# Wars of Attrition

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These notes discuss "war of attrition" models. These models are used to study industry shakeouts in industrial organization, rent-seeking and lobbying in political economy. The first war of attrition models are actually from evolutionary biology, where they were used to study conflict. As you saw from the last problem set, war of attrition models are closely related to auction models, in that they feature several players competing for a prize or set of prizes by expending resources.

We'll start with a quick overview of a war of attrition between two players with known values. Then we'll talk about the case where they either have a known value or, with small probability, are a type that never quits. Then we'll talk about games with more than two players, all of whose values are known. Then two players whose values are drawn from a distribution. Then more than two players with values drawn from a distribution.

## 1 Two Players, Known Values

Suppose there are two players competing for a single object. Both players start out competing and can drop out at any time. The game ends when one of the players drops out. Assume player 1's value is known to be $v_1$ and player 2's value is known to be $v_2$. Each player incurs a cost of fighting equal to $c$ per unit of time. A strategy for player $i$ species a drop-out time $t_i$, or alternatively a distribution $G_i(\cdot)$ over drop out times. The former is a pure strategy; the latter is a mixed strategy.

One kind of equilibrium in this game is for one player to exit immediately, while the other never exits. These asymmetric pure strategy equilibria are efficient in the sense that no resources are "wasted" fighting for the prize.

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There is also a mixed strategy equilibrium in which both player fight for a positive expected time. In a mixed equilibrium, $j$ must be just indifferent between dropping out at $t$ and waiting to drop out at $t + dt$. This means that:

$$c \cdot dt = v_j \cdot \frac{g_i(t)}{1 - G_i(t)} \cdot dt = v_j \cdot \lambda_i(t) \cdot dt,$$

where $c \cdot dt$ is the flow cost to staying in, and $v_j \cdot \lambda_i(t) \cdot dt$ is the flow benefit, equal to the probability $i$ drops out in the interval between $t$ and $t+dt$ times the value to winning. Here $g_i(t)/(1-G_i(t)) = \lambda_i(t)$ is player $i$'s "hazard rate" of quitting or her "exit rate." In equilibrium, player 1 exits with probability $c/v_2$ per unit time, while player 2 exists with probability $c/v_1$.

**Proposition 1** *In the unique symmetric mixed strategy equilibrium, player $i$ uses the strategy: $G_i(t) = 1 - e^{-\lambda_i t} = 1 - e^{-c/v_j}$.*

Note that the key in the mixed equilibrium is the ratio of benefit to cost. If player $i$'s value of winning doubles, but so does her cost of fighting, she is no stronger or weaker (assuming risk-neutrality), so neither player's equilibrium mixed strategy will change.

Something is a bit odd about this mixed equilibrium. For example, if $c = 1$, $v_1 = 5$ and $v_2 = 10$ then the implication is that player 1 exits at half the rate of player 2. So player 1 wins the war with probability 2/3. This is strange because player 2 has higher value. So maybe a more realistic equilibrium is for player 1 to drop out immediately.

Are there other equilibria? For instance, can we have an equilibrium where the players fight but if time $T$ is ever reached one player drops out for sure (for example because he will run out of resources)? No. If player $i$ necessarily had to drop out at $T$, he would lose for sure in equilibrium, so he would do better to just drop out at the beginning.

## 2 Many Players, Known Values

Let's generalize to the case where there are $N$ identical prizes and $N + K$ contestants in the war of attrition. Again all contestants start out competing, and each contestant can drop out at any time. We assume that contestants observe the exit behavior of their fellow contestants. The game ends as soon as only $N$ contestants remain. Assume the contestants incur a common flow cost $c$ to competing and have known values $v_1, ..., v_{N+K}$.

It is easy to see that there are lots of equilibria, both pure and mixed. One set of (pure) strategy equilibria is for all exit to take place immediately.

For example, with five players competing for two prizes, it can easily happen that three players exit immediately, with the other two not intending to exit for a long time.

It is also possible for there to be a mixed strategy equilibrium where some deterministic exit takes place at either the beginning or the end of the game. For instance, with five players and two prizes, it is possible that two players could exit for sure at the beginning and three could continue on fighting for the two prizes. We can also have more than one person exit at the end. If five players $A, B, C, D$ and $E$ are competing for two prizes, there could be an equilibrium where everyone starts off by staying in, but then coordinates on the two winners being $B$ and $C$ if $A$ is the first to drop out, $C$ and $D$ if $B$ is the first to drop out, and so on. In this equilibrium, all five would randomize at first; then after one person exited two others would follow immediately.

