) \geq y) &= \sup_\theta \left(y\theta - \int_0^\infty \chi\left(\theta e^{-s}\right) ds\right) \\ &= y\theta^* - \int_0^{\theta^*} \frac{\chi(u)}{u} du. \end{aligned}$$

Proof. All what we have to do is to identify the rate function. Theorem 2.2.2., p. 25, of Zajic (1993) indicates that $\alpha(\Gamma(\cdot/\alpha), \Lambda(\cdot/\alpha))_{\alpha>0}$ satisfies a LDP with good rate function

$$I(x,y) \triangleq \begin{cases} \int_0^\infty \sup_{\eta,\theta} (\dot{x}(s)\eta + \dot{y}(s)\eta - \psi(\eta,\theta))\, ds & \text{if } x,y \in AC_0 \\ \infty \text{ otherwise} \end{cases}.$$

This implies (combining the results of Puhalskii and Whitt (1997) and Russell (1998)) that $(\alpha\Lambda(\Gamma^{-1}(\cdot/\alpha)))_{\alpha>0}$ satisfies a full LDP with good rate function

$$J(x) \triangleq \begin{cases} \int_0^\infty \sup_\theta (\dot{x}(s)\theta - \chi(\theta))\, ds & \text{if } x \in AC_0 \\ \infty \text{ otherwise} \end{cases}.$$

This expression, combined with the contraction principle, yields the first part of the theorem. Hence, we only need to show that if $y > \lambda/\gamma$ and $y\theta^* = \chi(\theta^*)$, then

$$\begin{aligned} I(y) &= \sup_{\theta}\left(y\theta - \int_0^\infty \chi\left(\theta e^{-s}\right) ds\right) \\ &= y\theta^* - \int_0^{\theta^*} \frac{\chi(u)}{u} du. \end{aligned}$$

First, observe that integration by parts yields

$$\begin{aligned} &\inf_{x\in AC_0;\ y=\int_0^\infty e^{-s}x(s)ds}\left\{\int_0^\infty \sup_{\theta}\left(\theta\dot{x}(s) - \chi(\theta)\right) ds\right\} \\ =\ &\inf_{x\in AC_0;\ y=\int_0^\infty e^{-s}\dot{x}(s)ds}\left\{\int_0^\infty \sup_{\theta}\left(\theta\dot{x}(s) - \chi(\theta)\right) ds\right\}. \end{aligned}$$

Also, note that for every $s \in \mathbb{R}$

$$\sup_{\theta}\left(\theta\dot{x}(s) - \chi(\theta)\right) = \sup_{\theta}\left(\theta e^{-s}\dot{x}(s) - \chi\left(\theta e^{-s}\right)\right).$$

In particular, we have that for $x \in AC_0$ and $y = \int_0^\infty e^{-s}x(s)\,ds$

$$\begin{aligned} \int_0^\infty \sup_{\theta}\left(\theta\dot{x}(s) - \chi(\theta)\right) ds &= \int_0^\infty \sup_{\theta}\left(\theta e^{-s}\dot{x}(s) - \chi\left(\theta e^{-s}\right)\right) ds \\ &\geq \sup_{\theta}\int_0^\infty \left(\theta e^{-s}\dot{x}(s) - \chi\left(\theta e^{-s}\right)\right) ds \\ &= \sup_{\theta}\left(y\theta - \int_0^\infty \chi\left(\theta e^{-s}\right) ds\right). \end{aligned}$$

Consequently,

$$I(y) \geq \sup_{\theta}\left(y\theta - \int_0^\infty \chi\left(\theta e^{-s}\right) ds\right).$$

Now, if $y > \lambda/\gamma = \dot{\chi}(0)$, then

$$\sup_{\theta\geq 0}\left(y\theta - \int_0^\infty \chi\left(\theta e^{-s}\right) ds\right) \geq 0.$$

On the other hand, for every $\theta > 0$, we have (by making the change of variables $\theta e^{-s} = u$)

$$\int_0^\infty \chi\left(\theta e^{-s}\right) ds = \int_0^\theta \frac{\chi(u)}{u} du.$$

Therefore, by first order optimality conditions we have that (using the convexity of the rate function)

$$\sup_{\theta\geq 0}\left(y\theta-\int_0^\infty \chi\left(\theta e^{-s}\right)ds\right)=y\theta^*-\int_0^{\theta^*}\frac{\chi\left(u\right)}{u}du>0.$$

Finally consider the function $x_*(s)$ such that $\dot{\chi}\left(\theta^* e^{-s}\right)=\dot{x}_*(s)$ and $x(0)=0$. Note that

$$\begin{aligned}\int_0^\infty e^{-s}\dot{x}_*(s)\,ds &= \int_0^\infty e^{-s}\dot{\chi}\left(\theta^* e^{-s}\right)ds\\ &= \frac{-1}{\theta^*}\int_0^\infty d\chi\left(\theta^* e^{-s}\right)=\frac{\chi\left(\theta^*\right)}{\theta^*}=y.\end{aligned}$$

Hence, we have that

$$\begin{aligned}I(y) &= \inf_{x\in AC_0;\ y=\int_0^\infty e^{-s}x(s)ds}\left\{\int_0^\infty \sup_\theta\left(\theta\dot{x}(s)-\chi(\theta)\right)ds\right\}s\}\\ &\leq \int_0^\infty \sup_\theta\left(\theta e^{-s}\dot{\chi}\left(\theta^* e^{-s}\right)-\chi\left(\theta e^{-s}\right)\right)ds\\ &= y\theta^*-\int_0^{\theta^*}\frac{\chi(u)}{u}du=\sup_\theta\left(y\theta-\int_0^\infty\chi\left(\theta e^{-s}\right)ds\right),\end{aligned}$$

which yields the conclusion of the theorem.

