Exact Simulation of Random Structures Depending on Infinite Future Information & Applications to Max-Stable Processes

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What is the Goal

Present techniques to simulate processes that involve information from "infinite" future.

Application: Optimal Exact Simulation of Max-stable Fields & Density Estimation

Main Application of the Talk: Max-stable Fields

- **Example 1:** $M(\cdot)$ can be represented as

$$M(t) = \sup_{n \geq 1}\{-\log(A_n) + Z_n(t)\},$$

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- $A_n = n$-th arrival of Poisson process with rate 1 (independent of $Y_n(\cdot)$).
- Brown-Resnick, de Haan, Engelke, Kabluchko, Schlather, Smith, Penrose...

Additional Applications

- **Example 2:** Suppose that $S_n = \Delta_1 + ... + \Delta_n$ is a mean-zero multidimensional random walk

$$M_n = \max\{S_k - \mu(k) : k \geq n\},$$

where $\mu(k)$ grows faster than $O\left(k^{1/2+\epsilon}\right)$ for some $\epsilon > 0$.

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- Propp & Wilson '96, Kendall '98,...

Additional Applications

- **Example 3:** Stochastic Differential Equations, pick $\Delta_n = 2^{-n}$,

$$X_n\left(\left(k+1\right)\Delta_n\right) = X_n\left(k\Delta_n\right) + b\left(X_n\left(k\Delta_n\right)\right)\left(B\left(\left(k+1\right)\Delta_n\right) - B\left(k\Delta_n\right)\right)$$

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- Let $X_n\left(\cdot\right)$ be continuous piecewise linear interpolation.
- Given ε, simulate $N\left(\varepsilon\right)$ such that **with probability 1**

$$\sup_{0\leq t\leq 1}\left\|X_{N(\varepsilon)}\left(t\right) - X\left(t\right)\right\| \leq \varepsilon. \tag{1}$$

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- Note that $N\left(\varepsilon\right)$ is NOT a stopping time adapted to $\mathcal{F}_n = \sigma\left(B\left(k\Delta_n\right) : 0 \leq k \leq 2^n - 1\right)$.
- Finding a piecewise linear (or constant) process that satisfies (1) is *Tolerance Enforced Simulation* (TES).

How is TES Related To Exact Estimation?

- Suppose that $X_{N(\varepsilon_n)}(\cdot)$ can be obtained for $0 < \varepsilon_n \to 0$.

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- Let $F(\cdot)$ be ANY sample path functional of $X(\cdot)$.
- Assume for simplicity that $F(\cdot) \geq 0$ is Lipschitz in uniform norm.
- Let $T > 0$ have density $g(\cdot)$ & independent of $X(\cdot)$ & $X_{N(\varepsilon_n)}(\cdot)$

$$
\begin{aligned}
E(F(X)) &= E\int_0^\infty I(F(X) > t)\,dt \\
&= E\int_0^\infty \frac{I(F(X) > t)}{g(t)} g(t)\,dt \\
&= E\left(\frac{I(F(X) > T)}{g(T)}\right).
\end{aligned}
$$

How is TES Related To Unbiased Simulation?

- **Conclusion:**

$$Z = \frac{I\left(F\left(X\right) > T\right)}{g\left(T\right)}$$

is unbiased estimator of $E\left(F\left(X\right)\right)$.

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$$Z = \frac{I(F(X) > T)}{g(T)}$$

is unbiased estimator of $E(F(X))$.

- A lot sample $I(F(X) > T)$ since this 1 or 0 variable (answer "YES" or "NO").
- Continue increasing $n \longleftarrow n+1$ until

$$\begin{aligned} F\left(X_{N(\varepsilon_n)}\right) &> T + \kappa\varepsilon_n \quad \text{or} \\ F\left(X_{N(\varepsilon_n)}\right) &< T - \kappa\varepsilon_n, \end{aligned}$$

where κ is the Lipschitz constant of $F(\cdot)$.

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- Introduce a sequence of "records breakers".
- Records broken only finitely many times.
- Locate Record Breakers: YES or NO question ("will there be a next record?").
- Relevant future information encoded on finitely many YES or NO questions.

Applications Treated in Several Papers

- B. & Chen (2012): TES for Brownian and RBM.

