

REPRESENTATION OF THE ORIENTATIONS OF SHAPES *

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A geometrical model is put forward as underlying superficially quite different data from discrimination and mental transformation tasks and, also, as a possible framework for an eventual account of the perception of shape. The model was evaluated in an experiment in which participants indicated whether two picture-plane views of a polygon with an experimentally determined degree of symmetry under 180° rotation were the same or different in orientation. Multidimensional scaling based on the obtained discriminative reaction times yielded a four-dimensional solution that, in agreement with the proposed model, describes possible transformations of the planar polygons. We suggest that the constraints governing possible transformations of objects have become so internalized that we unconsciously represent these possible motions upon the visual presentation of any object.

The knowledge that is likely to have become most deeply and thoroughly internalized in our perceptual systems during biological evolution are those facts about the world (a) that have been most invariant in the world, and (b) that are most simply characterizable and hence easily internalized. A fundamental invariant that can be simply characterized is the way in which objects can rigidly transform in relation to ourselves in space (Shepard 1981b, 1984).

The knowledge that describes the possible transformations of rigid objects is far more stable than the knowledge that describes the particular shapes of objects in our environment. The predators, competitors, foods, weapons, shelters, and so forth that have been most significant to our ancestors and ourselves have changed markedly during our immense evolutionary history; and each of these objects

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would require a vast number of parameters to describe its range of characteristic shapes, postures and nonrigid movements. Yet even non-rigid objects can often be described as jointed assemblages of approximately rigid segments, e.g. trunk, upper arm, lower arm, finger, etc. (see Marr 1982). Moreover, nonrigid visual objects will exhibit transformations that are approximately rigid, relative to an observer who is actively exploring it through eye movements, head movements, and circumlocation (Shepard and Cooper 1982). Such rigid transformations have always been characterizable terms of just six parameters – three of relative translation and three of relative rotation. We suggest, therefore, that our seeing of objects and their motions is automatically constrained and guided by perceptual mechanisms embodying, at a deep level, our evolutionarily acquired wisdom about rigid transformations in Euclidean three-dimensional space.

The assignment of a fundamental role to such spatial transformations in visual perception is a natural extension of the earlier emphasis by Cassirer (1944), Gibson (1950, 1966), and others of the importance of invariance under transformations and, particularly, under those transformations induced in the optic array by the free movements of the perceiver (Gibson 1966, 1977). It is also consonant with Garner's (1974) characterization of figural goodness in terms of the size of the class of figures that are equivalent under translations, rotations, and reflections.

From a theoretical standpoint, the goal of characterizing our knowledge of the ways in which objects transform in space is much more general than the goal of characterizing how we perceive individual static objects, because it applies to the perception of all possible objects, in all possible positions. At the same time this goal is, from a methodological standpoint, much more restricted because it requires the specification of only six rather than a virtually unlimited number of parameters.

There is, furthermore, the possibility that the theory that we develop for the representation of such transformations will in turn afford us a more general way of addressing the traditionally prior problem of the perception of static objects. For the constraints governing the possible transformations in three-dimensional space may have become so deeply internalized that we unconsciously represent these possible motions upon the visual presentation of any object. These dynamic representations may not only underlie our immediate intuition as to how any objects move, but they may also underlie our appreciation of the shape

of the object itself. For, that shape is specifiable in terms of a kind of autocorrelation of the object with itself under these possible transformations (cf., Shepard 1981a, b; Uttal 1975; and, again, Garner 1974).

The existence of this internalized knowledge about spatial transformations is most strikingly demonstrated in the phenomenon of apparent motion. When alternately presented with two different views of the same object, we experience a single object rigidly moving back and forth over a definite trajectory in space (see Bundesen et al. 1981, 1983; Farrell 1983; Foster 1975; Shepard 1984; Shepard and Judd 1976). In the absence of any physically presented motion, the particular path that we perceive reflects our internalized knowledge of the constraints governing rigid displacements of an object.

In this paper we hypothesize that although there are an infinite number of paths over which the object could have moved from the one position to the other, the one path that is actually experienced on any one occasion is selected from a very small number of alternatives each of which is in some sense especially simple or, as we shall assume, short. In order to make this notion of shortest paths precise, we have to specify an appropriate metric of psychological distance for the abstract space of possible positions of an object in physical space. The physical space is, of course, three-dimensional and Euclidean, but, because there are six degrees of freedom of position in physical space, the abstract space of possible positions is six-dimensional. Moreover, because the three rotational degrees of freedom of position are circular, this six-dimensional space is non-Euclidean. This space can, however, be regarded as a curved six-dimensional surface embedded in a seven-dimensional Euclidean space. Within this curved surface or *manifold*, as it is called, each point corresponds to a particular position of the object in physical space, and any path connecting two points within the surface corresponds to a possible rigid motion of the object from the one position to the other.