Our final result is an exact LDP formulated in the continuous setting for processes with a Markovian structure. We adopt the setting of Subsection 5.2, in which a suitably time homogeneous Markov process $Y=(Y(s):s\geq 0)$ with generator A was introduced. We also assume that Λ is a Levy process independent of Y. The desired exact LDP for

$$\alpha D(\alpha)=\alpha\int_0^\infty \exp\left(-\alpha\Gamma(t)\right)d\Lambda(t)$$

provided in the next theorem gives support to the following approximation (valid when $x>>\lambda/\gamma$)

$$P_y\left(D>x\right)\approx\gamma^{1/2}u\left(y_0,-\chi\left(\theta^*\right)\right)\frac{\exp\left(\theta^*\dot{\chi}\left(\theta^*\right)c\left(\theta^*\right)\right)}{\theta^*\sqrt{\pi\chi''\left(\theta^*\right)}}\exp\left(-I(x)/\gamma\right),$$

where $u(y,\cdot)$ and $\chi''(\cdot)$ are defined as in Subsection 5.2 via the generalized eigenvalue problem (15), $c(\theta^*) = E_{\widetilde{\pi}}\left(u_\theta\left(\widetilde{Y}(1), -\chi(\theta^*)\right)/u\left(\widetilde{Y}(1), -\chi(\theta^*)\right)\right)$ (with $\widetilde{\pi}(dy) = \widetilde{\gamma}(y)\pi(dy)/E_\pi\widetilde{\gamma}(Y(1))$) and

$$I(x) = x\theta^* - \int_0^{\theta^*} \frac{\chi(u)}{u}du,$$

with $\chi(\theta^*) = \theta^* x$.

Theorem 7 *Suppose that Y is geometrically ergodic and that $\Lambda(1)$ is non-lattice with $\phi_\Lambda(\theta) = E\exp(\theta\Lambda(1)) < \infty$ for all $\theta \in \mathbb{R}$. Then, if $x > \lambda/\gamma$ and $c(\theta^*) = E_{\widetilde{\pi}}\left(u_\theta\left(\widetilde{Y}(1), -\chi(\theta^*)\right)/u\left(\widetilde{Y}(1), -\chi(\theta^*)\right)\right)$ ($\widetilde{\pi}(dy) = \widetilde{\gamma}(y)\pi(dy)/E_\pi\widetilde{\gamma}(Y(1))$)*

$$\exp(I(x)/\alpha)P_y(\alpha D(\alpha) > x) \sim \frac{\alpha^{1/2}}{u(y, -\chi(\theta^*))}\frac{\exp(\theta^*\dot{\chi}(\theta^*)c(\theta^*))}{\theta^*\sqrt{\pi\chi^{(2)}(\theta^*)}} \text{ as } \alpha \searrow 0,$$

where $\chi(\theta) = -\psi_\Gamma^{-1}(-\psi_\Lambda(\theta))$, θ^* satisfies $\chi(\theta^*) = \theta^* x$ and $u(y,\theta)$ ($u(y,0) = 1$) solves the eigenvalue problem

$$\frac{1}{\widetilde{\gamma}(y)}(Au)(y,\theta) = \left(\frac{-\psi_\Lambda(\theta)}{\widetilde{\gamma}(y)} + \chi(\theta)\right)u(y,\theta).$$

Proof. Consider the family of probability measures P_y^* defined as

$$dP_y^* = \exp(\theta^* D(\alpha) - \psi(\theta^*, \alpha))dP_y,$$

where $\psi(\theta^*, \alpha) = \log E\exp(\theta^* D(\alpha))$. Note that

$$\begin{aligned}&\exp(I(x)/\alpha)P_{y_0}(\alpha D(\alpha) > x)\\ =\;&\exp(I(x)/\alpha)E_{y_0}^*(1(\alpha D(\alpha) > x)\exp(\psi(\theta^*, \alpha) - \theta^* D(\alpha))).\end{aligned}$$

Now, observe that (since $\theta^* > 0$)

$$I(x)/\alpha - x\theta^*/\alpha = -\int_0^{\theta^*}\frac{\chi(u)}{u}du = -\int_0^\infty \chi\left(\theta^* e^{-\alpha s}\right)ds.$$

On the other hand, from the proof of Lemma 5 we have that for all $\theta \in \mathbb{R}$,

$$\begin{aligned}&\exp\left(\int_0^\infty \chi\left(\theta e^{-\alpha t}\right)dt\right)u(y_0, -\chi(\theta)) \qquad (22)\\ =\;& E\exp\left(\int_0^\infty \psi_\Lambda\left(\theta e^{-\alpha\Gamma(t)}\right)dt - \alpha\int_0^\infty \frac{e^{-\alpha u}u_\theta\left(\widetilde{Y}(u), -\chi(\theta e^{-\alpha u})\right)\dot{\chi}(\theta e^{-\alpha u})}{u\left(\widetilde{Y}(u), -\chi(\theta e^{-\alpha u})\right)}du\right).\end{aligned}$$