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- Several other applications...
- **Today: B., Dieker, Liu, Mikosch (2015): Exact sampling and TES for Max-stable Processes**.

- Wang & Stoev (2011) Conditional sampling for spectrally discrete max-stable random fields.

Literature Review on Simulation of Max-stable Processes

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- Dieker & Mikosch (2014) Exact simulation of Brown Resnick random fields.

On Exact Simulation of Brown-Resnick Fields

- Dieker & Mikosch (2014): **IF** $Z_n(\cdot)$ **stationary increments and** $E\exp(Z_n(t)) = 1$

$$e^{M(t_i)} = \sup_{n\geq 1}\left\{\frac{d}{A_n}\cdot\frac{\exp(Z_n(t_i - T_n))}{\sum_{k=1}^{d}\exp(Z_n(t_k - T_n))}\right\},$$

where $\{T_n\}_{n\geq 1}$ is i.i.d. uniform on $\{t_1, ..., t_d\}$ <− locations in advance.

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- Complexity $O(d)$ points of $Y_n(\cdot)$ for each t_i.
- **Complexity** $O(d \times C(d))$ **where** $C(d)$ **= Complexity of sampling** $(Y_n(t_1),...,Y_n(t_d))$.

Intrinsic Complexity of Exact Sampling on Compact Domains

Is sampling $M(t_1), ..., M(t_d)$ basically as "easy" as sampling $Z_1(t_1), ..., Z_1(t_d)$?

Answer: Yes! This is what we mean by optimality (total complexity $O(C(d))$)

Our goal next is to explain how & use it for density estimation...

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- Concept introduced in B. & Chen (2012).
- See also: ε-strong simulation G. Roberts, Beskos, Peluchetti, Murray, Pollock, Johansen...

Example of TES: Brownian Motion

- Consider Brownian Motion

$$Z_n(t) = \sum_{m=0}^{\infty} \lambda_m \Lambda_m(t) W_m(n).$$

where $W_m(n)$'s are i.i.d. $N(0,1)$ and $\lambda_m = 2^{-(j+1)/2}$ assuming $m = 2^{j-1} + k \geq 1,\ k = 0, 1, ..., 2^{j-1} - 1$ and $\lambda_0 = 1$.

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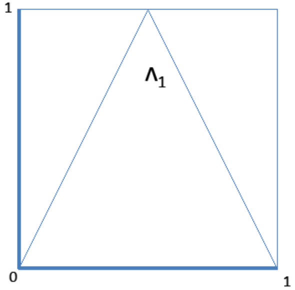
- Also, $\Lambda_0(t) = t$

$$\Lambda_1(t) = (1/2 - |t - 1/2|)\, I(t \in [0,1]),$$

and $\Lambda_n(t)$ are translations and dilations of $\Lambda_1(t)$ along dyadic points...

Wavelet Decomposition



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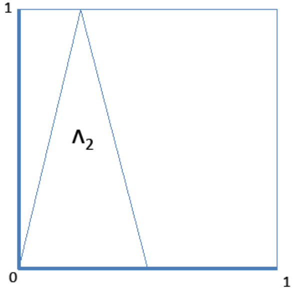

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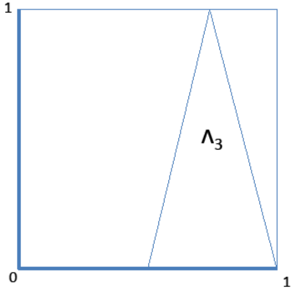

Tolerance Enforced Simulation: B. and Chen (2012)

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- $\mathbf{m}(\varepsilon) = 2^J \geq N + 1/\varepsilon^2 \geq 1/\varepsilon^2$ <– *after last record breaker*.

$$
\begin{aligned}
\sum_{n=\mathbf{m}(\varepsilon)}^{\infty} \lambda_j \Lambda_j(t)|W_j| &\leq \sum_{n=\mathbf{m}(\varepsilon)}^{\infty} \lambda_j \Lambda_j(t)\, r(j) \\
&= \sum_{j=J}^{\infty} r(j)\, 2^{-(j+1)/2} \sum_{k=0}^{2^j-1} \Lambda_{2^j+k}(t) \\
&= \sum_{j=J}^{\infty} \rho(j+3)^{1/2}\, 2^{-(j+1)/2} \leq 5\rho\varepsilon\sqrt{\log(1/\varepsilon)}.
\end{aligned}
$$

- **SIMULATE** W_j's jointly with times $R_m = \min\{n > R_{m-1} : |W_j| > r(j)\}$; $R_0 = -1$ **(PENDING)**
- Get $N = \max\{R_k : R_k < \infty\}$ (last record breaker).

