

Importance Sampling Methodology for Multidimensional Heavy-tailed Random Walks

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RESIM

- **Introduction**
- The One Dimensional Case
- Multidimensional Case
- Markov Random Walks
- Conclusions

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- ① *Illustrate our methodology for multidimensional regularly varying random walks.*
- ② *Discuss optimality properties of our proposed simulation procedure.*

Initial Notation

- Let X_1, X_2, \dots are iid regularly varying in R^d (to be discussed \rightarrow TBD).
- $EX_i = \eta \in R^d$ and A is an appropriate subset of R^d .
- Given $b > 0$ we write $bA = \{ba : a \in A\}$.
- $S_n = X_1 + \dots + X_n, (S_0 = s)$.
- $T_{bA} = \inf\{n \geq 0 : S_n \in bA\}$.
- Object of interest:

$$u_b(s, f) = E(f(S_0, S_1, \dots, S_{T_{bA}}, T_{bA}) I(T_{bA} < \infty)),$$

for any $f(\cdot)$ such that $0 < \delta_f \leq f \leq \delta_f^{-1}$ and $u_b(s, 1) \rightarrow 0$ as $b \nearrow \infty$ (TBD).

Efficiency and Performance Guarantee

Given any f for which there is $\delta_f \in (0, \infty)$, so that $\delta_f \leq f \leq \delta_f^{-1}$ estimate

$$u_b(s, f) = E(f(S_0, S_1, \dots, S_{T_{bA}}, T_{bA}) \mathbf{1}(T_{bA} < \infty)),$$

with good relative precision.

- The relative error of an estimator Z for $u_b(s, f)$ is

$$\text{Rel. Error} = \frac{(\text{Var}(Z))^{1/2}}{u_b(s)} + \left| \frac{EZ}{u_b(s)} - 1 \right|$$

- Suppose $EZ = u_b(s, f)$ and that $\text{Var}(Z) = O(u_b(s, f)^2)$, then we say that Z is *strongly efficient*.

- If Z is unbiased, then sampling n iid replications of Z gives an estimator $\hat{u}_b(n) = n^{-1} \sum_{j=1}^n Z_j$ such that

$$P(|\hat{u}_b(n) - u_b(s, f)| \geq \varepsilon u_b(s, f)) \leq \frac{\text{Var}(Z)}{n\varepsilon^2 u_b(s, f)^2}.$$

- So, it takes $O\left(\varepsilon^{-2} \delta^{-1} \text{Var}(Z) / u_b(s, f)^2\right)$ replications to achieve ε -relative error with $(1 - \delta) \cdot 100\%$ confidence.
- Any procedure for estimating $u_b(s, f)$ (for arbitrary f) must consider $(S_k : 0 \leq k \leq T_{bA})$ (this requires on average $O(E_s(T_{bA} | T_{bA} < \infty))$ operations).

- *In summary, the best performance that one can expect for ε -relative precision and $(1 - \delta) \cdot 100\%$ confidence based on iid replications of a given estimator involves*

$$O(\varepsilon^{-2} \delta^{-1} E_s(T_b | T_b < \infty))$$

operations.

An algorithm that achieves such performance is said to be optimal

- Introduction
- **The One Dimensional Case**
- Multidimensional Case
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The One Dimensional Case

- There is a rich large deviations theory for heavy-tailed random walks based on *subexponential* rv's

$$P(X_1 + X_2 > b) = 2P(X_1 > b)(1 + o(1))$$

as $b \rightarrow \infty$.

- *Focus on an important class* of a subexponential distributions is *the class of regularly varying distributions* (basically *power-law* type)

$$P(X_1 > t) = t^{-\alpha} L(t)$$

for $\alpha > 1$ and $L(t\beta) / L(t) \rightarrow 1$ as $t \nearrow \infty$ for each $\beta > 0$.

The One Dimensional Case

- Let $EX_i = \eta < 0$ and set $A = [1, \infty)$, (we write $T_b = T_{bA}$) estimate ruin (Pakes, Veraverbeker, Cohen... see text of Asmussen '03)

$$u_b(s, 1) = u_b(s) = P_s(T_b < \infty).$$

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- **Strategy:** use importance sampling consider the Markov kernel $K(\cdot)$

$$\begin{aligned} K(s_0, s_1 + ds_1) &= r^{-1}(s_0, s_1) P(s_0 + X_1 \in s_1 + ds_1) \\ K(s_0, s_1) ds_1 &= r^{-1}(s_0, s_1) f_{X_1}(s_1 - s_0) ds_1 \end{aligned} \quad (1)$$

for a positive function $r(\cdot)$. ((1) valid in presence of densities).