Which implies that

$$\begin{aligned} & \exp\left(\psi\left(\theta,\alpha\right)-\int_0^\infty \chi\left(\theta e^{-\alpha s}\right)ds\right) \\ \sim\ & u\left(Y_0,-\chi\left(\theta\right)\right)\exp\left(c\left(\theta\right)\right)\triangleq\xi\left(y_0,\theta\right). \end{aligned} \tag{23}$$

as $\alpha\searrow 0$. Therefore, we have that

$$\begin{aligned} & \exp\left(I\left(x\right)/\alpha\right)P_{y_0}\left(\alpha D\left(\alpha\right)>x\right) \\ \sim\ & \xi\left(y_0,\theta^*\right)E^{\alpha}_{y_0}\left(1\left(D\left(\alpha\right)-x/\alpha>0\right)\exp\left(-\theta^*\left(D\left(\alpha\right)-x/\alpha\right)\right)\right). \end{aligned} \tag{24}$$

The strategy is then to develop an Edgeworth expansion for $\sqrt{\alpha}\left(D\left(\alpha\right)-x/\alpha\right)$ under $E^*_{y_0}$. Using the same steps as in the proof of Lemma 5 we can obtain a description of the local behavior $\psi^*_\alpha\left(\theta\right)\triangleq\log E^*_{y_0}\exp\left(i\theta\sqrt{\alpha}\left(D\left(\alpha\right)-x/\alpha\right)\right)$. In fact, we can obtain

$$\psi^*_\alpha\left(\theta\right)=-\theta^2\frac{\chi^{(2)}\left(\theta^*\right)}{4}+\sqrt{\alpha}\left(c_1 i\theta+c_2\left(i\theta\right)^3\right)+o\left(\sqrt{\alpha}\right)$$

(uniformly on $\theta\in(-\delta,\delta)$ for some $\delta>0$). The coefficients c_1 and c_2 can actually be computed but their values are not relevant for purposes of developing sharp large deviations. The coefficient $\chi^{(2)}\left(\theta^*\right)/4$ comes from the development of

$$\begin{aligned} & \int_0^\infty\left(\chi\left(\left(\sqrt{\alpha}\theta+\theta^*\right)e^{-\alpha u}\right)-\chi\left(\theta^* e^{-\alpha u}\right)-\sqrt{\alpha}e^{-\alpha u}x\theta\right)du \\ =\ & \theta\sqrt{\alpha}\int_0^\infty\left(\dot{\chi}\left(\theta^* e^{-\alpha u}\right)-x\right)e^{-\alpha u}du+\theta^2\int_0^\infty\frac{2\alpha\chi^{(2)}\left(\theta^* e^{-\alpha u}\right)e^{-2\alpha u}}{4}du \\ & +o\left(\sqrt{\alpha}\right). \end{aligned} \tag{25}$$

Indeed, since

$$\int_0^\infty\dot{\chi}\left(\theta^* e^{-\alpha u}\right)e^{-\alpha u}du=-\frac{1}{\theta^*\alpha}\int_0^\infty d\chi\left(\theta^* e^{-\alpha u}\right)=\frac{\chi\left(\theta^*\right)}{\alpha\theta^*}=\frac{x}{\alpha},$$

we obtain that the coefficient multiplying θ in (25) vanishes and, thus, $\psi^*_\alpha\left(\theta\right)\sim-\theta^2\chi^{(2)}\left(\theta^*\right)/4$ as stated. We also must show that $\left|\phi^*\left(\theta,\alpha\right)\right|=\left|E^*_{y_0}\exp\left(i\theta D\left(\alpha\right)\right)\right|=o\left(\sqrt{\alpha}\right)$ uniformly on compact sets not containing the origin. The key observation to prove this condition is to note that

$$\phi^*\left(\theta,\alpha\right)=\frac{E\exp\left(\int_0^\infty\psi_\Lambda\left(\left(i\theta+\theta^*\right)\exp\left(-\alpha\Gamma\left(t\right)\right)\right)dt\right)}{E\exp\left(\int_0^\infty\psi_\Lambda\left(\theta^*\exp\left(-\alpha\Gamma\left(t\right)\right)\right)dt\right)}.$$

Next, observe that (23) implies that there exists a positive constant $C < \infty$ such that

$$|\phi^*(\theta,\alpha)| \leq C|E\exp\left(\int_0^\infty \psi_\Lambda\left((i\theta+\theta^*)e^{-\alpha\Gamma(t)}\right) - \chi\left(\theta^* e^{-\alpha t}\right)dt\right)|. \qquad (26)$$