We obtain $5\rho\varepsilon\sqrt{\log(1/\varepsilon)}$ *guaranteed uniform error with* $O(E(N + 1/\varepsilon^2)) = O(1/\varepsilon^2)$ *complexity (optimal).*

Simulation of the Crucial Quantities

- Consider $R_1 = \min\{n \geq 1 : |W_j| > r(j)\}$ and let $p_1 = P(R_1 = \infty)$. How to sample $Ber(p_1)$?

$$
\begin{aligned}
p_1 &\leq P(R_1 > m) := U(m) = \prod_{n=1}^{m} P(|W_j| \leq r(j)) \\
p_1 &= U(m) \cdot \prod_{n=m+1}^{\infty} P(|W_j| \leq r(j)) \\
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- **Let $V \sim U(0,1)$ and decide $V < p_1$ using "loop" $m \longleftarrow m+1$: Eventually finish when**

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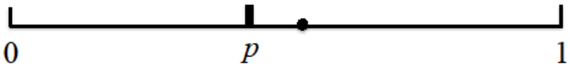
- Since

$$P(R_1 = m) = U(m-1) - U(m),$$

if $V > U(m)$ and $D(m) < V < U(m-1)$, then $R_1 = m$.

Locating Record Breaker IF IT Ever Happens



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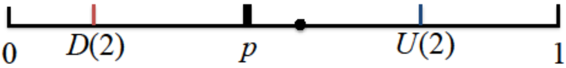

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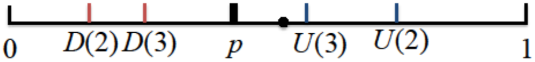

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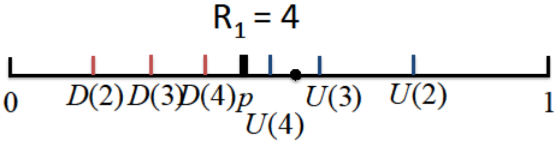

Summary: TES for Brownian Motion

Algorithm: Output $\mathbf{m}(\varepsilon)$ jointly with W_n's
Step 0: Set, ε, $\rho = 4$, $G = 2\left\lceil \varepsilon^{-2} \right\rceil$, $\mathcal{R} = []$.
Step 1: Set $U = 1$, $D = 0$. Simulate $V \sim U(0,1)$.
Step 2: While $U > V > D$, set $G \leftarrow G + 1$,
$U \leftarrow P(|W_1| \leq \rho\sqrt{\log G}) * U$ and $D \leftarrow (1 - G^{1-\rho^2/2})U$.
Step 3: If $V \leq D$, $\mathcal{R} = [\mathcal{R}, G]$ and return to Step 1.
Step 4: If $V > U$, $\mathbf{m}(\varepsilon) = G$, $\mathcal{R} = [\mathcal{R}, G]$.
Step 5: If $j \in S$, W_j has law $(W \mid |W| > \rho\sqrt{\log(n)})$ (else given $|W| \leq \rho\sqrt{\log(n)})$.

Theorem (B. & Chen '12)

The algorithm outputs a wavelet approximation with guaranteed ε error (with probability one) in uniform norm with complexity $O\left(\varepsilon^{-2}\log(1/\varepsilon)\right)$.

Comments

- Technique generally applicable to

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if $Y_n(\cdot)$ are independent, fully simulatable, and

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- Levy processes, fractional Brownian motion, ...