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- The importance sampling estimator is:**

$$Z = \prod_{j=1}^{T_b-1} r(S_j, S_{j+1}) I(T_b < \infty),$$

the S_n 's are simulated under $K(\cdot)$.

The One Dimensional Case

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- **Moral of the story:**
 - *Select an importance sampler that mimics the behavior of such conditional distribution.*

The One Dimensional Case

Theorem

(Asmussen and Kluppelberg): Conditional on $T_b < \infty$, we have that

$$\left(\frac{S_{uT_b}}{T_b}, \frac{S_{T_b} - b}{b}, \frac{T_b}{b} \right) \Longrightarrow (\eta u, Z_1, Z_2)$$

on $D(0, 1) \times R \times R$ as $b \nearrow \infty$, where Z_1 and Z_2 are Pareto with index $\alpha - 1$.

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- **Interpretation:** Prior to ruin, the random walk has drift η and a large jump of size b occurs suddenly in $O(b)$ time...
- So, given that a jump hasn't occurred by time k , then $S_k \approx \eta k$ and the chance of reaching b in the next increment *given* that we eventually reach b ($T_b < \infty$)

$$\frac{P(X > b - \eta k)}{\int_0^\infty P(X > b - \eta u) du} \approx \frac{-\eta P(X > b - \eta k)}{\int_b^\infty P(X > s) ds} = O\left(\frac{1}{b}\right).$$

The One Dimensional Case

- **Family of changes-of-measure:** Here s is the current position of the walk, f is the density (NOTE $p(s)$ and $a \in (0, 1)$)

$$f_{X|s}(x|s) = p(s) \frac{f_X(x) I(x > a(b-s))}{P(X > a(b-s))} + (1-p(s)) \frac{f_X(x) I(x \leq a(b-s))}{P(X > a(b-s))}$$

- In other words, $s_0 = s$ and $s_1 = s_0 + x$

$$r(s_0, s_1)^{-1} = p(s_0) \frac{I(s_1 - s_0 > a(b-s_0))}{P(X > a(b-s_0))} + (1-p(s_1)) \frac{I(s_1 - s_0 \leq a(b-s_0))}{P(X \leq a(b-s_0))}$$

- Introduced by Dupuis, Leder and Wang '06 for finite sum...

- **Lyapunov Inequalities for Variance Control:**

Lemma (B. & Glynn '07)

Suppose that there is a positive function $g(\cdot)$ such that

$$E_s^K \left(\frac{g(S_1) r(s, S_1)^2}{g(s)} \right) = E_s \left(\frac{g(S_1) r(s, S_1)}{g(s)} \right) \leq 1$$

for all $s \leq b$ and $g(s) \geq 1$ for $s > b$. Then,

$$E_s^K Z^2 = E_s^K \left(\prod_{j=1}^{T_b-1} r(S_j, S_{j+1})^2 I(T_b < \infty) \right) \leq g(s).$$

The One Dimensional Case

- **Testing the Inequality:**

- We wish to achieve *strong efficiency*, so we pick (for some $\kappa > 0$)

$$g(s) = \min \left(\kappa \left(\int_{b-s}^{\infty} P(X > u) du \right)^2, 1 \right).$$

- Pick (*note that we need to select $\theta, \kappa > 0$ to force Lyapunov ineq.*)

$$p(s) = \theta \frac{P(X > b-s)}{\int_{b-s}^{\infty} P(X > s) du}$$

- **Testing the Inequality on $g(s) < 1$ (note that $g \leq 1$):**

$$\begin{aligned} & E_s \left(\frac{g(S_1) r(s, S_1)}{g(s)} \right) \\ = & \frac{E(g(s+X); X > a(b-s)) P(X > a(b-s))}{p(s) g(s)} \\ & + \frac{E(g(s+X); X \leq a(b-s)) P(X \leq a(b-s))}{(1-p(s)) g(s)} \end{aligned}$$