Now consider the (geometrically ergodic) Markov process $\widetilde{Y} = \left(\widetilde{Y}(s) : s \geq 0\right)$ with generator $\widetilde{A} = \frac{1}{\widetilde{\gamma}}A$ (compare the proof of Lemma 5, where this process was introduced). Let us define the probability measure $\widetilde{P}$ acting the sigma-field generated by $\widetilde{Y}$ as

$$d\widetilde{P} = M_\infty(\theta^*)\,dP,$$

where $M_\infty(\theta^*)$ is the last element of the bounded martingale $M^* = (M_t(\theta^*) : 0 \leq t \leq \infty)$ defined as

$$\begin{aligned} M_t(\theta^*) &= \frac{u\left(\widetilde{Y}(t), -\chi(\theta^* e^{-\alpha t})\right)}{u\left(Y_0, -\chi(\theta^* e^{-\alpha t})\right)}\exp\left(\int_0^t \frac{\psi_\Lambda(\theta^* e^{-\alpha t})}{\widetilde{\gamma}\left(\widetilde{Y}(t)\right)}dt - \int_0^t \chi\left(\theta^* e^{-\alpha t}\right)dt\right) \\ &\quad \exp\left(-\alpha\int_0^t \theta^* e^{-\alpha t}\frac{u_\theta\left(\widetilde{Y}(t), -\chi(\theta^* e^{-\alpha t})\right)}{u\left(\widetilde{Y}(t), -\chi(\theta^* e^{-\alpha t})\right)}\dot{\chi}\left(\theta^* e^{-\alpha t}\right)dt\right). \end{aligned}$$

(This martingale was also introduced in the proof of Lemma 5, where it has been indicated how the martingale property follows from Lemma 2, p. 82 of Skorohod, Hoppensteadt and Salehi (2001)). Note, therefore, that

$$\begin{aligned} & E\exp\left(\int_0^\infty \psi_\Lambda\left((i\theta+\theta^*)e^{-\alpha\Gamma(t)}\right) - \chi\left(\theta^* e^{-\alpha t}\right)dt\right) \\ = \; & \widetilde{E}\exp\left(\int_0^\infty \psi_\Lambda\left((i\theta+\theta^*)e^{-\alpha\Gamma(t)}\right) - \int_0^t \psi_\Lambda\left(\theta^* e^{-\alpha\Gamma(t)}\right)dt\right)Z(\alpha), \end{aligned}$$

where $B_1 < |Z(\alpha)| < B_2$ for some constants $0 < B_1 < B_2 < \infty$. This implies, using the bound (26), that

$$|\phi^*(\theta,\alpha)| \leq CB_2\widetilde{E}|\exp\left(\int_0^\infty \left(\psi_\Lambda\left((i\theta+\theta^*)e^{-\alpha\Gamma(t)}\right) - \psi_\Lambda\left(\theta^* e^{-\alpha\Gamma(t)}\right)\right)dt\right)|.$$

From this bound it is easy to see that $|\phi^*(\theta,\alpha)| = o(\sqrt{\alpha})$ by noting that for every $\eta_2 \in \mathbb{R}$, $\exp(\psi_\Lambda(i\cdot+\eta_2) - \psi_\Lambda(\eta_2))$ is the characteristic function of $\Lambda(1)$ under the obvious exponential change of measure and that $\delta t \leq \Gamma(t) \leq Kt$.for positive finite constants δ and M. With these elements on hand, the corresponding Edgeworth expansion for $\sqrt{\alpha}\,(D(\alpha) - x/\alpha)$ under $E^{*}_{y_0}$ follows routine steps as in the proof of Theorem 3. Therefore, we obtain that

$$\begin{aligned}
& E^{\alpha}_{y_0}\left(1\left(D(\alpha) - x/\alpha > 0\right)\exp\left(-\theta^{*}\left(D(\alpha) - x/\alpha\right)\right)\right) \\
= \; & \sqrt{\alpha}\int_0^{\infty}\frac{\exp(-x)\exp\left(-\alpha x^2/\left(\chi^{(2)}(\theta^{*})\,\theta^{*}\right)\right)}{\theta^{*}\sqrt{\pi\chi^{(2)}(\theta^{*})}}dx \\
& +\sqrt{\alpha}\int_0^{\infty}\exp\left(-\theta^{*}x/\sqrt{\alpha}\right)p(x)\exp\left(-x^2/\chi^{(2)}(\theta^{*})\right)dx \\
& +\int_0^{\infty}\exp\left(-\theta^{*}x/\sqrt{\alpha}\right)G_{\alpha}(dx),
\end{aligned}$$

where $p(x)$ in the second term above represents a polynomial of degree 3 and $G_{\alpha}(dx)$ is a signed measure such that $\|G_{\alpha}(dx)\| = o(\sqrt{\alpha})$. Hence, using the Dominated Convergence Theorem and the stated property on the total variation of G_{α}, we obtain that

$$\frac{1}{\sqrt{\alpha}}E^{\alpha}_{y_0}\left(1\left(D(\alpha) - x/\alpha > 0\right)\exp\left(-\theta^{*}\left(D(\alpha) - x/\alpha\right)\right)\right) \to \frac{1}{\theta^{*}\sqrt{\pi\chi^{(2)}(\theta^{*})}}.$$

Combining these estimates with (24) yields the conclusion of the theorem.

5.5.2 The discrete time setting

The goal now is to obtain the LDP for the discrete time case. The following set of assumptions are analogous to those stated at the end of the previous section, and their validity has been verified in many cases (including under Markovian and strong mixing assumptions; see Zajic (1993), chapters 3 and 4).

