

Uniform Renewal Theory with Applications to Expansions of Random Geometric Sums

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July, 2005.

Abstract

Consider a sequence $X = (X_n : n \geq 1)$ of iid random variables (r.v.'s) and a geometrically distributed r.v. M with parameter p and independent of X . The r.v. $S_M = X_1 + \dots + X_M$ is called a geometric sum. In this paper, we obtain asymptotic expansions for the distribution of S_M as $p \searrow 0$. If $EX > 0$ the asymptotic expansion is developed in powers of p and it provides higher order correction terms to Renyi's theorem, which states that $P(pS_M > x) \approx \exp(-x/EX)$. On the other hand, if $EX = 0$ the expansion is given in powers of \sqrt{p} . We apply the results to obtain corrected diffusion approximations for the M/G/1 queue. These expansions follow in a unified way as a consequence of uniform renewal theory results that are also developed in this paper.

1 Introduction

Consider a sequence $X = (X_k : k \geq 1)$ of independent and identically distributed (iid) random variables (rv's) and let M be a geometrically distributed random variable independent of X . That is,

$$P(M = k) = p(1 - p)^{k-1} = pq^{k-1}; \quad k \geq 1.$$

Renyi's theorem for geometric sums of random variables establishes that if $0 < EX_1 < \infty$, then

$$P(pS_M > x) = \exp(-x/E(X_1)) + o(1) \tag{1}$$

as $p \searrow 0$, where $S_M = X_1 + \dots + X_M$ is a geometric sum. One of the main objectives of this paper is to provide (under regularity conditions to be discussed in Sections 2 and 3) higher order correction terms (in powers of p) to approximation (1). The asymptotic expansions that we shall develop are closely related to so-called "corrected diffusion approximations" (CDA's). CDA's for random walks were introduced by Siegmund (1979), who provided a correction for the Brownian approximation to the distribution of the maximum of a random walk (rw) with small negative drift. In order to see the

connection to CDA's note that a strong invariance principle (see Csorgo and Revesz (1981)) indicates that if $EX^2 < \infty$, then

$$S_{\lfloor t \rfloor} = (EX)t + (\text{Var}(X))^{1/2} B(t) + o(t^{1/2}) \quad (2)$$

as $t \nearrow \infty$, where $B(\cdot)$ is a standard Brownian motion. As a consequence, (1) follows directly from (2) by letting $t = M$ and using the fact that $pM \Rightarrow \text{Exp}(1)$ as $p \searrow 0$. Note, in addition, that when $EX = 0$ (2) also implies that

$$p^{1/2} S_M \Rightarrow \sigma T / 2^{1/2}, \quad (3)$$

where T follows the Double exponential or Laplace's distribution (c.f. Kalashnikov (1997)). Here we also develop higher order correction terms that complement the weak convergence result (3). As we shall see, in the mean zero case the asymptotic expansion developed is obtained in powers of $p^{1/2}$, which may have been expected given the scaling present in (3). However, it is interesting to remark that both asymptotic expansions (for the case of positive mean and the zero mean case) can be obtained via essentially the same techniques in a unified way. In particular, the asymptotic expansions developed here follow as a consequence of a uniform renewal theorem of independent interest that is also obtained in this paper. This uniform renewal theorem is closely related to previous results in the literature. We mention, for instance, the work by Siegmund (1979), Borovkov and Foss (2000), and Fuh (2004). A crucial difference between our uniform renewal theorem and previous related results is that we can allow the mean of the increment distribution underlying the associated renewal equation to be arbitrarily close to zero. This feature is required in order to obtain the asymptotic expansions in the zero mean case corresponding to (3).

As we have indicated, the results developed here can be seen as CDA's for geometric sums. We offer correction terms in the same spirit as Siegmund's CDA's. The theory we develop here also covers the situation in which the increment distribution fails to have exponential moments and plays a key role in our development elsewhere of CDA's for the maxima of rw with heavy-tailed distributions. Our theory here also has the simplifying characteristic that the form of the expansion is given in terms of moments of the X_i 's, which we assume here are readily computable. In the setting of maxima of rw, the corresponding X_i 's are the ascending ladder heights and the computation of the moments from the underlying rw increment distribution is non-trivial; see the papers by Chang and Peres (1997), who address this problem for the case of Gaussian increments and Blanchet and Glynn (2005a) for general strongly non-lattice distributions with exponential moments.

The rv S_M is utilized in many applied probability settings. For example, in queueing theory, it is well known (by appealing to the ascending ladder heights representation for the maximum of random walk) that the steady-state waiting time distribution of the standard single server queue can be represented as a geometric sum (c.f. Asmussen (1987) or Kalashnikov (1997), Section 1.3.3). In insurance risk

theory, the ruin probability in the renewal model can also be expressed as a tail probability of a geometric sum increments (see Asmussen (2001) or Kalashnikov (1997), Section 1.3.4). Finally, in the context of reliability models, the first break-down time of a system that consists of an operating element, $N - 1$ unloaded redundant elements and M identical repair units, can also be expressed as a geometric sum such as S_M (refer also to Kalashnikov (1997), Section 1.3.5). Other applications include program debugging and the total reward until visiting a rare set in a Markov setting. (See the book on geometric sums by Kalashnikov (1997) for additional details.)

The asymptotic regime that we study here (when p is close to zero) arises naturally in applications contexts such as those previously discussed. For instance, in the queueing example mentioned in the previous paragraph, this setting corresponds to the so-called heavy traffic regime in which the server utilization is close to 100%; in the risk insurance context, p close to zero describes the setting in which the security margin, included in the risk premium received by the insurance company, is close to zero. Finally, in the reliability example, p close to zero reflects a setting with a low break-down rate. In several of these examples, the distribution of the increments X_k depends on p as well. We must therefore develop a theory that can handle this dependence. As an important application of the results developed in this paper (in particular Theorem 4) Blanchet and Glynn (2005b) develop approximations for the maximum of random walk with heavy-tailed increments. These approximations are very useful in some of the applied settings mentioned at the beginning of our discussion (e.g. the steady-state distribution of the single server queue and the ruin probability in the insurance context).

The contributions of this paper are the following:

a) We obtain uniform rates of convergence in the renewal theorem when the underlying distributions of the renewals belong to a general class. These uniform rates of convergence are of independent interests and are utilized to obtain the results that we describe next.

b) When the increments - i.e. the X_i 's - have positive mean and are strongly non-lattice (see the definition in Sections 2 and 3) we develop asymptotic expansions for the distribution of pS_M . These expansions are given in powers of p .

c) If $EX_i = 0$ (also under strongly non-lattice assumptions) we develop asymptotic expansions for the distribution of the r.v. $p^{1/2}S_M$. In this case the expansion is derived in powers of $p^{1/2}$.

d) The expansions indicated in b) and c) can be applied to the case in which the distribution of the increments varies as $p \searrow 0$, a case that is of interest in applications to queueing theory and insurance risk theory.

e) We develop corrected diffusion approximations for the steady-state waiting time of the M/G/1 queue (or the ruin probability in the classical risk model) for both light-tailed and heavy-tailed service time distributions (or claim sizes in the insurance context).

The rest of the paper is organized as follows. In Section 2 we introduce

a uniform renewal theorem that we shall use repeatedly in the rest of the paper. In Section 3 the asymptotic expansions for $P(pS_M > x)$ are derived for the case in which the X_i 's are non-negative and possess a finite moment generating function. Section 4 develops the asymptotic expansions for non-negative increments under the existence of at least finite second moment. Our expansions are extended to the case of increments with positive mean and taking values in the whole real line in Section 6. In Section 7 we obtain the expansion for zero mean increment distributions. Finally, in Section 8, we apply our results to develop corrected diffusion approximations (CDA's) for the M/G/1 queue.

2 A Uniform Renewal Theorem

As we indicated in the introduction, uniform renewal theory will be the main tool that we shall use to derive our stated asymptotic expansions for Renyi's theorem. In particular, the next result will be key in the completion of our program.

Theorem 1 *Let \mathcal{F} be a family of distribution functions. For each $F \in \mathcal{F}$, let $E_F(\cdot)$ be the expectation operator associated to $F \in \mathcal{F}$, and define $E_F g(\tau) \triangleq \int_{-\infty}^{\infty} g(t) F(dt)$ for each continuous and bounded function $g : \mathbb{R} \rightarrow \mathbb{R}$. Suppose that the family \mathcal{F} is uniformly strongly non-lattice, (i.e. the corresponding characteristic functions $\chi_F(\lambda) = E_F \exp(i\lambda\tau)$ satisfy*

$$\inf_{F \in \mathcal{F}} \inf_{|\lambda| > \varepsilon} |1 - \chi_F(\lambda)| > 0, \quad (4)$$

and $E_F(\tau) > 0$ and $E_F(|\tau|) < \infty$. Then, $U_F(t) \triangleq \sum_{n=0}^{\infty} F^{*n}(t)$ satisfies the following.

1. If $\sup_{F \in \mathcal{F}} E_F \exp(\eta|\tau|) < \infty$ for some $\eta > 0$, then we can find positive constants K_1, c independent of F such that

$$E_F^2(\tau) \left| U_F(t) - \left(\frac{t}{E_F\tau} + \frac{E_F\tau^2}{2E_F^2\tau} \right) 1(t \geq 0) \right| \leq K_1 e^{-ct},$$

for $t \geq 0$ and

$$E_F^2(\tau) U_F(t) \leq K_1 e^{ct} + E_F(\tau) K_1 e^{cE_F(\tau)t}$$

for $t < 0$.

2. Moreover, under the condition $\sup_{F \in \mathcal{F}} E_F |\tau|^{\varepsilon+2} < \infty$ for $\varepsilon \geq 0$, we obtain that

$$\sup_{F \in \mathcal{F}} \mu_F^4 \left| U_F(t) - \tilde{U}_F(t) \right| \leq o\left(|t|^{-\lfloor \varepsilon+2 \rfloor} \log(|t|)\right)$$

as $|t| \rightarrow \infty$, where $\tilde{U}_F(t)$ is defined as follows. First define

$$H_1^F(t) = \begin{cases} \int_t^\infty (1 - F(s)) ds, & t \geq 0 \\ \int_t^\infty F(s) ds, & t < 0 \end{cases},$$

also let

$$H_2^F(t) = \begin{cases} -\int_t^\infty H_1^F(s) ds & t \geq 0 \\ \int_{-\infty}^t H_1^F(s) ds & t < 0 \end{cases}.$$

Then, $\tilde{U}_F(t)$ satisfies

$$\tilde{U}_F(t) = \begin{cases} t/E_F\tau + E_F\tau^2/2E_F^2\tau + H_2^F(t)/E_F^2\tau + H_1^F * H_1^F(t)/E_F^3\tau & t \geq 0 \\ H_2^F(t)/E_F^2\tau + H_1^F * H_1^F(t)/E_F^3\tau & t < 0 \end{cases}.$$

The proof of this result is given at the end of this section.

Condition (4) is crucial in order to guarantee the validity of the uniform renewal results indicated above. Often, it will be useful to verify the following alternative version of (4).