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 \leq & \frac{P(X > a(b-s))^2}{p(s) g(s)} + \frac{E(g(s+X); X \leq a(b-s))}{(1-p(s)) g(s)} \\
 \approx & \frac{a^{-\alpha} P(X > a(b-s))}{\theta \kappa \int_{b-s}^{\infty} P(X > u) du} + 1 + 2(\eta + \theta) \frac{P(X > (b-s))}{\left(\int_{b-s}^{\infty} P(X > u) du \right)}
 \end{aligned}$$

- **Corresponding Algorithm:**
- Select $a \in (0, 1)$, then choose θ and κ based on Lyapunov inequality
- AT EACH TIME STEP TEST
 - IF $g(s) < 1$ apply Imp. Sampling according $p(s)$ - mixture
 - ELSE do NOT apply I.S. and continue until hitting.
- OUTPUT PRODUCT OF LIKELIHOOD RATIOS Z

- **Termination of the algorithm:** Intuitively, what can go wrong?
- Let M be the time until we get one large jump given $S_0 = 0$

$$\begin{aligned} P(M > tb) &\approx \prod_{j=0}^{kb} (1 - p(-j\eta)) \approx \exp\left(-\sum_{j=0}^{tb} \frac{\theta(\alpha-1)}{b-\eta j}\right) \\ &\approx \exp\left(-\int_0^{tb} \frac{\theta(\alpha-1)}{b-\eta s} ds\right) = \left(\frac{1}{1-\eta t}\right)^{\theta(\alpha-1)/(-\eta)}. \end{aligned}$$

So, if $\theta < -\eta/(\alpha-1)$ then EXPECTED TERMINATION TIME COULD BE INFINITE!

- **Optimal Selection of Parameters:**

Lemma

For each $\varepsilon_1 > 0$ there exists $\delta_1, m > 0$ such if $\theta = -\eta - \delta_1$ and $a = 1 - \delta_1$ then, if $\alpha > 2$,

$$\left| E_0^K T_b - E_0 (T_b | T_b < \infty) \right| \leq \varepsilon_1 E_0 (T_b | T_b < \infty) \text{ for all } b \geq m$$

Corollary

The importance sampling algorithm based on the previous selection of parameters is **optimal**.

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- BUT, as we saw, this can give infinite expected hitting time

- **Introducing a Controlled Bias:**

Note that

$$\begin{aligned}P_0(T_b < \infty) &= P_0(T_b < T_{-\beta b}) + P_0(T_{-\beta b} < T_b, T_b < \infty) \\ &\leq P_0(T_b < T_{-\beta b}) + P_{-\beta b}(T_b < \infty) \\ &\leq P_0(T_b < T_{-\beta b}) + g(-\beta b)^{1/2}.\end{aligned}$$

So, relative bias

$$0 \leq 1 - \frac{P_0(T_b < T_{-\beta b})}{P_0(T_b < \infty)} \leq \frac{g(-\beta b)^{1/2}}{P_0(T_b < \infty)} \leq \frac{g(-\beta b)^{1/2}}{P_0(T_b < T_{-\beta b})}$$

- **Complexity Count:**

$$\text{RELATIVE BIAS} = O\left(\beta^{-\alpha/2}\right).$$

- Want relative bias less than $\varepsilon/2$. Then, total complexity count (for ε -relative error with $(1 - \delta) \cdot 100\%$ confidence)

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- Or letting $\beta(b) \nearrow \infty$ one gets a complexity count of order

$$O\left(\varepsilon^{-2}\delta^{-1}\beta(b)E_0\left(T_b \mid T_b < \infty\right)\right).$$

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The Multidimensional Case

Assumptions on X

- **Multidimensional regular variation:** There exists a (Radon) measure $\mu(\cdot)$ such that

$$\frac{P(X \in Ab)}{P(|X| > b)} \implies \mu(A)$$

(sense of vague convergence) for $A \in \mathcal{B}(R^d \setminus \{0\})$. Resnick (2007), Hult et al (2005),...

- **Example:** X follows a standard t with α degrees of freedom, then

$$P(X \in bA) = \int_A \frac{m}{(1 + y^T y / \nu)^{(\alpha+d)/2}} dy,$$
$$\mu(A) = \int_A \frac{dy}{(y^T y)^{(\alpha+d)/2}}$$

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- **Example:** For the t distribution $\sigma(\cdot)$ is the uniform distribution on the unit sphere.