We hypothesize that when this abstract six-dimensional manifold has been combined with the appropriate metric, the simplest motions of objects – the ones that tend to be experienced in apparent motion – will be those paths that cannot be made shorter by any small variations of the path (Shepard 1977, 1981b). In differential geometry such external paths are called *geodesic* paths. (They are like strings that, while confined to the curved surface, are pulled tight between the two end points.) Such paths are analogous to great circles on the curved

surface of the earth. Because the manifold is non-Euclidean, pairs of points that are not close together (corresponding to positions of the object that are quite different in orientation) can often be connected by distinct geodesic paths. This is a reflection of such facts as that an object can be rotated, from one orientation into another quite different orientation, in either of two opposite directions. Just as two points on the earth's surface can be connected by either of two (opposite) great circles, one path may be shorter than the other. But each will be an external or geodesic path in the sense that any small variations in the path will only make it longer. The manifold of possible positions thus provides us with a precise way of distinguishing between three perceptually different types of transformations: (a) rigid motions, which correspond to any continuous path falling within the curved manifold, (b) simplest rigid motions, which correspond to paths within the manifold that are also geodesic, and (c) nonrigid transformations, which correspond to paths passing outside the curved manifold – for example, short-circuit paths cutting directly through the higher-dimensional Euclidean embedding space (much as a straight line might connect two remote points on the two-dimensional surface of the earth by cutting directly through its three-dimensional interior).

To the extent that this abstract manifold of possible positions, when endowed with the psychologically appropriate metric, accounts for perceptual phenomena such as those arising from real and apparent motion, we can regard the geometrical structure of this manifold as a representation of our implicit knowledge about the ways in which objects move in three-dimensional space. And, to the extent that the psychologically appropriate metric corresponds to the natural metric for rigid physical motions, we can regard this implicit knowledge as an internalization of the kinematic geometry that has always constrained the relative physical motions of objects in our world (Shepard 1984). Accordingly, we can presume that whatever the neurophysiological details of our perceptual mechanism may eventually be found to be, they will necessarily embody, at the appropriate level of description, the metric structure of this abstract manifold.

In line with our earlier remarks about the fundamental role of transformations in perception, we propose that this same structure will not only describe the psychological data on apparent motion, but will also constrain other phenomena governed by the relations between objects that differ by rigid transformations. Thus the paths of mental

transformations such as mental rotations (Shepard and Metzler 1971; Shepard and Cooper 1982) should correspond to the same geodesic or shortest trajectories in this abstract space. Moreover, just as the time to discriminate stimuli depends quite generally on the psychological similarity between them (see the overviews in Podgorny and Shepard 1983; Shepard 1981a; Welford 1960), we conjecture that the time to distinguish whether two identical objects are in the same orientation will depend on some measure of distance between the points corresponding, in this same manifold, to the two presented positions. However, in this latter case, the appropriate distances may not be the geodesic distances within the curved manifold but the short-circuit distance through the embedding space. Moreover, the measured critical time, rather than exhibiting a linear increase with distance, as in the case of imagined transformations and apparent motion, should manifest a nonlinear decrease with distance (Shepard 1978).

In this paper we are particularly concerned with the intimate relationship between the representation of possible motions and the shape of an object that arises whenever the object possesses any symmetries or approximation to symmetry. This relationship is important because most objects of interest to us do possess some approximate bilateral or circular symmetry. Indeed, according to the autocorrelation theory of shape representation already mentioned, *any* shape can be characterized by the way its self-similarity varies with extent or angle of transformation or, in other words, by its degrees of approximation to the various possible symmetries.

The most natural way to capture such approximations to symmetry in the geometry of the manifold of possible position is by means of a suitable deformation of that manifold – a deformation that brings points corresponding to very different orientations (in which the object nevertheless appears quite similar) close together in, for example, the higher-dimensional embedding space. For example, because a rectangle becomes identical to itself under 180 degree rotations, points that were diametrically opposite in the original manifold of positions must be brought into coincidence by deforming the manifold to accommodate for the symmetry. Because a rectangle becomes somewhat more similar (though not identical) to itself under 90 degree rotations, points corresponding to positions differing by 90 degrees must additionally be brought more or less close together in the embedding space. By shortening the short-circuit path through the embedding space, such a defor-

mation can account for the increased time needed to discriminate between orientations in which an object appears similar – for example between nearly square rectangles differing by exactly 90 degrees.

In order to proceed, we must now become more specific about the geometrical structure of the manifold of positions and, in order to attempt an initial experimental exploration of the complication of approximate symmetries, we shall narrow attention to the simpler case of two-dimensional shapes and then rotational transformations within the two-dimensional picture plane. Our focus on rotational transformations is motivated (a) by the especially close entanglement between those transformations and the representation of shape, (b) by the fact that in the natural metric of the six-dimensional manifold of possible positions, translations (which can also be regarded as infinitesimal rotations) are essentially negligible, and (c) by the perhaps corresponding empirical observation (long ago foreseen by Mach 1886), that the times required for imagined translations evidently are generally short compared with the time required for imagined rotations.

