Exact Sampling and Steady-state Simulation of Reflected Brownian Motion

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Outline

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2. Multidimensional RBM: What is it?
3. Exact Simulation of RBM
4. Remark on Multidimensional SDEs Sampling
5. Unbiased Steady-state Estimation of RBM
6. Conclusions

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1. Multidimensional Reflected Brownian Motion (RBM): What is it and why do we care?

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2. Exact Sampling of RBM
3. A Remark about Sampling of Multidimensional SDEs
4. Multilevel Monte Carlo for steady-state analysis of RBM

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- **Model:** Solution to a pair $(Y(\cdot), L(\cdot))$ satisfying

$$dY(t) = dX(t) + RdL(t), \; Y(0) = y_0$$
$$Y(t) \geq 0 \text{ componentwise}$$
$$dL_j(t) \geq 0 \text{ non-decreasing for } j = 1, ..., d$$
$$Y_j(t)\, dL_j(t) = 0 \text{ i.e. } L_j \text{ increases only when } Y_j(t) = 0$$

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- **Skorokhod problem**: Existence and Uniqueness guaranteed (Harrison-Reiman '81)

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3. k-th service arriving in the i-th station: $V_i(k)$ (independent and identically distributed for each station)
4. Total jobs arriving at station i up to time t

$$J_i(t) := \sum_{k=1}^{N_i(t)} V_i(k)$$

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$$dY_i(t) = dJ_i(t) - r_i dt + \sum_{j=1}^{d} Q_{j,i} r_j dt$$

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- In matrix notation $X(t) = J(t) - (I - Q^T) rt$

$$dY(t) = dX(t) + (I - Q^T) dL(t),$$

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$$L_i(t) = r_i \int_0^t I(Y_i(s) = 0)ds.$$

- **This is a Skorokhod problem with input data** $X(\cdot)$.

Multidimensional RBM: Why do we care?

- **FACT 1 (Harrison - Reiman '81)**

$Y(\cdot)$ is Lipschitz continuous function of $X(\cdot)$ in uniform norm. WE'LL WRITE $Y(\cdot\,;X)$

CONSEQUENCE: By CLT (functional) a *really* large class of queueing systems can be approximated by RBM

RBM is one of the most important models in stochastic Operations Research!

Multidimensional RBM: Open problems and what we do

- Open Problems:

1. Can one evaluate the transition distribution of RBM?
2. Can one evaluate the steady-state distribution of RBM?

Our Contributions:

1) First exact sampler for RBM

2) Unbiased steady-state estimation

Outline

Exact Simulation of Diffusions: Why standard approach doesn't work

- Beskos & Roberts '04, Beskos, Roberts, and Papaspiliopoulos '06, Chen and Huang '12

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- Étoréa & Martinez (2011): *One* dimensional reflected diffusions.

Exact Simulation of Diffusions: Our contribution here

Theorem (B. and Murthy '14)

One can sample exactly $Y(t)$ for a multidimensional RBM with finite termination time.

- **Remark 1:** Methodology extends to multidimensional reflected diffusions of the form

$$dY(t) = \nabla u(Y(t))\,dt + dB(t) + dL(t), \quad Y(0) = y_0.$$

Forget about RBM and let's explain the key idea...

- Sample $Y = Z + \Delta$, with Z and Δ independent.
- Suppose Z's density is NOT known.
- Suppose that Z_n can be simulated so that $|Z_n - Z| < 1/n$ with probability 1.
- Suppose Δ has Lipschitz continuous density $f_\Delta(\cdot)$ with support on $[-a, a]$.

Exact Simulation of RBM: Use Following Facts

- **FACT 1:** $Y(\cdot\,;X)$ is Lipschitz in $X(\cdot)$. That is for $K > 0$ computable

$$\max_{t\in[0,1]} \left|Y(t;X) - Y(t;X')\right| \leq K \max_{t\in[0,1]} \left|X(t) - X'(t)\right|.$$