The Multidimensional Case

Assumptions on $EX \in R^d$ and A

- **Assumption (CONE):** There are linearly independent vectors $v_1^*, v_2^*, \dots, v_m^* \in R^d$ ($\|v_i^*\| = 1$) and $\delta^* > 0$ such that $v_i^{*T} \beta \leq -\delta^*$ for all $1 \leq i \leq m$ and for all $z \in A$ there is at least one i for which $v_i^{*T} z \geq \delta$.

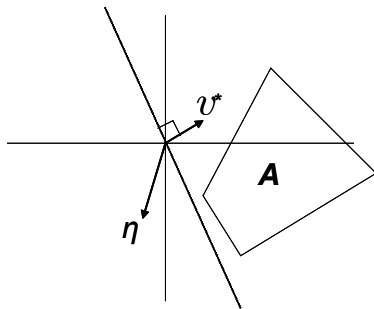
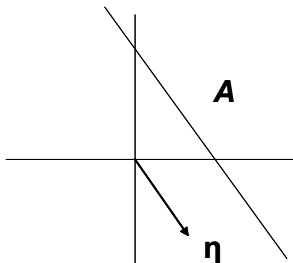


Diagram illustrating Assumption A or a two dimensional random walk

The Multidimensional Case

Assumptions on $EX \in R^d$ and A

- Previous assumption rules out the following situation:



- Here the drift goes parallel to A and the process will eventually hit A (this is the analogue of having $EX = 0$ in $d = 1$).

Assumptions on $\mu(\cdot)$ and A

- Avoid degenerate situations where A is "very thin" relative to $\mu(\cdot)$ (example A a line in the case of a t distribution)...

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- Avoid degenerate situations where A is "very thin" relative to $\mu(\cdot)$ (example A a line in the case of a t distribution)...
- **Assumption (POS):** Assume that $\alpha > 1$ (regularly varying index) and there exists $\delta_1 > 0$ such that

$$\inf_{0 \leq h \leq \delta_1} \mu(A - h\eta) > 0$$

Ruin Probabilities: (Hult, Lindskog, Mikosch and Samorodnitsky '05, Hult and Lindskog '06)

$$\begin{aligned} P_0(T_{Ab} < \infty) &\sim \int_0^\infty P(X + t\eta \in bA^*) dt \\ &\sim bP(|X| > b) \int_0^\infty \mu(A^* - t\eta) dt \end{aligned}$$

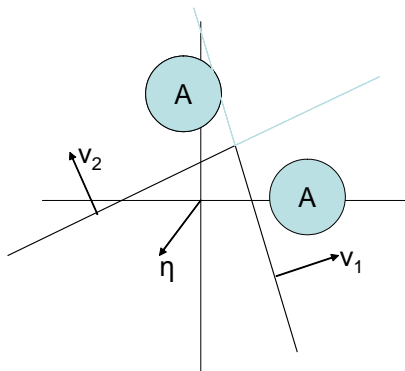
as $b \nearrow \infty$, where $A^* = \{A - t\eta : t \geq 0\}$

Remark: By assumption POS

$$\int_0^\infty \mu(A^* - t\eta) dt \in (0, \infty).$$

The Multidimensional Case

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- $$q_{X|s}(x) = p(s) \frac{f_X(x) I(\cup_{i=1}^m C_i^a(s, b))}{P(\cup_{i=1}^m C_i^a(s, b))} + (1 - p(s)) \frac{f_X(x) I(\cap_{i=1}^m \bar{C}_i^a(s, b))}{P(\cap_{i=1}^m \bar{C}_i^a(s, b))}.$$

Selection of Lyapunov Inequality:

- Heuristic (fluid) analysis suggests

$$\begin{aligned} P_s (T_{bA} < \infty) &\approx \int_0^\infty P(X + s + \eta t \in bA) dt \\ &\leq \sum_{i=1}^m \int_0^\infty P(v_i^{*T} (X + s + \eta t) \geq b\delta^*) dt \\ &= \sum_{i=1}^m \frac{1}{-v_i^{*T} \eta} G_i (b\delta^* - v_i^{*T} s), \end{aligned}$$

where

$$G_i (b\delta^* - v_i^{*T} s) = \int_{b\delta^* - v_i^{*T} s}^\infty P(v_i^{*T} X > u) du$$

