Splitting for Heavy-tailed Stochastic Processes

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Columbia University

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Outline

Goal of the talk

- **Our Goal**: Study splitting strategies for heavy-tailed systems.

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Light-tailed Systems: Random Walk Dynamics

- Assume $\psi(\theta) = \log E \exp(\theta X_i)$ finite around zero $<-$ X_i's are increments of random walk.

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- **Moral:** *Efficient* splitting easier than *efficient* importance sampling

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- **Moral:** *Simplicity comes with an price in (somewhat modest) efficiency loss...*

Motivating question

What about heavy tails? Is "efficient splitting" somehow easier than "efficient importance sampling" even if there is some (modest) loss in efficiency?

Outline

Importace Sampling Benchmark

- Correct family of importance sampling distributions

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\begin{aligned}
f(x|s) &= p_0 \frac{f_X(x)\, I(x \leq (b-s) - A(s))}{P(X \leq (b-s) - A(s))} \\
&+ p_1 \frac{f_X(x)\, I(x \in ((b-s) - A(s), a_1(b-s)])}{P(X \in ((b-s) - A(s), a_1(b-s)])} \\
&+ \sum_{k=1}^{m-1} p_{k+1}; \frac{f_X(x)\, I(x \in (a_k(b-s), a_{k+1}(b-s)])}{P(X \in (a_k(b-s), a_{k+1}(b-s)])} \\
&+ p_{m+1} \frac{f_X(b-s-x)\, I(x \in (a_m(b-s), b-s-A(b-s)]}{P(X \in (a_m(b-s), b-s-A(b-s)])} \\
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- $A(\cdot)$ is a sublinear function called auxiliary function, a_k's recursively defined, m smaller for heavier tails.

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- Specification depends on asymptotic behavior of hazard rate function.
- General: Blanchet and Liu (2012); Pareto case: Dupuis, Leder, Wang (2009).
- **Moral:** *Importance sampling depends on fine structure of hazard rates, and needs somewhat complex specification...*
- *BUT:* strongly efficient – often asymptotically zero relative error.

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Splitting Initial Thoughts and Considerations

- Traditional splitting: At time k, if particle random walk reaches *milestone* C_k, replace particle walk is r **children** & reweight each children using
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- Key property: Rewards walks that behave according to conditional distribution given rare event of interest...
- **Problem:** In heavy tailed systems "normal" behavior is typical conditional behavior prior to extreme event... (i.e. no warning).

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Splitting for Heavy-tailed Walks: Two Approaches

- Branching processes in "hazard rate" space:

$$\begin{array}{rcl} \mathbf{P}\left(X_1+...+X_n>b\right) & \longleftarrow & \textbf{PROBLEM OF INTEREST} \\ \text{WE'LL USE:} & & P\left(X>t\right)=P\left(\Lambda^{-1}\left(V\right)>t\right) \\ V & \sim & Exp\left(1\right). \\ \text{Pareto example:} & & \Lambda^{-1}\left(v\right)=\exp\left(v/\alpha\right)-1. \end{array}$$

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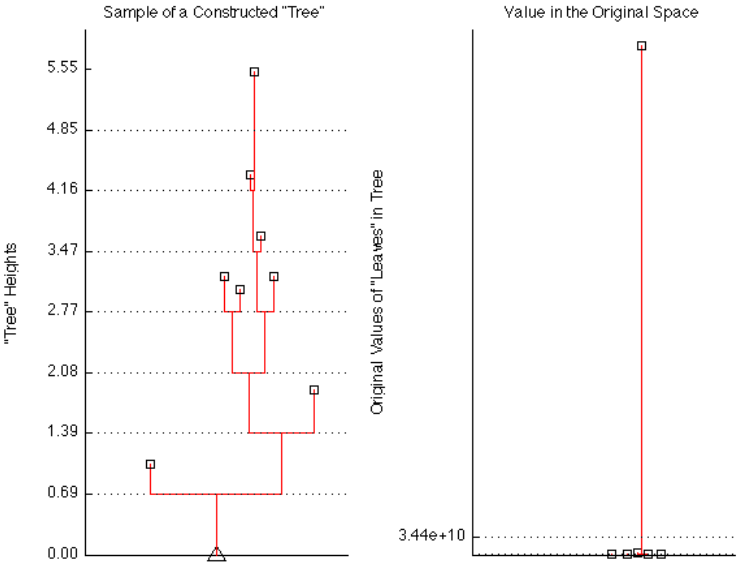
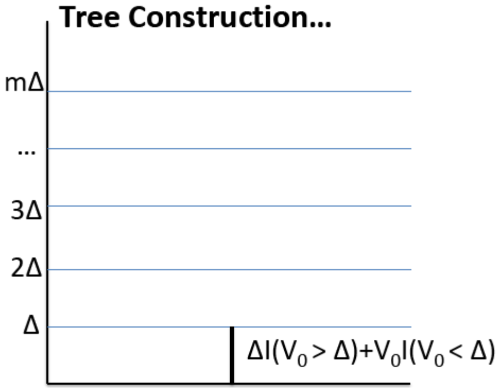
- New (not today): Branching and conditional Monte Carlo: Based on "normal" behavior prior to jump...

