Monte Carlo Methods for Spatial Extremes

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What is our Goal?

Our goal is to enable efficient Monte Carlo of extreme events in space...

Our focus here is on Max-stable Fields.

Elements of Extreme Value Analysis

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- **Textbook Example:** Given storm surges observed over the last 100 yrs in NYC, what's the maximum height of a storm surge which gets exceeded only about once in 1000 yrs?
- **Answer:** One must *extrapolate*...

Extreme Value Theory (EVT): The IID case

Theorem (Fisher-Tippet-Gnedenko)

Suppose $X_1, ..., X_n$ *is an IID sequence and define* $Z_n = \max\{X_1, ..., X_n\}$. **If there exists** $\{(a_n, b_n)\}_{n\geq 1}$ **deterministic numbers such that**

$$Z_n \overset{D}{\approx} b_n M + a_n,$$

then M **must be a max-stable distribution.**

Max-stable Distributions

- A max-stable r.v. M is characterized by the fact that if $M_1, M_2, ..., M_n$ are i.i.d. copies of M then

$$M \overset{D}{=} \max_{i=1}^{n} \left(M_i - a_n \right) / b_n.$$

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- We also have that

$$\begin{aligned} P(M \le x) &= \exp\left(-(1+\gamma x)^{-1/\gamma}\right) \quad 1+\gamma x > 0. \\ P(M \le x) &= \exp(-\exp(-x)); \gamma = 0 \text{ (the Gumbel case)}. \end{aligned}$$

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- **But in many cases it is important to account for spatial dependence...**

What is a Catastrophe Bond?

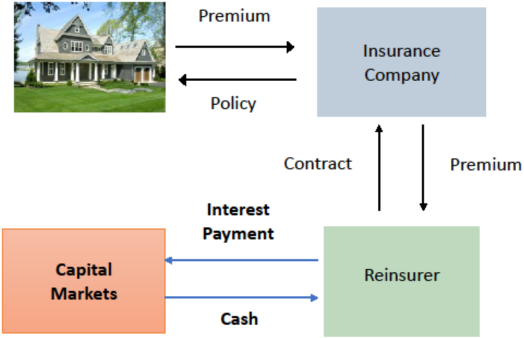
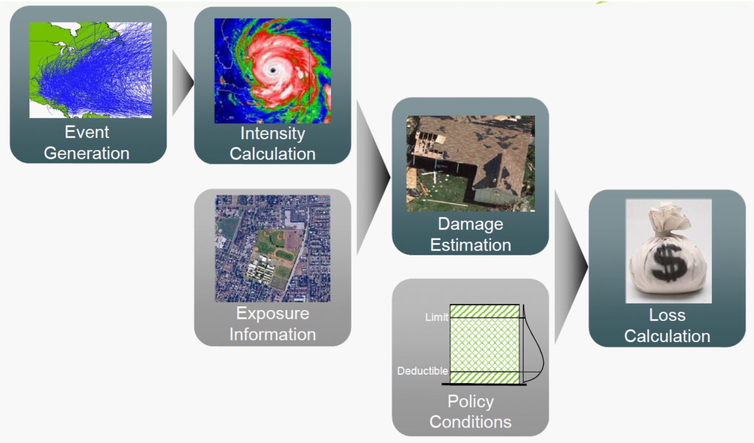


Diagram Illustrating Catastrophe Bonds Risk Calculation



Case Study: MetroCat Re Ltd

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- The parametric trigger is based on weighted average of maximum water levels in several locations.

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for i.i.d. $M_1(\cdot), ..., M_n(\cdot)$ of $M(\cdot)$ and some $a_n(\cdot), b_n(\cdot)$.

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- How to construct max-stable fields?

