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1 Agenda

2 Multidimensional RBM: What is it?

3 Exact Simulation of RBM

4 Remark on Multidimensional SDEs Sampling

5 Unbiased Steady-state Estimation of RBM

6 Conclusions
1. Multidimensional Reflected Brownian Motion (RBM): What is it and why do we care?
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3 A Remark about Sampling of Multidimensional SDEs
Agenda

1. Multidimensional Reflected Brownian Motion (RBM): What is it and why do we care?
2. Exact Sampling of RBM
3. A Remark about Sampling of Multidimensional SDEs
4. Multilevel Monte Carlo for steady-state analysis of RBM
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Multidimensional RBM?

- Multidimensional RBM: solution to a certain SDE with constraints.
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  1. \((X(t) : t \geq 0)\) Brownian Motion drift \(\mu\) covariance \(\sigma\).
  2. Matrix \(R = I - Q^T\) where \(Q\) is a substochastic matrix and zeroes in the diagonal.
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1. \((X(t) : t \geq 0)\) Brownian Motion drift \(\mu\) covariance \(\sigma\).
2. Matrix \(R = I - QT\) where \(Q\) is a substochastic matrix and zeroes in the diagonal.

**Model:** Solution to a pair \((Y(\cdot), L(\cdot))\) satisfying

\[
dY(t) = dX(t) + RdL(t), \quad Y(0) = y_0
\]

- \(Y(t) \geq 0\) componentwise
- \(dL_j(t) \geq 0\) non-decreasing for \(j = 1, \ldots, d\)
- \(Y_j(t) dL_j(t) = 0\) i.e. \(L_j\) increases only when \(Y_j(t) = 0\)
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**Skorokhod problem:** Existence and Uniqueness guaranteed (Harrison-Reiman '81)
Forget RBM for a moment let’s build intuition using Stochastic Fluid Networks (Kella ’96)
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  3. $k$-th service arriving in the $i$-th station: $V_i(k)$ (independent and identically distributed for each station)
  4. Total jobs arriving at station $i$ up to time $t$

\[
J_i(t) := \sum_{k=1}^{N_i(t)} V_i(k)
\]
Multidimensional RBM: What does it mean?

- Assume service rate $r_i$ at the $i$-th server.
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- Assume service rate $r_i$ at the $i$-th server.
- $Y(t) \in \mathbb{R}^d$: System workload at time $t$

$$dY_i(t) = dJ_i(t) - r_i dt + \sum_{j=1}^{d} Q_{j,i} r_j dt$$

$$+ r_i I(Y_i(t) = 0) dt - \sum_{j=1}^{d} Q_{j,i} r_j I(Y_j(t) = 0) dt$$
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- In matrix notation $X(t) = J(t) - (I - Q^T)rt$

$$dY (t) = dX (t) + (I - Q^T) dL (t),$$

$$L_i (t) = r_i \int_0^t I(Y_i (s) = 0) ds.$$
Multidimensional RBM: What does it mean?

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- $Y(t) \in \mathbb{R}_+^d$: System workload at time $t$

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dY_i(t) = dJ_i(t) - r_i \, dt + \sum_{j=1}^{d} Q_{j,i} r_j \, dt
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- In matrix notation $X(t) = J(t) - (I - Q^T)rt$

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dY(t) = dX(t) + (I - Q^T) dL(t),
\]
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\]

- This is a Skorokhod problem with input data $X(\cdot)$. 