Another Application of Record-Breaker Technique

- Consider random walk, say $\tau_i > 0$, are i.i.d.

$$\begin{aligned} S_n &= \tau_1 + ... + \tau_n - nv, \\ S_0 &= 0, \quad \& \;\; E(S_n) < 0. \end{aligned}$$

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$$\begin{aligned} S_n &= \tau_1 + ... + \tau_n - nv, \\ S_0 &= 0, \quad \& \;\; E(S_n) < 0. \end{aligned}$$

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- Record breakers = ascending ladder heights.
- $T_0 := 0$ and for $k \geq 1$

$$\begin{aligned} R_k &= \inf\{n \geq T_k : S_n - S_{T_k} > 0\}, \\ T_k &= \inf\{n \geq R_{k-1} : S_n - S_{R_{k-1}} \leq 0\}. \end{aligned}$$

Locating Record Breaker IF IT Ever Happens

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$$E \exp\left(\theta^* S_n\right) = 1.$$

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where V is $U\left(0,1\right)$ independent of $S_{R_1} > 0$.

- **Conclusion: We CAN answer YES or NO to "will there be a record breaker?" (Keep in mind that $E_{\theta^*} S_n > 0$).**

Locating Record Breaker IF IT Ever Happens

- Moreover, for each $f(\cdot)$ bounded

$$
\begin{aligned}
& E\left(f\left(R_1, S_1, \ldots S_{R_1}\right) | R_1 < \infty\right) \\
= \quad & \frac{E_{\theta^*}\left(f\left(R_1, S_1, \ldots, S_{R_1}\right) \exp\left(-\theta^* S_{R_1}\right)\right)}{P\left(R_1 < \infty\right)} \\
= \quad & \frac{E_{\theta^*}\left(f\left(R_1, S_1, \ldots, S_{R_1}\right) I\left(V \leq \exp\left(-\theta^* S_{R_1}\right)\right)\right)}{P\left(V \leq \exp\left(-\theta^* S_{R_1}\right)\right)}.
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- Moreover, for each $f(\cdot)$ bounded

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- **So, IF** $V \leq \exp\left(-\theta^* S_{R_1}\right)$ **(i.e.** $R_1 < \infty \Longrightarrow$ **YES there is Record Breaker) AND** $S_1, \ldots, S_{R_1}$ **from** P_{θ_*} **follows the law of** $S_1, \ldots, S_{R_1}$ **given** $R_1 < \infty$**.**

Another Application of Record Breakers

In both examples, answer to "Will there be a Record Breaker?" also gives the actual location of the Record Breaker (two birds with one stone!)

Application to Max-Stable Processes

- Split in two independent pieces: For any $k \geq 1$

$$
\begin{aligned}
& \max_{n \leq k} \{ - \log \left(\frac{A_n}{n} \right) - \log (n) + Z_n (t) \}. \\
\leq \quad & \max_{n \leq k} \{ - \log \left(\frac{A_n}{n} \right) \} + \max_{n \leq k} \{ - \log (n) + Z_n (t) \}.
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- Contribution of A_n easily handled (B. & Sigman (2011)).

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$$\max_{n \leq k} \{-\log\left(\frac{A_n}{n}\right) - \log(n) + Z_n(t)\}.$$
$$\leq \max_{n \leq k} \{-\log\left(\frac{A_n}{n}\right)\} + \max_{n \leq k} \{-\log(n) + Z_n(t)\}.$$

- Contribution of A_n easily handled (B. & Sigman (2011)).
- Contribution of $Z_n(\cdot)$ can be done using two approaches: TES & direct record breaking analysis.

Application to Max-Stable Processes

- Exact simulation (& TES) for

$$\max_{n\leq k}\{-\log(n)+Z_n(t)\}\vee\max_{n\geq k}\{-\log(n)+Z_n(t)\}$$

can be done as we now explain.

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  can be done as we now explain.

- **CRUCIAL**: For compact $\mathcal{C}$

$$\sup_{t \in \mathcal{C}} \max_{n \ge k}\{-\log(n) + Z_n(t)\} = O(-\log(k)) \to -\infty.$$

Application to Max-Stable Processes

- Recall that $Z_n(t) = \sum_{m=0}^{\infty} \lambda_m \Lambda_m(t) W_m(n)$

$$
\begin{aligned}
& \sum_{m,n} P\left(|W_m(n)| > \rho \log^{1/2}(m+1) + \rho \log^{1/2}(n+1)\right) \\
\leq \quad & C \sum_{m,n} \exp\left(-\rho^2 \frac{\left(\log^{1/2}(m+1) + \log^{1/2}(n+1)\right)^2}{2}\right) \\
\leq \quad & C \sum_{m} \exp\left(-\rho^2 (\log(m+1))\right) \sum_{n} \exp\left(-\rho^2 (\log(n+1))\right).
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- Thus a pair (m, n) such that

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is **valid record breaker** & finitely many of them.