Characterizations of Max-stable Fields

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$$M(t) = \sup_{n \geq 1} \{-\log(A_n) + X_n(t)\},$$

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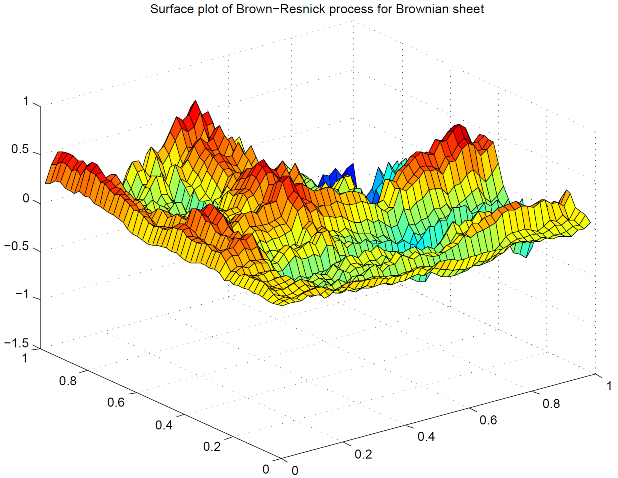
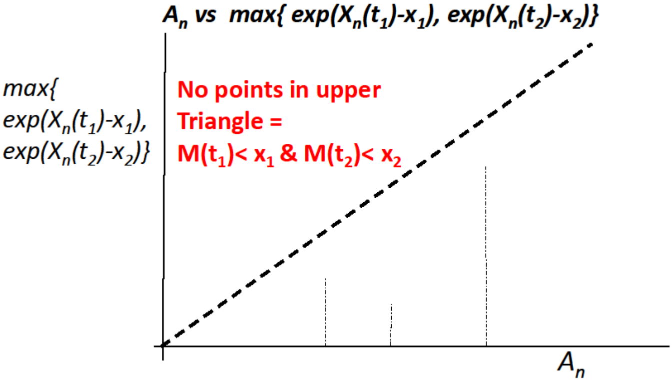
- Brown-Resnick, de Haan, Smith, Schlather, Kabluchko,...

This is How a Max-Stable Field Looks Like



Computing Joint CDFs

$$P\left(e^{M(t_1)} \leq e^{x_1}, e^{M(t_1)} \leq e^{x_2}\right) = P\left(\text{Poisson r.v. } = 0\right)$$
$$= \exp\left(-E\max\left\{\exp\left(X_n\left(t_1\right) - x_1\right), \exp\left(X_n\left(t_2\right) - x_2\right)\right\}\right)$$



Max-Stable Fields: Computational Challenges

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- Joint densities for $M(t_1), \ldots, M(t_d)$ quickly become intractable ($d = 10$ contains more than 10^5 terms).
- Likelihood methods applicable in low dimensions only.
- In the end we want to efficiently evaluate quantities such as

$$P\left(\int_T w(t)\, M(t)\, dt > b\right), \quad \text{<–}\textbf{CAT Bond Example}$$

$$f_{M(t_1),\ldots,M(t_k)}(x_1, \ldots, x_k),$$

$$f_{M(t_1),\ldots,M(t_k)|M(s_1),\ldots,M(s_d)}(x_1, \ldots, x_k \mid z_1, \ldots, z_d).$$

Our Contributions

Theorem 1 (B., Dieker, Liu, Mikosch '16): Algorithm for sampling $M(t_1), \ldots, M(t_d)$ with virtually the same asymptotic complexity as sampling $X_1(t_1), \ldots, X_1(t_d)$ as $d \to \infty$.

In other words, $M(t_1), \ldots, M(t_d)$ *is basically as "easy" as* $X_1(t_1), \ldots, X_1(t_d)$...

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- **Theorem 2** (B. and Liu '16): Construction of a finite variance $W(y_1, ..., y_d)$ such that

$$f_{M(t_1),...,M(t_d)}(y_1, ..., y_d) = E(W(y_1, ..., y_d)).$$

The estimator takes $O\left(\varepsilon^{-2}\log^2(1/\varepsilon)\right)$ samples to achieve ε error.

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- **Theorem 3** (B. and Liu '16): Construction of $W(y_1, ..., y_k, z_1, ..., z_d)$ such that

$$\begin{aligned} & f_{M(t_1),...,M(t_k)|M(s_1),...,M(s_d)}(y_1, ..., y_k \mid z_1, ..., z_d) \\ = \; & E(W(y_1, ..., y_k, z_1, ..., z_d)), \end{aligned}$$

similar complexity as Theorem 2.