Lemma 1 *Let \mathcal{F} be a family of distribution functions. For each $F \in \mathcal{F}$, let $E_F(\cdot)$ be the expectation operator associated to $F \in \mathcal{F}$. Suppose that the family \mathcal{F} is tight and that the corresponding characteristic functions $\chi_F(\lambda) = E_F \exp(i\lambda\tau)$ satisfy*

$$\inf_{F \in \mathcal{F}} \inf_{|\lambda| \geq 1} |1 - \chi_F(\lambda)| > 0. \quad (5)$$

Then, the family \mathcal{F} is uniformly strongly non-lattice in the sense that (4) is satisfied.

Remark Note that if $\sup_{F \in \mathcal{F}} E_F|\tau| < \infty$, then the family \mathcal{F} is tight. Therefore, given that we must verify the moment assumptions 1. and 2. in order to apply Theorem 1, it suffices to just verify condition (5) instead of (4)

Proof of Lemma 1. Fix $\varepsilon > 0$ and note that

$$\inf_{F \in \mathcal{F}} \inf_{|\lambda| \geq \varepsilon} |1 - \chi_F(\lambda)| = \inf_{F \in \mathcal{F}} \min \left(\inf_{|\lambda| \geq 1} |1 - \chi_F(\lambda)|, \inf_{\varepsilon \leq |\lambda| \leq 1} |1 - \chi_F(\lambda)| \right),$$

so we just have to show that

$$g_F = \inf_{\varepsilon \leq |\lambda| \leq 1} |1 - \chi_F(\lambda)|$$

satisfies $\inf_{F \in \mathcal{F}} g_F > 0$. Suppose that this is not the case, then there exists a sequence of distributions $(F_n : n \geq 1)$ in \mathcal{F} such that $g_{F_n} \rightarrow 0$ as $n \rightarrow \infty$. Since \mathcal{F} is tight, there must be a subsequence of distributions $(F_{n_k} : k \geq 1)$ such that $F_{n_k} \Rightarrow G$ (G

may not necessarily be in \mathcal{F}). It follows then that $\chi_G(\lambda) = 1$ for some $\lambda \in [\varepsilon, 1]$, which implies that G is lattice. On the other hand, we have that

$$0 < \lim_{k \rightarrow \infty} \left| 1 - \chi_{F_{n_k}}(\lambda) \right| = |1 - \chi_G(\lambda)|,$$

which implies that G is non-lattice, which is a contradiction. Thus, we must have $\inf_{F \in \mathcal{F}} g_F > 0$ and the proof is complete. ■

Theorem 1 is of independent interest, we therefore shall discuss its content and related results in the literature. Uniform renewal theorems have been previously studied in the literature. Siegmund (1979) provides a result similar to part 1. However, Siegmund's result only covers the case of positive increments and the underlying family of distributions enjoys a particular parameterization through exponential families. The fact that Theorem 1 is developed on the whole line with an explicit dependence on the mean of the increment distributions underlying the renewal function (i.e. μ_F) will be crucial in our developments, specially for the case in which the $E(S_M) = 0$. Fuh (2004) also studies uniform renewal theorems on the whole line under the same assumptions as in Theorem 1, however, there is no dependence on the mean of the increment distributions of the renewal function in his estimates. Theorem 1 is key for our purposes because we are interested in understanding how the distribution of S_M changes in general - even when we have zero mean increments distributions. Borovkov and Foss (2000) also developed a uniform renewal theorem under non-lattice assumptions as those imposed here, again, they consider the case in which the mean of increment distributions is bounded away from zero and therefore that theory does not apply directly to our current situation.

2.1 Proof of Theorem 1

The proof of Theorem 1 utilizes results from Carlsson (1983) and Stone (1965a) that we write here for completeness.

Lemma 2 (Stone (1965a)) *Assume that $\mu \triangleq EX > 0$ and set $F(\cdot) = P(X \leq \cdot)$. In addition, let $\chi(\lambda) = E \exp(\lambda X)$ and suppose that X is strongly non-lattice in the sense that*

$$\inf_{|\lambda| > \varepsilon} |1 - \chi(\lambda)| > 0.$$

Finally, define $U(x) = \sum_{n=0}^{\infty} F^{*n}(x)$ and set

$$V(x, h, a) = E(U(x + h - aZ) - U(x - aZ)),$$

where $Z \sim N(0, 1)$. Then,

$$\begin{aligned} & V(x, h, a) \\ &= h/2\mu + h/(2\pi) \int_{-\infty}^{\infty} \operatorname{Re} e^{-ix\lambda} (1 - e^{-ih\lambda}) e^{-a^2\lambda^2/2} / (ih\lambda(1 - \chi(\lambda))) d\lambda. \end{aligned}$$

Moreover, $V(x, h, a) - h/\mu = O(e^{-rx})/a$ for some $r > 0$ uniformly for h and a in bounded sets.

Proof. This is just the statement given in Equation (7) of Stone (1965a). ■

In order to state the next result of Carlsson (1983) we need to recall some basic definitions. We say that a continuous function is of rapid decrease if its absolute value decreases to zero faster than any polynomial. We say that $g : \mathbb{R} \rightarrow \mathbb{R}$ is a tempered function if $g(x) = \mu(-\infty, x]$ and the measure μ is in the (analytic) dual space of $\mathcal{S}(\mathbb{R})$ - this dual space is also known as the Laurent Schwartz space of tempered distributions and is denoted by $\mathcal{S}'(\mathbb{R})$. In turn, $\mathcal{S}(\mathbb{R})$ denotes the space of functions of rapid decrease that possess derivatives of all orders and these derivatives are also of rapid decrease. Note that $\mathcal{S}'(\mathbb{R})$ is the analytic dual of $\mathcal{S}(\mathbb{R})$, which means that we are just considering continuous linear functionals on $\mathcal{S}(\mathbb{R})$. Of course, we are assuming an analytical structure on $\mathcal{S}(\mathbb{R})$ that induces the continuity of a linear functional in $\mathcal{S}'(\mathbb{R})$. This analytical structure is given through the metric

$$d(f, f') = \sum_{r,m} 2^{(-r+m)} \frac{\max_{q \leq r} \sup_{x \in \mathbb{R}} (1+x^2)^m |f^{(q)}(x)|}{1 + \max_{q \leq r} \sup_{x \in \mathbb{R}} (1+x^2)^m |f^{(q)}(x)|},$$

and we say that $\mu \in \mathcal{S}'(\mathbb{R})$ if whenever $f_n \rightarrow f$ we have $\int f_n d\mu \rightarrow \int f d\mu$ (see, for example, Malliavin et al (1995) p. 149).

Lemma 3 (Carlsson (1983)) Define \mathcal{M}_b to be class of those non-decreasing functions $f : \mathbb{R} \rightarrow [1, \infty)$ that satisfy $f(2x) \leq bf(x)$ and let g be a tempered function such that

$$\sup_{x \leq y \leq x+1/T} (g(y) - g(x)) \leq K/m_T(x), \text{ for some } m_T \in \mathcal{M}_b.$$

Also, assume that $\hat{g}(\theta) = \int_{-\infty}^{\infty} e^{-i\theta s} dg(s)$ has its support outside $[-T, T]$. Then,

$$|g(x)| \leq CK/m_T(x),$$

where C depends on b but not T for $T \geq 1$.

Proof. This is Lemma 1 of Carlsson (1983). ■

Proof of Theorem 1. We first shall prove part 1. First let us define $\mu_F = E_F \tau$ and $v_F = E_F \tau^2$. The idea is to use Lemma 2 (under the notation defined there) to first estimate $V(x, h, a)$. More precisely, we note that

$$\begin{aligned} & \int_{-\infty}^{\infty} e^{-ix\lambda} (1 - e^{-ih\lambda}) (ih\lambda)^{-1} e^{-a^2\lambda^2/2} / ((1 - \chi_F(\lambda))) d\lambda \\ = & \int_{-\infty}^{\infty} e^{-ix\lambda} (1 - e^{-ih\lambda}) (ih\lambda)^{-1} e^{-a^2\lambda^2/2} / (-i\mu_F\lambda + v_F\lambda^2/2) d\lambda \end{aligned} \quad (6)$$

$$\begin{aligned} & + \int_{-\infty}^{\infty} e^{-ix\lambda} (1 - e^{-ih\lambda}) (ih\lambda)^{-1} e^{-a^2\lambda^2/2} \\ & (1/(1 - \chi_F(\lambda)) - 1/(-i\mu_F\lambda + v_F\lambda^2/2)) d\lambda. \end{aligned} \quad (7)$$

Observe that integrand corresponding to (7) is analytic on a strip defined by the imaginary axis and a parallel line to this axis (both to the right or the left). This is because we are removing the singularity of $1/(1 - \chi_F(\lambda))$ at zero by subtracting $1/(-i\mu_F\lambda + v_F\lambda^2/2)$. This step is different from that of Stone (1965a), we have added the term including v_F because we are also interested in the case in which μ_F can be arbitrarily small. Subtracting the term $1/(-i\mu_F\lambda + v_F\lambda^2/2)$ and using the uniformly strongly non-lattice assumption allows us to apply Cauchy's theorem in order to rewrite, for some $c > 0$ sufficiently small (uniformly over the family \mathcal{F}), (7) as

$$\begin{aligned} & 1/h \int_{-\infty}^{\infty} e^{-(c+i\lambda)x} (1 - e^{-(c+i\lambda)h}) (c+i\lambda)^{-1} e^{a^2(c+i\lambda)^2/2} \\ & (1/(1 - \chi_F(c+i\lambda)) - 1/(-\mu_F(c+i\lambda) + v_F(c+i\lambda)^2/2)) d\lambda. \end{aligned}$$

Note that this previous expression is (using a form of the Riemann-Lebesgue lemma) of order $O(e^{-cx}(1 + |\log(a)|))$ as $x \rightarrow \infty$ uniformly for h in bounded sets, and uniformly over the family \mathcal{F} . Note that a completely analogous estimate can be obtained as $x \rightarrow -\infty$. On the other hand, the integral in (6) can be rewritten as

$$\int_{-\infty}^{\infty} e^{-ix\lambda} (1 - e^{-ih\lambda}) (ih\lambda)^{-1} e^{-a^2\lambda^2/2} / (-i\mu_F\lambda) d\lambda \quad (8)$$

$$\begin{aligned} & + \int_{-\infty}^{\infty} e^{-ix\lambda} (1 - e^{-ih\lambda}) (ih\lambda)^{-1} e^{-a^2\lambda^2/2} \\ & [1/(-i\mu_F\lambda + v_F\lambda^2/2) - 1/(-i\mu_F\lambda)] d\lambda. \end{aligned} \quad (9)$$