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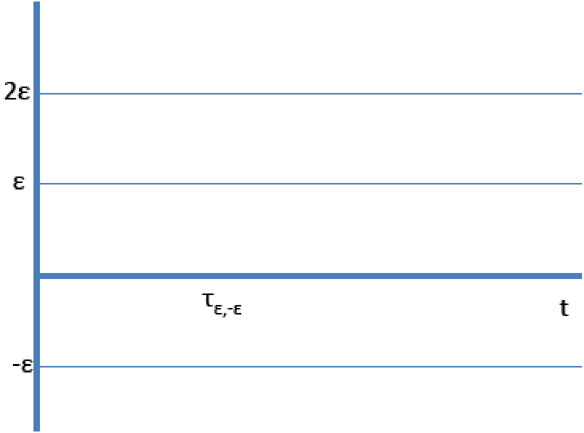
- **FACT 2:** $P(Y(t) > 0) = 1$ (deterministic t) and $Y(\cdot)$ is continuous.
- **FACT 3:** (Beskos, Peluchetti, Roberts '12 & *B. Chen '13*): Can simulate $X_\varepsilon(\cdot)$ piecewise linear such that **with probability one**

$$\max_{t\in[0,1]} |X(t) - X_\varepsilon(t)| < \varepsilon.$$

Exact Simulation of RBM: Using uniform simulation approximations

- **Simulate by stopping times...**



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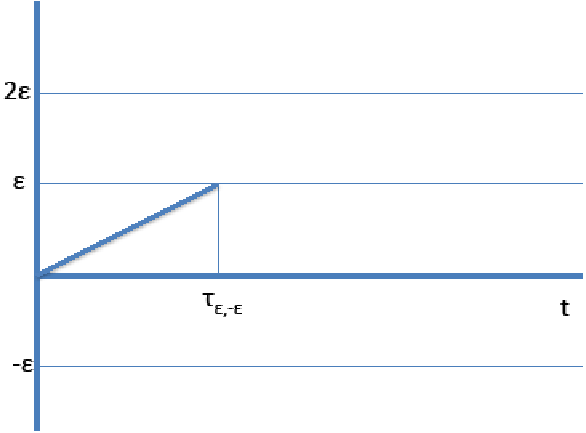

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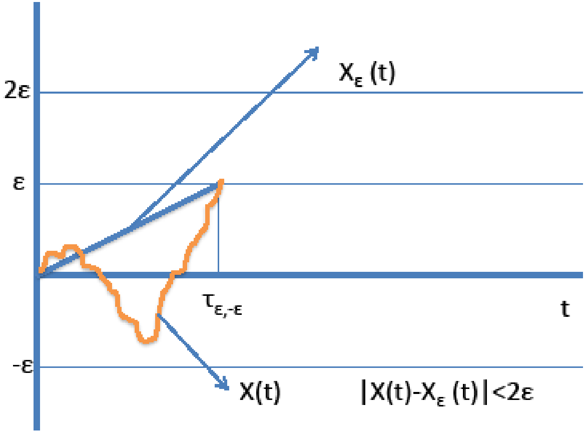

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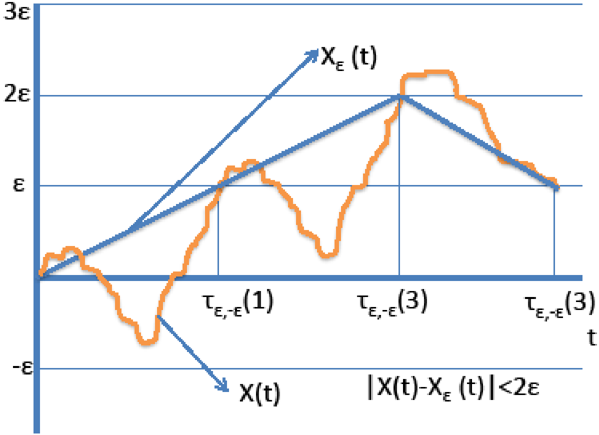
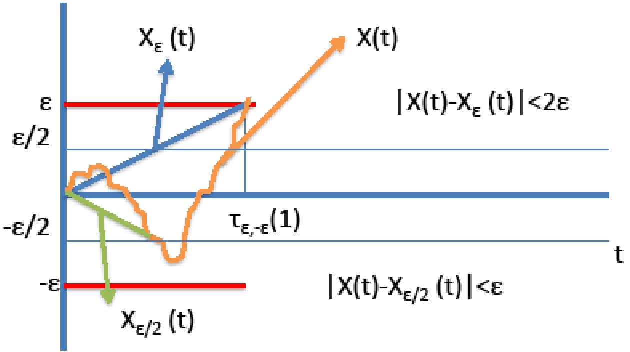


Exact Simulation of RBM: Using uniform simulation approximations

- **Refining $\varepsilon/2$: Sampling from conditional BESSEL BRIDGE $<$— Known transition density!**



Exact Simulation of RBM: Algorithm

- Simulate $X_{\varepsilon_1}(\cdot)$, $X_{\varepsilon_2}(\cdot)$,...,$X_{\varepsilon_N}(\cdot)$, $\varepsilon_N = 2^{-N}$ until $Y_{\varepsilon_N}(s) > 0$ for all $s \in [\tau_-, \tau_+]$ & $t \in [\tau_-, \tau_+]$.