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Exact sampling and Steady-state Simulation  
02/2015  7 / 34
FACT 1 (Harrison - Reiman ’81)

\[ Y(\cdot) \text{ is Lipschitz continuous function of } X(\cdot) \text{ in uniform norm. WE’LL WRITE } Y(\cdot; X) \]

CONSEQUENCE: By CLT (functional) a really large class of queueing systems can be approximated by RBM

RBM is one of the most important models in stochastic Operations Research!
Open Problems:

1. Can one evaluate the transition distribution of RBM?
2. Can one evaluate the steady-state distribution of RBM?

Our Contributions:

1) First exact sampler for RBM
2) Unbiased steady-state estimation
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Exact Simulation of Diffusions: Why standard approach doesn’t work

- Beskos & Roberts ’04, Beskos, Roberts, and Papaspiliopoulos ’06, Chen and Huang ’12

\[ dY(t) = \nabla u(Y(t)) \, dt + dB(t), \quad Y(0) = y_0 \]
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- **KEY ASSUMPTION:** Drift term \( \nabla u(Y(t)) \, dt \) CAN’T deal with \( dL(t) \).
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- **KEY ASSUMPTION:** Drift term \( \nabla u(Y(t)) \, dt \) **CAN’T** deal with \( dL(t) \).

Theorem (B. and Murthy ’14)

One can sample exactly $Y(t)$ for a multidimensional RBM with finite termination time.

Remark 1: Methodology extends to multidimensional reflected diffusions of the form

$$dY(t) = \nabla u(Y(t)) \, dt + dB(t) + dL(t), \quad Y(0) = y_0.$$
Forget about RBM and let’s explain the key idea...

- Sample $Y = Z + \Delta$, with $Z$ and $\Delta$ independent.
- Suppose $Z$’s density is NOT known.
- Suppose that $Z_n$ can be simulated so that $|Z_n - Z| < 1/n$ with probability 1.
- Suppose $\Delta$ has Lipschitz continuous density $f_\Delta(\cdot)$ with support on $[-a, a]$. 
FACT 1: \( Y(\cdot ; X) \) is Lipschitz in \( X(\cdot) \). That is for \( K > 0 \) computable

\[
\max_{t \in [0,1]} |Y(t; X) - Y(t; X')| \leq K \max_{t \in [0,1]} |X(t) - X'(t)|.
\]
FACT 1: $Y(\cdot; X)$ is Lipschitz in $X(\cdot)$. That is for $K > 0$ computable

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FACT 2: $P(Y(t) > 0) = 1$ (deterministic $t$) and $Y(\cdot)$ is continuous.
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FACT 2: \( P(Y(t) > 0) = 1 \) (deterministic \( t \)) and \( Y(\cdot) \) is continuous.

FACT 3: (Beskos, Peluchetti, Roberts ’12 & B. Chen ’13): Can simulate \( X_\varepsilon(\cdot) \) piecewise linear such that with probability one

\[
\max_{t \in [0,1]} |X(t) - X_\varepsilon(t)| < \varepsilon.
\]
Exact Simulation of RBM: Using uniform simulation approximations

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- Refining $\varepsilon/2$: Sampling from conditional BESSEL BRIDGE $\xleftarrow{\sim}$ Known transition density!
Exact Simulation of RBM: Algorithm

- Simulate $X_{\epsilon_1} (\cdot), X_{\epsilon_2} (\cdot), \ldots, X_{\epsilon_N} (\cdot)$, $\epsilon_N = 2^{-N}$ until $Y_{\epsilon_N} (s) > 0$ for all $s \in [\tau_-, \tau_+]$ & $t \in [\tau_-, \tau_+]$. 
Simulate $X_{\varepsilon_1} (\cdot), X_{\varepsilon_2} (\cdot), \ldots, X_{\varepsilon_N} (\cdot)$, $\varepsilon_N = 2^{-N}$ until $Y_{\varepsilon_N} (s) > 0$ for all $s \in [\tau_-, \tau_+]$ \& $t \in [\tau_-, \tau_+]$.

Stop in finite time: By FACT 2, $Y (t) > 0$ almost surely \& $Y (\cdot)$ continuous.
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Denote information $F_{N_0}(\tau_-) = \text{info. generated by } \{X_{\epsilon_{N_0}}(s) : s \leq \tau_\}$.
Simulate $X_{\epsilon_1}(\cdot), X_{\epsilon_2}(\cdot), ..., X_{\epsilon_N}(\cdot), \epsilon_N = 2^{-N}$ until $Y_{\epsilon_N}(s) > 0$ for all $s \in [\tau_-, \tau_+]$ & $t \in [\tau_-, \tau_+]$.