What we cannot have in equilibrium is for some player to exit for sure in the middle of the game. Why not? Consider someone who has a strategy that specifies mixing behavior contingent on who has already dropped out, and suppose the player's strategy involves immediate exit after some intermediate number of drop outs. For instance, in the five player, two prize case, imagine that all five start by mixing and $A$'s strategy (but no one else's) specifies dropping out immediately after the first guy drops out. Then $A$ will earn negative profits from this strategy, because she will stay in for a while and pay costs and then lose for sure. She would do better to drop out right at the beginning, so such a strategy cannot be part of the equilibrium.

# 3 Two players, Two Types

In the two player case, we observed that an odd phenomenon was that players who appeared strong (i.e. had high values) might have to drop out faster in the mixed strategy equilibrium, and might win less than half the time. A slight change in the model that pushes in the direction of getting the "sensible" equilibrium of the weaker player quitting at the beginning of the game is one in which there is a small possibility that each player has negative costs of continuing to play and so will never drop out.

Suppose that if $j$ wins the game, he receives a payoff of $W_j$, while his payoff is $L_j$ if he concedes first. As before, the game ends immediately once one player concedes. Therefore the "prize that $j$ gets from winning is $p_j = W_j - L_j$. Let the "interest rate" for player $j$ be $r_j$. Suppose that $j$ also has cash out-of-pocket costs per unit of time of $b_j$ until the game ends (i.e.

costs decline by that much after resolution of the war). We can thus write the total cost per unit time for $j$ of delaying resolution as $c_j = r_j L_j + b_j$.

Define $z_j$ as the probability that $j$ is a type who "irrationally" will never exit. For convenience define $\lambda_j = c_j/p_j$ as the probability that a rational $j$ must have of winning per unit of time to be indifferent to be exiting at different times along the path.

A strategy for $j$, $\hat{G}_j(\cdot)$ specifies the distribution of $j$'s quitting time *conditional* on $j$ being rational. Given $\hat{G}_j(\cdot)$, the probability that $j$ exits by time $t$, assessed from $i$'s perspective, will therefore be $G_j(\cdot) = (1 - z_j)\hat{G}_j(\cdot)$. We will look for a perfect bayesian equilibrium. To do this, it should be clear that we could in principle work with either $\hat{G}_j(\cdot)$ or $G_j(\cdot)$; we'll work with the latter. Our approach to solving for the equilibrium, as it was for auction models, will be to identify necessary conditions for equilibrium, then show that the behavior we identify is in fact an equilibrium.

First, a few basic facts. First, it cannot be the case for two times $t' > t$, we have $G_j$ constant on $[t, t']$ and increasing above $t'$. The reason is that $i$ would then strictly prefer to quit at $t+\varepsilon$ rather than in the interval $[t', t'+\varepsilon]$. But if $i$ never exited in the interval $[t', t' + \varepsilon]$ it could not be optimal for $j$ to exit in this interval rather than at $t'$. So therefore $G_i$ and $G_j$ must be strictly increasing until they equal $1 - z_i$ and $1 - z_j$.

A second point is that there cannot be a time $t > 0$ where $i$ exits with discrete probability. If there was, $j$ would strictly prefer to exit at $t+\varepsilon$ rather than at any time in some interval before $t$, which would contradict the claim above. A third point, is that although at time $t = 0$ one player could exit with discrete probability, both cannot do so in equilibrium, because then each would have an incentive to wait a second and win the game with positive probability at very low cost.

Third, because $G_i$ and $G_j$ must be strictly increasing up to $1 - z_i$ and $1 - z_j$, but cannot have atoms, the rational $i$ and $j$ types must be mixing after time zero. For "rational" $j$'s to be indifferent between exiting and staying in, $i$ must exit at a constant rate $\lambda_j$. This implies that at some point during the game, a time $T_i$ will be reached at which point all rational $i$'s will have exited. After $T_i$, $i$ will only be in the game if she is a type who never quits.

Fourth, both sides must reach the point where they will never again exit at the same time — that is, it must be the case that $T_i = T_j$ in equilibrium. The reason is that once all the rational $i$ types have exited, a rational $j$ has no reason to stay in the game. He would do better to quit immediately.