Let us refer to the integrals in (8) and (9) as J_1 and J_2 respectively. J_1 corresponds to the expression following Equation (7) of Stone (1965a). One can actually evaluate J_1 explicitly (as an inverse Fourier transform) and see that for every $c > 0$

$$\mu_F J_1 = \pm h/2 + O(e^{-c|x|}) \text{ as } x \rightarrow \pm\infty$$

uniformly over the family \mathcal{F} (compare Stone (1965a) next to his Equation (7)). Evaluation of the integral J_2 is also not difficult, in fact we have that

$$\begin{aligned} & 4h\mu_F^2 J_2 / (2\pi v_F) \\ & = P(x/2a \leq N(0,1) \leq (x+h)/2a) \\ & + P(N(0,1) > (x+2b_F a^2)/a) \exp(\mu_F(\mu_F a^2 + x)) \\ & - P(N(0,1) > (x+h+2b_F a^2)/a) \exp(\mu_F(\mu_F a^2 + x+h)), \end{aligned}$$

where $b = 2\mu_F/v_F$. We therefore have that $\mu_F^2 J_2 = O(e^{-cx})$ as $x \rightarrow \infty$ uniformly over the family \mathcal{F} and for a in bounded intervals, whereas $|\mu_F^2 J_2| \leq \mu_F K_1 e^{x\mu_F}$ as $x \rightarrow -\infty$, uniformly for a in bounded sets with K_1 independent of F . Consequently,

$$\mu_F^2 V_F(x, h, a) = h\mu_F + O(e^{-cx}(1 + |\log(a)|)),$$

as $x \rightarrow \infty$, where the term $O(e^{-cx}(1 + |\log(a)|))$ does not depend on the choice of F and similarly

$$\mu_F^2 V_F(x, h, a) \leq O(e^{cx}(1 + |\log(a)|)) + \mu_F K_1 e^{x\mu_F}$$

as $x \rightarrow -\infty$. Next we want to use an argument similar to that given by Stone (1965a) to estimate $U_F(x+1) - U_F(x) - \mu_F^{-1}$. The crucial step involved in providing a uniform version of this portion of Stone's argument is to observe that $K > 0$ (independent of F) can be chosen such that

$$\mu_F^2 (U_F(x+h) - U_F(x)) \leq K$$

uniformly over $|h| \leq 2$. Using this bound, it is not hard to see (just following the discussion between Equations (8) and (9) of Stone (1995a) that

$$\begin{aligned} & V_F(x + e^{-cx}, 1 - 2e^{-cx}, e^{-cx}/x) \\ & \leq U_F(x+1) - U_F(x) \\ & \leq V_F(x - e^{-cx}, 1 + 2e^{-cx}, e^{-cx}/x), \end{aligned}$$

where $c > 0$ is taken uniformly over the family \mathcal{F} . Which implies that

$$\mu_F^2 (U_F(x+1) - U_F(x) - \mu_F^{-1}) = O(e^{-cx}), \quad (10)$$

uniformly over \mathcal{F} as $x \rightarrow \infty$, and

$$\mu_F^2 |U_F(x+1) - U_F(x)| = O(e^{cx}) + \mu_F K_1 e^{x\mu_F} \quad (11)$$

also uniformly over \mathcal{F} as $x \rightarrow -\infty$. Next, we proceed to show that

$$\mu_F^2 (U_F(x) - (x\mu_F^{-1} + v_F/(2\mu_F)) 1(x \geq 0)) = o(1) \quad (12)$$

as $x \rightarrow \infty$, uniformly over \mathcal{F} . To show (12) we follow closely the ideas of Carlsson (1983) who applies Lemma 3 to a modification of

$$G_F(x) \triangleq U_F(x) - (x\mu_F^{-1} + v_F/(2\mu_F)) 1(t \geq 0) + S_F(x)/\mu_F^2.$$

More precisely, Carlsson (1983) shows that the Fourier transform \widehat{G}_F of G_F is given by

$$\widehat{G}_F(\lambda) \triangleq (\chi_F(\lambda) - 1 + i\lambda\mu_F)^2 (i\lambda)^{-1} (i\lambda\mu_F)^{-2} (1 - \chi_F(\lambda))^{-1},$$

(see the end of Section 3 of Carlsson (1983)). The strategy then is to consider $G_F^T(\cdot)$ and $G_F^*(\cdot)$ via

$$\begin{aligned} G_F^T(x) & \triangleq G_F(x) - \frac{1}{2\pi} \int_{-T}^T \widehat{G}_F(\lambda) \exp(i\lambda x) d\lambda, \\ & \triangleq G_F(x) - G_F^*(x), \end{aligned}$$

note that $\widehat{G}_F^T(\lambda)$ is zero for $|\lambda| \leq T$ (this prepares the way for Lemma 3). It follows that we can find a constant K (independent of F) such that, for $x \geq 1$,

$$\sup_{x \leq y \leq x+1/T} \mu_F^2 |S_F(x) - S_F(y)| \leq \mu_F^2 (S(x) - S(x+1/T)) \leq K/T.$$

Also observe the $\mu_F G_F^*(\cdot)$ is bounded (for fixed T) and that

$$\limsup_{x \rightarrow 0} \sup_F |\mu_F^2 G_F^*(x)| = 0 \tag{13}$$

(this last step follows using the strongly non-lattice condition, the form of \widehat{G}_F and a variant of the Riemann-Lebesgue lemma). These observations, combined with our previous estimates concerning $U_F(x+h) - U_F(x)$ yield that

$$\sup_{x \leq y \leq x+1/T} \mu_F^2 (G_F^T(x) - G_F^T(y)) \leq K/T$$

for some positive constant $K > 0$ (independent of F). Consequently, Lemma 3, applied to G_F^T , together with (13) yield that

$$\limsup_{x \rightarrow \infty} \sup_F \mu_F^2 |G_F(x)| \leq K/T.$$

Since T was arbitrary, we obtain (12). This combined with (10) and (11) gives part 1.

Part 2 follows the same steps as in Carlsson (1983), the key assumption again is the uniform strongly non-lattice condition (4). The Fourier inversion expressions are provided by Carlsson (1983) for each fixed F . In particular, the Fourier transform of \widetilde{U}_F (see the definition in part 2 of the present theorem)

$$\Gamma_F(\lambda) = (\chi_F(\lambda) - 1 + i\lambda\mu_F)^4 \mu_F (i\lambda\mu_F)^{-5} (1 - \chi_F(\lambda))$$

and the estimates again involve applications of Lemma 3. Carlsson's argument follows after multiplying Γ_F by μ_F^4 and using the strongly non-lattice assumption just as we described for the case of G_F in part 1 above. ■

3 Non-Negative Geometric Sums with Exponential Moments

In this section and through out the rest of the paper, we suppose that X_1 is strongly non-lattice in the sense that its characteristic function, $g(\lambda) = E \exp(i\lambda X_1)$, satisfies that for every $\varepsilon > 0$

$$\inf_{|\lambda| > \varepsilon} |1 - g(\lambda)| > 0 \tag{14}$$

or, equivalently, that $\overline{\lim}_{|\lambda| \rightarrow \infty} |g(\lambda)| < 1$ (see Siegmund (1985) p. 176).

In this section we assume that $X = (X_n : n \geq 1)$ is an iid sequence of non-negative rv's such that $\phi(\theta) \triangleq E \exp(\theta X_1) < \infty$ for some $\theta > 0$ and set $S_n = X_1 + \dots + X_n$ (with $S_0 = 0$). The main result of this section is the following.

Theorem 2 *Suppose that X_1 has strongly non-lattice distribution and that $\phi(\eta) \triangleq E \exp(\eta X) < \infty$ for some $\eta > 0$. Then, for some $a > 0$, and as $x/p \rightarrow \infty$,*

$$P(pS_M > x) = \exp\left(-x\tilde{\theta}/p + r(p)\right) + O(\exp(-ax/p)), \quad (15)$$

where $\tilde{\theta}$ is the unique non-negative solution to $E \exp(\tilde{\theta} X_1) = q^{-1}$ (which exists if $p > 0$ is small enough), and

$$\exp(r(p)) = \frac{p}{q^2 \tilde{\theta} \phi'(\tilde{\theta})} \triangleq c(p). \quad (16)$$

Remark 1 Since the error term indicated in (15) contains explicit information about the error term in terms of x , the theorem holds even if we allow x to depend on p , as long as $x(p)/p \rightarrow \infty$.

Remark 2 We decided to state our approximation in terms of $r(p)$ because later we shall be interested in suggesting approximations that are given as asymptotic expansions powers of p . Using the asymptotic expansions for $r(\cdot)$ instead of $c(\cdot)$ preserves the positivity of our suggested approximation.

We shall provide the mathematical support behind this theorem during the course of the present section. However, before doing so, let us discuss some of the implications of this result and its connections to related literature. Theorem 2 provides rigorous support for the approximation

$$P(S_M > y) \approx \exp\left(-y\tilde{\theta} + r(p)\right). \quad (17)$$

Note that, for a fixed value p , the evaluation of $\tilde{\theta}$ and $r(p)$ is a straightforward numerical task, and thus (17) can be easily implemented to provide an accurate approximation for the distribution of S_M . In fact, (17) corresponds to the so-called Cramer-Lundberg approximation that has been successfully applied in applications to insurance and queueing. Cramer-Lundberg's asymptotic result is derived, for fixed $p > 0$, as $y \rightarrow \infty$. Theorem 2 indicates that Cramer-Lundberg's approximation remains valid uniformly for $p > 0$ sufficiently small. A careful analysis of the rate of convergence in the standard Cramer-Lundberg asymptotic is crucial in order to study the distribution of S_M parametrically as $p \searrow 0$.

Once we know that (17) holds uniformly for small values of p , then we can easily provide (via the implicit function theorem) an expansion for $\tilde{\theta}$ in powers of p , and, as a consequence, for $r(p)$. We record these observations in our next proposition.

Proposition 1 Let $\tilde{\theta}$ and r be defined as in Theorem 2. Then, $\tilde{\theta}(p)$ and $r(p)$ can be expanded in absolutely convergent power series in $p \in [0, \delta]$ for some $\delta > 0$.

Proof. The fact that the series converges absolutely just follows by applying the inverse function theorem (for complex valued functions) to the analytic extension of $\phi(\cdot)$ at zero. ■

If our interest is to provide accurate approximations for the distribution of S_M for many different small values of p , an asymptotic expansion such as that suggested by Theorem 2 and the previous proposition for $P(S_M > \cdot)$ will typically provide a more convenient approximation (in terms of the computational cost) than (17). Let us then provide the elements of the expansion described in Proposition 1, thereby giving the expansion of $P(S_M > x/p)$ in powers of p . Proposition 1 indicates that

$$\tilde{\theta}(p) = \sum_{k=1}^{\infty} \tilde{\theta}^{(k)}(0) p^k / k!, \text{ and } r(p) = \sum_{k=0}^{\infty} r^{(k)}(0) p^k / v.$$

For notational convenience let us write $\tilde{\theta}^{(k)}(0)/k! = \gamma_k$ and $r^{(k)}(0)/k! = \xi_k$. We know that $\tilde{\theta}^{(1)}(0) = 1/EX_1$ and $r(0) = 0$, the rest of the γ_k 's and ξ_k 's can be easily computed via the implicit function theorem. For instance, $2E^3 X_1 \gamma_2 = E^2 X_1 - EX_1^2$, $6E^5 X_1 \gamma_3 = 3EX_1^2 Var(X_1) - 3EX_1(EX_1^3 - E^3 X_1)$. Consequently, $2E^2 X_1 \xi_1 = 3E^2 X_1 - EX_1^2$ and $24E^4 X_1 \xi_2 = 15E^2 X_1^2 - 6EX_1^2 E^2 X_1 - 8EX_1^3 EX_1 + 23E^4 X_1$. A similar expansion for $\tilde{\theta}(\cdot)$ has been obtained by Abate, Choudhury and Whitt (1995) in connection to heavy-traffic asymptotics for queues. We shall discuss these types of applications in Section 8 below. For completeness we provide a set of recursive equations to compute the γ_k 's.