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- $\Delta = X(t) - X(\tau_-)$ is increment of conditional Bessel bridge so KNOWN density $f_\Delta(\cdot)$
- **RESULT:** $f_\Delta(\cdot)$ **is Lipschitz continuous with support inside** $[-2^{-N_0+1}, 2^{-N_0+1}]$**.**

Exact Simulation of RBM: Algorithm

- Apply acceptance rejection: Let $f_{Y(t)}(\cdot)$ be density of $Y(t)$ given $\mathcal{F}_{N_0}(\tau_-)$
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- Propose Z from uniformly on $[Y_{\varepsilon_N}(\tau_-) - K2^{-N_0}, Y_{\varepsilon_N}(\tau_-) + K2^{-N_0}]$ then accept Z as a sample from $f_{Y(t)}(\cdot)$ IF
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- BIG problem $Y(\tau_-)$ is unknown... is it really?

Exact Simulation of RBM: Algorithm

- Key observations:

$$Law\left(\Delta|\sigma(\cup_{k=N_0}^{\infty}\mathcal{F}_k\left(\tau_-\right))\right) = Law\left(\Delta|\mathcal{F}_{N_0}\left(\tau_-\right)\right)$$

and missing information to finally evaluate $Y\left(\tau_-\right)$ is inside $\sigma(\cup_{k>N_0}^{\infty}\mathcal{F}_k\left(\tau_-\right))$.

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- So, you can continue refining $X_{\varepsilon_{N+1}}, X_{\varepsilon_{N+2}}, \ldots$to get $Y_{\varepsilon_{N+1}}\left(\tau_-\right)$, $Y_{\varepsilon_{N+2}}\left(\tau_-\right)$, $Y_{\varepsilon_{N+3}}\left(\tau_-\right)\ldots$ using Lipschitz continuity of $f_\Delta\left(\cdot\right)$ eventually

$$V \leq \frac{1}{C\left(N_0\right)} f_\Delta\left(Z - Y_{\varepsilon_L}\left(\tau_-\right)\right) - \frac{\widetilde{K}}{C\left(N_0\right)}\varepsilon_L \text{ -> ACCEPT}$$

OR

$$V \geq \frac{1}{C\left(N_0\right)} f_\Delta\left(Z - Y_{\varepsilon_L}\left(\tau_-\right)\right) + \frac{\widetilde{K}}{C\left(N_0\right)}\varepsilon_L \text{ -> REJECT}$$

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- **Since $\varepsilon_n \to \infty$, algorithm must finish in finite time!**

Outline

Remark on Multidimensional SDE Sampling

- **Important open problem in theory of Monte Carlo: Sample $Y(1)$ where**

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can be used to sample from $Y(1)$ exactly.

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- **B. Chen, and Dong (2015) provides the first algorithm that achieves (2) for the SDE (1). Algorithms uses theory of rough paths.**

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- Complexity for RBM, fixed ε, $\Omega(1/\varepsilon^k)$ for $k > 2$.
- Budhiraja, Chen, and Rubenthaler '12, only for $Ef(Z(\infty))$ with smooth $f(\cdot)$.

Steady-state Simulation of RBM: Our contribution here

Theorem (B., Chen, and Glynn '15)

Assume $\left(1 - Q^T\right)^{-1} EX(1) < 0$ *(stability). Let* $f(\cdot)$ *be Lipschitz continuous we construct an estimator* Z *such that*

$$EZ = Ef(Y(\infty))$$

and $Var(Z) < \infty$*. Moreover, the complexity of providing confidence intervals for* $Ef(Y(\infty))$ *with* ε *error and* δ *confidence is*

$$O\left(\frac{1}{\varepsilon^2}\log\left(\frac{1}{\varepsilon}\right)^2 \times \frac{1}{\delta}\right).$$

Steady-state Simulation of RBM: How we do it?

- Let $Y(t, y_0; X_{0:t})$ = value of RBM at t given $Y(0) = y_0$. Suppose $Y(0) = 0$ and $f(0) = 0$.

$$
\begin{aligned}
Ef(Y(\infty)) &= \sum_{n=0}^{\infty} E(f(Y(n+1, y_0; X_{0:n+1}) - f(Y(n, y_0; X_{0:n})) \\
&= \sum_{n=0}^{\infty} E(f(Y(n, Y(1); \mathbf{X}_{1:n}) - f(Y(n, y_0; \mathbf{X}_{1:n})) \\
&= E\left(\frac{f(Y(M, Y(1); \mathbf{X}_{1:M}) - f(Y(M, y_0; \mathbf{X}_{1:M})}{p(M)}\right) \\
&\leq KE\left(\frac{\| Y(M, Y(1); X_{1:M}) - Y(M, y_0; X_{1:M}) \|}{p(M)}\right),
\end{aligned}
$$

where M is a r.v. with probability mass function $p(m)$.