Psychological representation of the orientations of a partially symmetric object

The orientation of an object has three degrees of freedom, two specify the direction of an axis (corresponding to the latitude and longitude of an axis passing through the center of a globe), and one specifies the angle through which the object is rotated around that axis, from some fixed reference orientation. We can then represent that angle of rotation by a distance from the center of the globe along that specified axis, with clockwise rotations up to 180 degrees represented in one direction along that axis and counterclockwise rotations up to 180 degrees represented in the opposite direction along that same axis. Thus we can see that the manifold of all possible orientations correspond, topologically, to the surface and interior of a sphere (of radius 180 degrees), with diametrically opposite points on the surface identified (regarded as the same point).

Fig. 1 illustrates, by means of the orientations of a cube, one two-dimensional slice through this three-dimensional closed manifold. Notice that diametrically opposite points, as claimed, correspond to identical orientations of the cube. Of course, this one slice does not

include all orientations; there are infinitely many other slices. Moreover, this flat representation of the slice is only topologically valid. Metrically, each such slice is more accurately conceived of as hemispherical but with, again, opposite points regarded as the same point. There is in fact a natural metric of this space, which is equivalent to the natural metric of the orthogonal group $SO(3)$ (see Foster 1975). In this metric, geodesic paths are like great circles on the surface of a sphere, each corresponds to a rigid rotation of the object about some axis, and distance within the manifold corresponds simply to the angle of that rotation.

When we try to consider the deformations of this three-dimensional curved manifold needed to accommodate approximations to various symmetries in the object, visualization of the manifold becomes difficult. In order to simplify things, and in order to facilitate comparison with empirical data, we shall from here on restrict attention to the

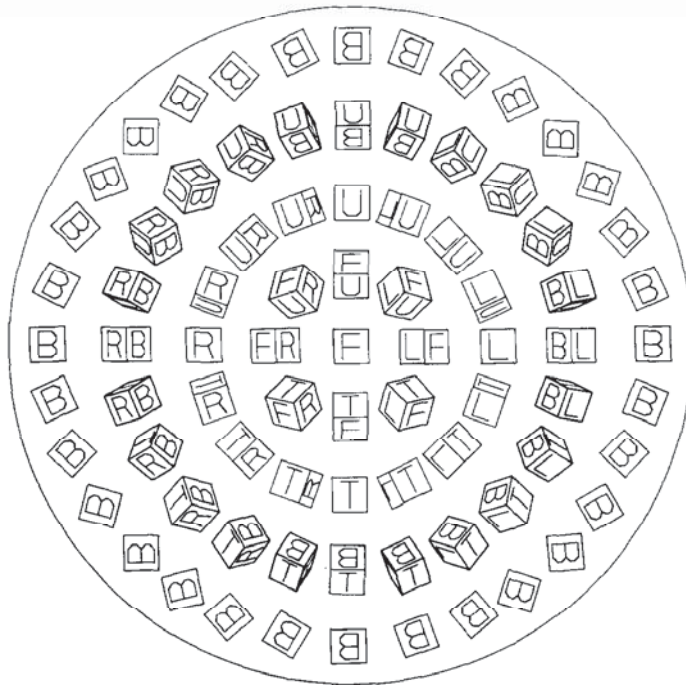


Fig. 1. Topological representation of a two-dimensional slice through the three-dimensional manifold of possible orientations.

much simpler case of two-dimensional polygons and their possible orientations in the plane. Because we are focusing only on differences in orientation, the kind of symmetry that will be most relevant is symmetry under rotation. We shall consider only the simplest of these; namely, symmetry under 180 degrees rotation – and various degrees of approximation to such symmetry.

We have only to consider a single geodesic or great circle path in the three-dimensional manifold – namely, that corresponding to different orientations of the polygon about an axis perpendicular to the plane and passing through the center of the polygon. If the object were completely asymmetrical, this path could be metrically represented by a simple circle, representing the perceptual fact that such an object becomes less and less similar to itself as it is rotated up to 180 degrees and then more and more similar again until it becomes identical to itself at 360 degrees. If, however, the object is completely symmetric under 180 degrees rotation, points that were diametrically opposite on the original circle become identical. We obtain, thereby, a double-wound circle in which, as the object rotates through 360 degrees, the point representing the appearance of the object in each orientation passes around the circle twice, passing through the same point at 180 degrees as well as at 360 degrees.

The interesting case is that in which the object possesses only some approximation to 180 degrees rotational symmetry. The original circle must then be deformed into a curve in which points representing orientation differing by 180 degrees are neither opposite, as in the case of complete asymmetry, nor coincident, as in the case of complete symmetry. Such pairs of points must then all be at the same intermediate separation in the embedding space – a separation that can become arbitrarily small as the object is made more and more nearly symmetric. Such a curve can only be isometrically embedded in a four-dimensional Euclidean space, where the straight or short-circuit path between points corresponding to orientations differing by 180 degrees sweeps out the famous one-sided surface of Möbius.

Fig. 2 provides a concrete illustration of the geometrical manifold based on the polygons generated for our earlier study of rotational apparent motion (Farrell and Shepard 1981). We shall use this representation in the experiment we report here on the time to discriminate differences in orientation. The leftmost curve in the middle of the figure is the original simple circle of orientations for an asymmetric object, the

rightmost curve is the double-wound circle for a completely symmetric object, and the middle curve is a crude two-dimensional portrayal of the curve in four-dimensional space for a partially symmetric object.