Proposition 2 For $n \geq 1$ and each $k \leq n$, the constants $(\gamma_k : 1 \leq k \leq n)$ can be computed by solving recursively the following set of equations (note that the k th equation is linear in the γ_k and it only depends on the γ_j 's for $j \leq k$).

$$\sum_{m=1}^k \frac{EX_1^m}{m!} \sum_{\{n_1+\dots+n_m=k-m, n_1, \dots, n_m \geq 0\}} \prod_{j=1}^m \gamma_{n_j+1} = 1, \text{ for } 1 \leq k \leq n.$$

Proof. The proof follows directly by applying the implicit function theorem. The details are omitted. ■

Remark As an immediate consequence note that, having computed the γ_j 's, the constants $(\xi_k : 0 \leq k \leq n-1)$ can be easily obtained through a Taylor expansion up to order n of $r(p) = \log(c(p))$ around $p = 0$. We have chosen to state our expansions in terms of $r(\cdot)$ rather than $c(\cdot)$ because we want to assure positivity of our suggested parametric approximation for any value of p .

Theorem 2 therefore provides the means to develop an algorithm, that can be implemented easily, for computing an asymptotic expansion for the tail probability

$P(S_M > x/p)$ in powers of p which gives the desired corrected diffusion approximation in the present geometric sum setting.

We now discuss the mathematical development of Theorem 2. Define $N(t) = \sup\{n \geq 0 : S_n \leq t\}$. Observe that, for each non-negative integer m , $\{S_m > x\} = \{N(x) < m\}$. Thus, combining the independence between X and M with the fact that $P(M > m) = q^m$, we can write

$$P(S_M > x) = P(N(x) < M) = E(P(N(x) < M | X)) = E(q^{N(x)}).$$

We therefore have shown the next proposition (c.f. Kalashnikov (1997)).

Proposition 3

$$P(S_M > x) = Eq^{N(x)}.$$

The previous proposition implies that in order to study $P(pS_M > x)$ it suffices to study the behavior of $Eq^{N(x/p)}$ for $x > 0$ and small $p > 0$. A renewal theoretic argument yields the following (defective) renewal equation

$$Eq^{N(t)} = P(X_1 > t) + q \int_{[0,t)} Eq^{N(t-s)} P(X_1 \in ds). \quad (18)$$

Note that, if $p > 0$ small enough and $E \exp(\eta X_1) < \infty$ for some $\eta > 0$, the equation

$$\phi(\tilde{\theta}) = E \exp(\tilde{\theta} X_1) = 1/q \quad (19)$$

has a unique solution $\tilde{\theta} > 0$. Therefore, (18) can be transformed into the non-defective renewal equation

$$e^{\tilde{\theta}t} Eq^{N(t)} = e^{\tilde{\theta}t} P(X_1 > t) + \int_{[0,t)} e^{\tilde{\theta}(t-s)} Eq^{N(t-s)} F_{\tilde{\theta}}(ds),$$

where $F_{\tilde{\theta}}(ds) = qe^{\tilde{\theta}s} P(X_1 \in ds)$. Renewal theory then implies that

$$e^{\tilde{\theta}t} Eq^{N(t)} = \int_{[0,t)} e^{\tilde{\theta}(t-s)} P(X_1 > t-s) U_{\tilde{\theta}}(ds), \quad (20)$$

where $U_{\tilde{\theta}}(t) = E_{\tilde{\theta}}(N(t) + 1)$, and under $P_{\tilde{\theta}}$ the X_i 's are iid with distribution $F_{\tilde{\theta}}$. Using results by Stone (1965a) it is not hard to verify that, for fixed but small $p > 0$

$$\left| e^{\tilde{\theta}t} Eq^{N(t)} - \frac{1}{E_{\tilde{\theta}} X_1} \int_{[0,t)} e^{\tilde{\theta}s} P(X_1 > s) ds \right| \leq K(p) \exp(-a(p)t). \quad (21)$$

Note that Stone's estimates provide exponential rates of convergence for fixed $p > 0$, but it does not say anything about the behavior of $K(p)$ and $a(p)$ in (21) for small

values of p . In contrast, as the proof that follows next indicates, Theorem 1 allows us to set $t = x/p$ and obtain an exponential rate of convergence, thereby controlling the behavior of $K(p)$ and $a(p)$ as $p \searrow 0$.

Proof of Theorem 2. The previous argument leads us to equation (20). We now verify that the assumptions in Theorem 1 are satisfied. Let us define $g_\theta(\lambda) \triangleq E_\theta \exp(i\lambda X_1) = qE \exp\left(\left(i\lambda + \tilde{\theta}\right) X_1\right)$. Using the implicit function theorem on (19) it follows easily that $\tilde{\theta} = p/EX_1 + O(p^2)$. As a consequence, the following inequality can be easily derived for all $p > 0$ sufficiently small and some $M_1 \in (0, \infty)$

$$|g_\theta(\lambda) - g(\lambda)| \leq M_1 p.$$

Hence, we conclude that it is possible to pick $\delta > 0$ sufficiently small so that

$$\inf_{p \in [0, \delta]} \inf_{|\lambda| \geq 1} |1 - g_\theta(\lambda)| \geq \inf_{p \in [0, \delta]} \inf_{|\lambda| \geq 1} |1 - g(\lambda)| - M_1 \delta > 0,$$

which verifies condition (5) in Lemma 1. Finally, also because $\tilde{\theta} = O(p)$, it is possible to pick $p > 0$ small enough so that $E_\theta \exp(\eta X_1) = qE_\theta \exp\left(\left(\eta + \tilde{\theta}\right) X_1\right) < \infty$ for some $\eta > 0$. We now can apply Theorem 1 to equation (20) and obtain

$$\begin{aligned} & \left| e^{\theta x/p} E q^{N(x/p)} - \frac{1}{E_\theta X_1} \int_0^\infty e^{\theta s} P(X_1 > s) ds \right| \\ & \leq \frac{1}{E_\theta X_1} \int_{x/p}^\infty e^{\theta s} P(X_1 > s) ds + \end{aligned} \quad (22)$$

$$\left| \frac{1}{E_\theta X_1} \int_{[0, x/p]} e^{\theta(x/p-s)} P(X_1 > x/p - s) V(ds) \right|, \quad (23)$$

where, $V(t)$ is a function that we are introducing here and it corresponds to the left hand side of 1 in Theorem 1), therefore $|V(t)| = O(e^{-at})$ for some $a > 0$. The integral in (22) is easily seen to be bounded by $Ke^{-ax/p}$ for some finite constants $K, a > 0$ (assuming that $p > 0$ is sufficiently small). We just need to analyze the integral in (23). Integration by parts yields

$$\begin{aligned} & \int_{[0, x/p]} e^{\theta(x/p-s)} P(X_1 > x/p - s) V(ds) \\ & = V(x/p) P(X_1 > 0) - e^{\theta x/p} P(X_1 > x/p) V(0) \end{aligned} \quad (24)$$

$$+ \tilde{\theta} e^{\theta x/p} \int_{[0, x/p]} V(s) e^{-\theta s} P(X_1 > x/p - s) ds \quad (25)$$

$$+ e^{\theta x/p} \int_{[0, x/p]} V(s) e^{-\theta s} P(X_1 > x/p - ds). \quad (26)$$

The absolute value of (24) is also bounded by $Ke^{-ax/p}$ for some finite constants $K, a > 0$. For the integral (25) observe that

$$\begin{aligned}
& \tilde{\theta} e^{\tilde{\theta}x/p} \left| \int_{[0, x/p)} V(s) e^{-\tilde{\theta}s} P(X_1 > x/p - s) ds \right| \\
& \leq \tilde{\theta} e^{\tilde{\theta}x/p} \int_{[0, x/2p)} |V(s)| e^{-\tilde{\theta}s} P(X_1 > x/p - s) ds \\
& \quad + \tilde{\theta} e^{\tilde{\theta}x/p} \int_{[x/2p, x/p)} |V(s)| e^{-\tilde{\theta}s} P(X_1 > x/p - s) ds \\
& \leq \tilde{\theta} e^{\tilde{\theta}x/p} P(X_1 > x/(2p)) M + \tilde{\theta} e^{\tilde{\theta}x/p} \int_{[x/2p, \infty)} |V(s)| ds.
\end{aligned}$$

Since $\tilde{\theta} = O(p)$ and X_1 has exponential moments, we conclude that the previous expression is bounded by $Ke^{-ax/p}$ (for appropriate positive constants K and a). The treatment for integral (26) is very similar to that of (25). Thus, we conclude that

$$Eq^{N(x/p)} = \frac{\int_0^\infty e^{\tilde{\theta}s} P(X_1 > s) ds}{E_{\tilde{\theta}} X_1} + O(\exp(-ax/p)). \quad (27)$$

In order to recover the required expression for $c(p)$, note that

$$E_{\tilde{\theta}} X_1 = q \int_{[0, \infty)} s e^{\tilde{\theta}s} P(X_1 \in ds) = q\phi'(\tilde{\theta}).$$

On the other hand, using integration by parts and the definition of $\tilde{\theta}$, we see that

$$\int_0^\infty e^{\tilde{\theta}s} P(X_1 > s) ds = \left(\phi(\tilde{\theta}) - 1 \right) / \tilde{\theta} = p / (q\tilde{\theta}).$$

Combining the previous last two identities together into (27) yields equation (15). The analytic properties of $\tilde{\theta}$ follow directly from the implicit function theorem. It is easy to see that $r(\cdot)$ is well defined at zero (i.e. that the right hand side of (16) is strictly positive when p is close to zero). However, it is almost immediate to verify that $c(p)$ is real analytic at the origin with $c(0) = 1$. This implies the real analyticity of r and the conclusion of the theorem. ■

4 Non-Negative Geometric Sums with Heavy Tails

In this section we also assume that $X = (X_n : n \geq 1)$ is an iid sequence of non-negative rv's, and in contrast to the previous section here we relax the existence of exponential moments assumption. Here we just require $EX_1^{\alpha+2} < \infty$ for some