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$$
\begin{aligned}
Ef\left(Y\left(\infty\right)\right) &= \sum_{n=0}^{\infty} E(f(Y\left(n+1, y_0; X_{0:n+1}\right) - f(Y\left(n, y_0; X_{0:n}\right))) \\
&= \sum_{n=0}^{\infty} E(f(Y\left(n, Y\left(1\right); \mathbf{X}_{1:n}\right) - f(Y\left(n, y_0; \mathbf{X}_{1:n}\right))) \\
&= E\left(\frac{f(Y\left(M, Y\left(1\right); \mathbf{X}_{1:M}\right) - f(Y\left(M, y_0; \mathbf{X}_{1:M}\right)}{p\left(M\right)}\right) \\
&\leq KE\left(\frac{\|Y\left(M, Y\left(1\right); X_{1:M}\right) - Y\left(M, y_0; X_{1:M}\right)\|}{p\left(M\right)}\right),
\end{aligned}
$$

where M is a r.v. with probability mass function $p\left(m\right)$.

- Randomized multilevel MC (McLeish '2011, Glynn & Rhee '2013).

Steady-state Simulation of RBM: A key result

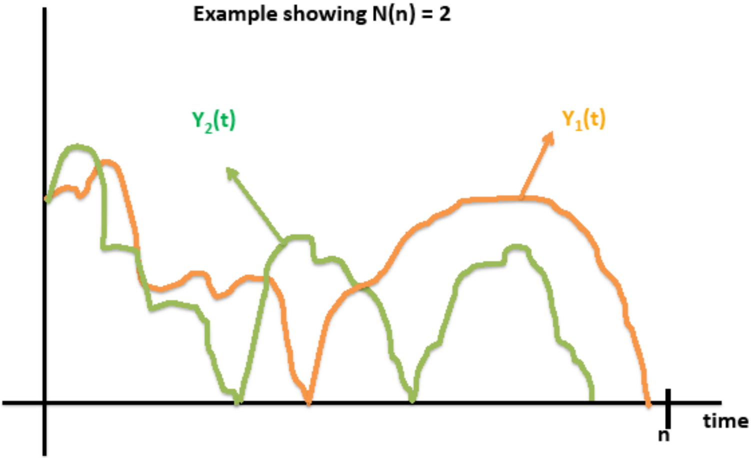
Lemma

Assume that Q irreducible (substochastic) and that $\left(I - Q^T\right)^{-1}\mu < 0$, *then*

$$\left\| Y\left(n, Y\left(1\right); X_{1:n}\right) - Y\left(n, y_0; X_{1:n}\right)\right\| \leq \rho^{N(n)},$$

for $\rho \in (0,1)$ *(depending on Q) and* $N(n)$ = *number of completed full cycles to zero in* $[0, n]$ *for process* $Y(\cdot)$.

Steady-state Simulation of RBM: How we do it?



Steady-state Simulation of RBM: Key insight in the proof

Proof.

[Proof Sketch] It turns out (*Mandelbaum and Ramanan (2010)*) that

$$\left\| D_{y_0} Y\left(n, y_0; X_{1:n}\right)^T \right\| \leq \| D_{i_1} D_{i_2} D_{i_3} \ldots D_{i_n} \| ,$$

where D_{i_k}'s looks like (for $i_k = 2$)

$$D_2 = \begin{pmatrix} 1 & 0 & 0 \\ Q_{2,1} & Q_{2,2} & Q_{2,3} \\ 0 & 0 & 1 \end{pmatrix}.$$

The order $i_1 \rightarrow i_2 \rightarrow \ldots \rightarrow i_n$ are visits to zero of ANY of the coordinates. Result follows from $Q^n \rightarrow 0$ as $n \rightarrow \infty$. □

Steady-state Simulation of RBM: Final Considerations

- Currently investigating the following

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- Currently investigating the following
- Application of coupling used here to other processes

Steady-state Simulation of RBM: Final Considerations

- Currently investigating the following
- Application of coupling used here to other processes
- Rates of convergence to stationarity and connections to product of random matrices

Outline

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- Presented first exact sampler for multidimensional RBM.
- Key idea builds on ε-approximations with path space with probability 1.
- First unbiased estimator for steady-state distributin of RBM.
- Key coupling connects to Lyapunov exponents and products of random matrices.