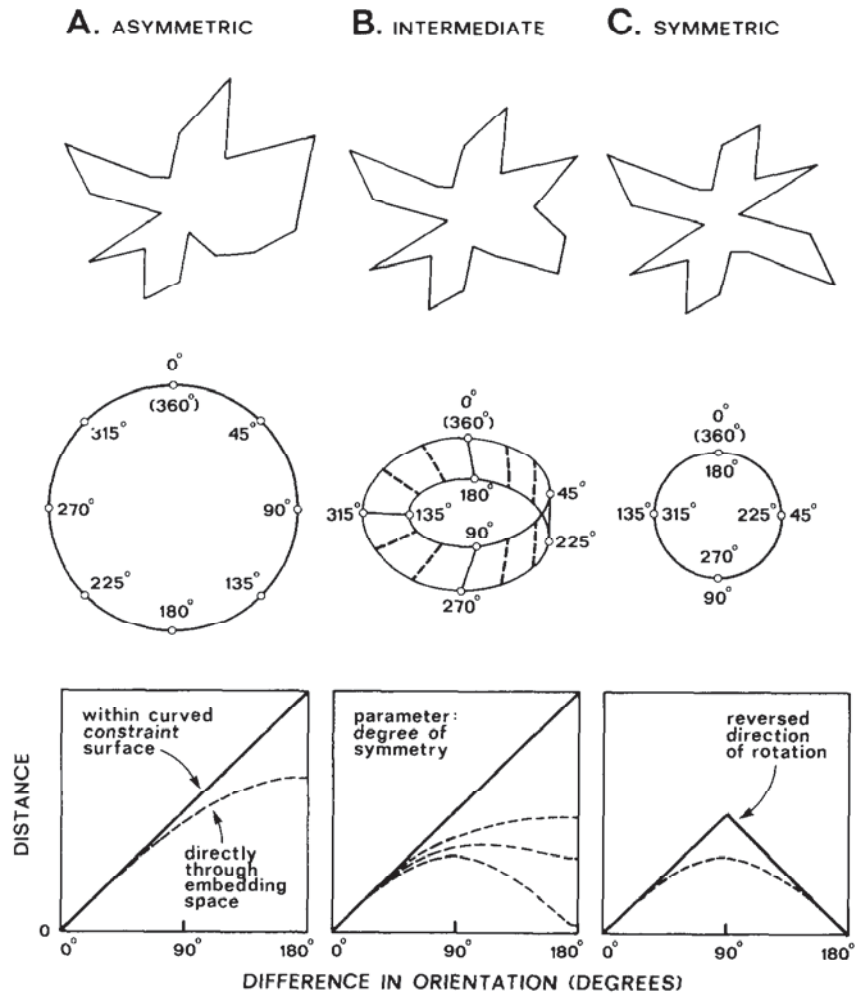


Fig. 2. Examples of polygons that are asymmetric (A), symmetric under 180° rotation (C), or intermediate between these (B) and, below each, a closed curve representing the set of perceived orientations of polygons with that degree of symmetry. The plots at the bottom show how distance of transformation depends on angular difference, for transformations that are rigid (solid lines) or nonrigid (dashed curves). (From 'Psychophysical Complementarity' by R.N. Shepard. In: M. Kubovy and J. Pomerantz (eds.), *Perceptual Organization*. Hillsdale, NJ: Lawrence Erlbaum, 1981. Copyright 1982 by Lawrence Erlbaum. Reprinted by permission.)

Pairs of prints connected by dashed lines in this middle curve correspond to orientations differing by 180 degrees.

Our two proposals are: (a) that critical times in mental rotation or apparent motion increase linearly with shortest distance along the curve (shown as solid lines in the graphs at the bottom), and (b) that times to discriminate differences in orientation decrease nonlinearly with direct distances through the embedding space (shown as dashed curved in the bottom graphs). Our first proposal is supported by our earlier experiment on apparent motion (Farrell and Shepard 1981). We now turn to an empirical evaluation of the second proposal and, also, to an attempt to recover the proposed four-dimensional structure from these discrimination times by means of multidimensional scaling.

Method

Stimuli

We presented each observer with 15 of the versions of the three basic polygons used by Farrell and Shepard (1981). In that experiment, we had first constructed three asymmetric polygons by placing one point a random distance out in each of 18 equally-spaced directions around an arbitrary center and then drawing straight line segments between these points in (20°) adjacent directions. We placed two of these points at 8 units above and below the center on the vertical axis of the shape. We then determined the distance out to the point in each of the 16 other directions from the center by randomly selecting a number between 1 and 17 (excluding 8) without replacement.

In effect, to create polygons having 180° rotational symmetry, we cut each of the asymmetric polygons in half along its vertical axis, and then attached each half to a duplicate version of itself that had been rotated 180° degrees. Thus we obtained three rotationally symmetric polygons from the left halves of the asymmetric polygons (called the left-side versions), and three rotationally symmetric polygons from the right halves of the asymmetric polygons (called the right-side versions).