$\alpha \geq 0$. If the moment generating function of the X_i 's exist, Theorem 2 corrects Renyi's approximation (1) by providing a full asymptotic expansion in powers of p with an exponential error term. In other words, Theorem 2 provides rigorous support for the parametric (in $p > 0$) approximation

$$P(S_M > x/p) \approx \exp\left(-x/EX_1 + \sum_{k=1}^{\infty} p^k (\xi_k - \gamma_{k+1}x)\right), \quad (28)$$

which is valid up to an error exponentially small as $p \searrow 0$. It is easy to see that γ_k and ξ_k depend on the first k and $(k+1)$ order moments of X_1 respectively. This suggests that, if $EX_1^{\alpha+2} < \infty$, the approximation

$$P(S_M > x/p) \approx \exp\left(-x/EX_1 + \sum_{1 \leq k \leq \alpha+1} p^k (\xi_k - \gamma_{k+1}x)\right) \quad (29)$$

should be more accurate than (1). Another (perhaps more natural) way of obtaining a formal expression such as (29) proceeds as follows. First define

$$\phi_\alpha(\theta) = 1 + \sum_{k \leq \alpha+2} EX^k \theta^k / k!.$$

Then find the smallest positive solution $\tilde{\theta}_\alpha$ to the equation

$$\phi_\alpha(\tilde{\theta}_\alpha) = 1/q. \quad (30)$$

Note that $\tilde{\theta}_\alpha$ exists and is well defined if $p > 0$ is sufficiently small. Finally, set

$$c_\alpha(p) = p / \left(q^2 \tilde{\theta}_\alpha \phi'_\alpha(\tilde{\theta}_\alpha)\right). \quad (31)$$

Using these elements, (29) is equivalent (up to quantities of order $o(p^{\alpha+1})$) to

$$P(S_M > x/p) \approx \exp\left(-x\tilde{\theta}_\alpha/p\right) c_\alpha(p). \quad (32)$$

Providing rigorous support for approximation (32) in the presence of heavy tails presents an additional mathematical complication. Note that a crucial ingredient in the proof of Theorem 2 is the existence of a root for equation (19). Which clearly indicates that the strategy followed in the proof of Theorem 2 is infeasible in the heavy tailed case. Our idea is then to proceed via truncation. Define the sequence $\bar{X} = (\bar{X}_k : k \geq 1)$ as $\bar{X}_k = X_k 1(X_k \leq x/p)$ and consider its associated random walk $\bar{S} = (\bar{S}_n : n \geq 0)$ (i.e. $\bar{S}_n = \bar{X}_1 + \dots + \bar{X}_n$ with $\bar{S}_0 = 0$). We first argue that this truncation has little impact on the distribution of the geometric sum over spatial scales of order $O(1/p)$ which are the ones the we are interested in studying here.

Lemma 4 *Suppose that $EX_1^\beta < \infty$ for $\beta \geq 1$, then*

$$|P(pS_M > x) - P(p\bar{S}_M > x)| \leq 2P(X_1 > x/p) / (qp) = o(p^{\beta-1}/x^\beta)$$

Proof. Note that

$$\begin{aligned} & q |P(pS_M > x) - P(p\bar{S}_M > x)| \\ & \leq p \sum_{k=1}^{\infty} q^k P(S_k > x/p; \bar{S}_k \leq x/p) + p \sum_{k=0}^{\infty} q^k P(\bar{S}_k > x/p; S_k \leq x/p) \\ & \leq 2p \sum_{k=1}^{\infty} q^k P\left(\max_{n \leq k} X_n > x/p\right) \leq 2p \sum_{k=1}^{\infty} kq^k P(X_1 > x/p) \\ & = 2P(X_1 > x/p) / p. \end{aligned}$$

■

The strategy to follow in order to provide rigorous support for the validity of (32) is perhaps clear now. Specifically, taking advantage of the fact that the increments \bar{X}_k have exponential moments, we want to deal with \bar{S}_M following the spirit of Section 3 by making use of uniform renewal theory. One of the basic steps that are involved in applying the same techniques used in Section 3 is solving the equation

$$E \exp(\bar{\theta} \bar{X}) = 1 / (1 - p). \quad (33)$$

It is not hard to see, using the implicit function theorem that $\bar{\theta} \sim p/EX$ as $p \searrow 0$. In fact, we have the following proposition.

Proposition 4 *Suppose that $|\theta| \leq pA$, then*

$$\phi_\alpha(\theta) = E \exp(\theta \bar{X}) + o(p^{\alpha+2}),$$

therefore, if $\bar{\theta}$ solves (33), then

$$\bar{\theta} = \theta_\alpha + o(p^{\alpha+2}).$$

Proof. Suppose that $|\theta| \leq pA$, for some fixed $A > 0$. Then,

$$E \exp(\theta \bar{X}) = E(\exp(\theta X); X \leq 1/p) + o(p^{\alpha+2}).$$

Also, note that

$$\begin{aligned} & \left| E(\exp(\theta X); X \leq 1/p) - 1 - \sum_{1 \leq k \leq \alpha+2} EX^k \theta^k / (k!) \right| \\ & \leq o(p^{\alpha+2}) + Ae^{xA} p^{[\alpha+2]+1} E(X^{[\alpha+2]+1}; X \leq x/p). \end{aligned}$$

Now, we claim that

$$p^{[\alpha+2]+1} E(X^{[\alpha+2]+1}; X \leq x/p) = o(p^{\alpha+2}).$$

In order to see this, we first use integration by parts to obtain

$$\begin{aligned} pE(X^{[\alpha+2]+1}; X \leq x/p) &= \int_0^{x/p} ps^{[\alpha+2]+1} P(X \in ds) \\ &= -P(X > x/p) (x/p)^{[\alpha+2]} x \end{aligned} \quad (34)$$

$$+ ([\alpha + 2] + 1) \int_0^x P(X > s/p) (s/p)^{[\alpha+2]} ds. \quad (35)$$

Expression (34) multiplied $p^{[\alpha+2]}$ is of order $o(p^{\alpha+2})$, hence we just have to show that the integral in (35) multiplied by $p^{[\alpha+2]}$ is of order $o(p^{\alpha+2})$. Equivalently, we must verify that

$$\int_0^x P(X > s/p) (s/p)^{\alpha+2} s^{[\alpha+2]-(\alpha+2)} ds \rightarrow 0$$

as $p \searrow 0$, which follows easily by dominated convergence since $0 \leq P(X > t) t^{[\alpha+2]} \leq c < \infty$, and

$$\int_0^x s^{[\alpha+2]-(\alpha+2)} ds < \infty.$$

goes to zero as $p \searrow 0$. The rest just follows from the implicit function theorem. ■

Finally, we provide the precise statement of our rigorous approximation in the context of heavy-tailed increments. (We say here that a non-negative random variable X_1 is heavy tailed if for every $\eta > 0$, $E \exp(\eta X_1) = \infty$.)

Theorem 3 *Assume that the distribution of X_1 is strongly non-lattice. Also, suppose that*

$$EX_1^{2+\alpha} < \infty$$

for $\alpha \geq 0$. Then,

$$P(pS_M > x) = \exp(-x\tilde{\theta}_\alpha/p) c_\alpha(p) + o(p^{\alpha+1}) \quad (36)$$

as $p \searrow 0$.

Remark In contrast to Theorem 2, note that the error term $o(p^{\alpha+1})$ does not have an explicit dependence on x . However, Theorem 3 still holds as long as $x = x(p)$ stays bounded as $p \searrow 0$.

The proof of this result will be provided at the end of this section.

As in the case of exponential moments, approximation (36) can be stated in power series form as we indicate next.

Corollary 1 *In the setting of Theorem 3,*

$$P(S_M > x/p) = \exp\left(-x/EX_1 + \sum_{1 \leq k \leq \alpha+1} p^k (\xi_k - \gamma_{k+1}x)\right) + o(p^{\alpha+1}),$$

where the γ_k 's and ξ_k 's are defined recursively via Proposition 2.

Proof. Follows from the implicit function theorem applied to (30), and then expanding (31) in powers of p . ■

We now are ready to provide the proof of Theorem 3. We do so by analyzing $P(p\bar{S}_M > x)$ just as we did in Theorem 2. Theorem 1 can also be applied here to obtain a suitable approximation for $P(p\bar{S}_M > x)$, as our next result shows and this is the strategy that we follow next.

Proof of Theorem 3. Let $\bar{N}(t) = \sup\{n \geq 0 : \bar{S}_n \leq t\}$, then, by virtue of Proposition 3 and Lemma 4 it suffices to compute $Eq^{\bar{N}(x/p)}$. Following similar steps as in the proof of Theorem 2 we obtain

$$e^{\bar{\theta}x/p} Eq^{\bar{N}(x/p)} = \int_{[0, x/p)} e^{\bar{\theta}(x/p-s)} P(X_1 > x/p - s; X_1 \leq x/p) \bar{U}_{\bar{\theta}}(ds). \quad (37)$$

The elements in equation (37) are indicated next. First, $\bar{\theta}$ is the solution to the equation

$$\bar{\phi}(\bar{\theta}) \triangleq E \exp(\bar{\theta} \bar{X}_1) = \frac{1}{q},$$

which clearly exists if p is small enough. In order to describe $\bar{U}_{\bar{\theta}}(s)$ define $P_{\bar{\theta}}(\cdot)$ via

$$P_{\bar{\theta}}(B) = q^n E(\exp(\bar{\theta} \bar{S}_n); 1(B)),$$

for every B in the sigma-field $\sigma(\bar{X}_1, \dots, \bar{X}_n)$. Next, we will show that

$$\bar{V}(s) \triangleq \bar{U}_{\bar{\theta}}(s) - \frac{s}{E_{\bar{\theta}} \bar{X}_1} - \frac{E_{\bar{\theta}} \bar{X}_1^2}{2E_{\bar{\theta}}^2 \bar{X}_1} - \frac{\int_s^\infty \int_t^\infty P_{\bar{\theta}}(\bar{X}_1 > u) dudt}{E_{\bar{\theta}}^2 \bar{X}_1}, \quad (38)$$

satisfies that $|\bar{V}(s)| = o(s^{-(\alpha+1)})$ as $s \nearrow \infty$ uniformly in $p > 0$ small enough. This follows from Theorem 1, as we now illustrate. (Note that the term \bar{V} in (38) includes the last two terms in the right hand side of the equation in the part 2 of Theorem 1.) Observe that $\bar{g}_p(\lambda) \triangleq E_{\bar{\theta}} \exp(i\lambda \bar{X}_1) = q E_{\bar{\theta}} \exp((i\lambda + \bar{\theta}) \bar{X}_1)$ satisfies

$$\begin{aligned} |\bar{g}_p(\lambda) - E \exp(i\lambda X_1)| &\leq |\bar{g}_p(\lambda) - E \exp(i\lambda \bar{X}_1)| + o(p^{\alpha+2}) \\ &\leq p |E \exp(i\lambda \bar{X}_1)| + \bar{\theta} E \bar{X}_1 + o(p^{\alpha+2}) = O(p). \end{aligned}$$

Since X_1 is strongly non-lattice, this implies that $\bar{g}_p(\cdot)$ satisfies the uniform strongly non-lattice condition (5). On the other hand, since $\bar{\theta} = O(p)$, we have that for all $p > 0$ small enough

$$E_{\bar{\theta}} \bar{X}_1^{\alpha+2} = qE \exp(\bar{\theta} X_1) \bar{X}_1^{\alpha+2} \leq ME \bar{X}_1^{\alpha+2} < ME X_1^{\alpha+2} < \infty.$$