ADL1 For each $\theta, \eta \in \mathbb{R}$, suppose that

$$g(\eta,\theta) \triangleq \sup_{n,k}\frac{1}{n}\log E\exp\left(\eta\sum_{j=k}^{n+k}Z_j + \theta\sum_{j=k}^{n+k}|X_j|\right) < \infty.$$

ADL2 If $0 = t_0 < t_1 < \ldots < t_m < \infty$ then

$$W_{\alpha,m} \triangleq \alpha\left(\left(\Gamma(t_i/\alpha) - \Gamma(t_{i-1}/\alpha)\right), \left(\Lambda(t_i/\alpha) - \Lambda(t_{i-1}/\alpha)\right)\right)_{i=1}^{m}$$

satisfies an LDP on $\mathbb{R}^{2m}$ with good rate function

$$I_m(z) = \sum_{i=1}^{m} (t_i - t_{i-1})\, I\left(\frac{z_i}{t_i - t_{i-1}}\right),$$

where $I(x_{1,}x_2)$ is the rate function governing the LDP of $n^{-1}(\Gamma(n), \Lambda(n))$

The strategy here is first to consider a related family of approximating processes $\left(\widetilde{\Gamma}, \widetilde{\Lambda}\right)$ defined via

$$\begin{aligned} \widetilde{\Gamma}(t) &\triangleq \sum_{k=1}^{\lfloor t \rfloor} Z_k + (t - \lfloor t \rfloor)\, Z_{\lfloor t \rfloor + 1}, \\ \widetilde{\Lambda}(t) &\triangleq \sum_{k=1}^{\lceil t \rceil} X_k + (t - \lfloor t \rfloor)\, X_{\lceil t \rceil + 1}. \end{aligned}$$

Theorem 2.1.1., p. 19, of Zajic (1993) establishes that $\alpha\left(\widetilde{\Gamma}(\cdot/\alpha), \widetilde{\Lambda}(\cdot/\alpha)\right)$ satisfies a full LDP under Stone's topology. (Note that $\lceil \cdot \rceil$ is being used here instead of $\lfloor \cdot \rfloor$ in the definition of $\widetilde{\Lambda}$, but it is straightforward to adapt Zajic's estimates in this setting. Also, recall that a full LDP is one that holds with a good and convex rate function. See Dembo and Zeitouni (1999) for the definition of good rate function.) Thus, Theorem 5 applies here yielding the full LDP for the corresponding normalized infinite horizon discounted reward

$$\alpha\widetilde{D}(\alpha) \triangleq \alpha\Psi\left(\widetilde{\Gamma}_\alpha^{-1}, \widetilde{\Lambda}_\alpha\right) = \alpha \int_{[0,\infty)} \exp(-u)\, \widetilde{\Lambda}_\alpha\left(\widetilde{\Gamma}_\alpha^{-1}(u)\right) du.$$

In view of this observation, the natural step is to show that $\alpha\widetilde{D}(\alpha)$ is suitably close to $\alpha D(\alpha)$ (in exponential scale) as $\alpha \searrow 0$. In other words, we would like to show that the families of random variables $\{\alpha\widetilde{D}(a)\}_{\alpha>0}$ and $\{\alpha D(\alpha)\}_{\alpha>0}$ are exponentially equivalent (i.e. that for each $\delta > 0$

$$\lim_{\alpha\to\infty} \alpha \log P\left(\left|\alpha\widetilde{D}(\alpha) - \alpha D(\alpha)\right| > \delta\right) = -\infty,$$

see Dembo and Zeitouni (1999), p. 130). With exponential equivalence on hand we would be able to conclude, by virtue of Theorem 4.2.13 of Dembo and Zeitouni

(1999), that a full LDP also holds for $\alpha D(\alpha)$ as $\alpha \searrow 0$. We will actually follow this program but we will utilize a different family of approximating processes. The reason is that the integral structure in the definition of $\widetilde{D}(\alpha) = \Psi\left(\widetilde{\Gamma}_\alpha^{-1}, \widetilde{\Lambda}_\alpha\right)$ allows us to take advantage of the nature of the Lebesgue measure to construct a family of approximating processes $\{\bar{\Lambda}_\alpha\}_{\alpha>0}$ which is more convenient for purposes of proving the exponential equivalence required. We, thus, define for each $\alpha > 0$, the continuous process $\bar{\Lambda}_\alpha$ as

$$\bar{\Lambda}_\alpha(t) \triangleq \sum_{k=1}^{\lceil t \rceil} X_k + U_\alpha(t),$$

where

$$U_\alpha(t) = \lceil t \rceil \frac{(t - (\lceil t \rceil - \alpha))}{\alpha} X_{\lceil t \rceil + 1} 1\left(t \in [\lceil t \rceil - \alpha, \lceil t \rceil)\right).$$

We now show that $\left(\alpha\widetilde{\Gamma}(\cdot/\alpha), \alpha\bar{\Lambda}_\alpha(\cdot/\alpha)\right)$ and $\left(\alpha\widetilde{\Gamma}(\cdot/\alpha), \alpha\widetilde{\Lambda}(\cdot/\alpha)\right)$ are equivalent from a large deviations standpoint.