We then constructed polygons of intermediate degrees of symmetry as follows: Placing each symmetric polygon over the asymmetric shapes from which it was derived, so that corresponding halves coincided, we simply linearly interpolated, for each direction from the center, between corresponding points on the symmetric and asymmetric polygon. For a particular polygon, these interpolated points were all chosen to fall either one-fourth, one-half, or three-fourths the distance from a point on the asymmetric polygons to the corresponding point on the symmetric polygon, yielding a resulting polygon of 25%, 50%, or 75% symmetry, respectively.

These procedures yielded a set of 27 distinct polygons: the three randomly generated original polygons, with 0% experimentally imposed rotational symmetry; and three left-side and three right-side versions at each of the following four levels of experimen-

tally imposed 180° symmetry: 25%, 50%, 75%, and 100%. However, the qualification 'experimentally imposed' is significant. Even a randomly generated polygon, unlike a perfect circle, must possess some appreciable, though haphazard, increments in self-similarity at certain angular disparities. In fact it is the functional dependence of these self-similarities on angle that characterizes the unique shape of such a polygon (Shepard 1981b, 1984). However, these haphazard increments in self-similarity, unlike the experimentally imposed increments, would not be expected to be common across an ensemble of independently generated polygons at a given level of experimentally imposed 180° symmetry. Illustrated, from left to right, across the top of fig. 2 are: (A) one of the three original polygons, (C) the left-side completely symmetric version of that polygon, and (B) its intermediate (50%) approximation to 180° rotational symmetry. (For the other polygons and other approximations to this type of symmetry, see Farrell and Shepard 1981: 479.)

The 15 versions used in the present experiment included three 'left-side' or three 'right-side' versions at each of the five experimentally imposed levels of 180° rotational symmetry, 0%, 25%, 50%, 75%, and 100%. This time, however, we displayed the two orientations of such a polygon simultaneously on the left and right sides of the computer-controlled CRT screen, rather than in sequential alternation in its center (as in the earlier study of apparent motion). The polygon on the left always appeared in its arbitrarily defined 'upright' orientation, while the same polygon appeared on the right either in the same orientation (on a random half of the trials) or rotated 30° , 60° , 90° , 120° , 150° , or 180° in a clockwise or counterclockwise direction (on the other half). The polygons subtended a visual angle of approximately 3.6° and their centers were separated by about 4.8° of visual angle.

Participants

The same eight observers who had already participated in the earlier experiment on apparent rotational motion (Farrell and Shepard 1981) returned for a half-hour experimental session in the present experiment.

Procedure

We instructed the participants to operate the right or left key on the response panel to indicate whether the two polygons presented on a given trial were in the same or in different orientations, respectively. We asked them to respond as soon as they could do so without making an error, following the appearance of each pair of polygons. We did not furnish them with feedback as to the correctness of each response, nor did we repeat trials on which they made errors. Their overall error rate was quite low, however, averaging under 2%. The computer recorded the reaction time for each pair and, following a two-second intertrial interval, presented the next pair. In this way, each participant proceeded through 360 pairs (3 polygons \times 5 levels of rotational symmetry \times 6 clockwise plus 6 counterclockwise plus 12 null angular disparities). All participants saw the same ('right-side' or 'left-side') versions of the polygons that they saw in the earlier experiment on apparent motion. The 360 pairs of polygons appeared to each

participant in random sequence, organized into three 120-trial blocks separated by optional periods of rest.

Results

Mean dependence of latency of 'different' response on angular disparity and degree of symmetry

The curves in fig. 3 show how the mean time to indicate that two polygons were in different orientations depended on the size of their orientational difference. Each of the five plotted curves is for a different level of 180° rotational symmetry of the polygons.

In accordance with the already noted general finding that the time required to detect any difference between stimuli decreases with the size of that difference, the time to detect a difference in orientation here decreased monotonically from nearly 800 msec at

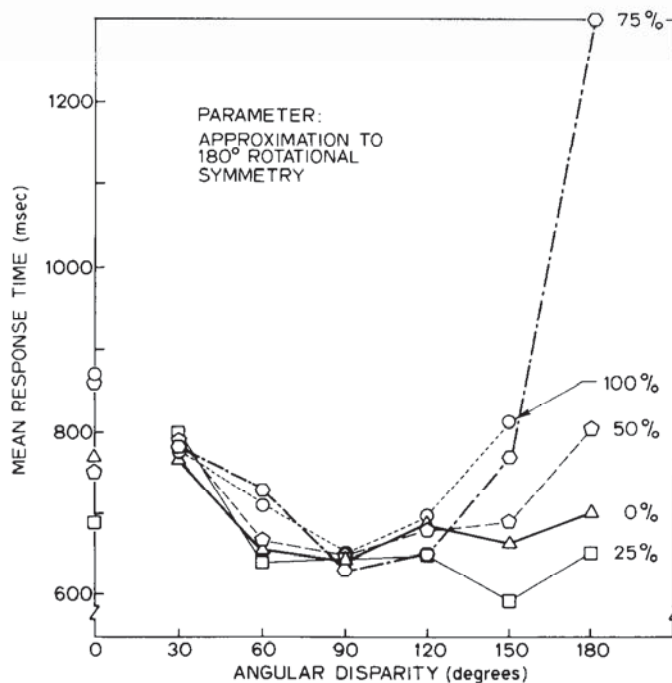


Fig. 3. Mean time required to indicate whether two polygons were in the same or different orientations plotted as a function of angular disparity for each of the five levels of rotational symmetry. ('Same orientation' times are plotted at 0° , on the left.)