This justifies the validity of representation (38). Furthermore, (38) implies that

$$Eq^{\bar{N}(x/p)} = \int_{[0, x/p]} \frac{e^{\bar{\theta}(x/p-s)} P(X_1 > x/p - s; X_1 \leq x/p)}{E_{\bar{\theta}} \bar{X}_1} ds \quad (39)$$

$$+ \int_{[0, x/p]} \frac{e^{\bar{\theta}(x/p-s)} P(X_1 > x/p - s; X_1 \leq x/p)}{E_{\bar{\theta}}^2 \bar{X}_1} \int_s^\infty P_{\bar{\theta}}(\bar{X}_1 > u) dud s \quad (40)$$

$$+ \int_{[0, x/p]} e^{\bar{\theta}(x/p-s)} P(X_1 > x/p - s; X_1 \leq x/p) \bar{V}(ds). \quad (41)$$

Let us denote by I_1 , I_2 , and I_3 the expressions (39), (40), and (41) respectively. We first show that $I_3 = o(p^{\alpha+1})$. To see this, we use integration by parts, the triangle inequality and the fact that $\bar{\theta} = O(p)$ to obtain

$$\begin{aligned} |I_3| &\leq |\bar{V}(x/p)| + M_1 \left| \int_{[0, x/p]} \bar{V}(s) de^{-\bar{\theta}s} P(X_1 > x/p - s; X_1 \leq x/p) \right| \\ &\leq |\bar{V}(x/p)| + M_1 \left| \int_{[0, x/p]} \bar{V}(s) e^{-\bar{\theta}s} P(X_1 > x/p - ds; X_1 \leq x/p) \right| \\ &\quad + M_1 \left| \int_{[0, x/p]} \bar{V}(s) e^{-\bar{\theta}s} P(X_1 > x/p - s; X_1 \leq x/p) ds \right|. \end{aligned} \quad (42)$$

The term $|\bar{V}(x/p)| = o(p^{\alpha+1})$. Now, observe that

$$\begin{aligned} &\left| \int_{[0, x/p]} \bar{V}(s) e^{-\bar{\theta}s} P(X_1 > x/p - ds; X_1 \leq x/p) \right| \\ &\leq \left| \int_{[0, x/2p]} \bar{V}(s) e^{-\bar{\theta}s} P(X_1 > x/p - ds; X_1 \leq x/p) \right| \\ &\quad + \left| \int_{[x/2p, x/p]} \bar{V}(s) e^{-\bar{\theta}s} P(X_1 > x/p - ds; X_1 \leq x/p) \right| \\ &\leq K_2 P(X_1 > x/(2p)) + K_1 \max_{1/2 \leq u \leq 1} |\bar{V}(ux/p)| \\ &= o(p^{\alpha+2}) + o(p^{\alpha+1}) = o(p^{\alpha+1}), \end{aligned} \quad (43)$$

for some constants K_1 and K_2 . The integral in (42) follows the same lines as (43). For I_2 we have

$$I_2 = \frac{1}{E_{\bar{\theta}}^2 \bar{X}_1} \int_{[0, x/p]} e^{\bar{\theta}(x/p-s)} P(X_1 > x/p - s) \int_s^\infty P_{\bar{\theta}}(\bar{X}_1 > u) dud s + o(p^{\alpha+1}). \quad (44)$$

Note that

$$\begin{aligned}
& \frac{1}{E_{\bar{\theta}}^2 \bar{X}_1} \int_{[0, x/p)} e^{\bar{\theta}(x/p-s)} P(X_1 > x/p - s) \int_s^{\infty} P_{\bar{\theta}}(\bar{X}_1 > u) dud s \\
& \sim \frac{1}{E^2 X} \int_0^{x/p} P(X_1 > x/p - s) \int_s^{\infty} P(X_1 > u) dud s \\
& \sim \frac{1}{E^2 X} \int_0^{x/p} P(X_1 > x/p - s) \int_s^{x/p} P(X_1 > u) dud s \tag{45}
\end{aligned}$$

$$+ \frac{1}{EX} \int_{x/p}^{\infty} P(X_1 > u) du. \tag{46}$$

Let J be the integral in (45) and note that the integral in (46) is of order $o(p^{\alpha+1})$. We therefore must show that $J = o(p^{\alpha+1})$. To see this just note that

$$\begin{aligned}
J &= \frac{1}{E^2 X} \int_0^{x/(2p)} P(X_1 > x/p - s) \int_s^{x/p} P(X_1 > u) dud s \\
&+ \frac{1}{E^2 X} \int_{x/(2p)}^{x/p} P(X_1 > x/p - s) \int_s^{x/p} P(X_1 > u) dud s \\
&\leq \frac{xP(X_1 > x/(2p))}{(2p)EX} + \int_{x/(2p)}^{x/p} P(X_1 > u) du \frac{1}{EX} \\
&\leq \frac{xP(X_1 > x/(2p))}{pEX} = o(p^{\alpha+1}),
\end{aligned}$$

which yields

$$I_2 = o(p^{\alpha+1}).$$

Finally, we analyze I_1

$$\begin{aligned}
& I_1 + o(p^{\alpha+1}) \\
&= \frac{1}{E_{\bar{\theta}} \bar{X}_1} \int_0^{x/p} e^{\bar{\theta}u} P(X_1 > u) du = \frac{1}{\bar{\theta} E_{\bar{\theta}} \bar{X}_1} \int_0^{x/p} P(X_1 \geq u) d e^{\bar{\theta}u} \\
&= o(p^{\alpha+1}) - \frac{1}{\bar{\theta} E_{\bar{\theta}} \bar{X}_1} + \frac{1}{\bar{\theta} E_{\bar{\theta}} \bar{X}_1} \int_0^{x/p} e^{\bar{\theta}u} P(X_1 \in du) \\
&= \frac{(1 - E \exp(\bar{\theta} \bar{X}_1))}{\bar{\theta} E (\exp(\bar{\theta} \bar{X}_1) \bar{X}_1)} + o(p^{\alpha+1}).
\end{aligned}$$

Lastly, the implicit function theorem yields that

$$\frac{p}{q^2 \bar{\theta} \phi'(\bar{\theta})} = \frac{(1 - E \exp(\bar{\theta} \bar{X}_1))}{q \bar{\theta} E (\exp(\bar{\theta} \bar{X}_1) \bar{X}_1)} = \exp\left(\sum_{k \leq \alpha+1} p^k \xi_k\right) + o(p^{\alpha+1}),$$

and

$$\bar{\theta} = \sum_{k \leq \alpha+2} p^k \gamma_k + o(p^{\alpha+2}).$$

This concludes the proof of the theorem. ■

5 Geometric Sums with Increment Distribution Depending on p

As we indicated before, many settings of interest demand treatment of the case in which the increment distributions are actually changing with p . This is the case if one wishes to develop CDA's for the time in system for the GI/G/1 queue and the probability of ruin for the renewal risk insurance process. Fortunately, Theorem 1 also provides a means to deal with the typical situations that arise in practice. To fix ideas, consider a family of probability measures $\mathcal{P} = \{P_p, p \in [0, \bar{\delta}] \text{ for some } \bar{\delta} > 0\}$. Suppose that, under each P_p , the random variables $(X_k : k \geq 1)$ form an iid sequence. Also, assume that the distribution of X_1 is uniformly strongly non-lattice with respect to \mathcal{P} (i.e. the characteristic functions $g_p(\lambda) = E_p \exp(i\theta X_1)$ satisfy condition (4)). In addition, suppose that one of the following conditions hold:

- A) for some $\eta > 0$, $\sup_{0 \leq p \leq \bar{\delta}} E_p \exp(\eta X_1) < \infty$ or
- B) $\sup_{0 \leq p \leq \bar{\delta}} E_p X_1^{2+\alpha} < \infty$, for some $\alpha \geq 0$.

Under this set of assumptions, we have the following analogue to Theorems 2 and 3.

Theorem 4 *Assume that the family $P_p, p \in [0, \bar{\delta}]$ is uniformly strongly non-lattice (see equation (4)). If condition A) above holds, then, there exist constants $K_1, K_2 > 0$ such that for $p > 0$ small*

$$|P_p(pS_M > x) - \exp(-\theta^* x/p + r_p(p))| \leq K_1 \exp(-K_2 x/p), \quad (47)$$

where $\theta^* = \theta^*(p)$ solves $\phi_p(\theta^*) \triangleq E_p \exp(\theta^* X_1) = 1/q$ and

$$\exp(r_p(p)) = \frac{p}{q^2 \theta^* \phi_p'(\theta^*)}.$$

Moreover, $\theta^*(p) = \sum_{k=1}^{\infty} p^k \gamma_k(p)$ and $r_p(p) = \sum_{k=1}^{\infty} p^k \xi_k(p)$ (where the $\gamma_k(p)$'s and $\xi_k(p)$'s depend on the first k and $(k+1)$ moments of X_1 under P_p respectively). Finally, if condition B) is in force, then

$$\left| P_p(pS_M > x) - \exp\left(-x/E_p X_1 + \sum_{1 \leq k \leq \alpha+1} p^k (\xi_k - \gamma_{k+1} x)\right) \right| = o(p^{\alpha+1}). \quad (48)$$

Proof. The proof parallels the arguments given in Theorems 2 and 3 using Theorem 1. The details are omitted. ■

Remark 1 Note that the γ_k 's and the ξ_k 's also depend on p . The previous result would yield the desired asymptotic expansion assuming that the problem at hand has enough structure, so that an asymptotic expansion of ξ_k 's and γ_k 's can be obtained. The expansion for the distribution of the all time maximum of a random walk with small negative drift given in Blanchet and Glynn (2005a) provides an example of the applicability of this previous result.

Remark 2 Just as we pointed out in Theorems 2 and 3, the estimate (47) applies uniformly in x , whereas (48) holds as long as $x = x(p)$ stays bounded.

6 Geometric Sums with Real-valued Increments

Some contexts demand looking at increment distributions that can take negative values, we refer the reader to the book by Kalashnikov (1997) for motivating applications. Our goal in this section is to show that completely analogous results to those presented in Section 3 even if we relax the assumption of non-negative increments. The strategy is a natural extension to that of Section 3. For simplicity, we shall assume that $E \exp(\eta |X_1|) < \infty$. Additional order correction terms can be obtained, just as in Theorem 3 via a truncation argument.