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- Use any convenient linear order of (m, n) (say lexicographic) to find them.

Application to Max-Stable Processes

- We conclude if $(\mathbf{m}, \mathbf{n})$ is last record breaker

$$
\begin{aligned}
|Z_n(t)| &\leq \sum_{m>\mathbf{m}}^{\infty} \lambda_m \Lambda_m(t) |W_m(n)| \\
&\leq \sum_{m>m_*}^{\infty} \lambda_m \rho \log^{1/2}(m) + \log^{1/2}(n) \sum_{m>m_*}^{\infty} \lambda_m \rho.
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\end{aligned}
$$

- Simply choose n large enough so that

$$
\log^{1/2}(n) \sum_{m>m_*}^{\infty} \lambda_m \rho < -\log(n).
$$

Application to Max-Stable Processes

Theorem (B., Dieker, Liu, Mikosch '15)

Algorithm outputs a wavelet approximation with guaranteed ε error (with probability one) in uniform norm with complexity $O\left(\varepsilon^{-2}\log(1/\varepsilon)\right)$.

- Applicable to fractional Brownian sheet (Dzhaparidze & van Zanten '05).

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- Applicable to fractional Brownian sheet (Dzhaparidze & van Zanten '05).
- But what if we don't have the wavelet expansion?

Application to Max-Stable Processes

Theorem (B., Dieker, Liu, Mikosch '15)

Suppose the following:

1) Sampling $Z_n(t_1), \ldots, Z_n(t_d)$ *with cost* $C(d)$.
2) $Z_n(\cdot)$ *is Hölder continuous.*
3) Can sample $Z_n(t_1), \ldots, Z_n(t_{d-1}) \mid Z_n(t_d) = z$ *with cost* $C(d)$.
4) $\{t_1, \ldots, t_d\} \subset C$ *compact.*
Then, can sample

$$M(t_1), \ldots, M(t_d)$$

with complexity $O\left(C(d)^{1+\varepsilon}\right)$ *for any* $\varepsilon > 0$.

Application to Max-Stable Processes

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- Define a record breaker at n if

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- Use algorithm by Adler, B., and Liu (2012) to optimally estimate

$$P(\max\{Z_n(t_1), ..., Z_n(t_d)\} > \log(n))$$

and sample

$$\{\max(Z_n(t_1), ..., Z_n(t_d)) > \log(n)\}$$

uniformly in d and n.

Combining with Malliavin Calculus for Max-Stable Processes

Theorem (B., Dieker, Liu, Mikosch '15)

Let N be the last record breaker for the Gaussian processes, then the density of $M := (M(t_1), ..., M(t_d))$ evaluated at $y = (y_1, ...y_d)$ satisfies

$$\begin{aligned} & p(y_1, ..., y_d) \\ = \; & E\left(\sum_{i=1}^{d} G_i(y - M) \sum_{n=1}^{N} C_{i\cdot}^{-1} \bar{Z}_n\right), \end{aligned}$$

where C is the covariance matrix of $Z_n := (Z_n(t_1), ..., Z_n(t_d))$ and $\bar{Z}_n(t_i) = Z_n(t_i) - E(Z_n(t_i))$, and

$$G_i(x_1, ..., x_d) = \kappa_d \frac{x_i}{\|x\|_2^d},$$

for an explicit constant κ_d.

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- Presented general techniques for exact simulation which are broadly applicable (e.g. perfect simulation, maxima of multidimensional random walks, SDEs, Levy processes, etc.)

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- Presented Optimal Exact Simulation & Tolerance Enforced Simulation (i.e. ε error in path space with probability 100% certainty) for max-stable fields.
- Presented unbiased Malliavin estimator for joint densities of max-stable processes.
- Key idea: define a sequence of finitely many record breakers & locate them with 0 - 1 questions (Bernoulli sampling).

Picture of a Max-Stable Gaussian Process