30° to about 650 msec at 90° for all five levels of 180° symmetry of the polygons. Moreover, as would be expected for a function that behaves reciprocally and hence asymptotically for larger differences, the curves for the essentially asymmetric polygons (of 0% and 25% symmetry), despite some fluctuations, remained at about the 650 msec-level beyond 90°.

The curves for increasing degrees of rotational symmetry, 50%, 75%, and 100%, however, exhibit increasingly marked upswings beyond 90°, yielding generally U-shaped functions. (No point is plotted for the 100% curve at 180°. Because the polygons are completely symmetric in this case, participants gave the 'same orientation' response. Their mean reaction time was the same as that shown for 100% symmetry at 0° disparity.) The curve for the 75% symmetric polygons shows a dramatic rise to an average reaction time for 180° that is over 1200 msec – nearly twice that for the asymmetric polygons at large angular disparities. The difficulty of detecting a 180° orientational difference is expected to become arbitrarily great as the degree of rotational symmetry of the polygon approaches 100%.

Mean dependence of latency of 'same' response on degree of symmetry

In this experiment, the two polygons presented on some trials were identical in orientation as well as in shape. The mean times that participants took to make the response indicating 'same orientation' are plotted vertically above 0° on the left in fig. 3. Notice that with the exception of the polygons of 0% symmetry, the decision times decreased from most to least symmetrical, with the 100% and 75% conditions well above the other three. Again, we expect that the more nearly a shape approximates 180° rotational symmetry, the more difficult will be the determination of whether it is right-side-up.

Reliability of the pattern of discrimination times in individual participants

With the exception of the more erratic data from one observer, the plots for individual participants are reasonably well represented by the group means plotted in fig. 3. Seven of the eight participants showed a mean decrease to 90°, while the remaining participant showed essentially no change. All eight participants yielded curves that increased as they approached 180° for both the 100% and the 75% symmetric conditions. The reliable upswing of the curves for the more symmetric polygons represents a consistent fanning out of the curves beyond 90°: From 120° to 180°, the average of the curves for the two most asymmetric condition (0% and 25%) did not go up for more than half of the eight participants, while the last point of the 75% curve was above all three of the remaining other curves for seven out of the eight observers. Correspondingly, the average 'same' time for the most symmetrical, 100% and 75% polygons was greater than the average 'same' time for the remaining, less symmetrical polygons for seven out of the eight participants.

Comparison of these discrimination times with the earlier transformation times

The discrimination times reported here furnish psychological measures of the perceptual similarities of each polygon to itself at the various orientational disparities that are

independent of and in some respects more direct than the transformation times obtained in our earlier experiment on apparent motion (Farrell and Shepard 1981). The consistency and relative magnitude of the upswing in discrimination time as the polygons approached 180° disparity and 180° rotational symmetry thus serves to confirm that our experimental manipulation of shape in that earlier study sufficiently dominated any uncontrolled residual approximations to symmetry inherited from the original, randomly generated polygons.

Moreover, the patterns of the chronometric data obtained from the two experiments can in some ways be regarded as inversions of each other – at least if we consider only the extreme cases of 0% and 100% symmetry of the polygons, and if we make allowances for the fact that transformation time varies directly with disparity while discrimination time varies inversely (and hence asymptotically) with disparity. Thus, for completely symmetric polygons, the earlier obtained transformation times went up to 90° and then down in a symmetric, inverted U pattern; while the discrimination times obtained here went down to 90° and then up in a similarly symmetric, upright U pattern. Moreover, for the asymmetric polygons, transformation time climbed linearly to 180° , while discrimination time dropped monotonically to 180° , effectively reaching asymptote around 90° .

If we examine the data for all levels of symmetry, however, we notice some clear violations of this simple relation of inversion between the two sets of data. In the present experiment, the curves for the intermediate cases generally fell in correspondingly intermediate positions between the curves for the extreme 0% and 100% cases, as those curves fanned out beyond 90° disparities. But in the earlier experiment, the curve for the nearly symmetric, 75% case consistently climbed well above the range delimited by the curves for the fully symmetric and asymmetric cases. Moreover, for that earlier experiment, but not for the present one, the curves for the more symmetric polygons tended to depart from the curves for the asymmetric polygons even before 90° . We explain these departures from a relation of inversion in terms of the notion that in the present experiment the participants had only to detect an angular difference; they did not, as in the earlier, transformation experiment, actually have to traverse it. Hence, the relevant distances in the model are different in the two cases: the geodesic curve in the case of apparent motion; the straight short-circuit path in the present case (see fig. 2).