First let us consider the case in which $EX_1 > 0$. Note that due to the memory-less property of the Geometric distribution, we obtain that

$$G(x) \triangleq P(S_M > x) = pP(X_1 > x) + qP(\tilde{Z} + X_1 > x),$$

where $S_M \stackrel{D}{=} \tilde{Z}$ and \tilde{Z} is independent of X_1 . Therefore,

$$G(x) = pP(X_1 > x) + q \int_{-\infty}^{\infty} G(x-s) P(X_1 \in ds).$$

As in Section 3, let $\tilde{\theta} > 0$ be such that $E \exp(\tilde{\theta} X_1) = 1/q$. Then,

$$e^{\tilde{\theta} x} G(x) = p \int_{-\infty}^{\infty} P(X_1 > x-s) e^{\tilde{\theta}(x-s)} U_{\tilde{\theta}}(ds), \quad (49)$$

where $U_{\tilde{\theta}}(t) = \sum_{n=0}^{\infty} P_{\tilde{\theta}}(S_n \leq t)$, and under $P_{\tilde{\theta}}$ the increments of the random walk $S = (S_n : n \geq 1)$, namely the X_i 's, are iid with distribution

$$P_{\tilde{\theta}}(X \in ds) = q \exp(\tilde{\theta} s) P(X \in ds).$$

We then obtain the next result which is the analog of Theorem 2.

Theorem 5 Suppose that X_1 has strongly non-lattice distribution, $EX_1 > 0$ and that $\phi(\eta) \triangleq E \exp(\eta|X|) < \infty$ for some $\eta > 0$. If $x > 0$, then there exists a $a > 0$ such that

$$P(pS_M > x) = \exp\left(-x\tilde{\theta}/p + r(p)\right) + O(\exp(-ax/p)),$$

as $p \rightarrow 0$, where

$$\exp(r(p)) = \frac{P}{q^2 \tilde{\theta} \phi'(\tilde{\theta})} \triangleq c(p). \quad (50)$$

If $x < 0$, then there exists a $a > 0$ so that

$$P(pS_M \leq x) = O(e^{ax/p}).$$

Finally,

$$P(S_M \leq 0) = \sum_{n=0}^{\infty} pq^{n-1} P(S_n \leq 0)$$

is real analytic in $p \in [0, \delta]$ for some $\delta > 0$.

Proof. Theorem 1 yields that

$$U_{\theta}(s) = I(s \geq 0) (s/E_{\theta}X_1 + E_{\theta}X_1^2/E_{\theta}^2X_1 + V_1(s)) + V_2(s)I(s < 0),$$

where $|V_1(s)|I(s \geq 0) + |V_2(s)|I(s < 0) = O(e^{-r|s|})$ as $|s| \rightarrow \infty$ for some $r > 0$ (uniformly over $p \in [0, \delta]$, for some $\delta > 0$). Hence, equation (49) implies

$$\begin{aligned} e^{\theta x/p} P(pS_M > x) &= p \int_{-\infty}^{\infty} P(X_1 > x/p - s) e^{\theta(x/p-s)} U_{\theta}(ds) \\ &= p \int_{-\infty}^{x/p} P(X_1 > u) e^{\theta u} du / E_{\theta}X_1 \end{aligned} \quad (51)$$

$$+ p \int_0^{\infty} P(X_1 > x/p - s) e^{\theta(x/p-s)} dV_1(s) \quad (52)$$

$$+ p \int_0^{\infty} P(X_1 > x/p + s) e^{\theta(x/p+s)} V_2(-ds). \quad (53)$$

It is not hard to see that the integrals (52) and (53) are of order $O(e^{-rx/p})$ for some $r > 0$ uniformly on $p \in [0, \delta]$ for some $\delta > 0$. Hence, we obtain that

$$\begin{aligned} e^{\theta x/p} P(pS_M > x) &= p \int_{-\infty}^{\infty} P(X_1 > u) e^{\theta u} du / E_{\theta}X_1 \\ &\quad + O(e^{-rx/p}) \end{aligned}$$

for some $r > 0$. On the other hand, integrating by parts we obtain

$$\int_{-\infty}^{\infty} P(X_1 > u) \tilde{\theta} e^{\theta u} du = \int_{-\infty}^{\infty} e^{\theta u} P(X_1 \in du) = 1/q.$$

Therefore

$$e^{\theta x/p} P(pS_M > x) = \frac{p}{\tilde{\theta} q E_{\tilde{\theta}} X_1} + O(e^{-rx/p}).$$

Equation (50) is obtained by noting that $E_{\tilde{\theta}} X_1 = q\phi'(\tilde{\theta})$. The behavior of $P(S_M \leq x/p)$ for $x < 0$ can be obtained using a similar analysis as the previous one. The analyticity of $P(S_M \leq 0)$ follows using Chernoff's bounds by noting that we can pick $\rho \in (0, 1)$

$$\begin{aligned} \sum_{n=0}^{\infty} p q^{n-1} P(S_n \leq 0) &\leq \sum_{n=0}^{\infty} p(1+p)^{n-1} P(S_n \leq 0) \\ &\leq \sum_{n=0}^{\infty} p(1+p)^{n-1} \rho < \infty \end{aligned}$$

as long as $p \in [0, \delta]$ for $\delta > 0$ sufficiently small. Hence, the series representation for $P(S_M \leq 0)$ converges absolutely and uniformly for p sufficiently small. This concludes the proof of the theorem. ■

7 Geometric Sums with Zero Mean Increments

The case in which $EX_1 = 0$ introduces qualitative differences. As we discussed in our introduction, it is not hard to see that $p^{1/2}S_M \Rightarrow \sigma 2^{-1/2}T$, where T follows the Laplace (or double exponential) distribution. As the scaling suggested by the previous weak convergence result, the expansion in this case is given in powers of $p^{1/2}$. The next theorem considers $P(p^{1/2}S_M > x)$ for $x > 0$, the case $x = 0$ is investigated separately.

Theorem 6 *Suppose that X_1 has strongly non-lattice distribution, $EX_1 = 0$ and that $\phi(\eta) \triangleq E \exp(\eta|X|) < \infty$ for some $\eta > 0$. If $x > 0$, then there exists a $a > 0$ such that*

$$P(p^{1/2}S_M > x) = \exp\left(-x\tilde{\theta}/p^{1/2} + r(p)\right) + O\left(\exp(-ax/p^{1/2})\right),$$

as $p \rightarrow 0$, where

$$\exp(r(p)) = \frac{p}{q^2 \tilde{\theta} \phi'(\tilde{\theta})} \triangleq c(p).$$

Proof. We proceed as in the argument given at the beginning of this section to obtain that

$$e^{\theta x} P(p^{1/2}S_M > x) = p \int_{-\infty}^{\infty} P(X_1 > xp^{-1/2} - s) e^{\theta(x-s)} U_{\tilde{\theta}}(ds), \quad (54)$$

where $U_{\theta}(t) = \sum_{n=0}^{\infty} P_{\theta}(S_n \leq t)$, and under P_{θ} the increments of the random walk $S = (S_n : n \geq 1)$, namely the X_i 's, are iid with distribution

$$P_{\theta}(X \in ds) = q \exp(\tilde{\theta}s) P(X \in ds),$$

again, $\tilde{\theta}$ is chosen at the unique non-negative root of the equation $E \exp(\tilde{\theta}X) = q^{-1}$.

Note that $\mu_{\theta} \triangleq E_{\theta}X_1 > 0$, so the representation given in (54) is valid by virtue of standard renewal theory. We can proceed just as in the proof of Theorem 5, we just have to be a bit careful here because $E_{\theta}X_1 = O(p^{1/2})$. Note that the integral in (54) can be written as

$$\begin{aligned} e^{\theta x/p} P(p^{1/2}S_M > x) &= p \int_{-\infty}^{\infty} P(X_1 > xp^{-1/2} - s) e^{\theta(x/p-s)} U_{\theta}(ds) \\ &= p \int_{-\infty}^{xp^{-1/2}} P(X_1 > u) e^{\theta u} du / E_{\theta}X_1 \end{aligned} \quad (55)$$

$$+ p \int_0^{\infty} P(X_1 > xp^{-1/2} - s) e^{\theta(x/p-s)} dV_1(s) \quad (56)$$

$$+ p \int_0^{\infty} P(X_1 > xp^{-1/2} + s) e^{\theta(x/p+s)} V_2(-ds), \quad (57)$$

where, by virtue of Theorem 1, $|V_1(s)| = O(e^{-rs})$ as $s \rightarrow \infty$ for some $r > 0$ (uniformly over $p \in [0, \delta]$, for some $\delta > 0$) and $|V_2(s)| \leq K\mu_{\theta}e^{-\mu_{\theta}rs}$ as $s \rightarrow -\infty$ for $K, r > 0$ (uniformly over $p \in [0, \delta]$, for some $\delta > 0$). Let us denote by J_1 , J_2 , and J_3 the integrals (55) to (57) respectively. Note that $\tilde{\theta} = O(p^{1/2})$, which implies that the treatment of J_1 proceeds just as that of (51) in the proof of Theorem 5. Moreover, J_2 is also very similar to (52). The analysis of J_3 deserves special attention because the decaying rate of $V_2(s)$ degrades as $p \searrow 0$. Observe, using integration by parts, that for p small enough,

$$\begin{aligned} J_3 &= p \int_0^{\infty} P(X_1 > xp^{-1/2} + s) e^{\theta(xp^{-1/2}+s)} V_2(-ds) \\ &= -pP(X_1 > xp^{-1/2}) e^{\theta xp^{-1/2}} V_2(0) \end{aligned} \quad (58)$$

$$+ p\tilde{\theta} \int_0^{\infty} V_2(-s) e^{\theta(xp^{-1/2}+s)} P(X_1 > xp^{-1/2} + s) ds \quad (59)$$

$$+ p\tilde{\theta} \int_0^{\infty} V_2(-s) e^{\theta(xp^{-1/2}+s)} P(X_1 \in xp^{-1/2} + ds). \quad (60)$$

Clearly, (58) and (59) are of order $O(e^{-rxp^{-1/2}})$ for some $r > 0$. Now, observe that (60) is bounded in absolute value by

$$p\tilde{\theta} \int_0^{\infty} \mu_{\theta} K e^{-r\mu_{\theta}s} e^{\theta(xp^{-1/2}+s)} P(X_1 \in xp^{-1/2} + ds). \quad (61)$$

Applying integration by parts to the previous integral allows us to conclude that we can find constants $K_1, K_2, r_1, m > 0$ (independent of p) such that (61) is bounded by

$$\begin{aligned} & K_1 e^{-r_x p^{-1/2}} + \mu_{\tilde{\theta}} p \tilde{\theta}^2 \int_0^\infty P(X_1 > x p^{-1/2} + s) K e^{-r \mu_{\tilde{\theta}} s} e^{\tilde{\theta}(x p^{-1/2} + s)} ds \\ \leq & K_1 e^{-r_x p^{-1/2}} + K_2 e^{\tilde{\theta} x p^{-1/2}} \int_0^\infty P(X_1 > x p^{-1/2} + u m \tilde{\theta}^{-1}) e^{-r_1 u} du \\ = & O\left(e^{-r_x p^{-1/2}}\right), \end{aligned}$$

where the bound on the second line above was obtained by using the change of variables $\tilde{\theta} s/m = u$, and noting that $\mu_{\tilde{\theta}}/\tilde{\theta} \rightarrow EX^2 > 0$. We therefore obtain that for some $r > 0$,

$$e^{\tilde{\theta} x/p^{1/2}} P(S_M > x/p^{1/2}) = \frac{p}{\tilde{\theta} q E_{\tilde{\theta}} X_1} + O\left(e^{-r_x x/p^{1/2}}\right).$$