A Quick Review of Current Methodology

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- Summary of computational complexity (BDML = Our method):

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$$\text{DEO} = \text{BDML} \quad \times d.$$

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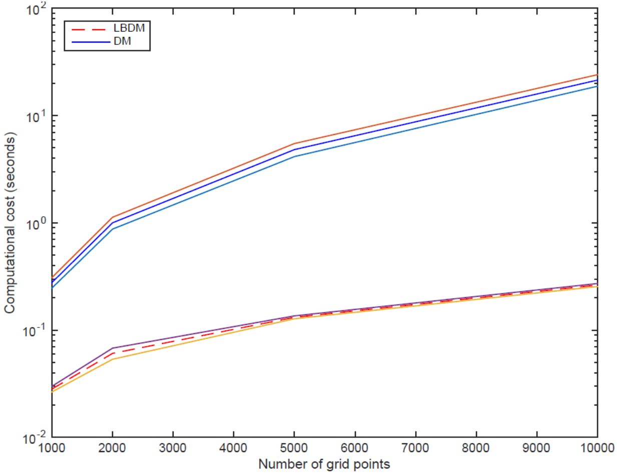
- For example if $X_n(\cdot)$ is Brownian motion our method takes $O(d)$ complexity to sample $M(t_1), ..., M(t_d)$. Both DM and DEO take at least $O(d^2)$ complexity.

Comparison vs Dieker & Mikosch

Log-log plot: Grid points vs time in seconds

RED = Our Method vs *BLUE = Alternative method.*



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- Note that for $m \geq \max(N_A, N_X)$
$$\max_{n \geq m} \{X_n(t_i) - \log(A_n)\} \leq -\frac{\log(m+1)}{2} + \log(2).$$

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$$\max_{n\geq m}\{X_n(t_i) - \log(A_n)\} \leq -\frac{\log(m+1)}{2} + \log(2).$$
- Need to detect when
$$-\log(m+1)/2 + \log(2) \leq \min_{i=1}^{d} X_1(t_i) - \log(A_1).$$

- If $X_n(\cdot)$ is continuous on the compact set T, then

$$\max_{t \in T} |X_n(t)| = O_p\left((\log(n))^{1/2}\right).$$

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$$A_n / n \longrightarrow 1.$$

- So detection of N_X and N_A should take $O(1)$ time as $d \longrightarrow \infty$.

The Milestone / Record Breaking Events Technique

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- Let us write $X_n = \{X_n(t_i)\}_{i=1}^d$ so $\|X_n\|_\infty = \max_{i=1}^d |X_n(t_i)|$.
- Let $\tau_0 = n_0 - 1$ (n_0 chosen later) and let

$$\begin{aligned}
\tau_{k+1} &= \inf\{n > \tau_k : \|X_n\|_\infty > \frac{1}{2}\log(n+1)\}, \\
N_X &= \sup\{\tau_k : \tau_k < \infty\}.
\end{aligned}$$

Can we sample Bernoulli with $P(\tau_1 < \infty)$? Can we simulate $X_1(\cdot), ..., X_{\tau_1}(\cdot)$ given $\tau_1 < \infty$?

Borrow from Rare Event Simulation Intuition

- Say n_0 is large: How does the rare event

$$\{\tau_1 < \infty\} = \bigcup_{n=n_0}^{\infty} \left\{ \|X_n\|_\infty > \frac{1}{2} \log (n+1) \right\} \quad \text{occur?}$$

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- **Approximate the optimal importance sampling distribution**

$$P(\tau_1 = n | \tau_1 < \infty) = \frac{P\left(\|X_n\|_\infty > \frac{1}{2}\log(n+1), \tau_1 > n-1\right)}{P(\tau_1 < \infty)}$$

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- This suggests a natural importance sampling strategy...

Will the First Record Breaking Event Occur?