Lemma 9 *The families* $\{\left(\alpha\widetilde{\Gamma}(\cdot/\alpha), \alpha\bar{\Lambda}_\alpha(\cdot/\alpha)\right)\}_{\alpha>0}$ *and* $\{\left(\alpha\widetilde{\Gamma}(\cdot/\alpha), \alpha\widetilde{\Lambda}(\cdot/\alpha)\right)\}_{\alpha>0}$ *are exponentially equivalent in* $C[0,\infty) \times C[0,\infty)$ *Stone's topology.*

Proof. It suffices to show the corresponding exponential equivalence for $\{\bar{\Lambda}_\alpha\}_{\alpha>0}$ and $\{\widetilde{\Lambda}_\alpha\}_{\alpha>0}$. Recall that Stone's topology is generated by the metric

$$d_\infty(x, y) = \sum_{k=1}^{\infty} 2^{-k} \frac{d_k(x, y)}{1 + d_k(x, y)},$$

where

$$d_T(x, y) = \sup_{0 \le t \le T} |x(t) - y(t)|$$

(see Zajic (1993) p. 20). Fix $\delta > 0$ small and choose $k_0 > \lceil -\log(\delta/2)/\log(2) \rceil$. Then, $\sum_{k=k_0}^{\infty} 2^{-k} < \delta/2$ and, noting that $d_k\left(\bar{\Lambda}_\alpha, \widetilde{\Lambda}_\alpha\right) \le \alpha \max_{1 \le k \le \lfloor t/a \rfloor} |X_k|$, we can write

$$\begin{aligned} P\left(d_\infty\left(\bar{\Lambda}_\alpha, \widetilde{\Lambda}_\alpha\right) > \delta\right) &\le P\left(d_{k_0}\left(\bar{\Lambda}_\alpha, \widetilde{\Lambda}_\alpha\right) > \delta/2\right) \\ &\le \lfloor k_0/\alpha \rfloor \max_{1 \le k \le \lfloor k_0/a \rfloor} P\left(|X_k| > 2^{-1}\delta/\alpha\right) \\ &\le \lfloor k_0/\alpha \rfloor \exp\left(-A2^{-1}\delta/\alpha\right) \max_{k \in \mathbb{N}} E\left(\exp\left(A2^{-1}\delta/\alpha\, |X_k|\right)\right), \end{aligned}$$

for every $A > 0$ (by virtue of assumption ADL2). Therefore, we conclude that

$$\lim_{\alpha\to\infty} \alpha \log P\left(d_\infty\left(\bar{\Lambda}_\alpha, \widetilde{\Lambda}_\alpha\right) > \delta\right) = -A2^{-1}\delta.$$

Letting $A \nearrow \infty$ yields the conclusion of the lemma.

The same strategy followed in the continuous case can now be applied to the pair $\left(\widetilde{\Gamma}_\alpha(\cdot), \bar{\Lambda}_\alpha(\cdot)\right)$ as the next proposition summarizes.

Proposition 6 *Under assumptions ADL1 and ADL2, the family of random elements* $\bar{\Lambda}_\alpha\left(\widetilde{\Gamma}_\alpha^{-1}(\cdot)\right)$ *satisfies a full LDP on the space of continuous function* $C[0,\infty)$ *endowed with Stone's topology. Moreover, the corresponding normalized infinite horizon discounted reward*

$$\alpha\bar{D}(\alpha) = \alpha \int_{[0,\infty)} \exp(-u)\, \bar{\Lambda}_\alpha\left(\widetilde{\Gamma}_\alpha^{-1}(u)\right) du$$

satisfies a LDP with good rate function

$$I(y) = \inf_{x\in AC_0} \{\int_0^\infty \sup_\theta (\theta\dot{x}(s) - \chi(\theta))\, ds : y = \int_0^\infty e^{-s} x(s)\, ds\}$$

where $\chi(\cdot)$ *is defined via* $\psi(-\chi(\cdot), \cdot) = 0$.

Proof. It follows just as Theorem 5.

We now are ready to show that $\alpha\bar{D}(a)$ is suitably close to $\alpha D(a)$ in exponential scale.

Lemma 10 *The families* $\{\alpha D(\alpha)\}_{\dot{\alpha}>0}$ *and* $\{\alpha\bar{D}(\alpha)\}_{\alpha>0}$ *are exponentially equivalent. In other words, for each* $\delta > 0$,

$$\overline{\lim}_{\alpha\to 0} \alpha \log P\left(\left|\alpha D(\alpha) - \alpha\bar{D}(\alpha)\right| > \delta\right) = -\infty.$$

Proof. Note that $\left\lceil\widetilde{\Gamma}^{-1}(t)\right\rceil = \Gamma^{-1}(t)$ for almost every t with respect to Lebesgue measure. Therefore, it follows that, for almost every t,

$$\alpha\Lambda\left(\Gamma^{-1}(t/\alpha)\right) - \alpha\bar{\Lambda}\left(\widetilde{\Gamma}^{-1}(t/\alpha)\right) = -\alpha U_\alpha\left(\widetilde{\Gamma}^{-1}(t/\alpha)\right).$$

As a result, we have (making the change of variables $\widetilde{\Gamma}(t/\alpha) = u/\alpha$) that

$$\left|\alpha D(\alpha) - \alpha \bar{D}(\alpha)\right| \leq \alpha \int_0^\infty W(u/\alpha)\, du,$$

where

$$W(u/\alpha) = \exp\left(-\alpha \widetilde{\Gamma}(u/\alpha)\right) |U_\alpha(u/\alpha)| Z(\lfloor u/\alpha \rfloor + 1).$$