Which yields, just as in Theorem 5, that there exists $a > 0$ such that

$$e^{\tilde{\theta} x/p^{1/2}} P(S_M > x/p^{1/2}) = \frac{p}{q^2 \tilde{\theta} \phi'(\tilde{\theta})} + O\left(e^{-a x/p^{1/2}}\right).$$

■

In order to recover the standard weak convergence result to a double exponential rv discussed before, observe that $\tilde{\theta} \sim p^{1/2} 2^{1/2}/E^{1/2}(X^2)$ and $\phi'(\tilde{\theta}) \sim (EX^2 2p)^{1/2}$. This implies, in particular, that $P(p^{1/2} S_M > x) \rightarrow \exp(-|x| 2^{1/2}/E^{1/2}(X^2))/2$, which is equivalent of the basic weak convergence result discussed before. Moreover, it is easy to see that in this case, $\tilde{\theta}(p)$ can be written as an absolutely convergent power series in $p^{1/2}$ for $p \in [0, \delta]$ with $\delta > 0$ small enough. To see this, let us write $\psi(\theta) = \log \phi(\theta)$ and note that θ satisfies

$$\psi(\tilde{\theta})^{1/2} = p^{1/2} (1 + p/2 + p^2/3 + \dots)^{1/2}.$$

Note that $\psi(\tilde{\theta})^{1/2}$ is a real analytic function on $[0, \delta]$ for $\delta > 0$ small enough and differentiable at 0 from the right with derivative equal to $EX^2 > 0$ which yields (using the inverse function theorem) the required expansion for $\tilde{\theta}(p)$. A system of equations completely analogous to that of Proposition 2 can be easily obtained here to retrieve the coefficients in the asymptotic expansion of $\tilde{\theta}$, the details are omitted. Using an expansion for $c(p)$ in powers of $\tilde{\theta}$ yields the desired asymptotic expansion in powers of $p^{1/2}$.

While the expansion for $P(p^{1/2}S_M > x)$ follows by taking advantage of the uniform renewal theory developed in Theorem 1, the case $P(p^{1/2}S_M \leq 0)$ demands a completely different strategy via the use of Edgeworth expansions. To make this computations rigorous we require, as an additional technical condition, that $E \exp(X^2/4) < \infty$. The general case can be developed by truncation, at the price of losing the absolute convergence of the expansion that is obtained assuming the technical condition. Since here we are just interested in illustrating the qualitative form of the expansion, we only provide a rigorous justification for this expansion under the technical condition discussed above.

Proposition 5 *In addition to the strongly non-lattice assumption, let us assume that $E \exp(X^2/4) < \infty$. Then,*

$$P(p^{1/2}S_M \leq 0) = \sum_{n=1}^{\infty} pq^{n-1}P(S_n \leq 0) \quad (62)$$

admits an absolutely convergent expansion in powers of $p^{1/2}$ for $p \in [0, \delta]$ and some $\delta > 0$.

Proof. For simplicity, throughout the proof we shall assume that $EX^2 = 1$. Using the Edgeworth expansion for $P(S_n \leq 0)$ we obtain

$$P(S_n \leq 0) = 1/2 + \sum_{j=1}^{\infty} n^{-j/2} p_j \quad (63)$$

where the p_j 's can be computed explicitly in terms of the cumulants of X_i (see Feller (1968) p. 536). On the other hand, Chang and Peres (1997) Theorem 2.1 yields that, for each $j \in \mathbb{N}$,

$$\sum_{n=1}^{\infty} q^n n^{-j/2} = \sum_{n \neq j/2-1}^{\infty} \zeta(j/2 - n) \frac{(\log(1-p))^n}{n!} \quad (64)$$

$$+ \frac{(\log(1-p))^{j/2-1} \pi (-1)^{j/2-1}}{\Gamma(j/2) \sin(\pi j/2)} I(j = \text{odd}), \quad (65)$$

where $\zeta(\cdot)$ is the Riemann zeta function. (Equation (19) of Chang and Peres (1997) indicates how to evaluate numerically $\zeta(j/2 - n)$.) We would like to combine (63) and (64) together and interchange the order of summations over j and n , then rearrange terms in powers of p . In order to carry over all these steps we need to study the radius of convergence of the series

$$\sum_{j=1}^{\infty} p_j z^j = p(z). \quad (66)$$

Recall that the sequence of p_j 's is obtained from expanding

$$\exp(\chi(\theta/n^{1/2})n) = \exp(-\theta^2/2) \exp\left(\sum_{j=1}^{\infty} n^{-j/2} \kappa_{j+2} (i\theta)^{j+2} / (j+2)!\right)$$

in powers of $n^{-1/2}$, taking inverse Fourier transforms and evaluating at zero. A term of the form $(i\theta)^k \exp(-\theta^2/2)$ gives rise to the Hermite polynomial of degree k (whose absolute value equals the absolute value of the k th derivative of standard Gaussian density). Call this polynomial $H_k(\cdot)$. It is not hard to see that $H_{2n}(0) = (2n)! / (n!2^n)$ ($H_{2n-1}(0) = 0$) for all $n = 0, 1, \dots$. Using Stirling's approximation it is not hard to verify that there exists a constant $B > 0$ such that for all $n, m \in \mathbb{N}$

$$Be^{2n} e^{2m} H_{2n}(0) H_{2m}(0) \geq H_{2(n+m)}(0).$$

So, we can analyze the convergence properties of the series (66) by looking at the radius of convergence of

$$\sum_{j=1}^{\infty} \kappa_{j+2} e^j H_{j+2}(0) z^{j+2} / (j+2)!,$$

which, in turn, can be analyzed in terms of the radius of convergence of

$$\sum_{j=1}^{\infty} \tilde{\mu}_{j+2} e^j H_{j+2}(0) z^{j+2} / (j+2)!, \quad (67)$$

where $\tilde{\mu}_n = E(|X_1|^n)$. Since we are assuming that X has tails at most as heavy as a Gaussian, we conclude that the series (67) has a strictly positive radius of convergence, which follows by comparison with the series

$$\sum_{j=1}^{\infty} e^j H_{j+2}^2(0) z^{j+2} / (j+2)! \leq \sum_{j=1}^{\infty} e^j E(|Z|^{2(j+2)}) z^{j+2} / (j+2)!,$$

where Z is standard Gaussian. We then obtain that

$$\begin{aligned} P(S_M \leq 0) &= \sum_{n=1}^{\infty} p q^{n-1} P(S_n \leq 0) = \frac{1}{2} + \frac{p}{q} \sum_{n=1}^{\infty} q^n \sum_{j=1}^{\infty} n^{-j/2} p_j \\ &= \frac{1}{2} + \frac{p}{q} \sum_{j=1}^{\infty} p_j \sum_{n=1}^{\infty} q^n n^{-j/2}. \end{aligned} \quad (68)$$

Furthermore, note that

$$p \sum_{j=1}^{\infty} p_j \sum_{n=1}^{\infty} q^n n^{-j/2} = p \sum_{j=1}^{\infty} p_j \sum_{n \neq j/2-1}^{\infty} \zeta(j/2 - n) \frac{(\log(1-p))^n}{n!} \quad (69)$$

$$+ p \sum_{j=1}^{\infty} p_j \frac{(\log(1-p))^{j/2-1} \pi (-1)^{j/2-1}}{\Gamma(j/2) \sin(\pi j/2)} I(j = \text{odd}) \quad (70)$$

Note that expanding (70) gives rise to powers of the form $p^{j/2}$ where j is odd. The double series in (69), as a function of p , can be analytically extended in a neighborhood of the origin in the complex plane by virtue of the analytic functions convergence theorem (see Rudin (1987), p. 214), and the expansion can be obtained directly from (69) in the natural way, by expanding $\log(1-p)$ and re-arranging the terms in powers of p . ■

8 An Application to Corrected Diffusion Approximations for the M/G/1 Queue

In this section we shall apply the results obtained in previous sections to develop corrected diffusion approximations (CDA's) for the time-in-system in the M/G/1 queueing model under first-in-first-out protocol, see Asmussen (1987). Using results by Siegmund (1979), a first order CDA for the M/G/1 queue was developed by Asmussen (1982) under exponential moments. In the heavy-tailed setting (assuming the existence of five moments in the underlying processing time distributions) Asmussen, S., and Binswanger, K. (1997) adapted the work by Hogan (1986) to provide a first order CDA for the time-in-system. In the next example, we provide, under weaker hypothesis, additional correction terms to the developments discussed above.

Example 1 (The M/G/1 queue) Suppose that customers arrive to a single-server queueing system according to a Poisson process with rate λ . The n^{th} customer requires an amount V_n of service time. Assume that the sequence $V = (V_n : n \geq 0)$ is iid and independent of the arrival process. Suppose that $EV^4 < \infty$ and that $\rho \triangleq \lambda EV < 1$. A well known identity to queueing theorists is the so-called Pollaczek-Khinchin formula (see Asmussen (1987)), which establishes that if W has the distribution of the steady-state waiting time in queue (exclusive of service), then

$$P(W > x) = \rho P(S_M > x),$$

where S_M is a geometric sum corresponding increments possessing distribution function $F(x) = \int_0^x P(V > s) ds / EV$ and geometric parameter $p = (1 - \rho)$. We are interested in obtaining an asymptotic expansion for the distribution of W as $\rho \nearrow 1$. This is the so-called heavy-traffic regime in which the systems is close to 100% utilization (a setting that often arises in applications). Let $v_j = EV^j$, then, a straightforward application of Theorem 3 indicates that

$$\begin{aligned} & P((1 - \rho)W > x) \\ &= \rho \exp(-x\gamma_0 + (\gamma_1 x + \xi_1)(1 - \rho) + (\gamma_2 x + \xi_2)(1 - \rho)^2) \\ & \quad + o((1 - \rho)^2), \end{aligned}$$

where

$$\begin{aligned}
\gamma_0 &= 2v_1/v_2, \\
\gamma_1 &= 2(3v_2^2 - 4v_3v_1)v_1/(3v_2^3) \\
\gamma_2 &= 2(6v_4v_1^2v_2 + 12v_3v_1v_2^2 - 16v_3^2v_1 - 3v_2^4)/(3v_2^5) \\
\xi_1 &= (3v_2^2 - 4v_3v_1)v_1/(6v_2) \\
\xi_2 &= (48v_4v_1^2v_2 + 24v_3v_1v_2^2 - 80v_3^2v_1^2 - 33v_2^4)/(36v_2^4).
\end{aligned}$$

The constants γ_k are also provided in Abate, Choudhury and Whitt (1995) for the more general GI/G/1 queue. Also, for the GI/G/1 queue also, under a different parameterization of the traffic intensity, Blanchet and Glynn (2005a) provide integral expressions (depending on the whole distribution of the inter-arrival and service times) for the ξ_k 's.

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