1. Sample K so that for $K \geq n_0$

$$Q\left(K = n\right) = \frac{\sum_{i=1}^{d} P\left(\left|X_n\left(t_i\right)\right| > \frac{1}{2}\log\left(n+1\right)\right)}{\sum_{k=n_0}^{\infty}\sum_{i=1}^{d} P\left(\left|X_k\left(t_i\right)\right| > \frac{1}{2}\log\left(k+1\right)\right)}.$$

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2. Define $h(n) = P(\tau_1 = n)/Q(K=n)$ then

$$P(\tau_1 < \infty) = E^{Q}(h(K)) = \sum_{n=n_0}^{\infty} P(\tau_1 = n).$$

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3. **If** $h(k) \leq 1$ **then** $I\{\tau_1 < \infty\}$ **is Bernoulli(**$h(K)$**) with** K **sampled under** Q**.**

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& \times P\left(\|X_k\|_\infty \le \frac{1}{2}\log(1+k) \ \forall\ n_0 \le k < n\right) \\
& \times \frac{P\left(\|X_n\|_\infty > \frac{1}{2}\log(n+1)\right)}{\sum_{i=1}^{d} P\left(|X_n(t_i)| > \frac{1}{2}\log(n+1)\right)} < 1.
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2. **Sample a Bernoulli with parameter:**

$$
\frac{P\left(\|X_n\|_\infty > \frac{1}{2}\log(n+1)\right)}{\sum_{i=1}^{d} P\left(|X_n(t_i)| > \frac{1}{2}\log(n+1)\right)}.
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- **Sampling from** $P\left(\cdot \mid \left\|X_n\right\|_\infty > \frac{1}{2}\log\left(n+1\right)\right)$ **by acceptance / rejection.**

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2. Given $J=j$, sample $X_n = \{X_n(t_i)\}_{i=1}^{d}$

$$\begin{aligned}
& Q\left(X_n(t_i) \in dx_i, \ldots, X_n(t_d) \in dx_d \mid J=j\right) \\
= \; & P\left(X_n(t_i) \in dx_i, \ldots, X_n(t_d) \in dx_d \mid |X_n(t_j)| > \frac{1}{2}\log(n+1)\right) \\
= \; & \frac{P\left(X_n(t_i) \in dx_i, \ldots, X_n(t_d) \in dx_d\right) \cdot I\left(|x_n(t_j)| > \frac{1}{2}\log(n+1)\right)}{P\left(|X_n(t_j)| > \frac{1}{2}\log(n+1)\right)}.
\end{aligned}$$

Computing the Likelihood Ratio

- Likelihood of X_n under Q

$$
\begin{aligned}
Q(X_n) &= \sum_j Q(X_n|J=j)\,Q(J=j) \\
&= \sum_j P(X_n)\,\frac{I\left(|x_n(t_j)| > \frac{1}{2}\log(n+1)\right)}{P\left(|X_n(t_j)| > \frac{1}{2}\log(n+1)\right)}\,Q(J=j) \\
&= P(X_n)\sum_j\left(\frac{I\left(|x_n(t_j)| > \frac{1}{2}\log(n+1)\right)}{P\left(|X_n(t_j)| > \frac{1}{2}\log(n+1)\right)}\right. \\
&\quad \left.\times\frac{P\left(|X_n(t_j)| > \frac{1}{2}\log(n+1)\right)}{\sum_{i=1}^{d} P\left(|X_n(t_i)| > \frac{1}{2}\log(n+1)\right)}\right) \\
&= P(X_n)\,\frac{\sum_j I\left(|x_n(t_j)| > \frac{1}{2}\log(n+1)\right)}{\sum_{i=1}^{d} P\left(|X_n(t_i)| > \frac{1}{2}\log(n+1)\right)}.
\end{aligned}
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Acceptance / Rejection Execution

- Likelihood of X_n under Q

$$\frac{dQ}{dP}\left(X_n\right) = \frac{\sum_{j=1}^{d} I\left(\left|X_n\left(t_j\right)\right| > \log\left(n+1\right)/2\right)}{\sum_{i=1}^{d} P\left(\left|X_n\left(t_i\right)\right| > \log\left(n+1\right)/2\right)}.$$

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- We conclude

$$\frac{dP\left(X_n \mid \left\|X_n\right\|_\infty > \frac{\log(n+1)}{2}\right)}{dQ}$$

$$=\frac{I\left(\left\|X_n\right\|_\infty > \frac{\log(n+1)}{2}\right)}{P\left(\left\|X_n\right\|_\infty > \frac{\log(n+1)}{2}\right)}\frac{dP}{dQ}\leq\frac{\sum_{i=1}^{d} P\left(\left|X_n\left(t_i\right)\right|>\frac{\log(n+1)}{2}\right)}{P\left(\left\|X_n\right\|_\infty > \frac{\log(n+1)}{2}\right)}.$$