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Note that

$$Y(t) = Y(\tau_-) + X(t) - X(\tau_-) \quad \text{AND}$$
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Simulate $X_{\varepsilon_1} (\cdot), X_{\varepsilon_2} (\cdot),..., X_{\varepsilon_N} (\cdot), \varepsilon_N = 2^{-N}$ until $Y_{\varepsilon_N} (s) > 0$ for all $s \in [\tau_-, \tau_+]$ & $t \in [\tau_-, \tau_+]$.

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Note that

$$Y (t) = Y (\tau_-) + X (t) - X (\tau_-) \quad \text{AND}$$

$$\Delta = X (t) - X (\tau_-) \text{ is increment of conditional Bessel bridge so KNOWN density } f_\Delta (\cdot)$$

RESULT: $f_\Delta (\cdot)$ is Lipschitz continuous with support inside $[-2^{-N_0+1}, 2^{-N_0+1}]$. 


Apply acceptance rejection: Let $f_{Y(t)}(\cdot)$ be density of $Y(t)$ given $\mathcal{F}_{N_0}(\tau_-)$

$$f_{Y(t)}(z) = f_{\Delta}(z - Y(\tau_-)).$$
Apply acceptance rejection: Let \( f_{Y(t)}(\cdot) \) be density of \( Y(t) \) given \( \mathcal{F}_{N_0}(\tau_\cdot) \)

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f_{Y(t)}(z) = f_\Delta(z - Y(\tau_\cdot)).
\]

We know that \( Y(t) \in [Y_{\epsilon N}(\tau_\cdot) - K2^{-N_0}, Y_{\epsilon N}(\tau_\cdot) + K2^{-N_0}] \) for computable \( K \) (FACT 1: Lipschitz continuity of Skorokhod map)
Apply acceptance rejection: Let $f_{Y(t)}(\cdot)$ be density of $Y(t)$ given $F_{N_0}(\tau_-)$

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Propose $Z$ from uniformly on $[Y_{\epsilon N}(\tau_-) - K2^{-N_0}, Y_{\epsilon N}(\tau_-) + K2^{-N_0}]$ then accept $Z$ as a sample from $f_{Y(t)}(\cdot)$ IF

$$V \leq \frac{1}{C(N_0)} f_{\Delta}(Z - Y(\tau_-)),$$

where $V$ is $U(0,1)$ independent of everything.
Exact Simulation of RBM: Algorithm

- **Apply acceptance rejection**: Let \( f_Y(t)(\cdot) \) be density of \( Y(t) \) given \( \mathcal{F}_{N_0}(\tau_-) \)

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f_Y(t)(z) = f_\Delta(z - Y(\tau_-)).
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- We know that \( Y(t) \in [Y_{\varepsilon N}(\tau_) - K2^{-N_0}, Y_{\varepsilon N}(\tau_) + K2^{-N_0}] \) for computable \( K \) (FACT 1: Lipschitz continuity of Skorokhod map)

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\]

where \( V \) is \( U(0, 1) \) independent of everything.

- **BIG problem** \( Y(\tau_) \) is unknown... is it really?
Key observations:

\[
\text{Law} \left( \Delta | \sigma(\bigcup_{k=N_0}^{\infty} \mathcal{F}_k (\tau_-)) \right) = \text{Law} \left( \Delta | \mathcal{F}_{N_0} (\tau_-) \right)
\]

and missing information to finally evaluate \( Y(\tau_-) \) is inside \( \sigma(\bigcup_{k>N_0}^{\infty} \mathcal{F}_k (\tau_-)) \).
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\]

and missing information to finally evaluate \( Y \left( \tau_- \right) \) is inside \( \sigma \left( \bigcup_{k=N_0}^\infty \mathcal{F}_k \left( \tau_- \right) \right) \).