Recovery of the manifold of perceived orientations from the obtained discrimination times

A basic assumption of the spatial model described in the introduction is that the conformation of the manifold of internal representations of possible orientations of an object is determined by the perceived similarities of that object to itself under all rotations. Since we suppose the discrimination times obtained here to be rather direct measures of these self-similarities, analysis of these data by multidimensional scaling (e.g. see Shepard 1980) should provide a basis for reconstructing the structure of this manifold for each level of experimentally imposed rotational symmetry. We do not have data for every pair of the presented orientations since one member of the pair was always in the arbitrarily defined ‘upright’ orientation. However, if we assume as first approximation that decision time depended only on the relative orientation of the two polygons in each pair, regardless of the absolute orientation of each polygon, we have a

basis for filling in estimates of all off-diagonal entries in the 12×12 symmetric matrix specifying the perceived similarity of each of the 12 presented orientations to each of the others for any given level of experimentally imposed symmetry of the polygons.

For any given entry in the matrix for a specified level of rotational symmetry, we simply use the mean time required to indicate that the two orientations (corresponding to that row and column of the matrix) were different. Each such mean is taken over all pairs of polygons at that level of symmetry and angle of disparity (whether clockwise or counterclockwise), and over all participants in the experiment. (The diagonal cells, which would correspond to 'same orientation' responses were ignored for the purposes of this analysis.) We constructed such a matrix for each of the four levels of symmetry of the polygons from 0% to 75%. A matrix for the 100% condition could not be completed in the same way because 'different' times were necessarily unavailable for the completely symmetric polygons at their indiscriminable 180° disparities.

According to the proposed spatial model (Shepard 1981b), application of Carroll's INDSCAL scheme for 'three-way' multidimensional scaling (Carroll and Chang 1970) to such a set of four matrices should reveal the underlying four-dimensional structure in the following form: For two of the four dimensions, (a) the structure, as projected onto the plane of those dimensions, should appear as a circle with the orientations of the polygons represented in order around that circle, and (b) the estimated weights of those dimensions should be greatest for the more asymmetric conditions (0% and 25%). For the remaining two dimensions, (a) the structure, as projected onto the corresponding plane, should appear as the circle of orientations 'double-wound' so that orientations differing by supplementary angles are superimposed, and (b) the weights of those dimensions should be greatest for the most symmetric condition included in the analysis (75%).

The results of this INDSCAL analysis are displayed in fig. 4. The times to make the 'different orientation' responses correlated 0.83 with the corresponding distances between points in the obtained four-dimensional solution as calculated on the basis of the dimension weights for each level of rotational symmetry. The obtained structure has, moreover, exactly the form predicted: The simple circle of orientations (top right) emerges as the projection of the obtained structure into the plane of Dimensions 3 and 4, which together account for 24% of the total variance. And the more complex, 'double-wound' circle in which supplementary orientations coincide (top left) emerges as the projection into the plane of Dimensions 1 and 2, which together account for 44% of the total variance. (The fact that the presented projections of the INDSCAL solution are so equally spaced and circular is a consequence of our use of the same data for all pairs of orientations differing by the same angle. If we had complete data, we would expect to obtain basically the same four-dimensional structure, but with some local random perturbations.)

In addition, the plots of the associated weights, presented in the lower portion of fig. 4, reveal that the most symmetric (75%) condition received weights that were greater on both Dimensions 1 and 2 and smaller on both Dimensions 3 and 4 than the weights for any of the other three conditions. Indeed, except for the most asymmetric (0%) condition (which, as in the earlier ordering with respect to 'same' times is also a bit out of place here), the ordering of the conditions with respect to their weights is as predicted.

What these estimated weights mean is (a) that the orientations of the asymmetric polygons are internally represented as on one large circle that can be embedded in a two-dimensional plane, (b) that symmetric polygons (by virtue of their self-identity under 180° rotation) are represented as on a double-wound circle that can also be embedded in a plane, and (c) that the polygons of intermediate degrees of symmetry are represented on an intermediate, double-loop curve that requires a four-dimensional embedding space and that opens out toward the single circle or collapses down toward the double one as the symmetry of the polygon approaches 0% or 100%, respectively. For the intermediate configurations of this curve, the distances between orientations, along the curve or through the four-dimensional embedding space, depend on the angular disparity of those orientations in the manner expected on the basis of the spatial model, as illustrated in fig. 2. (The equations for the curves exhibited in that figure have been given by Shepard (1981b: 316–318.))