Acceptance / Rejection Execution

- Likelihood of X_n under Q

$$\frac{dQ}{dP}(X_n) = \frac{\sum_{j=1}^{d} I\left(|X_n(t_j)| > \log(n+1)/2\right)}{\sum_{i=1}^{d} P\left(|X_n(t_i)| > \log(n+1)/2\right)}.$$

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$$\frac{dP\left(X_n \mid \|X_n\|_\infty > \frac{\log(n+1)}{2}\right)}{dQ}$$
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- By general acceptance / rejection theory we have:

$$\text{Probability of accepting} = \frac{P\left(\|X_n\|_\infty > \log(n+1)/2\right)}{\sum_{i=1}^{d} P\left(|X_n(t_i)| > \log(n+1)/2\right)}.$$

Summary

- We sampled $B = Bernoulli\left(h\left(n\right)\right) = I\left(\tau_1 < \infty\right)$ with $h\left(n\right)$:

$$
\begin{aligned}
h(n) &= \sum_{k=n_0}^{\infty} \sum_{i=1}^{d} P\left(|X_k(t_i)| > \frac{1}{2} \log(k+1) \right) \\
&\times P\left(\|X_k\|_\infty \leq \frac{1}{2} \log(1+k) \ \forall \ n_0 \leq k < n \right) \\
&\times \frac{P\left(\|X_n\|_\infty > \frac{1}{2} \log(n+1) \right)}{\sum_{i=1}^{d} P\left(|X_n(t_i)| > \frac{1}{2} \log(n+1) \right)} < 1.
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- Also sampled X_n given $\left\|X_n\right\|_\infty > \frac{1}{2}\log\left(n+1\right)$ by acceptance rejection:

$$
\text{Probability of accepting} = \frac{P\left(\left\|X_n\right\|_\infty > \log\left(n+1\right)/2\right)}{\sum_{i=1}^{d} P\left(\left|X_n\left(t_i\right)\right| > \log\left(n+1\right)/2\right)}.
$$

Exact Sampling Property

- **Also we actually obtain:**

$$\mathcal{L}aw_Q\left(X_1, ..., X_K, K \middle| B = 1\right) = \mathcal{L}aw_P\left(X_1, ..., X_{\tau_1}, \tau_1 \middle| \tau_1 < \infty\right).$$

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- Let's check this identity:

$$\begin{aligned}
& Q(X_1 \in dx_1, ..., X_n \in dx, K = n, B = 1) \\
= \; & Q\left(K = n\right) \sum_{k=n_0}^{\infty} \sum_{i=1}^{d} P\left(\left|X_k\left(t_i\right)\right| > \frac{1}{2}\log\left(k+1\right)\right) \\
& \times P\left(X_{n_0} \in dx, ..., X_{n-1} \in dx_{n-1} \mid \tau_1 > n-1\right) \\
& \times P\left(X_n \in dx | \left\|X_n\right\|_\infty > \frac{\log\left(n+1\right)}{2}\right) \\
= \; & P\left(\tau_1 = n\right) P\left(X_1 \in dx_1, ..., X_n \in dx \mid \tau_1 = n\right) \\
= \; & P\left(X_1 \in dx_1, ..., X_n \in dx, \; \tau_1 = n, \; \tau_1 < \infty\right).
\end{aligned}$$

How to Continue? What Drives the Complexity?

- Subsequently sample $\tau_2, ..., \tau_{L-1} = N_X$ until $\tau_L = \infty$.

