Bargaining and Reputation

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These notes consider Abreu and Gul's (2000) reputational model of bargaining. In the 1980s, following Rubinstein's paper, there was a big push to study sequential offer models of bargaining with incomplete information. The idea of these models is that (as in Rubinstein) engage in a series of monopoly offers. A central observation of this work was that the specific structure of the game (who makes the offers, and how much time elapses between offers), and the patience of the players was critically important. Abreu and Gul's model stresses the "strategic posture" of players who with slight probability may be committed to holding out for a certain share of the pie. The resulting model looks more like a war of attrition.

1 The Model

We will look at a simplified version of Abreu and Gul's model. There are two agents, who bargain over a pie of size 1. At time 0, player 1 makes an initial demand $a_1 \in (0,1)$. Player two then makes a demand $a_2 \in (0,1)$. If $a_1 + a_2 \leq 1$, the game ends immediately, if $a_1 + a_2 \geq 1$, we proceed to a concession game. The concession game takes place in continuous time $t \in [0,\infty)$. At each point in time, both players choose whether to concede or to hold out. If i concedes, he receives $1 - a_j$, while if his opponent j concedes, he receives a_i.

Each player may be either rational or, with probability z_i, irrational. If i is irrational, he insists from the start on a particular demand α_i. We suppose that $\alpha_i + \alpha_j > 1$, so if i and j are both irrational, they hold out forever and never agree. If player i is rational, he has discount rate r_i, so if agreement is reached at time t, and he gets a share a, his payoff is $e^{-r_i t}a$.

1.1 The Concession Game

Let us first consider the concession game that arises if, at time zero, both players always make their irrational demands α_1, α_2. We describe i's behavior in the concession game by a probability distribution over stopping times, $F_i(t) = \Pr[i$ will concede prior to $t]$, where we allow $F_i(0) > 1$, so i may concede immediately with positive probability.

Suppose player j's behavior is given by $F_j(t)$. We look for an equilibrium where player i mixes between conceding and not conceding. For a rational i to be indifferent, it must be that:

$$r_i\,(1-\alpha_j) = (\alpha_i - (1-\alpha_j))_i \frac{f_j(t)}{1-F_j(t)} = \alpha_i \lambda_j(t).$$

A similar equation holds for j. Thus, in such an equilibrium,

$$\lambda_i(t) = \lambda_i = \frac{r_j(1-\alpha_i)}{\alpha_i + \alpha_j - 1}.$$

Integrating up the hazard rate gives

$$F_i(t) = 1 - (1 - F_i(0))\, e^{-\lambda_i t}.$$

If i does not concede with positive probability at time 0,then $F_i(0) = 0$ and

$$F_i(t) = 1 - e^{-\lambda_i t}.$$

Now, observe that if i is irrational, he will never concede. It follows that $F_i(t) \leq 1 - z_i$ for all t. Define T_i to be the value of t that solves $1 - e^{-\lambda_i t} = 1 - z_i$,

$$T_i = -\frac{1}{\lambda_i} \log z_i.$$

Note that $T_i > T_j$ if and only if $\lambda_i < \lambda_j$. Let $T = \min\{T_1, T_2\}$.

Proposition 1 *There is a unique sequential equilibrium to the concession game, described as follows:*

- *If $\lambda_i \geq \lambda_j$, then i never concedes immediately, and concedes between $(0, T]$ at constant rate λ_i.*
- *If $\lambda_i < \lambda_j$, then i concedes immediately with probability $1 - z_i z_j^{-\lambda_i/\lambda_j}$, and concedes between $(0, T]$ at constant rate λ_i.*

- *After time T, both players are known to be irrational and never concede.*

Proof. (Sketch) The proof proceeds by observing that any sequential equilibrium pair (F_1, F_2) must have the following properties.

(i) A rational player will not hesitate to concede once he knows his opponent is irrational. Thus either $F_i(t) < 1 - z_i$ for all t, and $i = 1, 2$ or there is some $T < \infty$ such that $F_i(t) < 1 - z_i$ for all $t < T$ and $F_i(T) = 1 - z_i$ for $i = 1, 2$.

(ii) If F_i jumps at t, then F_j is constant at t. The reason is that j would always want to incur the rdt loss from waiting in order to enjoy the discrete chance of i conceding.

(iii) If F_i is constant between (t', t''), then so is F_j. If i will not concede between (t', t''), then if j plans to concede in this interval, he does better to concede immediately at t' rather than wait to some time $t > t'$.

(iv) There is no interval (t', t'') with $t'' < T$ on which F_i and F_j are constant. If so, i would do better to concede at $t'' - \varepsilon$ than to concede at t'', leading to contradiction.