Let us define

$$V_1 = \alpha \int_0^{t_0} W(u/\alpha)\, du \quad \text{and} \quad V_2 = \alpha \int_{t_0}^\infty W(u/\alpha)\, du,$$

and consider the sets

$$\begin{aligned}
A_1(t_0, \alpha, \varepsilon) &\triangleq \{\omega : \sup_{t>t_0} \left|\left(\alpha \widetilde{\Gamma}(t/\alpha) - \gamma t\right) t^{-1}\right| \leq \varepsilon\}, \\
A_2(t_0, \alpha, \varepsilon) &\triangleq \{\omega : \sup_{t>t_0} \left|(\alpha \Gamma(t/\alpha) - \gamma t) t^{-1}\right| \leq \varepsilon\}, \\
A_3(t_0, \alpha, m) &\triangleq \{\omega : \sup_{0 \leq t \leq t_0} \left|\alpha \widetilde{\Gamma}(t/\alpha) - \gamma t\right| \leq m\}, \\
A_4(t_0, \alpha, M) &\triangleq \{\omega : \alpha \sum_{k=1}^{\lceil t_0/\alpha \rceil} |X_k| \leq M\}, \\
A_5(t_0, \alpha, \varepsilon) &\triangleq \{\omega : \sup_{k>t_0/\alpha} \frac{|X_k|}{k} \leq \varepsilon\}.
\end{aligned}$$

For notational convenience, we will drop the arguments in the definitions of A_j, $1 \leq j \leq 5$. Using these definitions, we can write

$$\begin{aligned}
& P\left(\left|\alpha D(\alpha) - \alpha \bar{D}(a)\right| > \delta\right) \\
\leq\ & P\left(\alpha \int_0^\infty W(u/\alpha)\, du > \delta; \cap_{k=1}^5 A_k\right) + \sum_{k=1}^5 P(A_k^c).
\end{aligned}$$

Observe that if we write $K_1 = \exp(-m + \gamma)$ then, on $\cap_{k=1}^5 A_k$, we have

$$\begin{aligned}
& V_1 \\
\leq\ & \alpha K_1 \int_0^{t_0} |X(\lceil u/\alpha \rceil + 1)| Z(\lfloor u/\alpha \rfloor + 1) 1(u/\alpha \in [\lceil u/\alpha \rceil - \alpha, \lceil u/\alpha \rceil))\, du \\
\leq\ & \alpha^3 K_1 \sum_{k=1}^{\lceil t_0/\alpha \rceil} |X_{k+1}| Z_k \leq \alpha^3 K_1 \sum_{k=1}^{\lceil t_0/\alpha \rceil} |X_k| \sum_{k=1}^{\lceil t_0/\alpha \rceil} Z_k \leq \alpha^2 K_1 M \sum_{k=1}^{\lceil t_0/\alpha \rceil} Z_k.
\end{aligned}$$

On the other hand, also on $\cap_{k=1}^{5} A_k$, and for $t_0(\varepsilon, \gamma)$ suitably large, there exists a positive constant $K(\varepsilon, \gamma) < \infty$ such that

$$V_2 \leq \alpha K(\varepsilon, \gamma).$$

Thus, if $\alpha < \delta / (2K(\varepsilon, \gamma))$, we have that

$$P\left(\alpha \int_0^\infty W(u/\alpha)\, du > \delta; \cap_{k=1}^{5} A_k\right) \leq P\left(\alpha \sum_{k=1}^{\lceil t_0/\alpha \rceil} Z_k > \delta/(\alpha 2K_1 M)\right).$$

But we know that $\alpha \sum_{k=1}^{\lceil t_0/\alpha \rceil} Z_k$ satisfies an LDP (as $\alpha \to 0$) therefore, we must have that (for fixed ε, γ and large but fixed t_0)

$$\alpha \log P\left(\alpha \int_0^\infty W(u/\alpha)\, du > \delta\right) \to -\infty \text{ as } \alpha \searrow 0.$$

Now we analyze each $P(A_k^c)$ for each $1 \leq k \leq 5$. First, note that (by Lemma 7), t_0 can be chosen so that

$$\overline{\lim}_{\alpha \to 0} \alpha \log P(A_1^c((t_0, \alpha, \varepsilon))) \leq -t_0. \tag{27}$$

Because $(\alpha \Gamma(\cdot/\alpha) - \gamma \cdot)_{\alpha > 0}$ satisfies a full LDP on $D[0, \infty)$ endowed with the topology generated by the uniform convergence on compact sets (see Theorem 2.2.1 of Zajic (1993)), it follows that the same argument provided for the proof of condition (20), applies in this case as well. This implies that a bound such as (27) also applies for the set A_2^c. Observe that

$$\alpha \log P(A_3^c(t_0, \alpha, m)) \to -J(m),$$

for some convex good rate function $J(\cdot)$ (by definition of full LDP, see Dembo and Zeitouni (1999)). Now, for A_4, we can use Chebyshev's bound to obtain

$$\begin{aligned} \alpha \log P(A_4^c(t_0, \alpha, M)) &\leq \alpha \left\lceil \frac{t_0}{\alpha} \right\rceil \log E \exp\left(\sum_{k=1}^{\lceil t_0/\alpha \rceil} |X_k|\right) - M \\ &\leq \alpha g(0, 1) - M \to -M \quad \text{as} \quad \alpha \searrow 0. \end{aligned}$$