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- Total complexity turns out to be driven by n_0 such that

$$\sum_{k=n_0}^{\infty} \sum_{i=1}^{d} P\left(|X_k(t_i)| > \frac{1}{2}\log(k+1)\right) < 1$$
$$\Longrightarrow \quad n_0 = \exp\left(c\sqrt{\log(d)}\right) = O\left(d^{\varepsilon}\right) \text{ for any } \varepsilon > 0.$$

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Milestone / Record Breaking for Random Walks

- A similar technique used to sample *the last time* N such that

$$A_n < \frac{1}{2}n.$$

- In this case it suffices to sample A_n jointly with

$$\max_{m \geq n}\{A_m - m/2\}$$

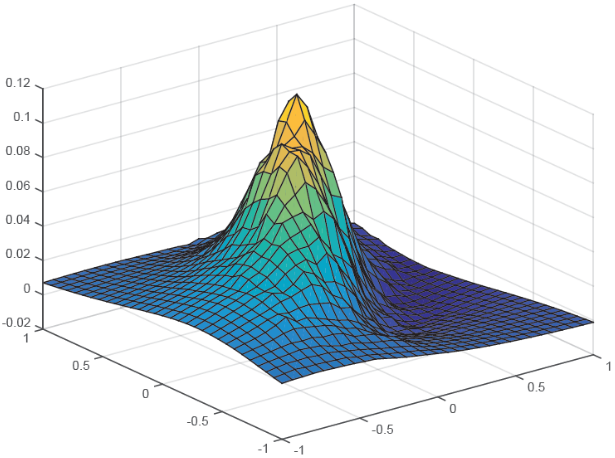
rare event simulation techniques can be applied (see B. & Wallwater (2014)).

Density Estimation

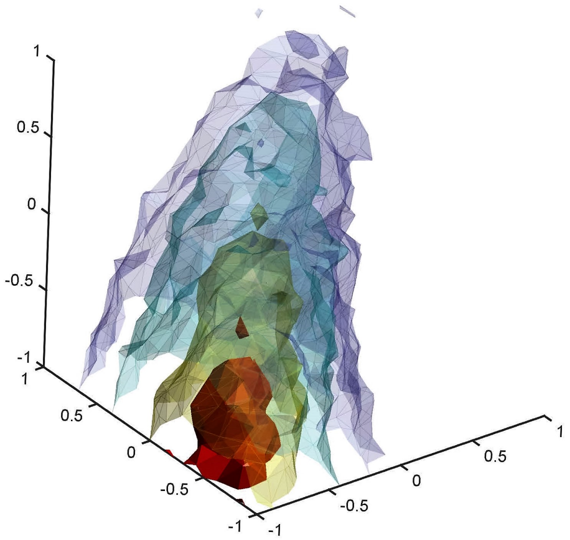
Want to design a finite variance estimator $W(y_1, ..., y_d)$ so that

$$E(W(y_1, ..., y_d)) = f_M(y_1, ..., y_d).$$

An Illustration of the Density Pictures



Some Pictures: 3-d Density



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formally if $G(\cdot)$ is the Poisson kernel we have

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- So, formally

$$\begin{aligned} \Delta u(x) &= \int f(y) \Delta G(x-y) dy = f(x) \\ \Delta G(x-y) dy &= \delta(x-y) dy \\ f_M(x_1, ..., x_d) &= E(\Delta G(x-M)). \end{aligned}$$

Applying Malliavin Thalmaier Formula

- For $d \geq 3$ we have that

$$\Delta G(x-y) = \sum_{i=1}^{d} \frac{\partial^2 G(x)}{\partial x_i^2}(x-y),$$

$$G_i(x) = \frac{\partial G(x)}{\partial x_i} = \kappa_d \frac{x_i}{\|x\|_2^d}, \quad G(x) = \kappa_d' \frac{1}{\|x\|_2^{d-2}}$$

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- Higher order derivatives of G implies higher variability...

- **Two ways to fix this: Integration by parts & Randomized Multilevel Monte Carlo.**

Improving Variance: Integration by Parts for Finite Maxima

- We claim

$$\frac{\partial G_i\left(x-M_n\right)}{\partial x_i}=-\frac{\partial G_i\left(x-M_n\right)}{\partial M_n\left(t_i\right)}=-\sum_{k=1}^{n}\frac{\partial G_i\left(x-M_n\right)}{\partial X_k\left(t_i\right)}.$$