This implies in equilibrium:

$$\frac{g_i(t)}{1 - G_i(t)} = \lambda_j,$$

and by integration:

$$G_i(t) = 1 - (1 - G_i(0))e^{-\lambda_j t}.$$

Moreover, there is some time $T = T_i = T_j$ where

$$G_i(T) = 1 - z_i \qquad \text{and} \qquad G_j(T) = 1 - z_j.$$

Beyond $T$, neither player ever exits.

What remains is to specify behavior at the beginning of the game, i.e. $G_i(0)$ and $G_j(0)$. There are three possibilities: either player 1 or player 2 drop out with some discrete probability at time 0, or neither player does so. As noted above, it cannot be that both players drop out with discrete probability at time 0.

To figure out which case we have, observe that:

$$G_i(T_i) = 1 - (1 - G_i(0))e^{-\lambda_j T_i} = 1 - z_i.$$

Solving out for $T_i$, we get

$$T_i = -\frac{1}{\lambda_j} \ln \frac{z_i}{1 - G_i(0)}.$$

We know that either $G_i(0) = 0$ or $G_j(0) = 0$ or both and also that $T_i = T_j$.

Therefore, we want to compare:

$$-\frac{1}{\lambda_j} \ln z_i \qquad \text{vs.} \qquad -\frac{1}{\lambda_i} \ln z_j$$

If these two expression are equal, we can have $G_i(0) = G_j(0) = 0$ and $T_i = T_j$. On the other hand, if the first expression is smaller, then we must have $G_i(0) = 0$ and $G_j(0) > 0$ chosen just large enough that $T_i = T_j$. Specifically, we have:

$$T_i = -\frac{1}{\lambda_j} \ln z_i = -\frac{1}{\lambda_i} \ln \frac{z_j}{1 - G_j(0)}.$$

So after some algebra we have

$$G_j(0) = 1 - z_j z_i^{-\lambda_i/\lambda_j}.$$

Of course, the reverse logic applies in the case where $G_j(0) = 0$ and $G_i(0) > 0$.

The preceding argument identifies a unique strategy profile that is potentially consistent with perfect bayesian equilibrium. It is not hard to show that the profile is in fact an equilibrium, giving the following result.

**Proposition 2** *There is a unique perfect bayesian equilibrium in which, assuming* $\lambda_i \ln z_i \geq \lambda_j \ln z_j$,

$$\begin{aligned} G_i(t) &= 1 - e^{-\lambda_j t} \\ G_j(t) &= 1 - z_j z_i^{-\lambda_i/\lambda_j} e^{-\lambda_i t} \end{aligned}$$

Relative to the complete information war of attrition, this model has at least two nice features. First, the equilibrium is unique. Second, the model has nice comparative statics. To expand on the latter point, suppose for instance that everything is equal for the two players except that $W_i > W_j$. In this case: $p_i > p_j$, so $\lambda_i < \lambda_j$, so in equilibrium the "weaker" player $j$ must drop out immediately with some positive probability. Similarly appealing comparative statics can be obtained with respect to $z_i$ and $z_j$ and the other parameters.

It is interesting to note that Abreu and Gul (2000) use this model to provide a war of attrition theory of bargaining. In their formulation, each player demands a share of a size one pie, so that $W_i$ is $i$ demand and $L_i = 1 - W_j$. They drop the time cost of bargaining $b_i$ and have just the opportunity cost of not settling $r_i L_i$ balanced against the flow probability of the other player conceding. There's more to the paper and I recommend it to anyone interesting in bargaining. Shin Kambe, a former GSB student, also has a nice paper in this vein (Kambe, 1998), as do Abreu and Pearce (2002).

# 4 Two players, Many Types

We now consider a model with two symmetric players with values drawn i.i.d. from a distribution $F(v)$, with a minimum value of 0 and positive density everywhere up to the maximum value. As before, players can stop fighting at any time and the game ends immediately once there is only a single player left. Each player has costs of 1 per unit of time. The bounded value of winning and positive cost of fighting will be enough to rule out the "commitment to fighting" types we saw in the previous model.

We are interested in finding a symmetric equilibrium for this game. Suppose such an equilibrium exists, where both players follows the strategy of exiting at time $T(v)$. Clearly to have an equilibrium, we must have $T(0) = 0$ because a type zero player will never win in equilibrium and hence could not expend positive resources fighting. It is also not hard to show that $T(v)$ must be continuous and strictly increasing.