So, you can continue refining \( X_{\varepsilon_{N+1}}, X_{\varepsilon_{N+2}}, \ldots \) to get \( Y_{\varepsilon_{N+1}} \left( \tau_- \right), \ Y_{\varepsilon_{N+2}} \left( \tau_- \right), \ Y_{\varepsilon_{N+3}} \left( \tau_- \right) \ldots \) using Lipschitz continuity of \( f_\Delta \left( \cdot \right) \) eventually

\[
V \leq \frac{1}{C \left( N_0 \right)} f_\Delta \left( Z - Y_{\varepsilon_L} \left( \tau_- \right) \right) - \frac{\tilde{K}}{C \left( N_0 \right)} \varepsilon_L \rightarrow \text{ACCEPT}
\]

OR

\[
V \geq \frac{1}{C \left( N_0 \right)} f_\Delta \left( Z - Y_{\varepsilon_L} \left( \tau_- \right) \right) + \frac{\tilde{K}}{C \left( N_0 \right)} \varepsilon_L \rightarrow \text{REJECT}
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Exact Simulation of RBM: Algorithm

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- So, you can continue refining \( X_{\epsilon_{N+1}}, X_{\epsilon_{N+2}}, \ldots \) to get \( Y_{\epsilon_{N+1}} (\tau_-), Y_{\epsilon_{N+2}} (\tau_-), Y_{\epsilon_{N+3}} (\tau_-) \) ... using Lipschitz continuity of \( f_{\Delta} (\cdot) \) eventually

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V \leq \frac{1}{C(N_0)} f_{\Delta} (Z - Y_{\epsilon_L} (\tau_-)) - \frac{\bar{K}}{C(N_0)} \epsilon_L \rightarrow \text{ACCEPT}
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- Since \( \epsilon_n \rightarrow \infty \), algorithm must finish in finite time!
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Important open problem in theory of Monte Carlo: Sample $Y(1)$ where

$$dY(t) = \mu(Y(t)) \, dt + \sigma(Y(t)) \, dB(t) \in \mathbb{R}^d$$ (1)
Remark on Multidimensional SDE Sampling

- **Important open problem in theory of Monte Carlo:** Sample $Y(1)$ where
  
  $$dY(t) = \mu(Y(t)) \, dt + \sigma(Y(t)) \, dB(t) \in \mathbb{R}^d \quad (1)$$

- Multidimensional RBM illustrates how
  
  $$\sup_{s \in [0,1]} |Y_\varepsilon(s) - Y(s)| \leq \varepsilon \quad (2)$$

  can be used to sample from $Y(1)$ exactly.
Remark on Multidimensional SDE Sampling

- **Important open problem in theory of Monte Carlo:** Sample $Y(1)$ where

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- Multidimensional RBM illustrates how

$$\sup_{s \in [0,1]} |Y_\varepsilon(s) - Y(s)| \leq \varepsilon$$  \hspace{1cm} (2)

can be used to sample from $Y(1)$ exactly.

- **B. Chen, and Dong (2015)** provides the first algorithm that achieves (2) for the SDE (1). Algorithms uses theory of rough paths.
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B. and Chen ’14: For RBM get $Y_\varepsilon(\infty)$ such that

$$|Y_\varepsilon(\infty) - Y(\infty)| \leq \varepsilon$$

with probability one.
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Complexity for RBM, fixed $\varepsilon$, $\Omega(1/\varepsilon^k)$ for $k > 2$. 

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Complexity for RBM, fixed $\varepsilon$, $\Omega(1/\varepsilon^k)$ for $k > 2$.