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- Let's check second equality:

$$\begin{aligned}
M_n\left(t_i\right) &= \max_{1\leq k\leq n}\left\{X_k\left(t_i\right)-a_k\right\}. \\
\sum_{k=1}^{n}\frac{\partial M_n\left(t_i\right)}{\partial X_k\left(t_i\right)} &= \sum_{k=1}^{n} I\left(M_n\left(t_i\right)=X_k\left(t_i\right)-a_k\right)=1. \\
\frac{\partial G_i\left(x-M_n\right)}{\partial X_k\left(t_i\right)} &= \frac{\partial G_i\left(x-M_n\right)}{\partial M_n\left(t_i\right)}\frac{\partial M_n\left(t_i\right)}{\partial X_k\left(t_i\right)} \\
\sum_{k=1}^{n}\frac{\partial G_i\left(x-M_n\right)}{\partial X_k\left(t_i\right)} &= \frac{\partial G_i\left(x-M_n\right)}{\partial M_n\left(t_i\right)}.
\end{aligned}$$

Improving Variance: Integration by Parts for Finite Maxima

- Integration by parts yields (with $\Sigma = cov(X_k)$)

$$E\left(\frac{\partial G_i(x - M_n)}{\partial X_k(t_i)}\right) = -E\left(G_i(x - M_n) \cdot e_i^T \Sigma^{-1} X_k\right)$$

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- Therefore

$$\begin{aligned} E\left(\frac{\partial G_i\left(x-M_n\right)}{\partial x_i}\right) &= -\sum_{k=1}^{n} E\left(\frac{\partial G_i\left(x-M_n\right)}{\partial X_k\left(t_i\right)}\right) \\ &= E\left(G_i\left(x-M_n\right)\cdot e_i^T\Sigma^{-1}\sum_{k=1}^{n} X_k\right) \\ f_{M_n}\left(x_1,\ldots,x_d\right) &= E\left(\sum_{k=1}^{n}\sum_{i=1}^{d} G_i\left(x-M_n\right) e_i^T\Sigma^{-1}X_k\right). \end{aligned}$$

Improving Variance: Infinite Horizon Maxima

- Let $\mathcal{F}_N$ be the information generated by the exact sampling procedure so that
$$M_N = M,$$
then we have that for $m \geq 1$
$$E\left(X_{N+m} | \mathcal{F}_N\right) = 0,$$
because given $n > N$, we have that $X_n\left(t_i\right)$ has a symmetric distribution.

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because given $n > N$, we have that $X_n\left(t_i\right)$ has a symmetric distribution.

- Thus

$$\begin{aligned} \lim_{n\to\infty} f_{M_n}\left(x_1, ..., x_d\right) &= f_{M_n}\left(x_1, ..., x_d\right) \\ &= E\left(\sum_{k=1}^{N}\sum_{i=1}^{d} G_i\left(x - M\right) e_i^T \Sigma^{-1} X_k\right). \end{aligned}$$

Continue Integrating by Parts

- The estimator

$$W\left(x-M\right)=\sum_{k=1}^{N}\sum_{i=1}^{d}G_{i}\left(x-M\right)e_{i}^{T}\Sigma^{-1}X_{k}.$$

Continue Integrating by Parts

- The estimator

$$W\left(x-M\right)=\sum_{k=1}^{N}\sum_{i=1}^{d}G_i\left(x-M\right)e_i^T\Sigma^{-1}X_k.$$

- One can continue obtaining

$$\begin{aligned}\text{quadratic weight} &\Rightarrow \text{finite variance } d\leq 3\\ \text{Cubic weight} &\Rightarrow \text{finite variance } d\leq 5\\ &\quad\ldots\end{aligned}$$

Applying Multilevel Monte Carlo

- Out starting point is estimator

$$W(x) = \sum_{k=1}^{N} \sum_{i=1}^{d} G_i(x - M)\, e_i^T \Sigma^{-1} X_k.$$

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- Randomized Multilevel Monte Carlo, introducing the differences

$$\begin{aligned} \Delta_n &= G_i^{n+1}(x - M) - G_i^n(x - M); \\ G_i^n(x - M) &= \kappa_d \frac{x_i - M(t_i)}{\|x - M\|_2^d + \|x - M\|_2 / \log(n+1)}. \end{aligned}$$

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- Ideas based on record-breaking events & rare-event simulation.
- Malliavin calculus ideas for first efficient density estimators for max-stable fields.