The Multidimensional Case

Lyapunov function to test:

- Heuristic (fluid) analysis suggests

$$g_b(s) = \min \left(\kappa h_b(s)^2, 1 \right) = O \left(b^2 P(|X| > b)^2 \right),$$

where

$$h_b(s) = \sum_{i=1}^m \frac{1}{-v_i^{*T} \eta} G_i \left(b \delta^* - v_i^{*T} s \right).$$

Theorem

Pick $p(s) = \theta P(\cup_{i=1}^m C_i^a(s, b)) / h_b(s)$ and $a \in (0, 1)$, then $\kappa, \theta > 0$ can be chosen so that $g_b(\cdot)$ satisfies the Lyapunov inequality. So, applying the proposed IS when $g_b(s) < 1$ gives a strongly efficient estimator.

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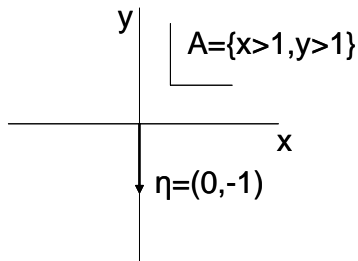
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 - Right, but now the problem is finite dimensional and static
 - Sometimes one can use the spectral measure to do the sampling
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 - As in $d = 1$ one can introduce a controlled bias
- Determination of the v_i^* 's when A is union of polyhedra through linear programs

The Multidimensional Case

Example: Random walk with t distributed increments and $\alpha = 3$ degrees of freedom. Relative bias no more than 5% with at least 97% confidence.



b	Est	SD	Sample size
10	$1.83e - 03$	$1.31e - 02$	20000
20	$4.28e - 04$	$2.54e - 03$	20000
50	$8.80e - 05$	$4.74e - 04$	20000

Theorem

Assume CONE and POS. In addition suppose that the sampling and the evaluation of $K(\cdot)$ can be done in $O(1)$ operations (as $b \nearrow \infty$). Then, state-dependent importance sampling via Lyapunov control requires

$$O\left(\varepsilon^{-2} \delta^{-1} \varepsilon^{-2/\alpha} b\right)$$

operations to achieve ε -relative error with $(1 - \delta) \cdot 100\%$ confidence.

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- *Select an appropriate change of measure.*
- *Use Lyapunov inequalities for variance / bias control and termination.*
- *NOTE the difference between light vs heavy-tailed... discrete nature (jumps for heavy tails) forces control at very short time scales!*
- **Goal: Apply Lyapunov inequalities at more than one time step**

Basic idea (again let $d = 1$):

- To simplify, let $(Y_n : n \geq 0)$ be an irreducible, aperiodic, finite state-space recurrent Markov chain

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- $E_{\pi} X_n < 0$ (but there maybe states $y = i$ for which $E_i X_n > 0$)

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- $E_{\pi} X_n < 0$ (but there maybe states $y = i$ for which $E_i X_n > 0$)
- Application of Lyapunov inequality **in one step** doesn't work (MIGHT NOT SEE $E_{\pi} X_n < 0$)

Basic idea (again let $\mathbf{d} = \mathbf{1}$):

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$$P(X_n > t | Y_n = k) = L_k(t) t^{-\alpha}$$

- $E_{\pi} X_n < 0$ (but there maybe states $y = i$ for which $E_i X_n > 0$)
- Application of Lyapunov inequality **in one step** doesn't work (MIGHT NOT SEE $E_{\pi} X_n < 0$)
- Test Lyapunov inequality and apply algorithm at time scales of order τ (for appropriately define stopping time τ).

- Introduction
- The One Dimensional Case
- Multidimensional Case
- Markov Random Walks
- **Conclusions**

- Discussed rare-event simulation for multidimensional random walks
 - Discrete nature (sudden jumps) forces control at short time scales
 - Variance control via Lyapunov functions obtained from fluid analysis
 - Biased estimators to control termination, but *bias is quantifiable*
 - Provably efficient algorithms & interplay between linear programming, affine functions and regular variation.
- Markov random walks (and other settings, small noise processes?) force control at slightly larger time scales \longrightarrow Lyapunov functions applied to τ transition kernel.