Finally, for A_5, we have

$$\begin{aligned} P\left(A_5^c\left(t_0,\alpha,\varepsilon\right)\right) &\leq \sum_{k>t_0/\alpha}^{\infty} P\left(\frac{|X_k|}{k}>\varepsilon\right) \\ &\leq \sum_{k>t_0/\alpha}^{\infty} \exp\left(-\varepsilon k\right) E\exp\left(|X_k|\right) \\ &\leq \frac{\exp\left(g\left(0,1\right)\right)\exp\left(-\varepsilon\left\lceil t_0/\alpha\right\rceil\right)}{1-\exp\left(\varepsilon\right)}. \end{aligned}$$

thus, for $\varepsilon > 0$ small but fixed,

$$\alpha \log P\left(A_5^c\left(t_0,\alpha,m\right)\right) \to -\varepsilon t_0.$$

Combining the previous estimates, we conclude that

$$\overline{\lim}_{\alpha\to 0}\alpha \log P\left(\left|\alpha D\left(\alpha\right)-\alpha \bar{D}\left(a\right)\right|>\delta\right) \leq -\left(2+\varepsilon\right)t_0 - M - J\left(m\right).$$

since $J\left(\cdot\right)$ is a convex good rate function, the previous quantity in the right hand side tends to infinity as $m, t_0, M \nearrow \infty$, which yields the proof of the lemma.

We are now in position to identify the rate function required to make practical use of approximation (18) and under which the LDP for $\alpha D\left(\alpha\right)$ holds. Define (as in Zajic (1993) p. 9)

$$\psi\left(\eta,\theta\right) = \lim_{n\to\infty}\frac{1}{n}\log E\exp\left(\eta\Gamma\left(n\right)+\theta\Lambda\left(n\right)\right) < \infty.$$

Theorem 8 *Suppose that ADL1 and ADL2 hold. Then, if $y > \lambda/\gamma$,*

$$\begin{aligned} &\lim_{\alpha\to\infty}\alpha\log P\left(\alpha D\left(\alpha\right)\geq y\right) \\ &= \inf_{x\in AC_0}\left\{\int_0^\infty \sup_\theta\left(\theta\dot{x}\left(s\right)-\chi\left(\theta\right)\right)ds : y = \int_0^\infty e^{-s}x\left(s\right)ds\right\}, \end{aligned}$$

where AC_0 is the space of absolutely continuous functions, defined on the interval $[0,\infty)$, that vanish at the origin. In addition, if there exists $\theta^* = \theta^*\left(y\right)$ such that $y\theta^* = \chi\left(\theta^*\right)$, then

$$\begin{aligned} \lim_{\alpha\to\infty}\alpha\log P\left(\alpha D\left(\alpha\right)\geq y\right) &= \sup_\theta\left(y\theta - \int_0^\infty \chi\left(\theta e^{-s}\right)ds\right) \\ &= y\theta^* - \int_0^{\theta^*}\frac{\chi\left(u\right)}{u}du. \end{aligned}$$

Proof. We know that $\{\alpha D(\alpha)\}_{\dot{\alpha}>0}$ and $\{\alpha \bar{D}(a)\}_{\alpha>0}$ are exponentially equivalent. On the other hand, Proposition 6 indicates that $\alpha \bar{D}(a)\}_{\alpha>0}$ satisfies a full LDP. Thus, By Theorem 4.2.13, p. 130, of Dembo and Zeitouni (1999) $\{\alpha D(\alpha)\}_{\dot{\alpha}>0}$ must also satisfy a full LDP with the same rate function. The identification of the rate function follows as in Theorem 6.

The corresponding exact large deviations asymptotic is provided under the iid setting described in Subsection 5.1. Under those conditions, if $x >> \lambda/\gamma$, we shall provide rigorous justification for the approximation

$$P(D > x) \approx \frac{\sqrt{\gamma}}{\theta^* \sqrt{\pi \chi''(\theta^*)}} \exp\left(-\left(x\theta^* - \int_0^{\theta^*} \frac{\chi(u)}{u} du\right)/\gamma\right),$$

where $x\theta^* = \chi(\theta^*)$, $\chi(\cdot)$ satisfies $\psi(-\chi(\cdot), \cdot) = 0$, and $\psi(\eta, \theta) = \log E \exp(\eta Z + \theta X)$. As usual, the approximation will be shown to hold in the regime of small interest rates. That is, we will show that the previous approximation is valid for the discrete time discounted reward $D(\alpha) = \sum_{k=0}^{\infty} \exp\left(-\alpha \sum_{j=0}^{k-1} Z_j\right) X_k$. The proof of the next theorem follows the same strategy as that of Theorem 7.

Theorem 9 *Suppose that $(X_k, Z_k)_{k\geq 0}$ is an iid sequence of random variables. Suppose that $Z_k > 0$ and that for all $\eta, \theta \in \mathbb{R}$ we have that $E\exp(\theta Z_1 + \eta X_1) < \infty$. In addition, assume that conditions AI2 and AI3 of Subsection 5.1 hold. Let $\chi(\theta)$ be defined as the solution to*

$$\psi(-\chi(\theta), \theta) = 0,$$

where $\psi(\eta, \theta) = \log E \exp(\eta Z_1 + \theta X_1)$. Suppose that $x > \lambda/\gamma$ and let θ^* be the solution of $x\theta^* = \chi(\theta^*)$. Then,

$$P(\alpha D(\alpha) > x) \sim \alpha^{1/2} \frac{\exp(-I(x)/\alpha)}{\theta^* \sqrt{\pi \chi^{(2)}(\theta^*)}} \text{ as } \alpha \searrow 0,$$

with $I(x) = x\theta^* - \int_0^{\theta^*} \frac{\chi(u)}{u} du$.

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