From *(i)–(iv)* it follows that F_i, F_j will be continuous and strictly increasing on $[0, T]$ or $[0, \infty)$. But if both are conceding with positive probability, then they must be conceding at hazard rates λ_i, λ_j as defined above, so $F_i(t) = 1 - (1 - F_i(0))e^{-\lambda_i t}$ as defined above. This rules out the latter case (where concession goes on indefinitely). Finally, the fact that both must stop conceding at the same time, so $F_i(t)$ must reach $1 - z_i$ at the same time T as $F_j(t)$ reaches $1 - z_j$, means that at least one of $F_i(0)$,$F_j(0)$ is strictly positive. By *(ii)*, they can't both be positive. So then $T = \min\{T_1, T_2\}$. But then, if $T_i \leq T_j$,

$$F_i(0) = 1 - e^{\lambda_i T_i} z_i = 0$$

while if $T_i > T_j$,

$$F_i(0) = 1 - e^{\lambda_i T_j} z_i = 1 - z_j^{-\lambda_i/\lambda_j} z_i.$$

So the unique equilibrium must be as described. *Q.E.D.*

Remark 1 *Note that $F_i(t)$ gives the probability i will concede prior to t. The probabilility that i will concede prior to t given that he is rational is higher, equal to $F_i(t)/(1 - z_i)$.*

1.2 Properties of Equilibrium

Equililibrium payoffs for a rational player i are

$$u_i = F_j(0)\alpha_i + (1 - F_j(0))(1 - \alpha_j)$$

Equilibrium has several interesting properties.

1. Bargaining is inefficient because there is delay in reaching agreement. For instance, if the model is symmetric, so $\alpha_i = \alpha_j$ and $r_i = r_j$, then $\lambda_i = \lambda_j$ and so $F_i(0) = F_j(0) = 0$. However, despite delay, there is only a small chance (z^2) of perpetual disagreement.

2. It is fairly natural to think of $T_i = -\frac{1}{\lambda_i}\log z$ as a measure of how "weak" player i is, since if $T_i > T_j$, then i will have to concede with positive probability right at the start, and the greater is T_i, the lower is i's payoff. Substituting for λ_i, we have:

$$T_i = -\frac{1}{\lambda_i}\log z_i = -\frac{\alpha_i + \alpha_j - 1}{r_j(1 - \alpha_i)}\log z_i$$

   So, for instance, i is weaker in the game when r_i is greater or z_i is smaller. In addition, i is weaker when either α_i or α_j is larger.

3. An interesting point, observed by Kambe (1999) is that if $z_i = z_j = z \to 0$,

$$T_i = -\frac{1}{\lambda_j}\log z \to \infty,$$

   but, if $\lambda_i > \lambda_j$, then $F_i(0) = 1 - z^{1-\lambda_i/\lambda_j} \to 1$, and so

$$u_i \to 1 - \alpha_j \text{ and } u_j \to \alpha_j.$$

   The "weak" player must concede immediately. We return to this below.

1.3 The Demand Game

Given the equililibrium for the concession game derived above, it is possible to characterize equilibrium in the demand game at time 0. If there is only one irrational type for each player, then it is fairly straightforward to see that both players will choose with probability one to mimic their irrational types at the start. In particular, if one player reveals rationality, but the other does not, then the player who has revealed rationality will concede immediately in the concession game.

With multiple irrational types for each player, then players will mix over their different irrational types. The crucial property of equilibrium is that i must obtain the same payoff from each irrational type. Since i's payoff depends on the posterior probability that he is irrational given his initial demand, equilibrium mixtures must reflect this. In particular, if i puts weight on some irratioinal demand α_i, he will put weight on all irrational demands $\alpha_i' > \alpha_i$.

Kambe (1999) and Abreu and Gul study the limit of this bargaining game as $z \to 0$. The key point (noted above) is that as $z \to 0$, if $\alpha_1 + \alpha_2 > 0$, then player i obtains his demand exactly if and only if:

$$\frac{r_i}{(1-\alpha_i)} < \frac{r_j}{(1-\alpha_j)}$$

and otherwise must concede immediately. By demanding

$$v_i = \frac{r_i}{r_i + r_j},$$

player i can ensure himself at least v_i, and j can similarly ensure v_j. Thus, in the limit as $z \to 0$, the equilibrium shares are $(v_i, v_j = 1 - v_i)$. Interestingly, this corresponds to the Nash bargaining solution.

2 Remarks

1. Note that two-sided reputation building is quite different than one-sided. If only one player can build a reputation, then if he is patient (or if moves are frequent), this asymmetric information tends to "take over" the game, as in Fudenberg-Levine. With two-sided reputation-building, neither player wants to reveal rationality, since this basically means admitting defeat. This generates the war of attrition type situation.

2. Abreu and Gul follow Kreps and Wilson (1982) and the literature on the war of attrition in studying the concession game in continuous time. But they also show that it is the limit of a sequence of models where players make offers in discrete time.

3. Abreu and Pearce (2006) use the Abreu-Gul and Kambe logic to study bargaining problems where players strategically interact during the

course of bargaining. For the model they consider, there is a Folk Theorem in the absence of reputational perturbations. They assume, however, that players first announce strategies for the bargaining game and with small probability become committed to these "bargaining postures". They show a remarkable generalization of Kambe's result. In the limit as the probability of commitment disappears, payoffs are given by the Nash bargaining solution with endogenous threat points, defined by Nash in his second paper on bargaining.

References

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