Budhiraja, Chen, and Rubenthaler ’12, only for $Ef (Z (\infty))$ with smooth $f (\cdot)$. 
Theorem (B., Chen, and Glynn ’15)

Assume \((1 - Q^T)^{-1} EX(1) < 0\) (stability). Let \(f(\cdot)\) be Lipschitz continuous we construct an estimator \(Z\) such that

\[
EZ = Ef(Y(\infty))
\]

and \(\text{Var}(Z) < \infty\). Moreover, the complexity of providing confidence intervals for \(Ef(Y(\infty))\) with \(\varepsilon\) error and \(\delta\) confidence is

\[
O \left( \frac{1}{\varepsilon^2} \log \left( \frac{1}{\varepsilon} \right)^2 \times \frac{1}{\delta} \right).
\]
Let $Y(t, y_0; X_{0:t}) = \text{value of RBM at } t \text{ given } Y(0) = y_0$. Suppose $Y(0) = 0$ and $f(0) = 0$. 

\[
Ef(Y(\infty)) = \sum_{n=0}^{\infty} E(f(Y(n+1, y_0; X_{0:n+1}) - f(Y(n, y_0; X_{0:n})))
\]

\[
= \sum_{n=0}^{\infty} E(f(Y(n, Y(1); X_{1:n}) - f(Y(n, y_0; X_{1:n})))
\]

\[
= E\left( \frac{f(Y(M, Y(1); X_{1:M}) - f(Y(M, y_0; X_{1:M}))}{p(M)} \right)
\]

\[
\leq KE\left( \frac{\|Y(M, Y(1); X_{1:M}) - Y(M, y_0; X_{1:M})\|}{p(M)} \right)
\]

where $M$ is a r.v. with probability mass function $p(m)$. 

Randomized multilevel MC (McLeish '2011, Glynn & Rhee '2013).
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= E\left( \frac{f(Y(M, Y(1); X_{1:M}) - f(Y(M, y_0; X_{1:M}))}{p(M)} \right)
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\leq KE\left( \frac{\|Y(M, Y(1); X_{1:M}) - Y(M, y_0; X_{1:M})\|}{p(M)} \right),
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where \( M \) is a r.v. with probability mass function \( p(m) \).

- Randomized multilevel MC (McLeish ’2011, Glynn & Rhee ’2013).
Lemma

Assume that $Q$ irreducible (substochastic) and that $(I - Q^T)^{-1} \mu < 0$, then

$$\| Y(n, Y(1); X_{1:n}) - Y(n, y_0; X_{1:n}) \| \leq \rho^{N(n)},$$

for $\rho \in (0, 1)$ (depending on $Q$) and $N(n) =$ number of completed full cycles to zero in $[0, n]$ for process $Y(\cdot)$. 
Example showing $N(n) = 2$
Proof.

[Proof Sketch] It turns out (Mandelbaum and Ramanan (2010) that

$$\| D_{y_0} Y (n, y_0; X_{1:n})^T \| \leq \| D_{i_1} D_{i_2} D_{i_3} \ldots D_{i_n} \|,$$

where $D_{i_k}$’s looks like (for $i_k = 2$)

$$D_2 = \begin{pmatrix} 1 & 0 & 0 \\ Q_{2,1} & Q_{2,2} & Q_{2,3} \\ 0 & 0 & 1 \end{pmatrix}.$$

The order $i_1 \rightarrow i_2 \rightarrow \ldots \rightarrow i_n$ are visits to zero of ANY of the coordinates. Result follows from $Q^n \rightarrow 0$ as $n \rightarrow \infty$. 

[QED]
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1. Application of coupling used here to other processes
2. Rates of convergence to stationarity and connections to product of random matrices
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Outline

1. Agenda
2. Multidimensional RBM: What is it?
3. Exact Simulation of RBM
4. Remark on Multidimensional SDEs Sampling
5. Unbiased Steady-state Estimation of RBM
6. Conclusions
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- Key idea builds on $\varepsilon$-approximations with path space with probability 1.
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- Key coupling connects to Lyapunov exponents and products of random matrices.