Now, for $T(v)$ to be an equilibrium, a type $v$ player must be just indifferent to exiting at time $T(v)$ and exiting slightly earlier. Given that the opponent uses the strategy $T(v)$ the benefit from waiting the last $dt$ up to $T(v)$ is equal to $v$ times the probability the opponent will exit in this interval, which $(f(v)/(1 - F(v)) \cdot (1/T'(v)) \cdot dt$. The cost is $dt$.

Therefore a necessary condition for $T(v)$ to be a symmetric equilibrium is that

$$T'(v) = vf(v)/[1 - F(v)] = vh(v).$$

Here $h(v)$ is the hazard rate of the value distribution.

If we have $N+1$ players competing for $N$ prizes (e.g. a group of penguins standing around until one jumps in the water to see if there are any sharks), each with a value of winning drawn i.i.d. from the distribution $F(\cdot)$, then the logic easily generalizes. A symmetric equilibrium $T(v)$ must satisfy $T(0) = 0$ and

$$T'(v) = Nvh(v).$$

An intuition for both these equations is that $T'(v)$ is the cost that must be paid to beat players of type $v$ while the benefit for someone of type $v$ (who must be indifferent to doing this in the symmetric equilibrium) is $vh(v)$ if there is one opponent and $Nvh(v)$ if there are $N$ opponents, the exit of any one of whom would make everyone else a winner.

Starting with the differential equation above and integrating up, we obtain the equilibrium concession times:

$$T(v) = T(0) + \int_0^v T'(x)dx = \int_0^v xh(x)dx.$$

As in the case of the first price auction, this is a necessary condition for a symmetric equilibrium. It is not hard to verify that $T(v)$ is a best-response to $T(v)$.

Also, just as we were able to solve the first price auction via a differential equation and via the envelope theorem, we can also derive this equilibrium using revenue equivalence, or at least the payoff equivalence that we obtain using the envelope theorem.

How does this work? In a second price auction between two players with value distributed i.i.d. from $F$, a player with a value of $v$ will have an expected payments of $\int_0^v xf(x)dx$. This must be equal her payment in the war of attrition. If the player wins the war of attrition he pays based on the exit time of the opponent; if he loses he pays based on his own exit time. So expected payments are $\int_0^v T(x)f(x)dx + T(v)[1-F(v)]$. Equating these two expected payments:

$$\int_0^v xf(x)dx = \int_0^v T(x)f(x)dx + T(v)[1-F(v)].$$

Differentiating both sides with respect to $v$ reduces to

$$vf(v) = T(v)f(v) + T'(v)[1-F(v)] - T(v)f(v)$$

and then to

$$T'(v) = vf(v)/[1-F(v)] = vh(v)$$

as before.

# 5 Many Players, Prizes, and Types

Okay, we're now ready to take a shot at solving for the symmetric equilibrium of a war of attrition with lots of players competing for lots of prizes, given that there is asymmetric information about values.

Suppose there are $N+K$ players competing for $N$ prizes. Suppose the prizes are identical, and each player $i$ has a value of winning drawn iid from $F(\cdot)$. Suppose that $F$ has support $[0,\overline{v}]$. Let's assume that each player has a flow cost 1 per unit of time while she is still competing. In contrast to the earlier models, let's also assume that if you drop out before the game ends, then in the interval between when you drop out and when the game ends you pay a cost $c$ per unit of time. We'll call the model with $c > 0$ the "generalized" war of attrition.

Why change the game? Well, the tricky part here is going to be getting people to play symmetrically and drop out as the game goes along. Remember that in the $N$ prize, $N+K$ player game all the exits would have to come at the beginning or the end. Here are a couple of interpretations of the $c$ cost:

1. Tim Bresnahan insists that everyone stay at the faculty meeting and pay attention until at least five people have agreed to be on the committee to review the econometrics comprehensive exam. Even if you "drop out" and agree to be on the committee, you're still stuck as long as everyone else (so $c = 1$).

2. Tim Bresnahan insists that everyone stay until he's got his five volunteers, but once you "drop out" and agree to be on the committee, you can tune out and write referee reports while sitting there (so $0 < c < 1$).