Splitting I: Branching Processes in Hazard Rate Space



Splitting Estimator: Construction



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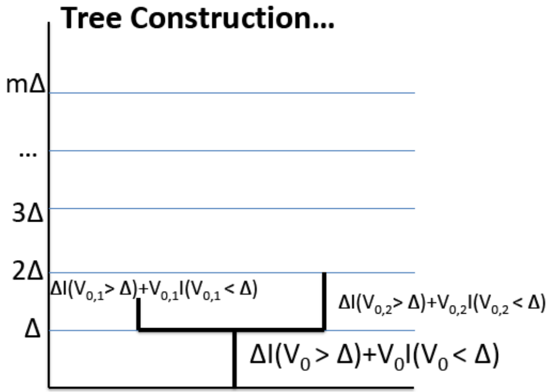

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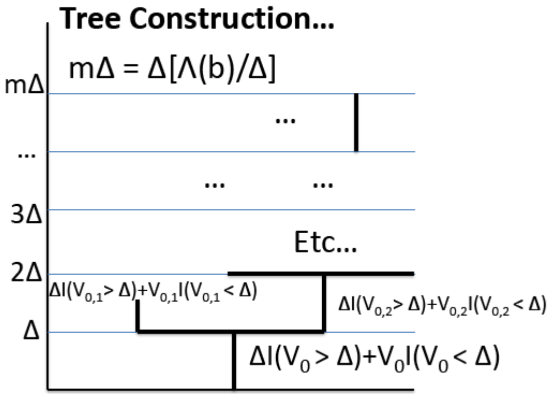

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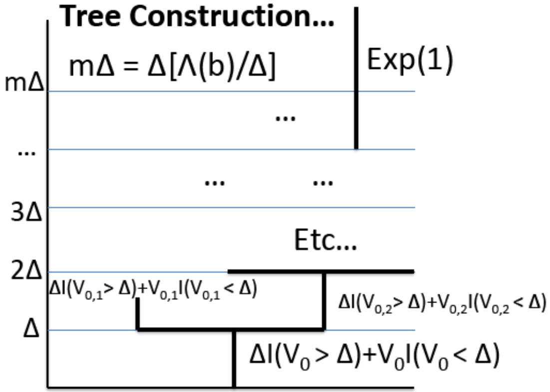

Splitting Estimator: Final Form

- Step 1: Let $\Pi_m(0) =$ Tree rooted at origin as discussed. Let

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\overline{\Pi}_1 &= \Pi_m(0) \\
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where $L_i(s)$ is the length produced in the i-th tree in the genealogy of s.

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- **Estimator:**

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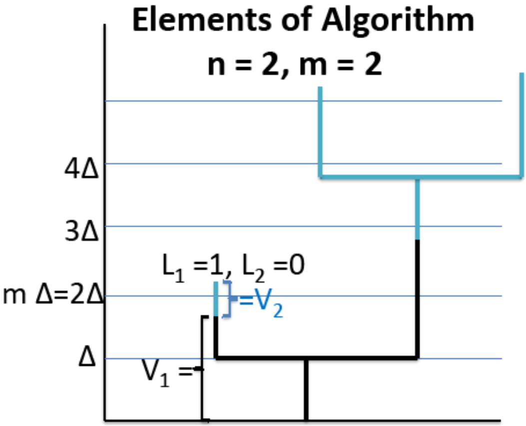
Splitting Estimator: Picture of Elements



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- Let's see... (elementary Galton-Watson processes)

$$\begin{aligned} E\left|\mathcal{L}(\Pi_{k+1}(0))\right| &\approx rP\left(V > \Delta\right) E\left|\mathcal{L}(\Pi_k(0))\right| \\ E\left|\mathcal{L}(\Pi_m(0))\right| &\approx \left(r\exp\left(-\Delta\right)\right)^m \approx r^m \exp\left(-\Lambda\left(b\right)\right) \\ \text{Recall } m &= \left[\Lambda\left(b\right)/\Delta\right]. \end{aligned}$$

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- How would you pick r and Δ?

Splitting Estimator: Efficiency Result

Theorem

Assuming $d\Lambda(t)/dt$ eventually non-increasing we obtain that for each $\varepsilon > 0$. Suppose that $r \exp(-\Delta) = 1$, then

$$E\left(\widehat{p}_b^2\right) \times E\left|\mathcal{L}(\overline{\Pi}_n)\right| = O(P\left(X_1 + ... + X_n > b\right)^{2-\varepsilon})$$

as $b \to \infty$.

Why Decreasing Hazard Rates?

- Where did we use heavy-tailed property?

$$\widehat{p}_b = \sum_{s\in\mathcal{L}(\overline{\Pi}_n)} r^{-L_1(s)-...-L_n(s)} I\left(\Lambda^{-1}(V_1(s)) + ...\Lambda^{-1}(V_n(s)) > b\right),$$

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- Condition on last common ancestor of s and u, say t

$$\begin{aligned} E(\mathbf{I}(s)\,\mathbf{I}(u)\,|\text{tree up to } t) &= P^2\Big(X_1 + \ldots + X_{n-k(t)} > b - l_1 - \ldots - \\ &= \exp\left(-2\Lambda\left(b - l_1 - \ldots - l_t\right)\right) \\ &\leq \exp\left(-2\Lambda(b) + 2\Delta L_1 + \ldots + 2\Delta L_t\right) \\ \text{Due to } \Lambda(b-x) &\leq \Lambda(b) - \Lambda(x) \ \text{ because of decreasing ha} \end{aligned}$$

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- Heavy tails: Splitting easier than importance sampling? (Uniform setup vs fine information on tails).
- Analysis of natural splitting estimator on hazard space.