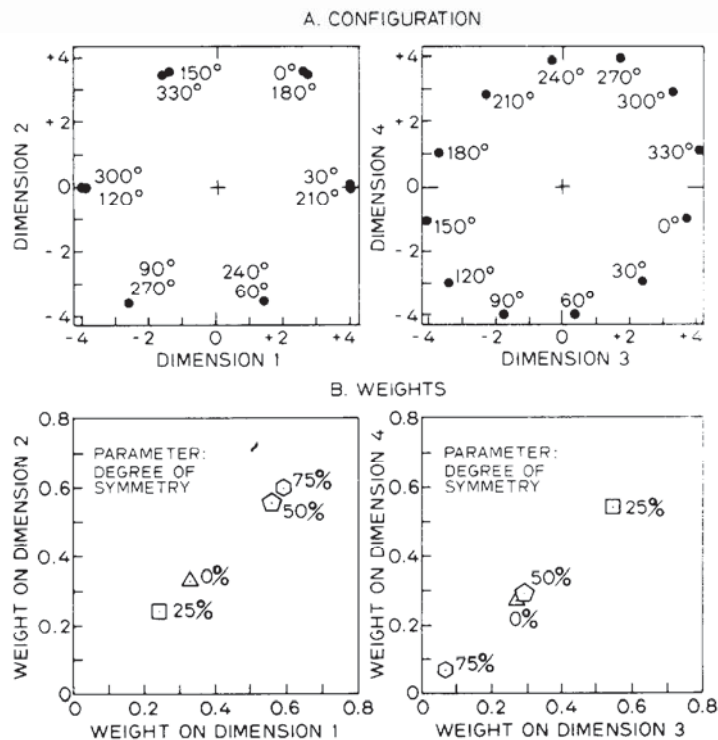


Fig. 4. Two projections of the four-dimensional solution obtained by applying INDSCAL to the discrimination times and, below, the corresponding plots showing the weights of these dimensions for the four analyzed levels of rotational symmetry.

Theoretical discussion

We regard the results reported here as furnishing support for the following ideas (earlier set forth by Shepard 1977, 1981): (a) The set of possible perceived orientations of a visual object are spatially representable on a closed manifold of specifiable form. (b) Symmetries and approximations to symmetries of the object can be accommodated by specifiable deformations of the manifold that bring previously remote points into closer proximity in a higher-dimensional embedding space. (c) Distances between points on such a deformed manifold (including both distances lying along geodesic paths within the curved manifold itself and direct or 'short-circuit' distances cutting straight through the embedding space) can account for two quite differently behaved chronometric data; namely, transformation times, which increase with distance, and discrimination times, which decrease with distance.

For the discrimination task reported here, we propose that it is the direct distances through the embedding space that determine perceived similarities and, hence discrimination times. Indeed, it is precisely the need to account for augmented self-similarities at certain angular departures that the manifold of possible orientations was deformed to bring previously remote points closer together. This could only be done by such a deformation because the geodesic paths within the manifold that correspond to rigid transformations must be left invariant in length to preserve the fact that a given rotation is still of the same magnitude regardless of the degree of perceived similarity that the resulting transformed shape has to the initial untransformed shape. Moreover, the successful application of the INDSCAL model, which fits Euclidean distances to the data, was predicted on this assumption that the discrimination times were determined by direct distances.

For the apparent motion task that we investigated earlier (Farrell and Shepard 1981), on the other hand, we propose that the geodesic distance within the constraint manifold, which corresponds to the extent of the rigid transformation, becomes the primary determiner of processing time (see, also, Shepard 1978: 53–55). Thus, although we are advancing the same spatial model to account for discrimination times and for transformation times, the two kinds of accounts differ in two ways: First, as noted in the introduction, whereas transformation times increase (often linearly) with distance, discrimination times decrease nonlinearly and asymptotically (toward a constant) with distance. Sec-

ond, as just noted, the relevant distances are different in the two cases, being geodesic distances within the curved manifold of orientations in the first case and direct distances through the embedding space in the second.

Our results do not, of course, compel the formulation of a theory in specifically geometrical terms. Moreover, we are still a long way from a satisfactory neurophysiological theory of the perception of shapes and the impletion of their rigid transformations in space. Still, whether or not the eventual theory retains the suggested geometrical terminology, it will have to provide an account for the sort of data that we have been obtaining from these discrimination and transformation tasks. To the extent that the theory is successful in doing this, it will presumably have to be in some part isomorphic to the geometrical model. On the assumption (reasonable for randomly generated shapes) that relative – not absolute – orientation is important, the results of our INDSCAL analysis (fig. 3) show that model to be in a real sense implied by our obtained discrimination times.

Finally, although we have focused here on what is a rather restricted case, namely, that of chronometric data for certain planar polygons differing in their orientations in the picture plane, we suggest that the type of spatial model put forward for this case is of potentially much greater generality. The spatial model, derived from data concerning how people compare static representations of objects, embodies a description of the possible trajectories of apparent motions of objects (Farrell and Shepard 1981). This generality suggests that the constraints governing the possible transformations in three-dimensional space have become so deeply internalized that we unconsciously represent these possible motions upon the visual presentation of any object. We believe that the model illustrates the intimate relationship between ‘seeing’ and ‘knowing’. Our seeing of objects and their motions is automatically constrained and guided by perceptual mechanisms embodying evolutionarily acquired knowledge about rigid transformations in Euclidean three-dimensional space.

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