3. Tim Bresnahan lets the volunteers leave immediately (so $c = 0$).

The last case is the exact extension of the models we've been looking at, as you can drop out immediately and still ensure a surplus of zero. This case turns out to be hard to solve (technically it won't have a symmetric PBE), but by solving the case for $c > 0$, we can take the limit of games as $c \to 0$ and basically recover what must happen when $c$ is very, very small. One reason to be interested in the $c = 0$ case is that it will be revenue equivalent to a Vickrey auction where $N + K$ bidders compete for $N$ prizes.

## 5.1 The Symmetric Equilibrium

We're going to look, as we said, for a symmetric PBE. To introduce a little notation, let $T(v, \underline{v}; k)$ denote the amount of time that type $v$ will wait to drop out in a subgame where *(i)* there are $N + k$ firms left and *(ii)* the lowest remaining type is $\underline{v}$.

To solve the model and characterize the symmetric PBE, let's start with the subgame where $K - 1$ guys have dropped out and we have $N + 1$ guys competing for $N$ prizes. In this case the $c$ is irrelevant because any drop-out will end the game immediately. In fact, we saw above that:

$$T'(v; \underline{v}, 1) = Nvh(v),$$

where $h(v) = f(v)/(1 - F(v))$ is the hazard rate of $F$. Remember the intuition is that at each moment the marginal firm with type $v$ faces the prospect of paying an extra $T'(v; \underline{v}, 1)$ to outlast any types between $v$ and $v + dv$, and equates these costs with the probability $Nh(v)$ of being a winner times the value $v$ of actually winning.

Of course, now we have to start with $\underline{v}$ dropping out immediately rather than type 0, so $T(\underline{v}; \underline{v}, 1) = 0$. Integrating up, we have:

**Lemma 3** *The unique symmetric perfect bayesian equilibrium of the subgame in which just one more exit is required to end the game is defined by:*

$$T(v; \underline{v}, 1) = \int_{\underline{v}}^{v} Nxh(x)dx.$$

Let's now state the main result before we give the intuition.

**Proposition 4** *The unique symmetric perfect bayesian equilibrium of the generalized war of attrition is defined by:*

$$T(v;\underline{v},k) = c^{k-1}\int_{\underline{v}}^{v} Nxh(x)dx.$$

Let's start by talking about case (1), where $c = 1$. In this case, the Proposition states that people won't condition their strategies on $k$ at all! Instead they'll just have in mind some stopping time $T(v;0;K)$ and stick with it regardless of who drops out when beforehand.

How can this be? The key insight is to realize that if you are not among the final $N+1$ players there will come a time shortly before you exit where you know with probability 1 you will not win (because there will not be 2 or more other exits in the short time before it is your equilibrium turn to leave). Therefore once we get near $v$'s drop-out time, her marginal decision about whether to stay in a bit longer or just drop out is not about whether she has a chance to win or lose but rather about whether she can shorten the game by dropping out more quickly. For $v$ to be willing to exit at the "right" time it must be that dropping out slightly earlier won't shorten the game. But how can the game's length be independent of when you exit (so that if you are "supposed" to leave at some time and there are 3 more people who need to drop out you don't do better by just waiting for three more to leave and then leaving instantly, and you also cannot gain by leaving as soon as it becomes apparent that you will not be a winner even though it is still a bit before your turn to leave)?

The answer is, there can be an equilibrium in which everyone's strategy is independent of how many other people have dropped out. What must be the strategies in such a game? We know that once there are $N+1$ players competing for $N$ prizes that each player will use the strategy $T'(v) = Nvh(v)$, and since their strategy is the same regardless of what others do, it must be that everyone plays according to that strategy from the very beginning of the game.

Now consider what happens in case (2), where costs drop from 1 to $c$ per unit of time after you exit and until the end of the game. Again, when it becomes apparent that you will lose your real option is to lose on time, or a moment earlier or later. But doing so cannot effect your total costs, otherwise you would go in the cheaper direction. So it must be that the game slows down by a factor $1/c$ after each player leaves. We know that the strategies followed in the last round when there are $N+1$ players left fighting for $N$ prizes (the prize being to stay off the committee in the example) are $T'(v) = Nvh(v)$, so the strategies in the next to last round must be $T'(v) = cNvh(v)$, while in the round before that the strategies must be $T'(v) = c^2Nvh(v)$ etcetera, or generally $T'(v) = c^{k-1}Nvh(v)$ where $k$ is the number of excess players in the game at the time.

This means, a bit more formally, that:

$$T'(v; \underline{v}, k) = c^{k-1}Nvh(v),$$

and we have the initial conditions $T(\underline{v}; \underline{v}, k) = 0$, so combining this together gives the equilibrium.

Finally, what about the case where $c \to 0$. In this case, even the round before the last becomes very fast and the round before that faster still. So basically people exit really quickly but efficiently in a very short time until the last $N + 1$ players. These guys then fight it out.

### 5.2 An Example with Numbers

Let's try a numerical example. Suppose there are three guys competing for one prize and that $c = 1/2$. Values are independent and uniform on [0,1]. Let's try to figure out the total expected length and cost of the game in a couple of different ways.

First, note that in the first round everyone is using the strategy

$$T'(v) = \frac{1}{2}vh(v).$$

Note that with 3 guys competing for two prizes the strategies would be:

$$T'(v) = 2vh(v),$$

which would be exactly four times as slow.

In this "3 for 2" case, the $c$ is irrelevant for the reasons we mentioned before (the game will end as soon as one guy drops out). So we're in revenue equivalence world — the "3 for 2" war of attrition is revenue equivalent to a "3 for 2" Vickrey auction. In such an auction, the two winners would expect to pay the value of the lowest guys, which is $1/4$. Therefore the total expected cost is $\frac{1}{4} \cdot 2 = \frac{1}{2}$.

Taking this observation to the war of attrition, note that the first round of the "3 for 1" generalized war of attrition goes four times as fast as the "3 for 2" war of attrition. Therefore, the expected cost in the first round must be $\frac{1}{8}$. The costs are paid by three players, so the expected length of the first stage is $\frac{1}{8}/3 = \frac{1}{24}$.

The second stage is easy because we have a "2 for 1" game. Strategies will be

$$T'(v) = vh(v).$$

Moreover, this subgame is revenue equivalent to a standard "2 for 1" Vickrey auction, so the winner must expect to pay the second highest value, i.e. the winner expects to pay $1/2$. The expected costs are spread across two players, so the expected length of the subgame is therefore $\frac{1}{4}$. Of course, the third guy who dropped out in the first stage also has a cost from this subgame, equal to $\frac{1}{4}c = \frac{1}{8}$. Therefore total costs in the game are $\frac{1}{8}$ in the first stage and $\frac{1}{2} + \frac{1}{8} = \frac{5}{8}$ in the second stage for a total of $\frac{3}{4}$.

Note that if we just had a regular Vickrey auction with three people competing for one prize the expected cost would be $\frac{1}{2}$, the expected value of the second highest bidder. Where does the extra $\frac{1}{4}$ come from? The answer is that it comes from the positive $c > 0$ cost that must be incurred even by losers!

That is, in a Vickrey auction a bidder with value 0 would simply get zero surplus. In the generalized war of attrition, however, a guy with a value of 0 drops out right away, but still has expected costs of $1/12$ (since the lower of the remaining two bidders would have a value averaging $1/3$, implying a length of the last stage of $1/6$ and a cost to the bottom bidder of $1/12$ if $c = 1/2$. This makes the expected surplus of the bottom type $S(0) = -1/12$. If we did the expected surplus for each bidder in this game using the envelope condition $S(v) = S(0) + \int_0^v p(x)dx$ we would see that the expected surplus of each type is therefore $1/12$ lower than in a second price auction. Adding up across three bidders, expected surplus is reduced by $3/12 = 1/4$ and total costs are that much higher than in the second price auction.

We can use revenue equivalence arguments of this sort more generally than in this numerical example. For instance, if there are $N + k$ bidders competing for $N + k - 1$ prizes everyone's strategy will be $T'(v) = Nvh(v)$ and the expected length of the game will be $(N + k - 1)E(v_{N+k})/(N + k)$ where $v_{N+k}$ is the expected value of the $N + k^{th}$ highest (and therefore lowest) remaining bidder, times the number of winners. With that expected length expected payments per winner would be $E(v_{N+k})$,which must be so by the RET. In the war of attrition as we go from $N + k$ to $N + k - 1$ remaining players people use the strategy $T'(v) = c^{k-1}Nvh(v)$ so therefore the expected length of the stage must be $c^{k-1}(N + k - 1)E(v_{N+k})/(N + k)$. By plugging into this formula for $k = 1, 2, 3, ...K$ and summing we can figure out the expected length of any game and then, somewhat more tediously, figure out the expected costs.

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